Department of Mathematics and Systems Analysis

Rank-based information in multi-attribute decision and efficiency analysis

Antti Punkka

Aalto University

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Abstract

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Abstract

Additive multi-attribute value models are widely employed in decision and efficiency analysis. Difficulties in specifying preferences for these models have motivated the development of methods that admit incomplete preference information, identify non-dominated alternatives and provide recommendations with heuristic decision rules. These methods accommodate many types of preference statements. Yet, several studies suggest that decision makers prefer to provide rank-based information rather than numerical statements.

First, this thesis defines the notion of incomplete ordinal information, which can capture statements about the relative importance of the attributes and about the achievement levels of alternatives. The thesis then develops an optimization model for identifying non-dominated alternatives when alternatives and preferences are characterized by incomplete ordinal information and possibly by other types of incomplete information. These forms of information can, for example, help stakeholders to arrive at a joint preference characterization.

Second, the thesis shows that the recommendations of many decision rules depend on the selected normalization of value functions. Motivated partly by this, the thesis develops optimization models to determine all the rankings the alternatives attain with the model parameters that are consistent with the stated incomplete information. The resulting ranking intervals help, for example, analyze how sensitive the alternatives' rankings are to the model parameters.

Third, the thesis introduces dominance relations and ranking intervals for the efficiency analysis of decision making units when efficiency is measured through ratios of multi-attribute output and input values, as in the original data envelopment analysis method. These relations and intervals, which can be computed with the optimization models developed in the thesis, make it possible to compare any two decision making units independent of what other units are included in the analysis and to analyze how sensitive the efficiency of a unit is to the output and input attribute weights.

Keywords decision analysis, additive value function, incomplete information, ordinal information, decision recommendations, efficiency analysis, data envelopment analysis

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Tiivistelmä

Tekijä
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Väitöskirjan nimi
Järjestysperustainen informaatio monitavoitteisessa päätös- ja tehokkuusanalyysissa

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Tiivistelmä

Additiivisia arvomalleja käytetään laajasti monikriteerisessä päätös- ja tehokkuusanalyysissa. Preferenssien täsmällisen määrittelyn vaikeudesta johtuen on kehitetty menetelmiä, jotka epätäydelliseen preferenssien kuvaukseen perustuen tunnistavat ei-dominoidut vaihtoehdot ja tuottavat suosituksia heuristisilla päätössäännöillä. Nämä menetelmät hyödyntävät erityyppisiä preferenssienilmaisutapoja. Useat tutkimukset ovat kuitenkin osoittaneet päätöksentekijöiden luonnehtivan preferenssejään mieluummin järjestysperustaisesti kuin numeroin.

Väitöskirjassa esitetään epätäydellinen järjestysperäinen informaatio, jolla voidaan mallintaa kriteerien keskinäistä tärkeyttä ja vaihtoehtojen kriteerikohtaisia ominaisuuksia. Tätä varten kehitetään optimointimalli, joka laskee päätössuositukset myös, kun preferenssejä ja vaihtoehtojen ominaisuuksia luonnehditaan samanaikaisesti muillakin tavoin. Epätäydellinen järjestysperäinen informaatio voi esimerkiksi auttaa päätösongelman sidosryhmiä yhteisen preferessien kuvauksen muodostamisessa.

Väitöskirjassa näytetään, että monien päätössääntöjen suositukset voivat riippua arvomallille valitusta normeerauksesta. Tämän ongelman ratkaisemiseksi kehitetään optimointimallit, jotka ratkaisevat kaikki vaihtoehdoille epätäydellisen informaation rajoissa mahdolliset järjestysluvut. Nämä järjestyslukuvaihteluvälit auttavat muun muassa tutkimaan sitä, kuinka herkkiä järjestysluvut ovat mallin parametreille.

Väitöskirjassa sovelletaan dominanssirelaatioita ja järjestyslukuvaihteluvälejä päätöksentekoyksiköiden tehokkuusanalyyseihin, joissa tehokkuutta mitataan monikriteeristen tuotos- ja panosarvojen suhteella, kuten alkuperäisessä DEA-menetelmässä. Nämä relaatiot ja vaihteluvälit, jotka voidaan ratkaista väitöskirjassa kehitetyillä optimointimalleilla, mahdollistavat kahden yksikön muista analyysin yksiköistä riippumattoman vertailun sekä analyysit siitä, kuinka herkkiä päätöksentekoyksiköiden tehokkuudet ovat tuotos- ja panoskriteerien painokertoimille.

Avainsanat päätösanalyysi, additiivinen arvofunktio, epätäydellinen informaatio, järjestysperustainen informaatio, päätössuositukset, tehokkuusanalyysi, data envelopment analysis (DEA)

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The Dissertation consists of the present summary article and the following papers:

[I] Salo, A., Punkka, A. (2005). Rank inclusion in criteria hierarchies. *European Journal of Operational Research* **163** 338–356.

[II] Punkka, A., Salo, A. (2012). Preference programming with ordinal information. *manuscript*, 30 pages.

[III] Punkka, A., Salo, A. (2012). Ranking intervals in additive value models with incomplete preference information. *manuscript*, 36 pages.

[IV] Salo, A., Punkka, A. (2011). Ranking intervals and dominance relations for ratio-based efficiency analysis. *Management Science* **57** 200–214.

Contributions of the author

In Paper [I], Punkka developed the computational algorithm, established most mathematical developments and proofs, and performed the computations. Salo proposed the idea for the paper and is the first author of the paper.

Punkka is the first author of Paper [II]. Punkka developed the mathematical formulations based on Salo's initial idea. The proofs and the examples are developed by Punkka.

Punkka proposed the idea for Paper [III] and is the first author of the paper. The mathematical developments, proofs and examples are by Punkka.

In Paper [IV], Punkka proposed the idea of ranking intervals and the ideas behind the computational models. The computational analyses were established by Punkka. Punkka and Salo contributed equally to the mathematical developments and proofs. Salo proposed the idea for the paper and is the first author of the paper.

Preface

This thesis has been made possible by many people who I have the privilege to acknowledge.

First of all, I wish to thank my supervising professor Ahti Salo for all cooperation regarding the thesis. His guidance, ideas, feedback, expertise in the field, and uncompromising attitude towards scientific writing have greatly contributed to the thesis. It's been a pleasure to be part of a research group which is led by an internationally renowned scholar in decision analysis.

To a great extent, the research in this thesis has also benefitted from discussions and brainstorming with my long-time colleague and good friend Dr. Juuso Liesiö. In the same vein, I would like to thank my ex-colleague and good friend Pekka Mild for the atmosphere he, Juuso and I created working together in room U232 with pretty similar research topics. Every doctoral student should have people like you to work with. I also wish to thank the preliminary examiners of the thesis, Professor Luis C. Dias and Professor David L. Olson.

I wish to thank Professor Raimo P. Hämäläinen for the opportunity to work in the Systems Analysis Laboratory. In addition, I'd like to recognize his efforts as the director of the Doctoral programme in systems analysis, decision making and risk analysis and, as the coordinator of this programme, I would like to thank him for all collaboration regarding the programme. I also want to thank all the colleagues I've had at the lab. It's very motivating to work with smart people like you. Special thanks for keeping the Friday floorball tradition alive, for the somewhat legendary camper trip across South Ostrobothnia, and to Ville Brummer, Anssi Käki and Eeva Vilkkumaa for keeping room U232 such a nice place to work in.

For some, science is a way of life. For me, it's not. Therefore, I'd like to thank all my friends for giving me something else to think about during my leisure time. I also wish to thank my parents Heikki and Irmeli for encouragement and support throughout my studies. Finally, I wish to thank my spectacular family. Anne, thank you for everything, especially for not questioning, whether it makes sense to continue studies after a master's degree. Miisa and Lotta, thank you for simply existing, and maybe even understanding that most adults go to work in the weekday mornings and letting me do so in a good mood.

Espoo, October 2012

Antti Punkka

Contents

1 Introduction

Evaluating the value or efficiency of a discrete set of alternatives often involves several criteria. Many methodologies, such as the outranking methods proposed by Roy (1968), the data envelopment analysis (DEA) by Charnes et al. (1978) and the analytic hierarchy process by Saaty (1980), represent advances in such multi-criteria evaluation. Yet, multi-attribute value theory (MAVT) by Keeney and Raiffa (1976) and Dyer and Sarin (1979) is unique in that it is based on an axiomatization of preferences, which establishes a solid theoretical background for multi-criteria evaluation and decision analyses. MAVT methods have received much attention both in literature and in applications, as Corner and Kirkwood (1991), Keefer et al. (2004), and Hämäläinen (2004) note in their reviews.

Based on MAVT applications, additive value functions, in particular, are transparent and easy-to-understand models for analyzing, and deriving decision recommendations in multi-criteria decision problems (e.g., Golabi et al. 1981, Kirkwood and Sarin 1985, Mustajoki et al. 2004, Ewing et al. 2006, Mild and Salo 2009). Such *value tree analysis* makes it possible to represent the objectives and the attributes that measure how alternatives achieve these objectives as a hierarchical 'tree'. Conventionally MAVT captures the decision maker's (DM's) preferences through tradeoff statements in terms of equally preferred (hypothetical) alternatives (Keeney and Raiffa 1976), or through direct evaluation of parameter values (Edwards 1977, Von Winterfeldt and Edwards 1986). Yet, difficulties in providing such preference statements have motivated the development of methods that accommodate incomplete information about the relative importance of the attributes and, moreover, about the alternatives' achievement levels with regard to the attributes (e.g., White et al. 1982, Weber 1987).

In many methods for incomplete specification of preferences, the DM expresses preferences with numbers, such as score intervals (White et al. 1982) or intervals for attribute weight ratios (Salo and Hämäläinen 1992). Several studies, however, suggest that ordinal comparison of actual or hypothetical alternatives is more suitable for eliciting the DM's preferences, because (i) alternatives' achievement levels are often described verbally due to lack of natural measurement scale (Larichev 1992), (ii) value judgements can be easier to express in words than through numbers (Sarabando and Dias 2009), and (iii) a group of DMs attempting to obtain a joint preference representation may disagree about the numerical statements (Kirkwood and Sarin 1985) or even the appropriate measurement scales (Grushka-Cockayne et al. 2008), but they may still agree on a rank-ordering of the attributes' relative importance or the alternatives' achievements with regard to the attributes. Indeed, Larichev (1992) and Edwards and Barron (1994) argue that numerical evaluation affects negatively the reliability of the analysis. Moshkovich et al. (2002) observed in their review that discrete scales with verbal explanations are often applied even, when the attributes have a natural numerical measurement scale. They argue that this is because procedures for numerical parameter estimation are time-consuming and not necessarily well understood by the DMs. Indeed, Moshkovich et al. (2002) conclude that ordinal information is less complex and expect it to more accurately reflect the DM's preferences. This view is shared by Larichev et al. (1995) who note that attempts to solve decision tasks through more 'exact' (quantitative) judgments may lead to erroneous results, thus suggesting use of ordinal judgments. Yet, ordinal information may need further quantification so that the precision of the preference specification better matches the intensions and 'true' preferences of the DM (Sage and White 1984) and, moreover, provides decision recommendations that discriminate between the alternatives.

With an incomplete specification of preferences and alternatives, there are typically several value functions and characterizations of the alternatives' achievement levels that are consistent with the stated information. Based on combinations of parameters that correspond to these value functions and achievement levels, the non-dominated (White et al. 1982; see also Hannan 1981) and potentially optimal alternatives (e.g. Hazen 1986) can be identified and proposed as 'good' decision candidates. Further decision support can be provided by applying heuristic decision rules that recommend a single alternative. Suggested rules are based on comparing (i) the 'sizes' of the parameter sets that favor an alternative (Eiselt and Laporte 1992), (ii) the magnitudes of value differences (Park and Kim 1997, Dias and Clímaco 2000, Salo and Hämäläinen 2001, Sarabando and Dias 2009) and (iii) sums of these value differences (Ahn et al. 2000). In addition to such rules, approaches to describe the alternatives' sensitivity to the DM's preference statements have been developed (e.g., Rios Insua and French 1991, Kämpke 1996, Butler et al. 1997).

The DEA method by Charnes et al. (1978) (referred to as CCR-DEA) resembles MAVT in that it models the efficiency of *decision making units* (DMUs) by examining the ratio of additive output value and additive input value. As its primary results, CCR-DEA distinguishes between *efficient* and *inefficient* DMUs. Further results are provided by *efficiency scores* that convey information about how efficient a DMU can *at best* be, when it is compared with all output and input weights to the DMU that is the most efficient with those weights. However, this measure does not discriminate between efficient DMUs. In addition, the efficiency score is based on one combination of weights, which is typically different for each DMU and also depends on what other DMUs are included in the analysis. These features have motivated the development of, for example, cross-efficiency analysis (Sexton et al. 1986) in which the DMUs' efficiencies are evaluated by using an aggregate measure that is based on several combinations of weights.

This dissertation extends possibilities of using ordinal information in value tree analysis and efficiency analysis. Specifically, Paper [I] introduces the notion of *incomplete ordinal information* which is specified through statements that associate a set of attributes or alternatives with a set of rankings. For example, the DM can state that attributes 1 and 2 are among the three most important ones in the preference model. Paper [II] develops a model to characterize the corresponding feasible region of value function parameters so that other kinds of statements can be used to complement ordinal statements, and to provide decision recommendations in this setting.

Paper [III] shows that many proposed decision rules and concepts for multi-parameter sensitivity analysis can exhibit *rank reversals* (Belton and Gear 1983) so that changing the normalization of the additive value functions can change the recommendations of these rules and the results of the sensitivity analyses. Furthermore, Paper [III] develops a model to compute all the rankings that the alternatives can attain under incomplete preference specification and characterization of alternatives. The resulting *ranking intervals* do not depend on the selected normalization. They can be used as complementary ordinal information alongside dominance relations and they help, for example, in conducting sensitivity analyses. Paper [IV] develops the ratio-based efficiency analysis methodology, which makes it possible to use the ordinal comparison concepts of dominance and ranking intervals to compare DMUs, when their efficiency is measured through ratios of additive output and input values, as in CCR-DEA.

The rest of this summary article is structured as follows. Section 2 discusses the relevant theory and methods for value tree analysis with incomplete information and data envelopment analysis. Section 3 summarizes the contribution of this dissertation. Section 4 discusses the implications of the methodological developments of the dissertation and outlines some ideas for future research.

2 Theoretical and methodological foundations

2.1 Additive value in multi-attribute value theory

Decision problems with several objectives are generally referred to as multi-criteria decision making problems. Miettinen (1999) divides these further into two categories (see also Korhonen et al. 1992): In multi-objective optimization, the problem's feasible solutions are in general implicitly defined, whereas multi-criteria decision analysis (MCDA) deals with problems with a finite number of predefined solution candidates. Methodologies for solving MCDA problems include for example (i) the analytic hierarchy process (AHP) by Saaty (1980) (see Ishizaka and Labib 2011 for a review), (ii) the outranking methods, such as the family of ELECTRE methods (see Roy 1968 for the seminal paper in French; see Roy 1991 and Roy and Vanderpooten 1996 for reviews), and the PROMETHEE methods (Vincke and Brans 1985; see Behzadian et al. 2010 for a review) and (iii) multi-attribute utility theory (MAUT) and multi-attribute value theory (MAVT) by Keeney and Raiffa (1976).

In MAVT, alternatives are described as vectors of attribute-specific achievement levels, and the DM's preferences are captured by a relation so that '$x = (x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n) = y$' is interpreted as "$x$ is preferred or indifferent to y" (Keeney and Raiffa 1976). The aim is to form a value function V which captures this relation so that $V(x) \geq V(y)$ if and only if $x \succeq y$. If both $x \succeq y$ and $y \succeq x$ hold, then the DM is indifferent between x and y, that is, they are equally preferred. Dyer and Sarin (1979) extend MAVT by presenting requisite conditions for comparing differences in the strength of preference between *pairs* of alternatives through relation '$\succeq_d$'. This establishes *measurable* value functions so that $V(x') - V(x) \geq V(y') - V(y)$ if and only if $x \rightarrow x' \succeq_d y \rightarrow y'$, that is, "the preference difference for x' over x is greater than or equal to the preference difference for y' over y". Measurable value functions are unique up to positive affine transformations. Hence, the rank-orderings of values and value differences do not depend on how the value function is normalized.

The form of a value function depends on the DM's preferences. If the requisite conditions – most notably mutual preference independence (Keeney and Raiffa 1976) and difference independence (Dyer and Sarin 1979) – hold, then the DM's preferences can be captured by a measurable additive value function $V(x) = \sum_{i=1}^{n} v_i(x_i)$, in which v_i is the attribute-specific value function for the i-th attribute. The additive value function is often represented in the normalized form $V^N(x) = \sum_{i=1}^{n} w_i v_i^N(x_i)$, in which positive *attribute weights* w_i reflect the value differences between predefined achievement levels x_i° and x_i^*, and v_i^N are normalized so that $v_i^N(x_i^\circ) = 0$ and $v_i^N(x_i^*) = 1$.

2.2 Preference elicitation and incomplete information

The assumptions of the additive value representation make it possible to elicit attribute-specific value functions independently of each other; for elicitation methods, see Keeney and Raiffa (1976) and Von Winterfeldt and Edwards (1986). The elicitation of attribute weights can be carried out by constructing pairs of equally preferred alternatives. These statements imply trade-offs between the attributes (Keeney and Raiffa 1976). Technically, these statements lead to a system of linear equalities from which weight ratios w_i/w_j can be solved. Methods that elicit weight ratios directly have also been proposed, for example the SMART method by Edwards (1977) and the subsequent SMARTS method by Edwards and Barron (1994), and the SWING method by Von Winterfeldt and Edwards (1986). The weights can be normalized to sum up to one, for example, to come up with numerical values for the weights.

Yet, complete specification of the value function parameters can be time-consuming (White et al. 1982) or require knowledge that is not available (Weber 1987). The DM may also be unable or unwilling to provide precise trade-off statements that are required for such a complete specification (Hazen 1986) or he may feel uncomfortable with giving them (Sage and White 1984). Complete specification can even be unneccessary, if less information would lead to an unequivocal decision recommendation. These reasons, among others, have motivated the development of methods that derive decision recommendations based on incomplete characterization of preferences and alternatives (e.g., White et al. 1982, 1983, 1984, Kirkwood and Sarin 1985, Weber 1985, Hazen 1986, Salo and Hämäläinen 1992, 2001; for reviews, see Weber 1987, Salo and Hämäläinen 2010).

Most of the methods for dealing with incomplete information build on two assumptions. First, the DM is able provide complete trade-off statements between alternatives that differ along a single attribute (Hazen 1986), that is, specify the attribute-specific value functions. Second, the DM is able to provide preference information about the relative importance of the attributes through statements $x \succeq y$, in which x and y are different with regard to two attributes (e.g., White et al. 1984, Kirkwood and Sarin 1985, Weber 1985, Hazen 1986, Pearman 1993, Malakooti 2000, Salo and Hämäläinen 2001). Also methods that employ incomplete weight or other value difference ratios (such as $v_2(x_2^*)-v_2(x_2^\circ) \leq [v_1(x_1^*)-v_1(x_1^\circ)] \leq 2[v_2(x_2^*)-v_2(x_2^\circ)]$; e.g., Salo and Hämäläinen 1992, 2001, Mustajoki et al. 2005), and methods that admit any kind of linear constraints on the weights (e.g., $0.4 \leq w_1 \leq 0.6$) or allow the DM to adjust the alternatives' achievement levels so that one becomes preferred to the other (e.g., Sage and White 1984, Park et al. 1996, Malakooti 2000) have been developed.

Incomplete information about the alternatives leads to constraints for feasible characterizations of the alternatives. Theoretically, such information corresponds to incompletely characterized achievement levels, for example, through intervals ($10 \leq x_1 \leq 15$; e.g., Sage and White 1984, Weber 1985, Salo and Hämäläinen 1992), direct evaluation of the alternatives' normalized attribute-specific values through intervals ($0.15 \leq v_1^N(x_1) \leq 0.2$; e.g., White et al. 1982), or ordinal pairwise comparisons of alternatives' attribute-specific values ($v_1(x_1) \geq v_1(y_1)$; e.g., Salo and Hämäläinen 2001).

Some methods elicit mostly ordinal information about the DM's preferences and alternatives. For example, the ZAPROS-LM method by Larichev and Moshkovich (1995) captures attribute-specific preferences by eliciting a ranking of a finite number of possible achievement levels, and admits preference information about the relative importance of the attributes through ordinal comparisons of hypothetical alternatives. The method of Kirkwood and Sarin (1985) admits a rank-ordering of hypothetical alternatives, whose overall values correspond to attribute weights (e.g., $w_1 \geq w_2 \geq \ldots \geq w_n$). In the ordered metric method of Pearman (1993), the DM ranks differences between these weights, too (e.g., $w_1 - w_2 \geq w_3 - w_4 \geq \ldots$). Park et al. (1996) extend this model to evaluation of alternatives. The models by Cook and Kress (1996, 2002) complement ordinal information by discrimination factors between attribute weights and alternatives' normalized attribute-specific values (e.g., $w_1 \geq w_2 + 0.02$).

2.3 Decision recommendations under incomplete information

White et al. (1982) propose that alternatives should be compared based on (pairwise) *dominance* so that an alternative dominates another if all its feasible characterizations are preferred to all of those of the latter one, with all value functions that are consistent with the preference information (see also Hannan 1981, Kirkwood and Sarin 1985, Hazen 1986, Salo and Hämäläinen 1992). Because the dominance relation is irreflexive, asymmetrical, and transitive (e.g., Weber 1987), the dominance relations among the alternatives under analysis can shown as a domination digraph (White et al. 1982).

Mathematically, incomplete information leads to linear inequalities on the attribute weights w_i, and the alternatives' normalized attribute-specific values $v_i^N(x_i)$ and defines a convex *feasible region* of these model parameters. Based on this idea of *set inclusion* (White et al. 1982), the feasible region includes the parameters that correspond to the DM's 'true' value function and alternatives' true achievement levels. The dominance relations can be solved by examining the alternatives' minimum and maximum value difference over the feasible region. Several algorithms for computing dominance relations have been developed. Especially the early ones are based on enumerating the extreme points of the feasible region (e.g., Kirkwood and Sarin 1985, Hazen 1986, Carrizosa et al. 1995, Cook and Kress 2002, Mustajoki and Hämäläinen 2005), but due to the recent growth in computational power the emphasis has shifted towards formulations of linear programs (LPs; see e.g., Ahn et al. 2000, Salo and Hämäläinen 2001, Kim and Han 2000, Park 2004).

With incomplete information, there can be several *non-dominated* alternatives. White et al. (1982, 1984) show that with the specification additional statements, there are fewer value functions or characterizations of alternatives which are compatible with the statements, and that this, in turn, can lead to fewer non-dominated alternatives. Liesiö et al. (2007) present conditions under which the set of non-dominated alternatives cannot be enlarged as a result of additional information. Some interactive preference elicitation methods – such as the PAIRS method by Salo and Hämäläinen (1992) – provide guidance to the DM in keeping new preference statements consistent with earlier ones. Some methods even suggest preference statements that could efficiently reduce the set of nondominated alternatives (Mustajoki and Hämäläinen 2005).

Potentially optimal alternatives, too, have been proposed as good candidates (see e.g., Hazen 1986, Weber 1987, Rios Insua and French 1991). For these alternatives there exists a feasible characterization of alternatives so that they have the highest value for some value function that is consistent with the DM's preference statements. LPs can be used to solve the potentially optimal alternatives (e.g., Hazen 1986, Rios Insua and French 1991, Lee et al. 2001, 2002, Park 2004).

In addition to identification of non-dominated and potentially optimal alternatives, determination of alternatives' rankings over the feasible region have been proposed. The model of Kämpke (1996) solves rank variability for a set of alternatives, when preferences are captured through holistic comparisons among these alternatives. Butler et al. (1997) simulate random value functions to explore the robustness of the alternatives' rankings. The flexible ranking approach by Köksalan et al. (2010) first estimates precise achievement levels for the alternatives and then determines the most favorable rankings for them, when attribute weights are constrained by linear inequalities.

To support the selection of a single (non-dominated or potentially optimal) alternative, heuristic decision rules and 'tighter' dominance concepts have been proposed. These rules include the *domain criterion* by Eiselt and Laporte (1992) (cf. *acceptability index* of Lahdelma et al. 1998), *weak dominance* by Park and Kim (1997) (equal to *minimax regret* rule by Salo and Hämäläinen 2001), *quasi-dominance* by Dias and Clímaco (2000) and related *quasi-optimality* and *quasi-dominance* rules by Sarabando and Dias (2009), and *maximax*, *maximin*, and *central values* rules by Salo and Hämäläinen (2001). Moreover, following the ideas of outranking methods, Ahn et al. (2000) propose the *net dominance value* to be used as a measure for a decision rule. Sarabando and Dias (2009) provide a comparison of such decision rules. In a related stream of proposed decision rules, heuristics have been developed to obtain 'representative attribute weights' from the feasible region, based on which the alternatives are then compared; see Stillwell et al. (1981) and Barron and Barrett (1996) for comparisons of such methods.

2.4 Ratio-based data envelopment analysis

The seminal work of Charnes et al. (1978) has preceded the development of a variety of *data envelopment analysis* (DEA) methods to compare decision making units (DMUs) that differ in the amounts of outputs they produce, and the amounts of inputs they use to produce the outputs. The original CCR-DEA method proposed by Charnes et al. (1978) models the efficiency of a DMU by its *efficiency ratio*, the ratio of additive virtual output value and additive virtual input value. It thus assumes constant returns to scale; DEA methods that assume variable returns to scale have been developed by Banker et al. (1984) and Charnes et al. (1985).

The CCR-DEA method is non-parametric in the sense that it identifies *efficient* (potentially optimal in MAVT literature) and *inefficient* DMUs based on the output and input data of the DMUs that are included the analysis. Yet, several models accommodate preference information through weight constraints (i) to provide results which are not based on weights that reflect too large a compensation of one output (or input) over another output (input) (Thompson et al. 1986), and (ii) to add discrimination among the DMUs by obtaining fewer efficient DMUs (e.g., Adler et al. 2002). In their review, Allen et al. (1997) distinguish between (i) assurance regions type I (Thompson et al. 1986), which are constraints on the relative values among different outputs or inputs, (ii) assurance regions type II, which apply constraints also between outputs and inputs (Thompson et al. 1990, Khalili et al. 2010), and (iii) absolute weight restrictions (Dyson and Thanassoulis 1988).

Technically, such preference information imposes linear constraints on the output and input weights, and thus resembles incomplete preference specification for additive value functions. Cooper et al. (1999, 2001) develop models that allow use of intervals in describing the DMUs' inputs and outputs. Other similarities between DEA and MCDA or MCDM have been discussed by several authors (e.g., Doyle and Green 1993, Stewart 1996, Athanassopoulos and Podinovski 1997, Joro et al. 1998). These observations have underpinned the development of methods that compare DMUs with the help of value functions, for example (e.g., Halme et al. 1999, Gouveia et al. 2008, de Almeida and Dias 2012).

In conventional CCR-DEA, the DMUs' efficiencies are characterized by evaluating them with the output and input weights that are most favorable to them, in the sense that their efficiency ratio divided by that of the most efficient DMU is maximized over the set output and input weights. As a result, the efficient DMUs are assigned an efficiency score of one, and inefficient DMUs' efficiency scores are between zero and one. The conventional DEA concepts thus do not discriminate among the efficient DMUs. According to Adler et al. 2002, this has partly motivated the development of models and efficiency measures that provide a full ranking for the DMUs. Of these, for example *super-efficiencies* indicate how much more efficient a DMU can be than the most efficient of other DMUs (Andersen and Petersen 1993). *Benchmark ranking* by Torgersen et al. (1996) is based on the extent to which a DMU affects other DMUs' efficiency scores. *Cross-efficiency* analysis by Sexton et al. (1986) differs from other concepts in that it evaluates DMUs' efficiency ratios with *several* combinations of output and input weights, and uses the average of these ratios in comparing the DMUs (see also Doyle and Green 1994). Cross-efficiency analysis indeed differs from the other above concepts in that it employs several different weights to evaluate the efficiency of a DMU. However, these weights are determined based on which specific DMUs are included in the analysis.

3 Results

3.1 Incomplete ordinal information in preference modeling

Paper [I] introduces the notion of incomplete ordinal information for capturing preference information. This information is obtained through paired statements of attributes and rankings; for example, the DM can state that attributes *cost* and *environmental aspects* are among the three most important attributes; or that either *cost* or *environmental aspects* is the most important attribute. The paper shows how the feasible region of attribute weights can be reduced by revising the provided preference statements. It also presents conditions under which this feasible region is non-convex. To compute decision recommendations over a non-convex feasible region, Paper [I] develops an algorithm to enumerate those attribute weights whose convex hull is equal to that of the feasible region, and shows how dominance relations can be determined by computing the alternatives' value differences at these points. This computational algorithm can be applied also in presence of common, absolute lower bounds for the attribute weights, and when alternatives' achievement levels are specified through intervals.

Paper [II] develops a computational model which makes it possible to give incomplete ordinal preference statements *also* about the *alternatives' performance* with regard to any set of attributes. For example, the DM can state that alternative A is among the two most preferred ones with regard to *environmental aspects*; or that either alternative A or B is the most preferred one in view of attributes *cost* and *environmental aspects* together. The corresponding feasible region is modeled with a set of linear constraints on the model parameters and auxiliary binary variables. This makes it possible to complement incomplete ordinal statements by any incomplete cardinal preference statements which correspond linear constraints on the model parameters. As a result, it is possible to admit incompletely specified attribute weight ratios or ordinal comparisons between alternatives' achievement levels in preference specification, for example. The number of binary variables employed in the mixed integer linear programs (MILPs) developed for solving decision recommendations depends on the given preference statements. For example, if the feasible region is convex, the optimization problems simplify from MILPs to LPs.

3.2 Rank-based results for value trees and CCR-DEA based efficiency analysis

Paper [III] focuses on ordinal results of value tree analysis under incomplete information. First, it shows that recommendations of some comparison concepts and decision rules that compare preference differences across value functions that describe different preferences (e.g., Eiselt and Laporte 1992, Park and Kim 1997, Dias and Clímaco 2000, Ahn et al. 2000, Salo and Hämäläinen 2001, Sarabando and Dias 2009) as well as sensitivity analysis results based on the size of the feasible region or distances within it (Rios Insua and French 1991, Lahdelma et al. 1998) can depend on how the additive value functions are normalized. These recommendations and results can thus exhibit rank reversals (Belton and Gear 1983) in the sense that changing the normalization of the value functions can change the relative ranking of two non-dominated alternatives. Furthermore, for maximax, maximin and weak dominance decision rules Paper [III] presents sufficient conditions, under which the normalization can always be selected so that a non-dominated alternative is favored over another.

Second, as a partial solution to this problematic phenomenon, Paper [III] develops MILPs for computing all rankings that the alternatives can attain over a convex feasible region of all those model parameters that correspond to the DM's incompletely specified preferences and incompletely characterized alternatives. Like dominance relations, the resulting *ranking intervals* do not depend on the selected normalization of the value functions.

Paper [IV] develops the *Ratio-based Efficiency Analysis* (REA) methodology, which follows the CCR-DEA method in that it models DMUs' efficiencies with their efficiency ratios. It differs from earlier methods in that it derives results based on, and for *all* feasible output and input weights, which fulfill possible statements about the relative values of different inputs and outputs in terms of assurance regions type I statements. REA extends conventional efficiency scores by developing LPs to compute *efficiency bounds*, which communicate how efficient a DMU can be related to a benchmark group of DMUs, for *all* feasible output and input weights. In addition to this generalization, REA adopts the ordinal comparison concepts of dominance and ranking intervals from the MCDA literature and develops MILPs and LPs for computing:

- What *rankings* can a DMU attain in comparison with other DMUs, based on the comparison of their efficiency ratios for *all* feasible output and input weights?
- Does a DMU *dominate* another DMU in the sense that its efficiency ratio is higher than or equal to that of the other for *all* feasible output and input weights?

The results provided by the REA methodology coincide with some well-known results of CCR-DEA-based methods as special cases. Specifically, (i) the best ranking of an efficient DMU is one, and, conversely, a DMU whose best ranking is one has efficiency score of one, (ii) if all DMUs are in the benchmark group, the upper efficiency bound of a DMU is equal to its efficiency score, (iii) if all other DMUs are in the benchmark group, the upper efficiency bound of a DMU is equal to its super-efficiency. The REA results offer new possibilities to set performance targets for the DMUs. For example, Paper [IV] develops MILPs to compute the smallest radial improvement in outputs required for a DMU to improve its best or worst ranking to some target ranking.

4 Discussion

Incomplete ordinal preference information has been used in modeling the relative importance of attributes in many applications, for example by Ojanen et al. (2005), Salo and Liesiö (2006), Mild and Salo (2009), and Mild (2006) (in Finnish; a very similar case study is found in Liesiö et al. 2007). In all of the above applications, the preference information has represented the preferences of a group of DMs (or, stakeholders). Indeed, incomplete ordinal information makes it possible to construct preference statements even from group members who disagree. For example, each DM can be asked to specify the two most preferred alternatives with regard to an attribute, after which the group's preferences are expressed by a statement that the two most preferred alternatives are among the ones specified by the group members.

If the DMs cannot agree on the attributes' numerical measurement scales, or if natural scales do not exist, one way to describe preferences between the alternatives is to divide them into classes for which numerical values – perhaps together with verbal expressions describing preferences between these classes – are assigned (e.g., Salo and Liesiö 2006, Könnölä et al. 2007). Such preference information ranks the classes, but the fixed numerical values do not necessarily reflect strength of preference between the classes. In addition, the DMs may be prepared to provide additional preference statements between the alternatives in the same class for example through pairwise comparisons. Incomplete ordinal information helps model such classification as ordinal information, yet making it possible to define bounds for the values associated with the classes and to constrain value differences between the classes. This way, incomplete ordinal information can be used to perform *ex ante* sensitivity analysis on the values associated with the classes, and to allow alternatives in the same class to differ in values. Such possibilities for preference elicitation can be particularly beneficial in large problems with dozens alternatives in which data is available for only some attributes. In these settings, the available data together with incomplete ordinal information with regard to the other attributes can be sufficient to establish dominance relations that reduce the set of non-dominated alternatives. This, in turn, can lead to resource savings as fewer alternatives need be thoroughly evaluated.

Ranking intervals are suitable for this kind of *screening* of alternatives, especially if the aim is to choose several alternatives (referred to as 'pick k out of n' by Stillwell et al. 1981). Indeed, Butler et al. (1997) note that multi-criteria analysis is often performed in order to select a subset of alternatives, and they suggest that the ranking intervals should be examined to get insights about the robustness of the alternatives' rankings. Specifically, the ranking intervals identify which alternatives are among the K most preferred ones (i) for all, (ii) for some, and (iii) for no combinations of feasible parameters. These results are obtained simultaneously for all 'budgets' K, thus making it possible to analyze how decision recommendations change as a function of the budget. The above categorization is closely connected to recent advances in multi-criteria *portfolio* decision analysis (Salo et al. 2011). More precisely – following the terminology of the robust portfolio modeling (Liesiö et al. 2007, 2008) – if feasible portfolios are characterized only by the number of alternatives they include, the ranking intervals identify core, borderline and exterior alternatives among all potentially optimal portfolios. Paper [III] illustrates this connection by revisiting an application by Könnölä et al. (2007).

Some fifteen years ago, Butler et al. (1997) noted that exploration of all feasible parameter combinations to compute ranking intervals would be "extremely tedious". In this regard, the MILPs developed in Paper [III] are computationally effective as they can compute the ranking intervals among hundreds of alternatives, as shown in the sensitivity analysis of university rankings in Paper [III]. The use of ranking intervals as a tool for multi-parameter sensitivity analysis is supported by the observation in Paper [III] that ranking intervals do not depend on the selected normalization of the value functions, unlike many other results. From the perspective of decision support, practitioners can be given a holistic view through these intervals, independently of the number of attributes.

The ranking intervals and efficiency bounds are novel concepts in CCR-DEA based efficiency analysis in that in addition to communicating how 'good' a DMU can be at best, they also provide information about how 'bad' it can be at worst. They can be used to compare efficient DMUs, unlike conventional efficiency scores, for example. More specifically, they can help identify (i) those efficient DMUs, which perform 'well' compared to other DMUs across the entire set of feasible weights, and (ii) those inefficient DMUs, which do not perform 'extremely badly' compared to other DMUs for any feasible weights. On the other hand, the results can help identify the ones, whose relative efficiency varies 'much' in the set of feasible weights. This may help identify the outputs and inputs that should be bettered in order to improve the worst possible ranking, for example.

Many efficiency measures, such as efficiency scores, cross efficiencies, and super-efficiencies, are computed relative to the other DMUs included in the analysis. These measures discriminate between the efficiencies of the DMUs only on the condition that the number of DMUs is large enough compared to the number of outputs and inputs (Cooper et al. 2000). Furthermore, they can exhibit rank reversals, if the set of DMUs included in the analysis is manipulated. These concerns do not apply to dominance relations which compare pairs of DMUs independently of any other DMUs

Although the proposed concepts for ratio-based efficiency analysis are new, efficiency scores, super-efficiencies, and division into efficient and inefficient DMUs are obtained as special cases of the new results. They are also intuitive in that additional preference information in terms of new weight constraints (i) keeps previous dominance relations intact, but can establish new ones, (ii) does not widen the ranking intervals or the intervals bound by the efficiency bounds, but can make them narrower. These appealing features together with the relations to earlier efficiency measures can catalyze the adoption of the REA methodology by researchers and practitioners.

The thesis suggests some future research directions. First, preference elicitation procedures that accommodate incomplete ordinal information should be designed and tested. These procedures should give the DMs the possibility to express their preferences with the accuracy they feel confident with, but deploy also more discriminative numerical information to obtain decision recommendations. One possibility could be to extend the classification procedure discussed in Section 4 so that it would admit incomplete assignments; for example, when evaluating research proposals, a proposal's attribute-specific performance could be evaluated to belong to either class 'excellent' or to class 'very good'.

Second, the REA methodology could be extended to admit interval-valued data about the DMUs (Cooper et al. 1999, 2001). Furthermore, some of the proposed results for REA could be applied to DEA models with other returns-to-scale assumptions, such as the BCC model by Banker et al. (1984).

Third, the observation that comparing value differences' magnitudes across value functions that describe *different* preferences can result in rank reversals has implications outside the scope of this thesis. For example, many simulation studies have used the average loss of value (or, utility) – which is effectively a sum of value differences over different value functions – to evaluate the quality of decision recommendations in comparing (i) attribute weight (Barron and Barrett 1996) and multi-attribute utility function approximations (Durbach and Stewart 2012) and (ii) multi-attribute value function elicitation procedures that are based on incomplete preference information (Salo and Hämäläinen 2001, Paper [I], Mustajoki et al. 2005). Furthermore, in the context of resource allocation, Liesiö et al. (2008) suggest that budgeting decisions could be based on the minimum value of the portfolio suggested by the maximin decision rule over different budgets. Thus, one research question raised by this thesis is how – if at all – should strengths of preferences between different value functions be measured?

And, subsequently, if such a measure were to be found, can it be used (i) to evaluate the robustness of the alternatives, (ii) to act as a basis for decision rules, and (iii) to characterize incompleteness of preference specification? Or is rank-based information all there is, when we are comparing alternatives across different value functions?

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Decision Aiding

Rank inclusion in criteria hierarchies

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Abstract

This paper presents a method called Rank Inclusion in Criteria Hierarchies (RICH) for the analysis of incomplete preference information in hierarchical weighting models. In RICH, the decision maker is allowed to specify subsets of attributes which contain the most important attribute or, more generally, to associate a set of rankings with a given set of attributes. Such preference statements lead to possibly non-convex sets of feasible attribute weights, allowing decision recommendations to be obtained through the computation of dominance relations and decision rules. An illustrative example on the selection of a subcontractor is presented, and the computational properties of RICH are considered.

Keywords: Multiple criteria analysis; Decision analysis; Hierarchical weighting models; Incomplete preference information

1. Introduction

Methods of multiple criteria decision making (MCDM) are widely employed in problems characterized by incommensurate objectives. Numerous successful MCDM applications have been developed in fields such as energy policy, environmental decision making and comparison of industrial investment opportunities (see, e.g., Corner and Kirkwood, 1991; Hämäläinen, 2004; Keefer et al., 2004). In MCDM applications, the decision problem is structured by associating measurable attributes with the objectives that are relevant to the decision maker (DM). In most methods—such as the Analytic Hierarchy Process (AHP; Saaty, 1980) and value tree analysis (Keeney and Raiffa, 1976)—the DM is also requested to supply weights as a measure for the relative importance of attributes.

In practice, the elicitation of precisely specified attribute weights may be difficult. This may be due to the urgency of the decision, lack of resources for completing the elicitation process, or conceptual difficulties in the interpretation of intangible objectives (see, e.g., Weber, 1987). In group settings, difficulties in determining attribute weights for the group's joint preference model may arise from differences in the group members' level of knowledge or their interpretation of what the relevant objectives mean (Hämäläinen et al., 1992).

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However, complete information about attribute weights is not always necessary in order to produce a decision recommendation. Together with the difficulties of producing a complete model specification, this realization has motivated the development of methods for dealing with incomplete information in hierarchical weighting models (see, e.g., Kirkwood and Sarin, 1985; Hazen, 1986; Weber, 1987; Salo and Hämäläinen, 1992; Salo, 1995; Kim and Han, 2000). Even though these methods differ in their details, they all (i) accommodate incomplete information about attribute weights and possibly other model parameters as well and (ii) provide more or less conclusive dominance relations concerning which alternatives are preferred to others.

In this paper, we extend the earlier literature on incomplete preference information by allowing the DM to specify subsets of attributes which contain the most important attribute or, more generally, to associate several rankings with a given set of attributes. Resulting method—called Rank Inclusion in Criteria Hierarchies (RICH)—generalizes the use of ordinal preference information in attribute weighting. In view of our theoretical and computational results, we believe that the RICH method is especially suitable for decision contexts where only rather few and easily elicited preference statements can be obtained before preliminary decision recommendations must be produced. Also, inspired by positive experiences from the deployment of internet-based decision aiding tools (e.g., Web-HIPRE; see Mustajoki and Hämäläinen, 2000; Lindstedt et al., 2001), we have already proceeded with the development of a user-friendly decision support tool for the RICH method. This tool—entitled *RICH Decisions*—is available free of charge to academic users (see `http://www.decisionarium.hut.fi`; Liesiö, 2002).

The remainder of this paper is structured as follows. Section 2 reviews earlier approaches to the analysis of incomplete information in hierarchical weighting models. Section 3 considers the use of incomplete ordinal information in the elicitation of attribute weights and the properties of resulting feasible weight regions. Section 4 presents a measure for the size of feasible regions, and Section 5 discusses the development of decision recommendations. Section 6 summarizes results from a simulation study on the computational properties of RICH. An illustrative example is given in Section 7, followed by concluding remarks in Section 8.

2. Earlier approaches to the analysis of incomplete information

In an early contribution on the modeling of incomplete information, Arbel (1989) discusses how the precise articulation of preferences through ratio statements can be extended to capture incomplete information about the relative importance of attributes. He models incomplete preference information through lower and upper bounds on the relative importance of attributes. These bounds correspond to linear constraints of linear programming (LP) problems from which the lower and upper bounds on the weight of each attribute can be obtained.

The PAIRS method (Preference Assessment by Imprecise Ratio Statements; Salo and Hämäläinen, 1992) extends Arbel's concepts to attribute hierarchies in which lower and upper bounds on the relative importance of attributes define a region of feasible weights at each higher-level attribute. Combined with possibly incomplete score information, such ratio-based information is processed by solving a series of hierarchically structured LP problems, in order to obtain bounds on the alternatives' overall values. The decision recommendations are based on the (*pairwise*) *dominance criterion* according to which alternative x_i is preferred to x_j if the overall value of x_i is higher than that of x_j, no matter how the weights are chosen from the feasible regions. If the available preference information does not lead to sufficiently conclusive dominance relations, the DM is requested to supply additional preference statements. PAIRS supports the consistency of the preference model through so-called *consistency bounds* which are presented to the DM before the elicitation of each new preference statement.

Analogous to PAIRS in many ways, the preference programming approach of Salo and Hämäläinen (1995) provides an *ambiguity index* which measures the incompleteness of a preference model. Salo (1995) extends the preference programming approach to group decision settings where several decision makers can supply incomplete preference information about (i) how the alternatives perform on the lowest-level attributes and (ii) how important the attributes are to the different DMs. These statements lead to linear constraints so that value intervals and dominance relations for the alternatives can be computed from LP problems. The potential of this approach has been explored in a study on traffic planning by Hämäläinen and Pöyhönen (1996), for instance.

The PRIME method (Preference Ratios in Multi-Attribute Evaluation; Salo and Hämäläinen, 2001) allows the DM to provide preference statements through holistic comparisons between alternatives, ordinal strength of preference judgments or ratios of value differences. Like PAIRS, PRIME provides information about the consistency of the DM's preference statements and dominance relations. Full support for PRIME is provided by the decision support tool *PRIME Decisions* which is available at `http://www.decisionarium.hut.fi`. PRIME Decisions employs value intervals and dominance structures to show intermediate results to the DM. It has been applied to the valuation of a high-technology firm, among others (Gustafsson et al., 2001).

Park and Kim (1997) give an extensive taxonomy of alternative ways to the elicitation of incomplete preference information in hierarchical weighting models. In particular, they distinguish between the following statements:

1. weak ranking: $\{w_i \geqslant w_j\}$,
2. strict ranking: $\{w_i - w_j \geqslant \alpha_i\}$,
3. ranking with multiples: $\{w_i \geqslant \alpha_i w_j\}$,
4. interval form: $\{\alpha_i \leqslant w_i \leqslant \alpha_i + \epsilon_i\}$,
5. ranking of differences: $\{w_i - w_j \geqslant w_k - w_l\}$ for $j \neq k \neq l$,

where $\alpha_i, \epsilon_i \geqslant 0\ \forall i$. Furthermore, they consider more general multi-criteria problems with incomplete probabilities, utilities and attribute weights. Although these problems may involve non-convex objective functions, approximate or even exact solutions can often be obtained by solving a series of LP problems.

Mármol et al. (1998) present an algorithm for computing the extreme points of the region of feasible attribute weights in two highly relevant cases (i.e., linear inequalities and weight intervals). They also examine the computational properties of their algorithm and establish conditions for introducing further linear relations which preserve the structure of the feasible region. A similar approach is taken by Puerto et al. (2000) who utilize the extreme points of the set of feasible weights in the implementation of three decision criteria (i.e., Laplace criterion, Wald's optimistic/pessimistic criterion, Hurwicz criterion).

Kim and Han (2000) extend the methods of Park and Kim (1997) to hierarchically structured attribute trees. In their model, the DM can place several kinds of linear constraints at any level of the attribute tree. These constraints are processed by an algorithm which can be invoked to obtain upper and lower bounds for the value of an alternative with regard to any attribute, subject to the assumption that the DM's preference statements remain consistent.

Fig. 1 presents a schematic diagram on the consecutive phases of the RICH method. In effect, this method is analogous to many others (e.g., PRIME; Salo and Hämäläinen, 2001) in that the DM can (i) interactively introduce new preference statements or revise earlier ones, and (ii) obtain tentative decision recommendations and information about the completeness of the currently available preference information. The key difference lies in the elicitation of attribute weights which are in the RICH method characterized through incomplete ordinal preference statements. At any phase of the process, results on (i) the alternatives' possible overall values, (ii) (pairwise) dominance structure of the alternatives, (iii) decision recommendations and (iv) information about the possible rankings of the attributes can be obtained from LP problems. After examining these results, the DM may either choose to accept one of the decision recommendations or continue with the specification of further preference information.

Except for the work of Park and Kim (1997)—in which combinations of incompletely specified

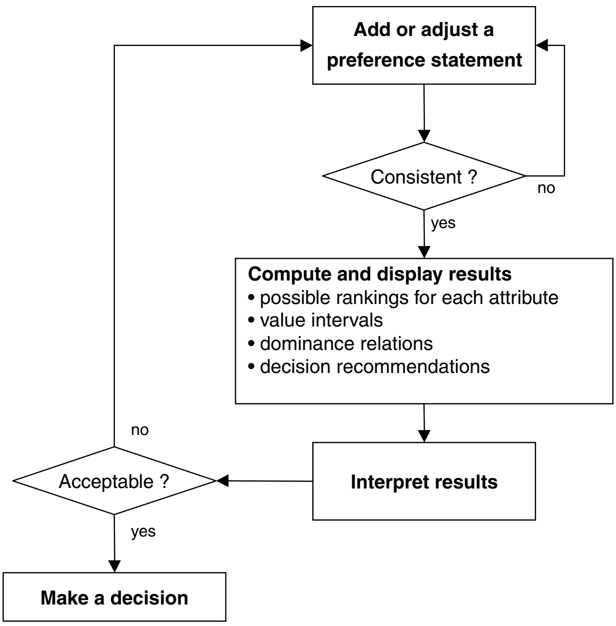

Fig. 1. Phases in the RICH method.

weights, probabilities and utilities are considered—a common feature of all earlier approaches is that the region of feasible attribute weights is convex and bounded by linear constraints. As we next move to the consideration of incomplete ordinal preference information, there is a significant difference in that the resulting feasible region may not be convex.

3. Formalization of incomplete ordinal information

Let $A = \{a_1, \dots, a_n\}$ be the set of relevant attributes in the decision problem. The importance of attribute a_i is measured by its *weight* $w_i \in [0, 1]$. By convention, the attribute weights are normalized so that they add up to one, i.e., $\sum_{i=1}^{n} w_i = 1$. *Alternatives* are denoted by x_j, $j = 1, \dots, m$. The performance of the jth alternative with regard to attribute a_i is measured by its *score* $v_i(x_j) \in [0, 1]$. The *overall value* of alternative x_j is given by $V(x_j) = \sum_{i=1}^{n} w_i v_i(x_j)$.

3.1. Weak orders, linear orders and rankings

Following several other approaches, we assume that the DM makes statements about the relative importance of attributes. These preferences are captured through a relation $\succeq$ on the set $A \times A$, in the understanding that $a_i \succeq a_j$ if and only if attribute a_i is at least as important as attribute a_j.

The relation is a *weak order* if it is comparable (i.e., $\forall a_i, a_j \in A$ either $a_i \succeq a_j$ or $a_j \succeq a_i$, or both) and transitive (i.e., if $a_i \succeq a_j$ and $a_j \succeq a_k$, then $a_i \succeq a_k$). If this relation is also antisymmetric (i.e., $\nexists a_i, a_j \in A, a_i \neq a_j$ such that $a_i \succeq a_j$ and $a_j \succeq a_i$), it is a *linear order*. In this case, each attribute $a \in A$ can be assigned a unique ranking $r(a) \in N = \{1, \dots, n\}$ such that $a_i \succeq a_j$ if and only if $r(a_i) < r(a_j)$. Thus, the ranking of the most important attribute is one, that of the second most important is two, and so on, until the least important attribute is reached, the ranking of which is n.

If $\succeq$ is a weak order, there is a possibility that two or more attributes are equally important. In this case, the attributes A can be partitioned into sets $A(1),\ldots,A(k)$ such that (i) $a_i \succeq a_j$, $a_j \succeq a_i$ if attributes a_i, a_j are in the same subset (i.e., $\exists A(l)$ such that $a_i, a_j \in A(l)$) and (ii) $a_i \succeq a_j$, $a_j \not\succeq a_i$ for any $a_i \in A(l)$, $a_j \in A(l+1)$. Nevertheless, the attributes can still be given rankings in $r(a_i)$, $r(a_j)$ such that $a_i \succeq a_j$ whenever $r(a_i) < r(a_j)$; but these rankings are not necessarily unique because permuting the rankings of attributes which belong to the same partition would lead to different rankings which still fulfil the above condition. Whatever the case, the ranking $r(a)$ implies that $r(a) - 1$ attributes are at least as important as the attribute a.

Formally, a rank-ordering r is a function from the set of attributes $A = \{a_1,\ldots,a_n\}$ onto the set N. The set of all possible rank-orderings r is denoted by R. Because each rank-ordering r is a bijection, the attribute with the ranking k is given by the inverse function r^{-1}, i.e., $a_i = r^{-1}(k) \Longleftrightarrow r(a_i) = k$. For example, if attribute a_3 is the second most important attribute, the ranking of a_3 is $r(a_3) = 2$ and $r^{-1}(2) = a_3$.

While linear and weak orders correspond to rank-orderings as indicated above, rank-orderings can be used directly in the elicitation of incomplete preference information. This can be helpful in situations where the DM does not provide a linear or weak order when considering the relative importance of the attributes: for example, if there are three attributes, the DM may state that the most important one is either the first or the second attribute, without taking a stance on which one of the two is the most important one. Among the six possible rank-orderings, four (i.e., $r = (r(a_1), r(a_2), r(a_3)) = (1,2,3), (1,3,2), (2,1,3)$ or (3,1,2)) are compatible with this statement which rules out the remaining two (i.e., (2,3,1) and (3,2,1)).

The above approach to preference elicitation can be formalized through (i) an attribute set $I \subseteq A$ and (ii) a set of rankings $J \subseteq N$ such that the rankings of attributes in I belong to J (subject to some qualifications discussed below). For instance, the example above corresponds to $I = \{a_1, a_2\}$ and $J = \{1\}$. Moreover, if I contains several attributes while the only ranking in J is one, it follows that the most important attribute must belong to I.

The attribute set I and the set of rankings J need not be equal in size. If the number of attributes is at least as large as that of possible rankings (i.e., $|I| \geqslant |J|$), the specification of these two sets is interpreted as the requirement that all attributes whose rankings belong to J are in the attribute set I. On the other hand, if there are fewer attributes than rankings (i.e., $|I| < |J|$), we require that for each attribute in I, the corresponding ranking is in the set J.

If a rank-ordering meets the above requirements, it is said to be *compatible* with the sets I and J. For example, if there are three attributes and the DM states that attribute a_2 is either the most important or the second most important attribute, then we have $I = \{a_2\}$ and $J = \{1,2\}$. The four rank-orderings that are compatible with these two sets are (2,1,3), (3,1,2), (1,2,3) and (3,2,1). Formally, rank-orderings that are compatible with an attribute set I and a set of rankings J are defined as follows:

Definition 1. If $I \subseteq A = \{a_1,\ldots,a_n\}$ and $J \subseteq N$, the set of compatible rank-orderings is

$$R(I,J) = \begin{cases} \{r \in R | r^{-1}(j) \in I \ \forall j \in J\}, & \text{if } |I| \geqslant |J|, \\ \{r \in R | r(a_i) \in J \ \forall a_i \in I\}, & \text{if } |I| < |J|. \end{cases}$$

3.2. Feasible regions

Because the attribute weights are non-negative and add up to one, they belong to the set

$$S_w = \left\{ w = (w_1,\ldots,w_n) \middle| \sum_{i=1}^{n} w_i = 1, \quad w_i \geqslant 0 \ \forall i \in N \right\}. \tag{1}$$

The weight vector $w \in S_w$ is *consistent* with the rank-ordering r if $w_i \geqslant w_j$ whenever $r(a_i) < r(a_j)$. Thus, the feasible region associated with $r \in R$ can be defined as

$$S(r) = \{w \in S_w | w_i \geqslant w_j \text{ for any } i,j \text{ such that } r(a_i) < r(a_j)\}. \tag{2}$$

For example, the feasible region implied by $r = (4,2,1,3)$ is $S(r) = \{w \in S_w | w_3 \geqslant w_2 \geqslant w_4 \geqslant$

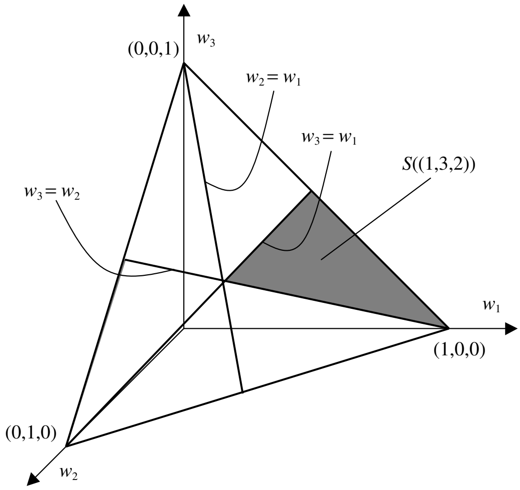

Fig. 2. The feasible region for $r = (1, 3, 2)$.

$w_1\}$. Fig. 2 illustrates the feasible region for $r = (1, 3, 2)$.

The region that corresponds to $R(I, J)$ is defined as the union of feasible regions that are associated with compatible rank-orderings, i.e.,

$$S(I,J) = \bigcup_{r \in R(I,J)} S(r).$$

In general, for a given $R' \subseteq R$, the corresponding feasible region is defined as $S(R') = \bigcup_{r\in R'} S(r)$. For example, Fig. 3 shows the feasible region associated with $R' = \{(3, 1, 2), (1, 3, 2)\}$, based on the statement that a_3 is the second most important one among three attributes (i.e., $I = \{a_3\}$, $J = \{2\}$).

An important special case is obtained when the DM specifies an attribute set I which contains the $p \leqslant |I|$ most important attributes. For brevity, we use $S_p(I)$ to denote the corresponding feasible region, $S_p(I) = S(I, \{1, \ldots, p\})$. In view of (1) and (2), this region is

$$S_p(I) = \{w \in S_w | \exists I' \subseteq I, |I'| = p, \text{ such that } w_k \geqslant w_i \ \forall a_k \in I', \quad a_i \notin I'\}. \tag{3}$$

It immediately follows that $S_p(I)$ can be written as $S_p(I) = \bigcup_{\{I'|I'\subseteq I \wedge |I'|=p\}} S_p(I')$. For example, Fig. 4 illustrates that in a case with three attributes, $S_1(\{a_1, a_2\})$ can be built as the union of $S_1(\{a_1\})$ and $S_1(\{a_2\})$.

3.3. Properties of feasible regions

We next examine several interesting properties of the feasible region based on attribute set I and the rankings J. Proofs are in Appendix A, unless otherwise stated.

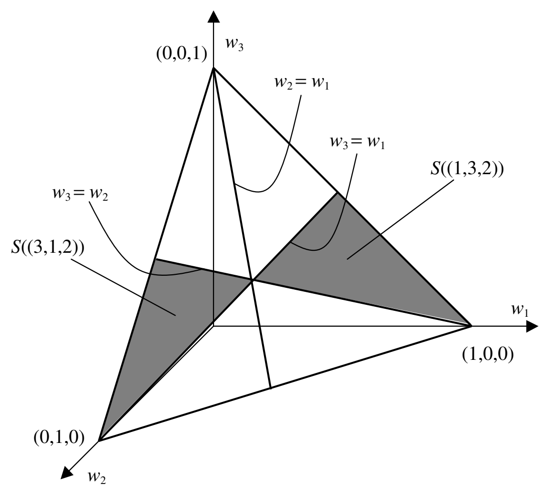

Fig. 3. The third attribute as the second most important one.

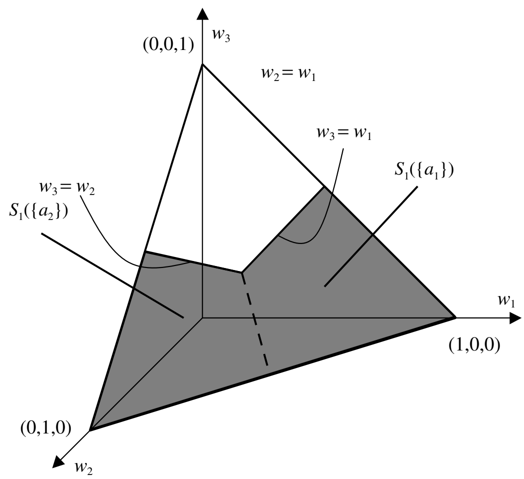

Fig. 4. A non-convex feasible region.

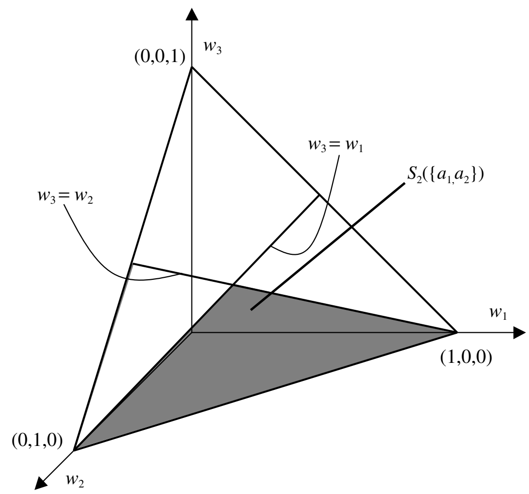

Fig. 5. A convex feasible region.

The feasible region $S_p(I)$ may not be convex (see Fig. 4). In fact, the feasible region is convex if and only if the number of rankings p is equal to the number of attributes in the set I; this result is stated in Theorem 1. Fig. 5 gives an example of the set $S_2(\{a_1, a_2\})$ in the case of three attributes. Here (and throughout this paper) '$\subset$' denotes a proper subset.

Theorem 1. *Let $I \subset A$ and $p \leqslant |I|$. Then $S_p(I)$ in* (3) *is convex if and only if $|I| = p$.*

Theorem 1 holds also when $I = A$. In this trivial case, $S_p(I) = S_w$ for $p \leqslant n$, because knowing that the p most important attributes come from the set of all attributes does not contain any preference information.

If two attribute sets I_1, I_2 are different but contain equally many attributes (p), the two feasible regions $S_p(I_1)$, $S_p(I_2)$—based on the requirement that the attributes in the sets I_1, I_2 are the p most important ones—have disjoint interiors.

Lemma 1. *If $I_1, I_2 \subset A$ such that $|I_1| = |I_2| = p$ and $I_1 \neq I_2$, then*
$\mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2)) = \emptyset$.

For a given attribute set I and a set of rankings J, the resulting feasible region is the same as that defined by the complement sets of I and J. Taking the feasible region in Fig. 4 as an example, the statement that the most important attribute is a_1 or a_2 (i.e., $S(\{a_1, a_2\}, \{1\})$) is equivalent to stating that attribute a_3 is either the second or third most important one (i.e., $S(\{a_3\}, \{2, 3\})$).

Theorem 2. *Assume that I and J are non-empty proper subsets of $A = \{a_1, \ldots, a_n\}$ and N, respectively. Then*
$S(I, J) = S(\overline{I}, \overline{J})$,
where $\overline{I} = A \setminus I$ and $\overline{J} = N \setminus J$ are the complement sets of I and J.

Several comparative results about feasible regions can be obtained. If there are more rankings in J than attributes in I, then, as stated in Theorem 3, increasing the number of attributes that are associated with these rankings reduces the size of the feasible region. Conversely, if there are more attributes in I than rankings in J, reducing the number of attributes in I makes the feasible region smaller.

Theorem 3. *Let I_1 and I_2 be non-empty attribute sets such that $|I_1|$, $|I_2| < n$ and $|J| < n$.*

(a) *If $|I_1|, |I_2| \leqslant |J|$, then $I_1 \subset I_2 \Longleftrightarrow S(I_2, J) \subset S(I_1, J)$.*

(b) *If* $|I_1|, |I_2| \geqslant |J|$, *then* $I_2 \subset I_1 \iff S(I_2, J) \subset S(I_1, J)$.

If there are fewer rankings in J than attributes in I, increasing the number of rankings leads to a feasible region that is a proper subset of the original one. Conversely, if there are more rankings in J than attributes in I, the feasible region becomes smaller if rankings are removed from the set J.

Theorem 4. *Let* J_1 *and* J_2 *are non-empty sets such that* $|J_1|, |J_2| < n$ *and* $|I| < n$.

(a) *If* $|J_1|, |J_2| \leqslant |I|$, *then* $J_1 \subset J_2 \iff S(I, J_2) \subset S(I, J_1)$.
(b) *If* $|J_1|, |J_2| \geqslant |I|$, *then* $J_2 \subset J_1 \iff S(I, J_2) \subset S(I, J_1)$.

The above results can be applied to examine how the feasible region $S_p(I)$—based on the requirement that the $p \leqslant |I|$ most important attributes are in the set I—changes due to incremental changes in the set I or the number p. That is, the feasible region $S_p(I)$ becomes smaller if

1. the attribute set I is extended to contain a larger number of the most important attributes; this means that p becomes larger (i.e., in Theorem 4, the set J_1 is extended to its proper superset $J_2 \supset J_1$), or
2. some attributes are removed from I without changing the number p; this means that the attributes that are removed from I are not among the p most important ones (i.e., in Theorem 3, the set I_1 is reduced to its proper subset $I_2 \subset I_1$).

The above results do not provide information on how 'large' the feasible regions are. We next turn to this issue, in order to provide guidance for eliciting statements which help reduce the size of the feasible region.

4. Measuring the completeness of information

Definition 1 and Eq. (2) suggest that a measure for the size of the feasible region $S(I, J)$ can be based on the number of compatible rank-orderings in the set $R(I, J)$. An appealing property of such a measure is that this number can be readily computed, as shown by Lemma 2 (here, we use the convention $0! = 1$).

Lemma 2. *The number of rank-orderings that are compatible with sets I and J is*

$$|R(I, J)| = \begin{cases} \frac{|I|!(n-|J|)!}{(|I|-|J|)!}, & \text{if } |I| \geqslant |J|, \\ \frac{|J|!(n-|I|)!}{(|J|-|I|)!}, & \text{if } |I| < |J|. \end{cases}$$

Proof. If $|I| \geqslant |J|$, there are $\binom{|I|}{|J|} = \frac{|I|!}{|J|!(|I|-|J|)!}$ different ways of choosing $|J|$ attributes from I. These $|J|$ attributes can be arranged in $|J|!$ ways while the remaining ones can be arranged in $(n - |J|)!$ ways, implying that there is a total of $\frac{|I|!(n-|J|)!}{(|I|-|J|)!}$ different rank-orderings. If $|I| < |J|$, the proof is similar, with the roles of I and J interchanged. □

The above lemma suggests a measure which is formally defined in the following theorem.

Theorem 5. *Let* $\mathscr{P}(R)$ *be the power set which contains all subsets of R. Then the function* $\varphi(\cdot)$, *defined for any* $R' \in \mathscr{P}(R)$ *as* $\varphi(R') = \frac{|R'|}{n!}$, *is a measure which maps the elements of* $\mathscr{P}(R)$ *onto the range* [0,1].

Table 1 shows the size of the feasible region (as measured by $\varphi(\cdot)$) for 10 attributes as a function of possible combinations of $|I|$ and $|J|$. The feasible region is smallest when (i) the attribute set and the ranking set are of equal size and (ii) they both contain (about) half as many elements as there are attributes (i.e., $|I| = |J| \approx \frac{n}{2}$). This means that bisecting the attributes into two sets—one which contains the $\frac{n}{2}$ most important attributes and one which contains the remaining $\frac{n}{2}$ less important attributes—effectively reduces the size of the feasible region.

Lemma 2 and Theorem 5 can be combined to obtain the following expression for the size of the feasible region $S_p(I)$.

Table 1
Size of the feasible region ($n = 10$)

$\|I\|$	$\|J\|$									
	1	2	3	4	5	6	7	8	9	10
1	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
2	0.2000	0.0222	0.0667	0.1333	0.2222	0.3333	0.4667	0.6222	0.8000	1.0000
3	0.3000	0.0667	0.0083	0.0333	0.0833	0.1667	0.2917	0.4667	0.7000	1.0000
4	0.4000	0.1333	0.0333	0.0048	0.0238	0.0714	0.1667	0.3333	0.6000	1.0000
5	0.5000	0.2222	0.0833	0.0238	0.0040	0.0238	0.0833	0.2222	0.5000	1.0000
6	0.6000	0.3333	0.1667	0.0714	0.0238	0.0048	0.0333	0.1333	0.4000	1.0000
7	0.7000	0.4667	0.2917	0.1667	0.0833	0.0333	0.0083	0.0667	0.3000	1.0000
8	0.8000	0.6222	0.4667	0.3333	0.2222	0.1333	0.0667	0.0222	0.2000	1.0000
9	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Corollary 1. *For any $I \subset A$ such that $|I| \geqslant p$,*

$$\varphi_n(R(I,\{1,\ldots,p\})) = \frac{|I|!(n-p)!}{(|I|-p)!n!} = \frac{\binom{|I|}{p}}{\binom{n}{p}}.$$

The measure $\varphi(\cdot)$ can be used for the purpose of analyzing how the size of $S_p(I)$ changes when attributes are removed from the set I. It turns out that the resulting comparative change is larger, the more attributes there are in I.

Lemma 3. *Assume that the attribute set I_2 is obtained from the set $I_1, |I_1| = k > p$ by removing one of the attributes in I_1 (i.e., $I_2 = I_1 \setminus \{a_k\}$ for some a_k). Then the size of the revised feasible region $S_p(I_2)$, relative to the initial feasible region $S_p(I_1)$, is*

$$Q(1,k,p) = \frac{\varphi(R(I_2,\{1,\ldots,p\}))}{\varphi(R(I_1,\{1,\ldots,p\}))} = \frac{k-p}{k}.$$

Proof. Corollary 1 leads to the quotient

$$Q(1,k,p) = \frac{\varphi(R(I_2,\{1,\ldots,p\}))}{\varphi(R(I_1,\{1,\ldots,p\}))} = \frac{(k-1)!(n-p)!}{(k-1-p)!n!}\,\frac{n!(k-p)!}{k!(n-p)!} = \frac{k-p}{k}. \qquad \square$$

Lemma 3 can also be applied to examine changes in the size of the feasible region when several attributes are removed from the initial set of attributes. That is, if the DM chooses to remove $l \leqslant k-p$ attributes from I, consecutive application of Lemma 3 gives

$$Q(l,k,p) = \prod_{i=0}^{l-1} \frac{k-p-i}{k-i} = \frac{(k-p)!(k-l)!}{(k-p-l)!k!} = \binom{k-p}{l} \Big/ \binom{k}{l}.$$

For example, if $p = 2$ and the DM removes four attributes from an initial set of seven attributes, the revised feasible region is $[(7-2)!(7-4)!]/[(7-2-4)!7!] = [5!3!]/[1!7!] = 1/7$ of the size of the initial feasible region.

5. Computation of decision recommendations

The elicitation of preferences through the specification attributes and corresponding rankings would usually take place iteratively so that each new statement is combined with the earlier ones to obtain a reduced feasible region (see Fig. 1). The feasible regions implied by these statements, i.e. intersections of the sets $S(I,J)$, are unions of elementary sets that correspond to complete rank-orderings (i.e., $S(r)$ for some $r \in R$). This has implications for the computational analysis of incomplete ranking information.

The development of decision recommendations based on dominance relations and decision rules does not presume that the feasible regions are unions of the elementary sets $S(r), r \in R$. Thus, in Sections 5.1 and 5.2 we only assume that the feasible region S is some non-empty subset of S_w in (1).

5.1. Dominance structures

Following Salo and Hämäläinen (1992), dominance relations for the alternatives can be established on the basis of (i) the value intervals that the alternatives can assume, subject to the requirement that the attribute weights belong to the feasible region, and (ii) the minimization of value differences between pairs of alternatives, as computed from the *pairwise bounds*

$$\mu_0(x_k, x_l) = \min_{w \in S} [V(x_k) - V(x_l)] = \min_{w \in S} \sum_{i=1}^{n} w_i [v_i(x_k) - v_i(x_l)]. \quad (4)$$

If the minimum in (4) is non-negative, the value of alternative x_k is greater than or equal to that of alternative x_l, no matter how the feasible weights are chosen. In this case, alternative x_k dominates x_l in the sense of pairwise dominance.

The computation of dominance relations does not presume that precise score information is available. For instance, if incomplete information about scores is available as intervals, the minimization problem (4) can be solved by first determining attribute-specific pairwise bounds $\mu_i(x_k, x_l)$ from the minimization problems

$$\mu_i(x_k, x_l) = \min[v_i(x_k) - v_i(x_l)].$$

These bounds can be inserted into (4) to replace the bracketed differences. In hierarchically structured value trees with attributes on several levels, the computation of pairwise bounds proceeds from the lower levels towards the topmost attribute (for details see Salo and Hämäläinen, 1992).

In many problems, it is plausible to require that all attributes are essential in the sense that they influence the alternatives' overall values. This can be modelled by requiring that the weight of each attribute is greater than some fixed lower bound $\epsilon < \frac{1}{n}$ (i.e., $w_i \geqslant \epsilon \ \forall i \in N$): for example, if—for the sake of convenience—the weight of each attribute is required to be at least one third of the average weight of an attribute, then ϵ would be $1/[3n]$. With the requirement of lower bounded attribute weights, the set S_w in (1) becomes

$$S_w(\epsilon) = \{w = (w_1, \ldots, w_n) \in S_w | w_i \geqslant \epsilon \ \forall i \in N\}. \quad (5)$$

These constraints help reduce the size of the feasible region so that dominance results for alternatives are more likely obtained. In a somewhat different but analogous setting, Cook and Kress (1990, 1991) consider the use of lower bounds on weight differences so that the weight of attribute a_i with ranking $r(a_i) = k$ exceeds the weight of the attribute a_j with ranking $k + 1$ by a certain gap ϵ; in this case, the inequality $w_i - w_j \geqslant \epsilon$ must hold.

5.2. Decision rules

Throughout the analysis, the DM can be offered tentative decision recommendations based on different *decision rules* (Salo and Hämäläinen, 2001). These rules are procedures for extrapolating a decision recommendation from a preference specification which is not complete enough to establish dominance results. Alternative decision rules include, among others, (i) the choice of an alternative with the largest possible overall value (i.e., *maximax* rule), (ii) the choice of an alternative for which the smallest possible value is largest (i.e., *maximin* rule), (iii) the choice of an alternative such that the maximum value difference to some other alternative is minimized (i.e., *minimax regret*), and (iv) the comparison of central values, computed for each alternative as the average of its smallest and largest possible values. Formally, these decision rules can be defined as follows:

$$\text{maximax}: \arg\max_{x_i} \left[\max_{w \in S} V(x_i) \right],$$

$$\text{maximin}: \arg\max_{x_i} \left[\min_{w \in S} V(x_i) \right],$$

$$\text{minimax regret}: \arg\min_{x_i} \left[\max_{x_k \neq x_i} \max_{w \in S} [V(x_k) - V(x_i)] \right],$$

$$\text{central values}: \arg\max_{x_i} \left[\max_{w \in S} V(x_i) + \min_{w \in S} V(x_i) \right].$$

Because these decision rules are based on the analysis of alternatives' overall values, the recommendations depend on the scores (i.e., $v_i(x_j)$). It is also possible to offer decision recommendations by choosing representative vectors from the feasible region S without considering scores. Here, possibilities include the computation of (i) central weights, defined by normalizing the vector $w_i' = \max_{w \in S} w_i + \min_{w \in S} w_i$, and (ii) the point of gravity of feasible region S. Also, the use of equal weights (i.e., $w_i = 1/n$) offers a benchmark against which the performance of any decision rule can be contrasted (Salo and Hämäläinen, 2001).

5.3. Computational issues

Because the feasible region $S(I,J)$ is not necessarily convex, the computation of dominance results may lead to linear optimization problems over non-convex sets. In principle, these problems can be solved by branch-and-bound algorithms or other suitable approaches (see, e.g., Taha, 1997). In particular, if the DM states that the p most important attributes are in the attribute set I, Lemma 1 implies that $S_p(I)$ can be decomposed into $|I|!/[p!(|I|-p)!]$ convex subsets with disjoint interiors. Each of these subsets could be dealt with as a separate subproblem, allowing dominance structures and decision recommendations to be derived by combining results from these subproblems.

Because the objective functions in the computation of value intervals, dominance structures and decision rules are linear, solving these optimization problems over the convex hull of the feasible region $S(I,J)$ leads to the same result as solving these problems over the feasible region. This approach is not attractive, however, because the determination of a minimal set of constraints through which this convex hull is characterized entails an additional computational effort.

An efficient approach to the determination of dominance relations and decision rules can be based on the realization that each feasible region $S(I,J)$ is the union of the sets $S(r)$, $r \in R(I,J)$. By construction, each such set is convex, and its extreme points are related to the rank-orderings r as stated in the following lemma (for the proof, see, e.g., Carrizosa et al., 1995).

Lemma 4. *Let $r \in R$ be a rank-ordering. Then the extreme points of the feasible region $S(r)$ in (2), $X(r)$, are*

$$X(r) = \text{ext}(S(r)) = \Big\{ w \in S_w \,|\, \exists k \in \{1, \ldots, n\} \text{ s.t. } w_i = \frac{1}{k}\ \forall r(a_i) \leqslant k, \quad w_i = 0\ \forall r(a_i) > k \Big\}.$$

Lemma 4 can be adapted to obtain the extreme points of $S_\epsilon(r) = S(r) \cap S_w(\epsilon)$:

$$\text{ext}(S(r) \cap S_w(\epsilon)) = \Big\{ w \in S_w \,|\, \exists k \leqslant n \text{ s.t. } w_i = \frac{1-(n-k)\epsilon}{k}\ \forall r(a_i) \leqslant k, \quad w_i = \epsilon\ \forall r(a_i) > k \Big\}.$$

Based on this result, the extreme points can be enumerated at the outset (on condition that the number of attributes is not too large). Then, as the DM supplies preference statements, the resulting list can be shortened by removing those extreme points that are not compatible with the DM's statements. At any stage of the analysis, value intervals, dominance structures and decision rules can be computed by inspection. For instance, the pairwise bound for alternatives x_k, x_l is obtained from $\mu_0(x_k, x_l) = \min_{r \in R'} \min_{w \in X(r)} \sum_{i=1}^{n} w_i [v_i(x_k) - v_i(x_l)]$, where R' is the set of rank-orderings that are compatible with the DM's preference statements.

6. A simulation study on the computational properties of RICH

To examine the computational properties of RICH, we carried out a simulation study in which the number of attributes was $n = 5$, 7, 10 and the number of alternatives was $m = 5$, 10, 15. The attribute weights were generated by assuming a uniform distribution over the set S_w. Because in many cases it is realistic to assume that the weight of each attribute is greater than some lower bound, simulation results are presented for the case where this bound was $\epsilon = 1/[3n]$, which seemed plausible enough.

Following Salo and Hämäläinen (2001), scores for the alternatives were defined under each attribute by (i) generating random numbers from a uniform distribution over [0,1] and by (ii) normalizing the resulting numbers under each attribute. This normalization was carried out through a linear mapping in which the random number of the best performing alternative was set to one and that of the worst performing alternative was set to zero.

Five thousand problem instances were generated for each problem type (as characterized by the number of attributes (n) and alternatives (m)). Each problem instance consisted of a full combination of weights and scores in an additive preference model. The alternative with the highest overall value, i.e.,

$$\arg\max_x V(x) = \sum_{i=1}^{n} w_i v_i(x)$$

will be referred to as the *correct choice*.

The simulation study was based on the following preference statements:

- A: The DM specifies the most important attribute only.
- B: The DM specifies the two most important attributes (without taking a stance on which one is more important than the other).
- C: The DM specifies a set of three attributes which contains the two most important attributes.

Even though other kinds of preference statements are also worth studying, these three statements are nevertheless indicative of different ways of expressing incomplete preference information through rank inclusion. The sizes of the respective feasible regions (see Table 2) indicate that statement B leads to a preference specification which is more informative than statement A or statement C. This is also in keeping with the theoretical results of Sections 3 and 4.

The preference statements—and corresponding feasible regions of attribute weights—were derived from the randomly generated weights as follows.

Table 2
Size of the feasible region

n	A	B	C	Complete rank-ordering
5	0.200	0.100	0.300	8.33×10^{-3}
7	0.143	0.048	0.143	1.98×10^{-4}
10	0.100	0.022	0.067	2.76×10^{-7}

Starting from the randomly generated weight vector w, the corresponding rank-ordering r was first derived. For instance, if the simulated weights of the five attributes were $w_1 = 0.09$, $w_2 = 0.30$, $w_3 = 0.18$, $w_4 = 0.20$ and $w_5 = 0.23$, the resulting rank-ordering was $r = (5, 1, 4, 3, 2)$.

For preference statement A, the feasible region was set equal to $S_1(\{r^{-1}(1)\})$. For preference statement B, the feasible region was defined analogously as $S_2(\{r^{-1}(1), r^{-1}(2)\})$. For the third preference statement C, the set of three attributes was defined by taking the union of the two most important attributes (i.e., $r^{-1}(1)$, $r^{-1}(2)$) and a third attribute from the remaining $n-2$ attributes. This third attribute a_i was selected at random by assuming a uniform distribution over the set $N \setminus \{r^{-1}(1), r^{-1}(2)\}$, whereafter the feasible region was defined as $S_2(\{r^{-1}(1), r^{-1}(2), a_i\})$.

Results based on the above preference statements were compared to those obtained on the use of (i) equal weights (i.e. $w_i = 1/n$, $\forall i \in N$) and (ii) complete rank-ordering (where the feasible region was set equal to $S(r) \cap S_w(1/[3n])$). The comparisons were made using four decision rules (maximax, maximin, central values and minimax regret) in conjunction with two measures of efficiency, i.e., (i) the average expected loss of value relative to the correct choice and (ii) the percentage of problem instances in which the decision rule lead to the identification of the correct choice. We also computed the average number of non-dominated alternatives that would remain after (i) the specification of the above three statements A, B, and C and (ii) the use of complete rank-ordering information.

Among alternative measures of efficiency, expected loss of value is arguably the most important as it indicates how great a loss of value the DM would incur, on the average, if he or she were to follow a particular decision rule (Salo and

Hämäläinen, 2001). For a given problem instance, the corresponding loss of value is obtained from

$$\mathrm{LV} = \sum_{i=1}^{n} w_i[v_i(x^*) - v_i(x')],$$

where w_i is weight of attribute a_i, x^* is the correct choice and x' is the alternative that is recommended by a particular decision rule. In our simulation study, averaging these terms over the entire sample lead to an estimate for the expected loss of value.

In the simulation results, the use of central values as a decision rule outperformed the other decision rules, wherefore the results are presented using this decision rule only (see also Salo and Hämäläinen, 2001). In particular, an analysis of the results in Table 3 supports the following conclusions:

- Among the three statements, statement B is the most efficient and C is the least efficient one with regard to all measures of efficiency. All the three preference statements A, B and C give better results than the use of equal weights.
- Changes in the number of attributes or alternatives do not reveal consistent trends in the expected loss of value. In comparative terms, statement A performs best when there are few attributes and alternatives, while the opposite holds for statement C. For preference statement B and complete rank-ordering information, changes in the expected loss of value are relatively small across the full range of problems.
- The percentage of problem instances in which the application of decision rules leads to the identification of the correct choice tends to decrease as the number of alternatives or attributes grows; this is because there is a higher chance that some other alternative (i.e., other than the correct choice) will be favored. The share of problem instances where the correct choice is identified increases with about 5% units when complete rank-ordering information is used instead of information about the two most important attributes only (i.e., statement B). For statement C, the corresponding difference is about 15% units.
- The percentage of non-dominated alternatives decreases as the number of alternatives increases. Increasing the number of attributes leads to a larger number of non-dominated alternatives. Statement B has the smallest percentage of non-dominated alternatives across the entire spectrum of problems, because the size of the feasible region is smallest for this statement.

7. An illustrative example

To further exemplify the application of RICH, we assume there is a main contractor who is about to choose a subcontractor for an engineering project at a construction site. The contractor chooses among competing subcontractors on the basis of five attributes: (i) ability to finish the project on schedule (i.e., *punctuality*), (ii) *quality* of work, (iii) overall *cost* of the contract, (iv) *references* from earlier engagements with the respective subcontractor, and (v) possibilities for introducing *changes* into the subcontract. These attributes are essential in the sense that the weight of each is greater than a positive lower bound ϵ, which in this example is set equal to $1/[3n] = 1/15 \approx 0.0667$.

The main contractor invites tenders from three potential subcontractors. Among these, the first (x_1) is a *large firm* which is punctual and offers its services at a reasonable cost. The second one (x_2) is *a small entrepreneur* who has had difficulties in completing the project tasks on schedule. The third subcontractor (x_3) is *a medium-sized firm* which is in many ways similar to the entrepreneur, except that it is more punctual.

Score information for the three subcontractors is generated as follows. Using the first attribute (i.e., punctuality) as a benchmark, the main contractor assigns 1.00 points to the best performance level and 0.00 to the worst performance level. Then, scores reflecting incomplete information about the subcontractors are generated using these ranges as a point of reference: thus, for the first attribute, the score of the large firm is given by the interval [0.80,1.00] while the score interval for the entrepreneur is [0.00,0.20]. For the other subcontractors and attributes, scores in the [0.00,1.00]

Table 3
Simulation results

n	m	Equal weights	A	B	C	Complete rank-ordering
Expected loss of value						
5	5	0.065	0.021	0.025	0.050	0.013
	10	0.062	0.024	0.023	0.048	0.014
	15	0.059	0.027	0.022	0.043	0.015
7	5	0.060	0.024	0.021	0.045	0.013
	10	0.061	0.027	0.023	0.041	0.015
	15	0.060	0.029	0.022	0.042	0.014
10	5	0.054	0.025	0.021	0.038	0.015
	10	0.054	0.030	0.023	0.039	0.015
	15	0.056	0.031	0.023	0.038	0.016
Percentage of correct choices						
5	5	61%	76%	76%	64%	81%
	10	53%	67%	70%	57%	76%
	15	50%	62%	66%	55%	72%
7	5	60%	72%	75%	64%	81%
	10	50%	63%	67%	57%	73%
	15	47%	59%	64%	53%	72%
10	5	58%	70%	72%	64%	77%
	10	49%	58%	65%	55%	71%
	15	44%	54%	60%	51%	66%

n	m	A	B	C	Complete rank-ordering
Percentage of non-dominated alternatives					
5	5	54%	53%	65%	41%
	10	37%	36%	49%	26%
	15	30%	27%	40%	19%
7	5	69%	62%	74%	46%
	10	52%	46%	60%	30%
	15	45%	37%	51%	23%
10	5	84%	75%	85%	51%
	10	71%	61%	75%	35%
	15	64%	53%	68%	28%

range are generated in the same way, recognizing that this range is used in interpreting the attribute weights (see Table 4). The subcontractors' scores of are assumed independent, i.e., the performance of a given subcontractor may assume all the scores within its respective interval, regardless of the other subcontractors' scores.

Assume that the DM confirms that the two most important attributes are among the three first attributes, i.e., punctuality (a_1), quality (a_2) and cost (a_3). Using the notation of Section 2, we have $p = 2$ and $I = \{a_1, a_2, a_3\}$ so that the feasible region is $S_2(\{a_1, a_2, a_3\})$. According to Lemma 2 and Theorem 5, the size of this region is $\varphi(S_2(\{a_1, a_2, a_3\}) = [3!(5-2)!]/[1!5!] = 3/10$, i.e., it covers 30% of the entire weight space $S_w(\epsilon)$ in (5).

To derive dominance results, the pairwise bounds $\mu_0(x_i, x_j)$ in (4) are computed. Towards this end, the pairwise bounds $\mu_i(\cdot,\cdot)$ are first computed

Table 4
Score intervals for the alternatives

	a_1	a_2	a_3	a_4	a_5
$v_i(x_1)$	[0.80,1.00]	[0.70,0.90]	0.80	0.40	0.70
$v_i(x_2)$	[0.00,0.20]	[0.50,0.70]	[0.40,0.60]	[0.20,0.60]	[0.30,0.90]
$v_i(x_3)$	0.60	[0.50,0.70]	0.60	[0.20,0.40]	[0.30,0.90]

with regard to each attribute (see Table 5). For example, because the score intervals of the first two subcontractors on the first attribute are [0.80,1.00] and [0.00,0.20], respectively, the pairwise bound $\mu_1(x_1, x_2)$ is $0.80 - 0.20 = 0.60$.

Next, for each pair of subcontractors, the weighted sum of pairwise bounds (4) is minimized over the feasible region $S_2(\{a_1, a_2, a_3\})$ which consists of three convex sub-regions $S_2(\{a_1, a_2\})$, $S_2(\{a_1, a_3\})$ and $S_2(\{a_2, a_3\})$. The results indicate that the first alternative (large firm) is better than the third (medium-sized enterprise), because the value difference $\sum_{k=1}^{5} w_k[v_k(x_1) - v_k(x_3)]$ is positive over the entire feasible region (see Table 6). No dominance relations are obtained for the two first subcontractors because the pairwise bounds $\mu_0(x_1, x_2)$, $\mu_0(x_2, x_1)$ are negative. Thus, the DM would be asked to supply further preference information, or to accept one of the recommendations based on decision rules.

Further insights can be obtained by examining the recommendations of three decision rules, i.e., maximax, maximin, and maximization of central values. For the maximax criterion, the decision recommendation is based on the comparison of largest possible values for each subcontractor, obtained as solutions to the linear problems $V_{\max}(x_i) = \max \sum_{k=1}^{5} w_k v_k^{\max}(x_i)$ subject to the requirement that $w \in S_2(\{a_2, a_3\})$ and $w_i \geqslant 1/15$, $i = 1, \ldots, 5$. The analysis is based on $S_2(\{a_2, a_3\})$, since elsewhere in the feasible region dominance relations are already obtained. Similarly, the minimum possible values are computed from $V_{\min}(x_i) = \min \sum_{k=1}^{5} w_k v_k^{\min}(x_i)$ subject to the same

Table 5
Attribute-specific pairwise bounds

	a_1	a_2	a_3	a_4	a_5
$\min[v_i(x_1) - v_i(x_2)]$	0.60	0.00	0.20	−0.20	−0.20
$\min[v_i(x_2) - v_i(x_1)]$	−1.00	−0.40	−0.40	−0.20	−0.40
$\min[v_i(x_1) - v_i(x_3)]$	0.20	0.00	0.20	0.00	−0.20
$\min[v_i(x_3) - v_i(x_1)]$	−0.40	−0.40	−0.20	−0.20	−0.40
$\min[v_i(x_2) - v_i(x_3)]$	−0.60	−0.20	−0.20	−0.20	−0.60
$\min[v_i(x_3) - v_i(x_2)]$	0.40	−0.20	0.00	−0.40	−0.60

Table 6
Pairwise bounds μ_0

	$S_2(\{a_1, a_2\})$	$S_2(\{a_1, a_3\})$	$S_2(\{a_2, a_3\})$	Min
$\mu_0(x_1, x_2)$	0.060	0.093	−0.007	−0.007
$\mu_0(x_2, x_1)$	−0.827	−0.827	−0.560	−0.827
$\mu_0(x_1, x_3)$	0.040	0.013	0.013	0.013
$\mu_0(x_3, x_1)$	−0.373	−0.373	−0.373	−0.373
$\mu_0(x_2, x_3)$	−0.520	−0.520	−0.387	−0.520
$\mu_0(x_3, x_2)$	−0.187	−0.160	−0.253	−0.253

Table 7
Maximax, maximin and central values

Alternative	V_{max}	V_{min}	V_{ave}
x_1	0.920	0.649	0.784
x_2	0.689	0.093	0.391
x_3	0.702	0.413	0.558

constraints. Finally, central values for the three subcontractors are obtained as the average $V_{ave} = [V_{max}(x_i) + V_{min}(x_i)]/2$.

Table 7 indicates that the maximum possible value for the large firm is greater than that for the small entrepreneur (0.920 > 0.689): thus, the large firm would be recommended by the maximax rule. The application of the maximin rule leads to the same conclusion (0.649 > 0.093). Because both maximax and maximin rules support it, the large firm outperforms the small entrepreneur according to the maximization of central values as well. Thus, it would be offered as a tentative decision recommendation.

Finally, we illustrate sensitivity analyses by assuming that (i) the DM states that quality and cost are the two most important attributes and that (ii) the DM wishes to know how large a weight the first attribute (i.e., punctuality) should have to establish a dominance relationship between the first two alternatives. The revised feasible region thus becomes $S_2(\{a_1, a_2\})$, in which no dominance relations between the two first alternatives were obtained. The feasible region is now defined by $I = \{a_2, a_3\}$, $J = \{1, 2\}$, the size of which is $\varphi(S_2(\{a_2, a_3\})) = 1/10$, i.e., one third of the original feasible region $S_2(\{a_1, a_2, a_3\})$. The question about the lower bound for the weight of the first attribute can be answered by maximizing w_1, subject to the constraint that the value of the large firm is not smaller than that of the small entrepreneur, i.e., $\mu_0(x_1, x_2) = 0$. Thus, we have a maximization problem $\max w_1$ subject to the constraints $\mu_0(x_1, x_2) = 0.6w_1 + 0.2w_3 - 0.2w_4 - 0.2w_5 = 0$, $w \in S_2(\{a_2, a_3\})$, and $w_i \geqslant 1/15\ \forall i = 1, \ldots, 5$. The solution to this problem is $w_1 \approx 0.0769$, which indicates that even a small increase in the lower bound for the weight of the first attribute (i.e., punctuality) would ensure that the large firm becomes preferred to the small entrepreneur.

8. Conclusion

The elicitation of precise statements about the relative importance of attributes can pose difficulties in the development of multi-attribute decision models. To some extent, these difficulties can be alleviated by allowing the DM(s) to provide incompletely specified rank-ordering information. In a natural way, such information constrains the attribute weights so that partial dominance results can be obtained even in the absence of complete preference information. An essential feature of such an approach is that the application of decision rules makes it possible to offer decision recommendations even when dominance concepts do not allow the most preferred alternative to be inferred.

One of the features of the proposed RICH method is that the DMs need not submit preference statements that would be more explicit than what they feel confident with. Thus, the DMs may remain ambiguous about their 'true' preferences. This, in turn, may lead to a decision support process which is more acceptable from the viewpoint of group dynamics than approaches where full preference information is solicited and communicated among the group members. For example, the decision recommendation can be produced under the assumption that the most important attribute in the group's aggregate preference model is an attribute that is regarded as the most important one by some group member.

From the viewpoint of applied work, the decision support tool *RICH Decisions* ©—which is available free-of-charge for academic users at `http://www.decisionarium.hut.fi`—is important because it provides full support for the RICH method and thus enables the development of case studies based on the proposed method. Such studies will be instrumental in assessing the benefits and disadvantages of incomplete ordinal preference information in challenging decision contexts, which in turn helps set directions for further theoretical research.

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Appendix A

Proof of Theorem 1. '$\Leftarrow$': Let $\lambda \in [0,1]$, choose any $w^1 = (w_1^1, w_2^1, \dots, w_n^1), w^2 = (w_1^2, w_2^2, \dots, w_n^2) \in S_p(I)$, and define $w^\lambda = \lambda w^1 + (1-\lambda)w^2$. We need to show that $w^\lambda \in S_p(I)$. Because $w_i^1, w_i^2 \geqslant 0$, it follows that $w_i^\lambda = \lambda w_i^1 + (1-\lambda)w_i^2 \geqslant 0$. Likewise, $\sum_{i=1}^n w_i^1 = 1$, $\sum_{i=1}^n w_i^2 = 1$ imply that $\sum_{i=1}^n w_i^\lambda = \sum_{i=1}^n (\lambda w_i^1 + (1-\lambda)w_i^2) = \lambda \sum_{i=1}^n w_i^1 + (1-\lambda)\sum_{i=1}^n w_i^2 = 1$. Finally, because $w_k^1 \geqslant w_i^1$ and $w_k^2 \geqslant w_i^2$ for all $a_k \in I$, $a_i \notin I$, it follows that $w_k^\lambda = \lambda w_k^1 + (1-\lambda)w_k^2 \geqslant \lambda w_i^1 + (1-\lambda)w_i^2 = w_i^\lambda$ if $a_k \in I$, $a_i \notin I$. Thus, $w^\lambda \in S_p(I)$.

'$\Rightarrow$': Assume that $p < |I|$ and choose some $I' \subset I$ such that $|I'| = p$. Since $|I| < n$, there exists an attribute $a_k \notin I$. Put $I_1 = I' \cup \{a_k\}$ define the weight vector w^1 by letting $w_l^1 = 1/(p+1), a_l \in I_1, w_l^1 = 0, a_l \notin I_1$. Next, choose attributes $a_i \in I \setminus I'$, $a_j \in I'$ and define the attribute set $I_2 = (I' \setminus \{a_j\}) \cup \{a_i\} \cup \{a_k\}$ and define w^2 by letting $w_l^2 = 1/(p+1)$, $a_l \in I_2$, $w_l^2 = 0$, $a_l \notin I_2$. By construction, $w^1, w^2 \in S_p(I)$ because they contain p elements in I that are greater than or equal to all the other elements. However, this is not true for the vector $w^3 = (1/2)w^1 + (1/2)w^2$ where $w_l^3 = 1/(p+1), a_l \in \{a_k\} \cup (I' \setminus \{a_i, a_j\}), w_l^3 = 1/(2(p+1))$, $a_l \in \{a_i, a_j\}, w_l^3 = 0$, otherwise. Thus, w^3 contains only $p-1$ elements in I that are larger than the other elements so that it does not belong to $S_p(I)$, which implies that $S_p(I)$ is not convex. □

Proof of Lemma 1. By assumption, there exist attributes a_k, a_l such that $a_k \in I_1 \setminus I_2$, $a_l \in I_2 \setminus I_1$. Let $\alpha > 0$ and define the vector δ so that $\delta_k = \alpha$, $\delta_l = -\alpha$, $\delta_i = 0$, $i \neq k, l$. If $\exists w \in \mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2))$, then for some $\varepsilon > 0$ the weight vectors $w^1 = w + \varepsilon\delta$ and $w^2 = w - \varepsilon\delta$ are also in $\mathrm{int}(S_p(I_1)) \cap \mathrm{int}(S_p(I_2))$. In particular, since $w^1 \in S_p(I_2)$, it follows that $w_l - \varepsilon\alpha \geqslant w_k + \varepsilon\alpha \Rightarrow w_l - w_k \geqslant 2\varepsilon\alpha > 0$. Also, since $w^2 \in S_p(I_1)$, it follows that $w_k - \varepsilon\alpha \geqslant w_l + \varepsilon\alpha \Rightarrow w_k - w_l \geqslant 2\varepsilon\alpha > 0$, in contradiction with the earlier inequality $w_l - w_k > 0$. □

Proof of Theorem 2. The cases $|I| \geqslant |J|$ and $|I| < |J|$ can be dealt with separately. First, if $|I| \geqslant |J|$ and $w \in S(I,J)$, then $w \in S(r)$ for some $r \in R(I,J)$. Thus, for some $I' \subseteq I, |I'| = |J|$ we have $r(I') = J$. But then $r(\overline{I'}) = \overline{J}$. Since $\overline{I} \subseteq \overline{I'}$, we have $r(\overline{I}) \subseteq \overline{J}$ and $r \in R(\overline{I}, \overline{J})$. Second, if $|I| < |J|$ and $w \in S(I,J)$, then $w \in S(r)$ for some $r \in R(I,J)$ such that $r(I) \subset J$. Because $|I| < |J|$, there exists a set I' such that $I \subset I', |I'| = |J|$ and $r(I') = J$. Thus we have $r(\overline{I'}) = \overline{J}$. By construction, $|\overline{I'}| = |\overline{J}|$ and $\overline{I'} \subset \overline{I}$ so that $r \in R(\overline{I}, \overline{J})$. This far it has been shown that $w \in S(I,J) \Rightarrow w \in S(\overline{I}, \overline{J})$. Since $\overline{\overline{I}} = I$, $w \in S(\overline{I}, \overline{J}) \Rightarrow w \in S(I,J)$. □

Proof of Theorem 3. Item (a). '$\Rightarrow$': If $w \in S(I_2, J)$, there exists a rank-ordering $r \in R(I_2, J)$ such that $w \in S(r)$ and $r(I_2) \subseteq J$. But since $I_1 \subset I_2$, it follows that $r(I_1) \subset J$. In addition, by Definition 1 this implies that $r \in R(I_1, J)$ and $w \in S(I_1, J)$. To prove that $S(I_2, J)$ is a *proper* subset of $S(I_1, J)$, we construct a rank-ordering r' such that $r'(I_1) \subseteq J$ and $r'(a_k) \notin J$ for some $a_k \in I_2$, $a_k \notin I_1$ (such an order exists because $|J| < n$). Then $r' \in R(I_1, J)$, but $r' \notin R(I_2, J)$.

'$\Leftarrow$': Take any $w \in S(I_2, J)$. Then there exists some $r \in R(I_2, J)$ such that $w \in S(r)$ and $r(I_2) \subseteq J$. Because $S(I_2, J) \subset S(I_1, J)$, it follows that $r \in R(I_1, J)$ and hence $r(I_1) \subseteq J$. Now, if $I_1 \not\subset I_2$, there exists some $a_k \in I_1$, $a_k \notin I_2$. Because $a_k \in I_1$, we have $i_k = r(a_k) \in J$. Also, since $|J| < n$, there is some a_l such that $i_l = r(a_l) \notin J$. By construction, this a_l is not in I_1 or I_2. Next, construct the rank-ordering r' so that $r'(a_k) = i_l$, $r'(a_l) = i_k$ and $r'(a_i) = r(a_i)$, $\forall i \neq k, l$. Then $r' \in R(I_2, J)$ but $r' \notin R(I_1, J)$. But this violates the assumption $S(I_2, J) \subset S(I_1, J)$, leading to a contradiction.

Item (b): By Theorem 2, $S(I,J) = S(\overline{I}, \overline{J})$. Thus, we have to prove that $I_2 \subset I_1 \iff S(\overline{I_2}, \overline{J}) \subset S(\overline{I_1}, \overline{J})$, which is equal to $\overline{I_1} \subset \overline{I_2} \iff S(\overline{I_2}, \overline{J}) \subset S(\overline{I_1}, \overline{J})$; but this follows from item (a) above. □

Proof of Theorem 4. Item (a). '$\Rightarrow$': If $w \in S(I, J_2)$, then there is a rank-ordering $r \in R(I, J_2)$ and an attribute set $I' \subseteq I$ such that $|I'| = |J_2|$ and $r(I') = J_2$. Next, define the set $I'' = \{a_i \in I' | r(a_i) \in J_1\}$. Because $J_1 \subset J_2$, we have $|I''| = |J_1|$ so that $r \in R(I, J_1)$; hence $w \in S(I, J_1)$ as well.

'$\Leftarrow$': Assume that $S(I,J_2) \subseteq S(I,J_1)$. By Theorem 2, this is equivalent to $S(\overline{I},\overline{J_2}) \subseteq S(\overline{I},\overline{J_1})$. From the assumptions it also follows that $|\overline{I}| \leqslant |\overline{J_1}|, |\overline{J_2}|$. Choose a $w \in S(\overline{I},\overline{J_2})$. There then exists a rank-ordering r such that $w \in R(\overline{I},\overline{J_2})$, i.e. $r(\overline{I}) \subseteq J_2$. Since $S(\overline{I},\overline{J_2}) \subseteq S(\overline{I},\overline{J_1})$, $r \in R(\overline{I},\overline{J_1})$ so that $r(\overline{I}) \subset J_1$, too. Contrary to the claim $J_1 \subseteq J_2$, assume that there is an i_k such that $i_k \in J_1, i_k \notin J_2$. Then the rank-ordering associates i_k with an attribute $a_k \in I$ (because $r(\overline{I}) \in \overline{J_1}$ and $r(\overline{I}) \in \overline{J_2}$). Also, choose an $a_l \in \overline{I}$ and define $i_l = r(a_l)$; by construction, $i_l \notin J_1$, $i_l \notin J_2$. Next, define a rank-ordering r' so that $r'(a_k) = i_l$, $r'(a_l) = i_k$ and $r'(a_i) = r(a_i)$, $\forall i \neq k, l$. Then $r' \in R(\overline{I},\overline{J_2})$, but $r' \notin R(\overline{I},\overline{J_1})$, which violates the assumption $S(I,J_2) \subseteq S(I,J_1)$.

Item (b): According to Theorem 2, $S(I,J) = S(\overline{I},\overline{J})$. Thus, we have to show that $J_2 \subset J_1 \iff S(\overline{I},\overline{J_2}) \subset S(\overline{I},\overline{J_1})$, which is equal to $\overline{J_1} \subset \overline{J_2} \iff S(\overline{I},\overline{J_2}) \subset S(\overline{I},\overline{J_1})$. This follows directly from item (a) above. □

Proof of Theorem 5. Clearly, $\varphi(\emptyset) = 0$. Assume that $R' \in \mathscr{P}(R)$ and that $R_1, \ldots, R_M \in R$ are disjoint sets of rank-orderings such that $R' = \biguplus_{i=1}^{M} R_i$. By construction, the intersection of any R_i and $R_j, i \neq j$ is empty; thus, $|R'| = \sum_{i=1}^{M} |R_i|$, which implies that $\varphi(R') = \frac{|R'|}{n!} = \sum_{i=1}^{M} \frac{|R_i|}{n!} = \sum_{i=1}^{M} \varphi(R_i)$. Finally, since the total number of different rank-orderings is $n!$, we have $\varphi(R) = 1$. □

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Preference programming with incomplete ordinal information

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Abstract:

This paper extends possibilities for analyzing incomplete ordinal information about the parameters of an additive value function. Such information is modeled through preference statements which associate sets of alternatives or attributes with corresponding sets of rankings. These preference statements can be particularly helpful in developing a joint preference representation for a group of decision-makers who may find difficulties in agreeing on numerical parameter values. Because these statements can lead to a non-convex set of feasible parameters, a mixed integer linear formulation is developed to establish a linear model for the computation of decision recommendations. This makes it possible to complete incomplete ordinal information with other forms of incomplete information.

Keywords: value tree analysis, incomplete information, ordinal information

1 Introduction

The exact specification of attribute weights and scores of an additive multiattribute value function (Keeney and Raiffa 1976) can be challenging due to many reasons, such as urgency of the decision or lack of resources (e.g. Weber 1987). A complete specification of parameters may not be needed, if less information would produce the same unequivocal decision recommendation or if the aim of the analysis is to discard inferior alternatives to establish a smaller set of good decision candidates. A complete specification can even be undesirable if the model seeks to capture multiple decision-makers' (DMs') preferences, for example. Indeed, several *Preference Programming* methods accommodate incomplete preference information through intervals of weight ratios or scores, for example. They thus allow the DMs to characterize their preferences with the degree of accuracy they are confident with. The resulting set of feasible parameters then contains the 'true preferences' (e.g., White et al. 1982, 1984, Salo and Hämäläinen 1992).

The assessment of multi-attribute alternatives usually requires subjective evaluations (Moshkovich et al. 2002). According to Larichev (1992), numerical evaluation can even affect negatively the reliability of the analysis. Indeed, Moshkovich et al. (2002) argue that ordinal statements – as ordinal comparisons of actual or hypothetical alternatives – tend to be more reliable, less time-consuming and better understood by the DMs. Thus, they are widely used in the elicitation of scores and attribute weights. One benefit of ordinal information is that it does not necessitate a quantitative measurement scale; for example, stakeholders' disagreements about appropriate quantitative measures can be resolved by using ordinal information, as in the application by Grushka-Cockayne et al. (2008). There are several methods which first elicit a rank-ordering for the attribute weights or apply Likert-type scales to classify alternatives, and then convert these ordinal statements into numerical point estimates (see Barron and Barrett 1996). The decision recommendations provided by such conversions can, however, be sensitive to the selected numerical point estimates.

The Preference Programming methods avoid these conversions by modeling pairwise comparisons between attribute weights (e.g., Kirkwood and Sarin 1985, Hazen 1986) and scores (Salo and Hämäläinen 2001) as constraints on the model parameters. Most of these methods assume that the DM provides either a full rank-ordering of the alternatives or the attributes, or that she makes several individual separate pairwise comparisons among them. In the RICH method (Salo and Punkka 2005), the relative importance of the attributes is modeled with *incomplete ordinal information*. The DM can, for example, state that attributes *cost* and *environmental aspects* are among the three most important ones. The method, however, does not capture other forms of preference statements about the relative importance of attributes and the alternatives' attribute-specific values can be evaluated through score intervals, only.

The Preference Programming model developed in this paper captures *incomplete ordinal information* about both the alternatives' scores and the attribute weights. Such information is elicited through statements which associate a set of alternatives or attributes with a set of rankings. For example, the DM can specify three alternatives of which one has the ranking one (i.e., it is the most preferred) with regard to attribute *environmental aspects*; or she can specify the two most important attributes; or state that a given alternative is among the three least preferred ones. Such preference statements can be particularly helpful in modeling a joint preference representation for a group of DMs who cannot agree on exact numerical parameter values. For example, if each group member is asked to specify the most preferred alternative with regard to environmental aspects, the group's preferences can be expressed by a statement that the most preferred alternative is among the ones that are specified by the group members. Similarly, for example the group's statement concerning an alternative's ranking can be formed by taking the union of its members' opinions. The form of the statements is flexible in that the statements can be given with regard to any subset of attributes and within any set of alternatives (or attributes). Taken together, incomplete ordinal information can be used to classify attributes or alternatives without assigning exact numerical values to these classes, for example.

Mathematically, the set of weights and scores that are consistent with incomplete ordinal statements can be non-convex, as shown by Salo and Punkka (2005) in the context of attribute weights. In this paper, the computational challenges caused by this non-convexity are resolved by employing binary variables to retain the linearity of the model. The decision recommendations can therefore be computed efficiently; in addition, it is possible to introduce even other forms of incomplete information about the alternatives' scores and the attribute weights. This possibility to incorporate both ordinal and numerical information in the same model improves possibilities for capturing the DM's preferences. For example, it makes it possible to exclude extreme realizations of model parameters so that the preference information model better matches the DM's true intensions, as noted by Sage and White (1984). Indeed, if only ordinal information about the alternatives is used, these extreme parameters can rule out dominance relations between alternatives.

The possibility to use both incomplete ordinal and numerical preference statements can be particularly helpful in problems which involve attributes without a natural measurement scale and are therefore difficult to evaluate numerically, or in 'large' problems in which the first phase of the decision support process is to screen the initial set of alternatives. In such cases, even a statement like "these 2 alternatives are among the 5 best ones" may help discard many alternatives through dominance relations so that not all alternatives need to be considered, thus leaving fewer alternatives for more detailed evaluation.

The rest of this paper is organized as follows. Section 2 discusses Preference Programming methods. Section 3 formalizes incomplete ordinal information and constructs the corresponding mixed integer linear formulation. Section 4 considers uses of incomplete ordinal information in decision support. An example is presented in Section 5. Section 6 concludes.

2 Preference Programming

The difficulties of specifying the parameters of an additive value function $\sum w_j v_j(x_j)$ have led to the development of elicitation approaches that accommodate incomplete information about the attribute weights or the alternatives' attribute-specific scores. In these approaches, incomplete information is often modeled through set inclusion, whereby the DM's preferences are captured through preference statements which impose constraints on the model parameters. These statements – such as intervals of scores, weights and weight ratios (e.g., Kirkwood and Sarin 1985, Salo and Hämäläinen 1992, 2001) and ordinal statements (e.g., White et al. 1984, Kirkwood and Sarin 1985, Hazen 1986, Pearman 1993) – lead to linear constraints on the scores and weights (for reviews, see Weber 1987, Salo and Hämäläinen 2010).

In Preference Programming, the (pairwise) dominance relation is used for the comparison of alternatives. Specifically, alternative x^i dominates x^k if its value is higher than or equal to that of x^k for all feasible parameters and strictly higher for some feasible parameters (e.g., White et al. 1982, Kirkwood and Sarin 1985, Hazen 1986). A dominated alternative should not be selected, because the alternative that dominates it yields a higher or equal value for all feasible parameters. A rational DM is thus interested in *non-dominated alternatives*. The set of non-dominated alternatives can become smaller, if the model parameters are further constrained (White et al. 1982). Many early methods provide tailored algorithms which examine the relevant extreme points of the feasible region to resolve dominance relations (e.g., White et al. 1982, Kirkwood and Sarin 1985, Hazen 1986). More generally, the non-dominated alternatives can be computed through linear programming.

Ordinal information consists of comparisons in which numbers are not used to describe strength of the preference. Kirkwood and Sarin (1985) present dominance conditions when strict or weak ordinal statements about the relative importance of the attributes are given. Pearman (1993) extends this work by presenting a computational model which uses an ordered metric that rank-orders attribute weight differences, too. Park et al. (1996) extend this approach by applying the ordered metric for scores.

Hazen (1986) establishes a mathematical relation between the non-dominated and the *potentially optimal alternatives* which have the greatest overall value with some feasible parameters. He also shows how the consistency of preference statements, dominance relations between alternatives, and potentially optimal alternatives can be solved when the preferences are expressed through strict ordinal comparisons of hypothetical alternatives.

The RICH method (Salo and Punkka 2005) introduces the notion of incomplete ordinal information to the elicitation of attribute weights. This information is elicited through statements, which associate a set of rankings with a set of attributes. These sets do not have to be equal in size; statements like "the two attributes a_1 and a_2 are among the three most important ones" or "either a_1 or a_2 is the most important among the three attributes" can be captured. Incomplete ordinal information has been applied for weight elicitation in several applications (e.g., Ojanen et al. 2005, Salo and Liesiö 2006, Liesiö et al. 2007, Mild and Salo 2009).

Because the feasible region for the attribute weights may be non-convex, the computation of decision recommendations in RICH is based on the enumeration of extreme points. This approach has limitations in that only ordinal information can be used to characterize the attribute weights while the alternatives' (normalized) scores can only be modeled as intervals or point estimates.

In the next section, the use of incomplete ordinal information is extended by developing a model for the evaluation of both attributes and alternatives. This model eliminates the above limitations and can therefore be used in conjunction with any other forms of incomplete information that correspond to linear inequalities on the model parameters.

3 Modeling incomplete ordinal information

3.1 Additive value, incomplete information and dominance

There are m alternatives $X^* = \{x^1, \ldots, x^m\}$ which are evaluated with regard to n attributes $A = \{a_1, \ldots, a_n\}$. Each alternative $x^i \in X^*$ is described by a vector of achievement levels $(x_1^i, \ldots, x_n^i)$. If the DM's preferences fulfill certain conditions, such as mutual preference independence and difference independence (Keeney and Raiffa 1976, Dyer and Sarin 1979), the alternatives' *overall values* can be modeled with a measurable additive value function

$$V(x^i) = \sum_{j=1}^{n} v_j(x_j^i), \tag{1}$$

in which $v_j(x_j^i)$ is the value of alternative x^i with regard to attribute a_j (i.e., *score*).

It is customary to choose the most and least preferred achievement levels x_j^* and x_j°, respectively, for each attribute a_j so that the scores of the alternatives in X^* fulfill $v_j(x_j^\circ) \leq v_j(x_j^i) \leq v_j(x_j^*)$ in which at least the other of the inequalities is strict. We follow the usual convention and normalize the value function so that $v_j(x_j^\circ) = 0, j = 1, \ldots, n$ and $V(x^*) = \sum_{j=1}^{n} v_j(x_j^*) = 1$.

After this normalization, the overall value (1) can be expressed

$$V(x^i) = \sum_{j=1}^{n} v_j(x_j^i) = \sum_{j=1}^{n} [v_j(x_j^*) - v_j(x_j^\circ)] \frac{v_j(x_j^i)}{[v_j(x_j^*) - v_j(x_j^\circ)]} = \sum_{j=1}^{n} w_j v_j^N(x_j^i), \tag{2}$$

in which the value difference $w_j = v_j(x_j^*) - v_j(x_j^\circ) = v_j(x_j^*)$ is the *attribute weight* of a_j and $v_j^N(x_j^i) = v_j(x_j^i)/v_j(x_j^*) \in [0,1]$ is the *normalized score* of x^i with regard to this attribute. The multiplication of these two terms in (2), however, leads to non-linearities and consequently poses computational challenges when analyzing incomplete preference information about the weights and scores. The following formulations are therefore based on representation (1), in the recognition that attribute weights and normalized scores are consistent with this representation through (2). For notational purposes, the set X^* includes all hypothetical alternatives that are employed in preference elicitation. Specifically, $X^* \supset \{x^{1*}, \ldots, x^{n*}\}$ for which $x_j^{i^*} = x_j^*$, if $i = j$ and $x_j^{i^*} = 0$ otherwise, because the overall values of these alternatives are equal to the attribute weights: $V(x^{j*}) = v_j(x_j^{j*}) + \sum_{l \neq j} v_l(x_l^{j*}) = v_j(x_j^*) = w_j$. The scores $s_{ij} = v_j(x_j^i)$ are recorded in the matrix

$$s = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mn} \end{pmatrix} = \begin{pmatrix} v_1(x_1^1) & \cdots & v_n(x_n^1) \\ \vdots & \ddots & \vdots \\ v_1(x_1^m) & \cdots & v_n(x_n^m) \end{pmatrix}.$$

The above normalization conditions are satisfied by scores that belong to the set

$$S_0 = \{s \in \mathbb{R}^{m \times n} \mid 0 \leq s_{ij} \leq w_j, i = 1, \ldots, m, j = 1, \ldots, n, \sum_{j=1}^{n} s_{j^*j} = 1\},$$

in which $w_j = v_j(x_j^*) = s_{j^*j}$.

In Preference Programming, incomplete information leads to constraints on feasible scores. For example the statement "the normalized score of x^1 with regard to attribute a_1 is between 0.5 and 0.6" would lead to the constraint $0.5 s_{1^*1} \leq s_{11} \leq 0.6 s_{1^*1}$. Even *hypothetical* alternatives can be utilized in preference statements: "overall, a hypothetical alternative x^3 with normalized scores $v^N(x_j^3) \in [0.5, 0.6]$ for all $j = 1, \ldots, n$ is at least as preferred as x^2, but not preferred to x^1" leads to constraints $\sum_{j=1}^n s_{2j} \leq \sum_{j=1}^n s_{3j} \leq \sum_{j=1}^n s_{1j}$ and $0.5 s_{j^*j} \leq s_{3j} \leq 0.6 s_{j^*j}$ $\forall$ $j = 1, \ldots, n$. The statement "the weight of attribute a_2 is not higher than the weight of attribute a_1" can be modeled with the constraint $V(x^{2*}) = s_{2^*2} = w_2 \leq w_1 = s_{1^*1} = V(x^{1*})$. Such statements reduce the set of feasible scores to $S \subset S_0$.

Generally, the alternatives' overall values are different for different feasible scores in S. The concept of dominance compares the overall values of two alternatives with all feasible scores (e.g., White et al. 1982):

Definition 1 *Let $x^i, x^k \in X^*$ and $\emptyset \neq S \subseteq S_0$. Alternative x^i dominates x^k if and only if $V(x^i) \geq V(x^k)\ \forall\ s \in S$ and $\exists s' \in S$ such that $V(x^i) > V(x^k)$.*

By definition, the alternatives in $X \subseteq X^*$ that are not dominated by any alternative in X are *non-dominated* alternatives (among X). It is not advisable to choose a dominated alternative, because the values of those alternatives that dominate it are at least as high for all feasible scores. If x^i dominates x^k with feasible scores S, then x^i dominates x^k also with feasible scores $S' \subset S$, unless their overall values are equal throughout S'. Thus, as a rule, dominated alternatives will remain dominated, even if further preference information is obtained (White et al. 1982). Dominance relations can be checked by linear programming. If the minimum of $[V(x^i) - V(x^k)] = \sum_{j=1}^{n} [s_{ij} - s_{kj}]$ over S is negative, then x^i does not dominate x^k. If it is positive, then x^i dominates x^k. If it is zero, computing the maximum of $[V(x^i) - V(x^k)]$ over S reveals whether there also exists a feasible score such that $V(x^i) > V(x^k)$.

The DM need not define all the alternatives that are used in preference elicitation or compared at the outset of the analysis. Introduction of a new alternative x^h, for which the normalization conditions $v_j(x_j^\circ) \leq v_j(x_j^h) \leq v_j(x_j^*)$ hold, introduces new score variables $s_{hj}, j = 1, \ldots, n$, but does not constrain the other scores $s_{ij}, i \neq h$. Thereby also all dominance relations between pairs of alternatives in $X \not\ni x^h$ remain unchanged if x^h is added to X^*. Preference statements that involve x^h can then result in new dominance relations also between the alternatives in X.

Because the set X^* includes alternatives whose overall values are equal to the attribute weights, all the results on capturing statements about the alternatives' values in Sections 3.2–3.5 can also be applied to the elicitation of attribute weights.

3.2 Incomplete ordinal information

Complete ordinal information of alternatives X is a full *rank-ordering* that associates a unique *ranking* with each alternative $x^i \in X$. Formally, if $X \subseteq X^*$ is a non-empty set of alternatives, a rank-ordering is a bijection $r : X \mapsto \{j \in \mathbb{Z} \mid 1 \leq j \leq |X|\}$ (Salo and Punkka 2005; throughout the paper $|B|$ denotes the number of elements in the set B). Rank-orderings are denoted by the convention $r(X) = (r_1, \ldots, r_{|X|})$ in which $r_k = r(x^l; X)$

is the ranking of the alternative with the k-th smallest index in X (i.e., $|\{x^i \in X \mid i < l\}| = k-1$). We use rank-orderings such that if the ranking of alternative x^k is smaller than that of x^i, then x^k is either strictly preferred to x^i or they are equally preferred. For example, if $X = \{x^1, x^4, x^5\}$, then the rank-ordering $r(X) = (1, 3, 2)$ corresponds to rankings $r(x^1; X) = 1$, $r(x^4; X) = 3$ and $r(x^5; X) = 2$, meaning that x^1 is the most preferred alternative of the three (or, more specifically, no other alternative is preferred to x^1), followed by x^5 and then x^4. The rank-orderings can be defined with regard to any non-empty attribute set $A' \subseteq A$.

Incomplete ordinal information refers to ordinal statements which associate a set of alternatives (denoted by I) with a set of rankings (J) (cf. attribute weight elicitation proposed by Salo and Punkka 2005). For example, if $I = \{x^1\}$ and $J = \{1\}$, then the ranking of alternative x^1 is one. Thus one such a statement does not specify unique rankings for all alternatives if there are more than two alternatives.

Definition 2 *(I, J, X, A') is an incomplete ordinal statement (IOS), if $\emptyset \neq A' \subseteq A$, $\emptyset \neq I \subseteq X \subseteq X^*$, and $\emptyset \neq J \subseteq \{j \in \mathbb{Z} \mid 1 \leq j \leq |X|\}$.*

In Definition 2, the set X includes the alternatives under comparison and A' is the set of attributes with regard to which this comparison is made. Whenever there is no risk of confusion, references to these sets are omitted by adopting the notational conventions $r(X) = r$ and $(I, J) = (I, J, X, A')$.

Definition 2 does not assume that sets I and J have equally many elements. Indeed, if the number of alternatives in I is equal to or less than the number of rankings in J (that is, $|I| \leq |J|$), then the ranking of each alternative in I is in J. For example, the statement that x^1 is either the best or the second best one of $\{x^1, x^2, x^3\}$ with regard to attribute a_4 is captured by the IOS $(I, J, X, A') = (\{x^1\}, \{1, 2\}, \{x^1, x^2, x^3\}, \{a_4\})$. This statement is compatible with four rank-orderings $r = (r(x^1; X), r(x^2; X), r(x^3; X))$, that is, $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$ and $(2, 3, 1)$.

On the other hand, if $|I| > |J|$, it is assumed that the rankings in J are attained by alternatives that are in I. For example, stating that some two alternatives among x^1, x^4 and x^7 are holistically the two most preferred alternatives in the set $X = \{x^1, \ldots, x^{10}\}$ can be modeled as $(I, J, X, A') = (\{x^1, x^4, x^7\}, \{1, 2\}, \{x^1, \ldots, x^{10}\}, A)$.

The rank-orderings that are compatible with an IOS (I, J, X, A') are denoted by $R(I, J, X, A')$ (or $R(I, J)$ for short) and defined as follows (Salo and Punkka 2005).

Definition 3 *Let (I, J, X, A') be an IOS. The set of compatible rank-orderings is*

$$R(I,J) = \begin{cases} \{r(X) \in R(X) \mid \forall\ j \in J\ \exists\ x^k \in I \text{ s.t. } r(x^k;X) = j\}, & \text{if } |I| \geq |J| \\ \{r(X) \in R(X) \mid r(x^k;X) \in J\ \forall\ x^k \in I\}, & \text{if } |I| < |J| \end{cases}, \quad (3)$$

in which $R(X)$ is the set of all rank-orderings defined on X.

3.3 Characterizing the feasible region

Statements about the alternatives' rankings constrain feasible scores. Let $v(x^i, A') = v(x^i) = \sum_{\{j \mid a_j \in A'\}} v_j(x^i_j)$ denote the value of alternative x^i with regard to attributes A'. The alternative with ranking 1 cannot have a lower value than the alternative with ranking 2, and so on: $r(x^i) < r(x^k) \Rightarrow v(x^i) \geq v(x^k)$. In other words, the ranking $r(x^i)$ implies that there are $r(x^i) - 1$ alternatives in X whose value is not strictly smaller and $|X| - r(x^i)$ alternatives, whose value is not strictly higher than that of x^i.

The scores that are consistent with a single rank-ordering $r(X)$ are

$$S(r(X)) = \{s \in S_0 \mid v(x^i) \geq v(x^j) \text{ if } r(x^i;X) < r(x^j;X),\ x^i, x^j \in X\}, \quad (4)$$

and the scores that fulfill statement (I, J, X, A') are

$$S(I,J) = \bigcup_{r \in R(I,J)} S(r(X)). \quad (5)$$

The set $S(I, J)$ is not necessarily convex (Salo and Punkka 2005) and consequently cannot be modeled through linear constraints on scores. We therefore develop a characterization of $S(I, J)$ that employs binary variables $y_j(x^i)$ to constrain compatible rank-orderings $R(I, J)$ such that $y_j(x^i) = 1 \Rightarrow r(x^i) \leq j$ and $y_j(x^i) = 0 \Rightarrow r(x^i) > j$ for any $r \in R(I, J)$. Binary variables $y_j(x^i)$ are defined for all $x^i \in X$, but the indices $j \in \{1, \ldots, |X| - 1\}$ that are employed depend on the statement.

First, assume that J consists of consecutive rankings: $J = \{j \mid \underline{j} \leq j \leq \overline{j}\}$, $\underline{j} \leq \overline{j}$. If $|I| < |J|$, the ranking of each alternative $x^i \in I$ is in J. Because J consists of consecutive rankings, this implies that the ranking of any $x^i \in I$ is at most $\overline{j}$, but not smaller than $\underline{j}$. With the above interpretation of the binary variables, this can be captured with constraints $y_{\overline{j}}(x^i) = 1$ and $y_{\underline{j}-1}(x^i) = 0$, which can be written $y_{\overline{j}}(x^i) - y_{\underline{j}-1}(x^i) = 1 \; \forall \; x^i \in I$. For example, if $I = \{x^1\}$, and $J = \{2, 3\}$, then the ranking of x^1 is two or three. That is, it is at most three ($y_3(x^1) = 1$), but not smaller than two ($y_1(x_1) = 0$).

If $|I| \geq |J|$, according to Definition 3 each ranking in J is attained by an alternative in I. Therefore, alternatives that are not in I have rankings that are either smaller than $\underline{j}$ or greater than $\overline{j}$. Thus, if the ranking of $x^i \in \overline{I} = \{x^i \in X \mid x^i \notin I\}$ is at most $\overline{j}$ (i.e., $y_{\overline{j}}(x^i) = 1$), it must also be smaller than $\underline{j}$ (i.e., $y_{\underline{j}-1}(x^i) = 1$). If, on the contrary, the ranking of $x^i \in \overline{I}$ is higher than $\overline{j}$ (i.e, $y_{\overline{j}}(x^i) = 0$), then it is also bigger than $\underline{j}$ (i.e., $y_{\underline{j}-1}(x^i) = 0$). Constraints $y_{\overline{j}}(x^i) = y_{\underline{j}-1}(x^i) \; \forall \; x^i \in \overline{I}$ capture this inference. For example, the statement $(I, J) = (\{x^1, x^2\}, \{2\})$ imposes the requirement that the ranking of x^i, $i \notin \{1, 2\}$, is not 2. If it is at most 2 (i.e., $y_2(x^i) = 1$), then it must also be smaller than 2 (i.e., $y_1(x^i) = 1$); but if it is higher than 2 (i.e., $y_2(x^i) = 0$), then it cannot simultaneously be smaller than 2 (i.e., $y_1(x^i) = 0$). Ranking 2 is then attained by x^1 or x^2, because r is a bijection. Figure 1 illustrates the above inference.

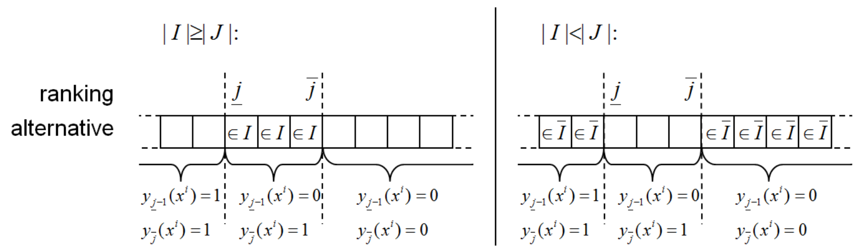

Figure 1: Modeling an IOS with the binary variables, when J consists of consecutive integers

To model the scores $S(I, J)$, we introduce a real-valued milestone variable z_j to distinguish between the values of alternatives with rankings j and $j + 1$. In view of (4), (5) and the above interpretation of the binary variables, if $y_j(x^i) = 1$, then $v(x^i) \geq z_j$; and if $y_j(x^i) = 0$, then $v(x^i) \leq z_j$. Lemma 1 presents a mixed integer linear characterization for $S(I,J)$ under the assumption that J consists of consecutive integers. All proofs are in the Appendix.

Lemma 1 *Let (I,J) be an IOS such that $J = \{j \mid \underline{j} \leq j \leq \overline{j}\}$, $\underline{j} \leq \overline{j}$, and let constant $M > 1$. Then $s \in S(I,J)$ if and only if $s \in S_0$ such that constraints*

$$\begin{aligned} z_j &\leq v(x^i) + (1 - y_j(x^i))M \quad \forall\ x^i \in X & (6)\\ v(x^i) &\leq z_j + y_j(x^i)M \quad \forall\ x^i \in X & (7)\\ \sum_{x^i \in X} y_j(x^i) &= j & (8) \end{aligned}$$

hold for $v(x^i)$ and some $z_j \in [0,1], y_j(x^i) \in \{0,1\}$, $j \in \{\underline{j}-1, \overline{j}\} \setminus \{0, |X|\}$ so that

$$\begin{cases} y_{\overline{j}}(x^i) = y_{\underline{j}-1}(x^i) & \forall x^i \in \overline{I}, \text{ if } |I| \geq |J| \\ y_{\overline{j}}(x^i) - y_{\underline{j}-1}(x^i) = 1 & \forall x^i \in I, \text{ if } |I| < |J| \end{cases} \quad (9)$$

also hold (with notational conventions $y_0(x^i) = 0, y_{|X|}(x^i) = 1$).

For example, the statement "alternatives x^2 and x^3 are the two most preferred among $\{x^1, x^2, x^3, x^4\}$ with regard to attribute a_1" is modeled with the IOS $(I, J, X, A') = (\{x^2, x^3\}, \{1, 2\}, \{x^1, x^2, x^3, x^4\}, \{a_1\})$ and can be characterized by constraints

$$z_2 \leq s_{i1} + (1 - y_2(x^i))M, \qquad s_{i1} \leq z_2 + y_2(x^i)M, \qquad y_2(x^i) \in \{0,1\} \quad \forall\ i = 1,2,3,4$$

$$y_2(x^1) + y_2(x^2) + y_2(x^3) + y_2(x^4) = 2,\ z_2 \in [0,1],\ y_2(x^1) = y_0(x^1) = 0,\ y_2(x^4) = y_0(x^4) = 0.$$

Because the only feasible solution for the binary variables is $(y_2(x^1), y_2(x^2), y_2(x^3), y_2(x^4)) = (0,1,1,0)$, the above constraints simplify to $z_2 \leq s_{21}$, $z_2 \leq s_{31}$, $z_2 \geq s_{11}$ and $z_2 \geq s_{41}$. Indeed, Lemma 1 does not employ binary variables to model convex feasible regions. According to Salo and Punkka (2005), $S(I,J)$ is convex if and only if $J = \{1, \ldots, |I|\}$ or $J = \{|X| - |I| + 1, \ldots, |X|\}$. With such sets J, the constraints (6)–(9) simplify to

$$\begin{aligned} &z_{|I|} \leq v(x^i)\ \forall\ x^i \in I, && z_{|I|} \geq v(x^i)\ \forall\ x^i \in \overline{I} && \text{if } J = \{1, \ldots, |I|\} \text{ or} \\ &z_{|X|-|I|} \geq v(x^i)\ \forall\ x^i \in I, && z_{|X|-|I|} \leq v(x^i)\ \forall\ x^i \in \overline{I}, && \text{if } J = \{|X| - |I| + 1, \ldots, |X|\}. \end{aligned}$$

The rationales for modeling $S(I,J)$ with $J = \{\underline{j}, \ldots, \overline{j}\}$ can be applied to model a general $S(I,J)$. Any set J can be partitioned into subsets J_k, which each consist of consecutive rankings. The *min-partition* of set J is the unique division of J into a minimal number of such sets J_k.

Definition 4 *Sets J_k, $k = 1, \ldots, K$, are the min-partition of non-empty $J \subset \{1, \ldots, |X|\}$, if $J = \cup_{k=1}^{K} J_k$, $\emptyset \neq J_k = \{j \mid \underline{j_k} \leq j \leq \overline{j_k}\}$ for some $\underline{j_k}, \overline{j_k}$, for all $k = 1, \ldots, K$, and $\overline{j_k} < \underline{j_{k+1}} - 1 \;\forall\; k = 1, \ldots, K-1$.*

For example, the min-partition of $J = \{1, 2, 4, 5, 7\}$ is $J_1 = \{1, 2\}, J_2 = \{4, 5\}, J_3 = \{7\}$.

Let sets J_k be the min-partition of J. If $|I| \geq |J|$, the rankings of $x^i \in \overline{I}$ are not in the sets J_k. Therefore, if the ranking of $x^i \in \overline{I}$ is at most $\overline{j_k}$, it must also be smaller than $\underline{j_k}$. If $|I| < |J|$, the ranking of $x^i \in I$ is in J_k for some $k \in \{1, \ldots, K\}$. Thus, if it is smaller than $\underline{j_k}$, it must at most $\overline{j_{k-1}}$. However, the ranking of $x^i \in I$ cannot be smaller than $\underline{j_1}$ or larger than $\overline{j_K}$. Theorem 1 presents a mixed integer linear formulation of $S(I,J)$ for a general IOS.

Theorem 1 *Let (I,J) be an IOS, sets J_k, $k = 1, \ldots, K$ the min-partition of J and $j_\cup(J) = \cup_{k=1}^{K} \{\underline{j_k} - 1, \overline{j_k}\} \backslash \{0, |X|\}$. Then, $s \in S(I,J)$ if and only if $s \in S_0$ such that constraints (6)–(8) hold for $v(x^i)$ and some $z_j, y_j(x^i) \in \{0,1\}$, for all $j \in j_\cup(J)$ so that*

$$\begin{cases} y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i) = 0 \quad \forall k = 1, \ldots, K, \forall x^i \in \overline{I}, & \text{if } |I| \geq |J| \\ y_{\overline{j_{k-1}}}(x^i) - y_{\underline{j_k}-1}(x^i) = 0 \quad \forall k = 1, \ldots, K+1, \forall x^i \in I, & \text{if } |I| < |J| \end{cases} \tag{10}$$

holds (with notational conventions $y_0(x^i) = 0, y_{|X|}(x^i) = 1$, $y_{\overline{j_0}}(x^i) = 0, y_{\underline{j_{K+1}}-1}(x^i) = 1$).

For example, the statement "alternative x^2 is either the most preferred or the least preferred with regard to a_1, among $X = \{x^1, \ldots, x^5\}$" is modeled with the IOS $(\{x^2\}, \{1, 5\}, X, \{a_1\})$. The min-partition of $J = \{1, 5\}$ is $J_1 = \{1\}, J_2 = \{5\}$ and the index set $j_\cup(J) = \{\underline{j_1} - 1, \overline{j_1}, \underline{j_2} - 1, \overline{j_2}\} \backslash \{0, |X|\} = \{0, 1, 4, 5\} \backslash \{0, 5\} = \{1, 4\}$. By Theorem 1, the constraints (6)–(8) become

$$z_1 \leq s_{i1} + (1 - y_1(x^i))M, \quad s_{i1} \leq z_1 + y_1(x^i)M \qquad \forall\; x^i \in X, \quad \sum_{i=1}^{5} y_1(x^i) = 1$$

$$z_4 \leq s_{i1} + (1 - y_4(x^i))M, \quad s_{i1} \leq z_4 + y_4(x^i)M \qquad \forall\; x^i \in X, \quad \sum_{i=1}^{5} y_4(x^4) = 4.$$

Because $|I| < |J|$, (10) yields the constraint

$$y_{\overline{j_1}}(x^2) - y_{\underline{j_2}-1}(x^2) = y_1(x^2) - y_4(x^2) = 0$$

with $k = 2$ and the redundant equations $y_{\overline{j_0}}(x^2) - y_0(x^2) = 0$ and $y_5(x^2) - y_{\underline{j_{K+1}}-1}(x^2) = 0$ with $k = 1$ and $k = K + 1 = 3$, respectively.

If there are P statements, the set of feasible scores is the intersection $\cap_{p=1}^{P} S(I^p, J^p, X^p, A^p)$ with independent variables $z_j^p, y_j^p(x^i)$ for the required indices j. However, if in some of these statements both X and A' are equal, the computational effort required to solve dominance relations in $\cap_{p=1}^{P} S(I^p, J^p, X^p, A^p)$ can be decreased.

3.4 Computational considerations

If $L > 1$ statements (I^l, J^l, X, A') are given among the same alternatives X, with regard to the same attributes A', these statements concern the same rank-orderings $R(X)$ and the same values $v_{A'}$. Then, the intended interpretation $y_j(x^i) = 1$ if and only if $r(x^i) \leq j$ allows the binary and milestone variables related to these statements to be modeled to depend on each other. This observation helps reduce the number of milestone and binary variables and makes it possible to impose more constraints on the binary variables. These modifications, in turn, improve computational performance.

More specifically, for any index j which is used in modeling statements (I^l, J^l, X, A') (that is, $j \in \cup_{l=1\ldots,L} j_\cup(J^l)$), only one set of variables z_j^l and $y_j^l(x^i)$ is needed. Furthermore, across all the L statements (I^l, J^l, X, A') the binary variables can be assumed non-increasing in the sense that $y_j(x^i) \leq y_{j'}(x^i)$ whenever $j' > j$. For example, constraint $y_1(x^i) \leq y_3(x^i)$ can be introduced for any x^i, because violating this constraint would require that the ranking of x^i is one $(y_1(x^i) = 1)$ and simultaneously higher than three $(y_3(x^i) = 0)$. Lemma 2 shows that for each $s \in \cap_{l=1,\ldots,L} S(I^l, J^l)$, there exists a solution $z_j, y_j(x^i)$ for $j \in \cup_{l=1}^{L} j_\cup(J^l), x^i \in X$ that fulfills these conditions and constraints (6)–(8) and (10).

Lemma 2 *Let (I^l, J^l, X, A'), $l = 1, \ldots, L$, be IOSs such that $\bigcap_{l=1,\ldots,L} R(I^l, J^l, X, A') \neq \emptyset$ and let $J_{\text{ind}} = \cup_{l=1}^{L} j_\cup(J^l)$. Let $S^L \subset S^0$ be a set of scores such that (6)–(8) hold for $v(x^i)$, $y_j(x^i) \in \{0, 1\}, z_j \in [0, 1]\ \forall\ x^i \in X, j \in J_{\text{ind}}$, so that constraints*

$$y_{j_1}(x^i) \leq y_{j_2}(x^i) \qquad \forall j_1 \in J_{\text{ind}} \setminus \{\max J_{\text{ind}}\}, j_2 = \min\{j \in J_{\text{ind}} \mid j > j_1\} \tag{11}$$

and (10) on statements (I^l, J^l)*,* $l = 1, \ldots, L$ *also hold. Then,* $\cap_{l=1}^{L} S(I^l, J^l) = S^L$*.*

According to Lemma 2, *monotonicity constraints* can be introduced for binary variables which relate to same sets X and A', and each index j requires at most one set of variables $z_j, y_j(x^i)$. For example, the two statements $(I^1, J^1) = (\{x^1, x^2, x^3\}, \{1, 2\}), (I^2, J^2) = (\{x^4, x^5, x^6\}, \{3\})$ can be modeled with the two indices in $J_{\text{ind}} = j_\cup(J^1) \cup j_\cup(J^2) = \{\min J^1 - 1, \max J^1\} \setminus \{0, |X|\} \cup \{\min J^2 - 1, \max J^2\} \setminus \{0, |X|\} = \{2\} \cup \{2, 3\} = \{2, 3\}$ instead of three. As a consequence, the number of binary variables drops from $3|X|$ to $2|X|$. The constraints $y_2(x^i) \leq y_3(x^i) \; \forall \; x^i \in X$ also reduce the number of possible binary variable combinations. Furthermore, the introduction of monotonicity constraints simplifies constraints (10) of Theorem 1 as follows.

Lemma 3 *Let* (I, J) *be an IOS and let (11) hold for* $J_{\text{ind}} = j_\cup(J)$*. Then, constraints (10) are equivalent to constraints*

$$\begin{cases} \sum_{x^i \in \overline{I}} \sum_{k=1}^{K} [y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i)] = 0, & if \; |I| \geq |J| \\ \sum_{x^i \in I} \sum_{k=1}^{K} [y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i)] = |I|, & if \; |I| < |J| \end{cases} .$$

3.5 Numerical statements and incomplete ordinal information

The feasible region that results from several incomplete ordinal statements $(I^l, J^l, X^l, A^{l\prime})$ always includes score matrix s such that $s_{ij} = s_{kj} = 0$ for all $j = 1, \ldots, n$. If sets X^l do not include alternatives whose values relate to the attribute weights, such equal scores are feasible even if the attribute weights are completely specified. Consequently, the minimum of the value difference $V(x^i) - V(x^k)$ is not positive, and dominance relations are not likely. For example, if the statements $(I^l, J^l, X^l, A^{l\prime})$ are consistent with each other and each of them evaluates alternatives with regard to one attribute (that is, sets A^l include only one attribute each), then x^i does not dominate x^k unless (i) all attributes are considered ($\cup_l A^l = A$) and (ii) x^i is ranked better than x^k with regard to all attributes. Indeed, numerical statements are needed to complement ordinal information to exclude such extreme realizations of parameters, as suggested by Sage and White (1984).

In practice, one often-used procedure to evaluate alternatives is to divide them into classes so that alternatives in 'class one' (alternatives I^1) are preferred to those in 'class two' (I^2), and so on. Such classification into p classes can be modeled with $p-1$ statements (I^k, J^k) in which $J^k = \{j \mid \sum_{l=0}^{k-1} |I^l| + 1 \leq j \leq \sum_{l=1}^{k} |I^l|\}$ for $k = 1, \ldots, p-1$, $|I^0| = 0$. Often this ordinal information is complemented by assigning equal values to all alternatives in the same class to represent value differences between the classes (e.g., Lindstedt et al. 2007, Salo and Liesiö 2006, Könnölä et al. 2007, see also Corner and Kirkwood 1991). Our framework makes it possible to relax the assumption that all alternatives in the same class have exactly the same value. Yet, it makes it possible to define intervals for the values that the alternatives in a class can have and to constrain value differences between the classes. Thus, incomplete ordinal information can be used for *ex ante* sensitivity analysis on the values associated with the classes, for example.

Technically, the alternatives' values and value differences can be bounded by constraints on the milestone variables. Variable z_j is a lower bound on the value of the alternative with ranking j – and consequently also on the values of alternatives with rankings smaller than j – and, similarly, an upper bound on the value of the alternatives with rankings $j+1, \ldots, |X|$. Specifically, the following numerical bounds help complement the above classification procedure, but they can technically be used to complement other statements, too.

a) A statement that the value of the alternative with ranking j is at least $\underline{b}_j$ can be modeled with the constraint $z_j \geq \underline{b}_j$.

b) A statement that the value of the alternative with ranking k is at most $\overline{b}_k$ can be modeled with the constraint $z_{k-1} \leq \overline{b}_k$.

c) A statement that the value difference between the alternatives with rankings j and k ($k > j$) is at most $\overline{d}_{j,k}$ can be modeled with the constraint $z_{j-1} - z_k \leq \overline{d}_{j,k}$.

d) A statement that the value difference between the alternatives with rankings j and k ($k > j$) is at least $\underline{d}_{j,k}$ can be modeled with the constraint $z_j - z_{k-1} \geq \underline{d}_{j,k}$, if $k - j > 1$. If $k - j = 1$, then the constraints (7) can be transformed into $v(x^i) + \underline{d}_{j,k} \leq z_j + y_j(x^i)M$.

If variables z_j are not used to model any IOS and therefore not defined, a redundant statement $(I, J) = (X, \{1, \ldots, j\})$ can be introduced to allow numerical statements that employ z_j and $y_j(x^i)$. The above bounds constrain the milestone variables and consequently the scores of all alternatives in X, as well. Because this leads to a smaller feasible region for scores, dominance relations hold also after the introduction of these constraints. Therefore, the dominated alternatives can be ignored in the specification of these bounds if the DM is not interested in how they compare with non-dominated ones.

4 Implications for decision support

The proposed model admits several kinds of ordinal preference statements as a complement to numerical information about the parameters of an additive value model. This is useful, because multi-criteria decision analysis methods are often applied to problems in which some attributes are described by numerical database entries, and for some attributes natural measurement scales do not exist (e.g., Mild and Salo 2009). Although numerical evaluation of the latter attributes is used and may even be required to reach a single non-dominated alternative, the use of incomplete ordinal information on these attributes can reflect the DM's preferences better and help identify and discard dominated alternatives. Indeed, *screening* is often the first task of a decision support process. It reduces the set of relevant alternatives and guides further work on data collection: As additional statements keep the already established dominance relations intact and may establish new ones, all evaluations and assessments can be focused on non-dominated alternatives. In effect, incomplete ordinal information offers novel possibilities for screening when only some attributes are described by numerical data or statements (cf. the example in Section 5). Discarding alternatives with incomplete ordinal information before thorough numerical assessment or data collection can save time and costs, because fewer alternatives need to be evaluated numerically.

In group settings, the DMs may have different opinions about the alternatives' performance or the relative importance of the attributes. In addition, their ability to understand and to answer the elicitation questions can differ. If the group seeks to establish a joint preference characterization, incomplete ordinal information can be used to model these different opinions. First, despite of the DMs' differences, they may still agree on 'less restrictive' statements, such as "cost is among the three most important attributes", or "alternative x^1 is not the most preferred". If the group's analysis is assisted by a facilitator, she can even suggest such statements based on the group's discussion. Ordinal information can even help the group focus its discussion on differences among their opinions instead of arguing over numerical parameters.

Second, the flexibility of ordinal statements makes it possible to employ several preference elicitation *processes* in group settings. For example, all group members $k = 1, \ldots, K$ can be asked to choose the three most preferred alternatives I_k, whereafter the group's preferences are described by the IOS $(\cup_k I_k, \{1, 2, 3\})$ so that all alternatives with 'votes' can be among the three most preferred alternatives. This group's statement can be then specified by removing alternatives from the set $I = \cup_k I_k$ – on the condition that more than three alternatives have received votes. If an alternative is among the three most preferred ones by all DMs, an additional statement $(\cap_k I_k, \{1, 2, 3\})$ can be introduced. Considering different attributes – or even all attributes holistically – with different sets J leads to additional statements and consequently reduces the set of feasible scores.

5 An example

We consider an illustrative example in which a student uses an additive model to help her select a business school for an MBA degree. The alternatives consist of the thirty US business schools which ranked best in the 2009 The Financial Times (FT) ranking (www.ft.com 2009). The student's aim is to discard some of these schools with an additive multi-attribute model and to then examine more closely the schools that remain non-dominated after her analysis.

The model consists of seven attributes, five of which are based on the data and the results of the FT 2009 annual business school ranking. The additional two *non-educational* attributes account for possibilities to pursue her hobbies. The evaluations with regard to these attributes have been added to this example to illustrate the application of subjective incomplete ordinal statements.

The student performs the analysis in two phases. First, she makes use of the information in the FT's data and results, and provides incomplete ordinal statements with regard to non-educational attributes. In the second phase, information on the non-educational attributes is further specified by acquiring more information about the schools that remain non-dominated after the preference statements in the first phase. The schools are denoted by $X = \{x^1, \dots, x^{30}\}$. The attribute-specific evaluations are explained in the following. The data is summarized in Table 1.

a_1: **FT overall ranking performance** is evaluated using the interval of the schools' overall rankings among X in 2007–2009 such that the best of the three rankings defines the lower bound $\underline{j_1}(x^i)$ and the worst ranking $\overline{j_1}(x^i)$ defines the upper bound of the ranking interval. This information is modeled with the preference statement $(I, J, X, A') = (\{x^i\}, \{j \mid \underline{j_1}(x^i) \leq j \leq \overline{j_1}(x^i)\}, X, \{a_1\})$ for each school. However, if school x^k outperforms school x^i in each of the three rankings, an additional constraint $v_1(x_1^k) \geq v_1(x_1^i)$ is introduced even if the ranking intervals overlap.

School x^1 outperforms all other schools in each of the three rankings and is assigned a $[0, 1]$-normalized score 1. No school is worst in all three rankings, and the student considers a fictitious school that ties ranking 30 each year to correspond to the least preferred level. Thus, constraint $z_{29} = 0$ is introduced. Comparing to these two points of reference, the student introduces two further milestones: all alternatives which are ranked among the ten best ones have normalized score of at least 0.5 and those outside of the top twenty cannot exceed score of 0.2. These statements lead to constraints $z_{10} \geq 0.5$ and $z_{20} \leq 0.2$.

a_2: **Salary expectation** is based on the *average alumni salary three years after graduation* (s_A in Table 1) and the *weighted salary* (s_W) measures of the FT ranking. The student models the salary expectation x_2^i as an interval bounded by these two measures. She uses a linear value function which is normalized so that the smallest of the above measures among X, $\min_{x^i \in X}\{S_A(x^i), S_W(x^i)\}$=\$92863, gets a normalized score of 0 and the largest, \$173935, gets 1.

a_3: **Alumni recommendations** are based on a query in which alumni were asked to name three business schools which would be their primary sources of recruitment. In the FT ranking, the results of this query have been converted into a rank-ordering (r_3). In this example, the student exploits this rank-ordering as ordinal information about the scores $v_3(x_3^i)$ (see (4)). The best-ranked and worst-ranked schools define the most and the least preferred levels, respectively.

a_4: **Employment expectations** are modeled in the FT ranking using the percentage of graduates that are employed within three months after graduation ($p(x^i)$). The share of students whose employment information is recorded in the FT data ($c(x^i)$) is known, but varies from school to school. The student does not make assumptions about the employment of the alumni whose information is not available. She therefore accounts for all possibilities between the extreme cases that none of them and all of them are employed. This assumption results in the intervals $x_4^i \in [c(x^i)p(x^i), c(x^i)p(x^i) + (1 - c(x^i))]$. She applies a linear value function such that the smallest employment rate among the schools (60 %) corresponds to the least preferred level and 100 % to the most preferred.

a_5: **Female students** is measured through the share of female students $x_5^i \in [0, 100\%]$ in the FT data. In all schools in X, the share of female students is below 50 %, and the student focuses on modeling the value between 0 and 50 %. In this range, the student prefers more female students to less. She is also indifferent between changes from 0 % to 15 % and from 15 % to 50 %. She approximates her preferences with $v_5^N(x_5) = ax_5^b$ in which a and b are defined so that $v_5^N(0.15) = 0.5$ and $v_5^N(0.5) = 1$.

a_6: **Hobby 1** is evaluated by the student. She divides the schools into five classes and constrains the normalized scores of the schools in these classes. *Excellent* schools are referred to by 'E' in Table 1 and their normalized scores fulfill $v_6^N \geq 0.9$. They are preferred to *very good* (VG; $v_6^N \geq 0.6$), which are preferred to *good* (G; $v_6^N \leq 0.8$), followed by *fair* (F; $v_6^N \geq 0.1$) and finally *poor* (P; $v_6^N \leq 0.1$). The schools that belong to the same class can have unequal scores. The most and least preferred levels are not necessarily achieved by any of the schools in X.

a_7: **Hobby 2** is evaluated by the student. Her evaluation is based on an article, in which three persons familiar with this hobby have named ten best and five worst universities of X with regard to this hobby. These lists are quite close to each other: there are eight schools (schools $I^{\mathrm{T10}} \subset X$) that belong to all three top ten lists and four schools (I^{B5}) that are assigned to bottom five in all three lists. Based on agreed schools, the student introduces preference statements $(I^{\mathrm{T10}}, \{1, \ldots, 10\})$ and $(I^{\mathrm{B5}}, \{26, \ldots, 30\})$. To exclude equal scores among all alternatives, she bounds the top ten schools' normalized scores from below by 0.7, which also serves as an upper bound for schools whose ranking is worse than ten. Similarly, the normalized score 0.1 distinguishes between the five least preferred and the other schools. The most and least preferred levels are not necessarily achieved by any of the schools in X.

The attribute weights are constrained by (i) two incomplete ordinal statements $(I, J) = (\{a_1, a_2, a_3\}, \{1, 2, 3\})$ and $(\{a_4, a_5\}, \{4, 5\})$, (ii) ordinal statements $w_2 \geq w_1$, $w_6 \geq w_7$ and (iii) lower bound $w_i \geq 0.05$ for all attributes.

Table 1: Business schools

	a_1				a_2		a_3			a_4			a_5	a_6	a_7
	$\underline{j_1}$	$\overline{j_1}$	s_A	s_W	$v_2^N(\underline{x_2^i})$	$v_2^N(\overline{x_2^i})$	r_3	p	c	$v_4^N(\underline{x_4^i})$	$v_4^N(\overline{x_4^i})$	x_5^i	$v_5^N(x_5^i)$	x_6^i	x_7^i
x^1	1	1	170210	169784	0.95	0.95	2	89 %	98 %	0.68	0.96	36 %	0.83	VG	
x^2	2	4	164783	163637	0.87	0.89	1	93 %	99 %	0.80	0.98	38 %	0.85	E	T10
x^3	2	3	168073	164310	0.88	0.93	6	92 %	93 %	0.64	0.84	32 %	0.77	VG	T10
x^4	3	4	173935	170340	0.96	1.00	3	93 %	99 %	0.80	0.98	36 %	0.83	F	
x^5	5	9	155811	156451	0.78	0.78	7	90 %	95 %	0.64	0.89	35 %	0.81	E	
x^6	6	7	147784	144125	0.63	0.68	8	93 %	96 %	0.73	0.91	41 %	0.89	P	T10
x^7	5	7	154340	150272	0.71	0.76	5	94 %	100 %	0.85	0.85	35 %	0.81	E	B5
x^8	7	8	158089	156124	0.78	0.80	9	94 %	100 %	0.85	0.85	33 %	0.79	E	
x^9	8	9	142948	140803	0.59	0.62	16	95 %	98 %	0.83	0.95	34 %	0.80	VG	T10
x^{10}	10	12	142378	142645	0.61	0.61	4	96 %	97 %	0.83	0.93	34 %	0.80	G	
x^{11}	11	14	128612	128692	0.44	0.44	12	92 %	97 %	0.73	0.93	39 %	0.87	VG	B5
x^{12}	12	13	132500	132522	0.49	0.49	11	90 %	93 %	0.59	0.84	34 %	0.80	E	T10
x^{13}	12	19	126320	126262	0.41	0.41	20	86 %	98 %	0.61	0.96	38 %	0.85	G	
x^{14}	14	16	135768	137215	0.53	0.55	13	91 %	100 %	0.78	0.78	29 %	0.73	G	T10
x^{15}	10	15	144949	141065	0.59	0.64	14	89 %	92 %	0.55	0.82	35 %	0.81	P	
x^{16}	14	16	138422	137699	0.55	0.56	10	91 %	98 %	0.73	0.95	30 %	0.75	P	T10
x^{17}	16	17	128056	127858	0.43	0.43	17	93 %	98 %	0.78	0.95	35 %	0.81	VG	
x^{18}	18	21	122386	121786	0.36	0.36	22	98 %	88 %	0.66	0.71	30 %	0.75	E	
x^{19}	19	23	101066	101066	0.10	0.10	29	60 %	100 %	0.00	0.00	27 %	0.70	P	T10
x^{20}	17	20	108404	108404	0.19	0.19	26	90 %	98 %	0.71	0.96	29 %	0.73	F	
x^{21}	18	21	118071	120198	0.31	0.34	15	86 %	99 %	0.63	0.98	28 %	0.72	P	
x^{22}	21	23	114618	115867	0.27	0.28	28	92 %	91 %	0.59	0.79	33 %	0.79	VG	
x^{23}	23	28	114189	114761	0.26	0.27	18	89 %	96 %	0.64	0.91	31 %	0.76	P	B5
x^{24}	22	24	116341	116773	0.29	0.29	19	96 %	99 %	0.88	0.98	21 %	0.61	E	
x^{25}	25	26	111254	109302	0.20	0.23	25	90 %	96 %	0.66	0.91	37 %	0.84	E	
x^{26}	26	30	95633	95633	0.03	0.03	30	89 %	95 %	0.61	0.89	33 %	0.79	G	
x^{27}	23	27	92863	92863	0.00	0.00	27	89 %	88 %	0.46	0.73	34 %	0.80	F	B5
x^{28}	26	28	109202	110768	0.20	0.22	24	90 %	98 %	0.71	0.96	25 %	0.67	P	
x^{29}	28	30	105532	105532	0.16	0.16	23	95 %	93 %	0.71	0.83	45 %	0.94	P	
x^{30}	24	30	98647	98647	0.07	0.07	21	98 %	100 %	0.95	0.95	21 %	0.61	P	

With the above preferences, nine schools $X^{ND1} = \{x^1, x^2, x^3, x^4, x^5, x^8, x^9, x^{10}, x^{12}\}$ remain non-dominated, and the student examines them more closely. She first acquires more information about the schools with regard to Hobby 2 (a_7) and states that among the non-dominated schools, x^1 and x^{10} are the two least preferred. This is captured by the statement $(\{x^1, x^{10}\}, \{8, 9\}, N^{ND1}, \{a_7\})$. She also thinks that x^2 is much better than other schools in X^{ND1} and constraints the normalized score of x^2 so that it exceeds those of others by at least 0.1: $v_7^N(x_7^2) - v_7^N(x_7^i) \geq 0.1$ for $x^i \in X^{ND1} \setminus \{x^2\}$. After these statements, the set of non-dominated schools is reduced to $X^{ND2} = \{x^1, x^2, x^3, x^4, x^5\}$.

After further examination of the schools X^{ND2} with regard to Hobby 1 (a_6), the student is willing to tighten the bounds on variables z_j. She bounds the fair schools' (x^4) normalized score to be at most 0.12 and the normalized scores of the very good schools (x^1 and x^3) to be at least 0.15 smaller than the those of the excellent schools (x^2, x^5), which are bounded from below by 0.98. After these specifications of preference information, x^5 becomes dominated. The student decides to visit the four remaining schools.

6 Conclusion

In response to the difficulties in eliciting numerical preference information for additive value models, we have developed a model for capturing incomplete ordinal information about the alternatives' values and the relative importance of attributes. Such information can be elicited through paired sets of alternatives (or attributes) and associated rankings. For example, statements like "alternatives 1 and 2 are among the three most preferred alternatives with regard to cost", or "alternative 1 is not the most preferred one with regard to environmental factors" can be modeled. The statements lead to a set of feasible scores.

Incomplete ordinal statements about the alternatives' values are flexible in that they can be given with regard to any set of attributes, among any set of alternatives. The DM can also match the accuracy of the statements with the available information or the level she considers appropriate in terms of how specific decision recommendations are needed or what the costs of data collection are.

Because incomplete ordinal information can result in a non-convex set of feasible scores, we have developed a mixed integer linear formulation to model these scores. In this model, the number of binary variables employed depends on the preference statements. For example, binary variables are not employed to model convex feasible regions.

The linearity of the formulation makes it possible to complement incomplete ordinal information by other statements that are modeled with linear constraints, such as point estimates or intervals of normalized scores, weights, and weight ratios; or by numerical bounds on the developed model's milestone variables. These possibilities help rule out unintended extreme realizations of the model parameters and thus help capture the DM's intended preference specification. Hence, the model treats numerical and ordinal information jointly without transforming numerical information into ordinal information or vice versa.

While the proposed model admits different kinds of incomplete ordinal statements, some preference elicitation questions can be easier to answer to than others. Furthermore, the statements differ in how conclusive dominance relations they produce. Hence, a possible direction for further research is to design elicitation procedures that address these issues by exploiting the characteristics of incomplete ordinal information. To avoid striving for unnecessarily strict statements and to allow different opinions in group settings, the elicitation procedures should admit 'loose' statements, which can be further specified by tightening the given statements or by introducing complementary numerical information along the lines discussed in Section 3.5. One specific elicitation procedure that could be examined in more detail is the classification procedure discussed in Section 4. The modeling ideas of incomplete ordinal information could be used in *incomplete classification*, in which alternatives can be assigned to several classes, for example: "This research proposal is either 'good' or 'great' with regard to international collaboration" (cf. e.g. Guo et al. 2007).

APPENDIX

Proof of Lemma 1: $\subseteq$: Take any $s \in S(I, J)$. Then $\exists\ r \in R(I, J)$ such that $s \in S(r)$, i.e., $v(x^i) \geq v(x^j)$ whenever $r(x^i) < r(x^j)$. Let $z_{\underline{j}-1} = v(r^{-1}(\underline{j}-1))$, $z_{\overline{j}} = v(r^{-1}(\overline{j}))$, and $y_j(r^{-1}(k)) = 1 \iff k \leq j$ for $j \in \{\underline{j}-1, \overline{j}\} \cap \{1, \ldots, |X|-1\}$. We show that $v, z_j, y_j(x^i)$ satisfy constraints (6)–(9).

Take any $x^i \in X, j \in \{\underline{j}-1, \overline{j}\} \cap \{1, \ldots, |X|-1\}$. If $r(x^i) \leq j$, then $z_j = v(r^{-1}(j)) \leq v(x^i) + (1 - y_j(x^i)M) = v(x^i)$ and thus (6) holds. Constraint (7) holds, because $y_j(x^i)M >$

$1 \geq v(x^i)$. If $r(x^i) > j$, then (7) holds because $v(x^i) \leq v(r^{-1}(j)) = z_j$. Constraint (6) holds as well, because $M > 1 \geq v(r^{-1}(j))$. Constraint (8) holds by construction, because $|\{x^i \in X \mid r(x^i; X) \leq j\}| = j$.

To show that (9) holds, assume first that $|I| \geq |J|$. Take any $j \in [\underline{j}, \overline{j}]$ and $x^i \in \overline{I}$. According to Definition 3, $r^{-1}(j) \notin \overline{I}$. Thus, if $r(x^i) \leq \overline{j}$ then also $r(x^i) \leq \underline{j} - 1$, in which case $y_{\overline{j}}(x^i) = y_{\underline{j}-1}(x^i) = 1$. If $r(x^i) > \overline{j}$, $y_{\overline{j}}(x^i) = y_{\underline{j}-1}(x^i) = 0$ by construction. Thus (9) is fulfilled. If $|I| < |J|$, take any $x^i \in I$. According to Definition 3, $r(x^i) \in [\underline{j}, \overline{j}]$. The above construction of binary variables yields $y_{\overline{j}}(x^i) = 1, y_{\underline{j}-1}(x^i) = 0$, whereby (9) is fulfilled.

$\supseteq$:

Let $v(x^i), z_j, y_j(x^i)$ be a feasible solution to (6)–(9) for all $x^i \in X$, $j \in \{\underline{j} - 1, \overline{j}\} \cap \{1, \ldots, |X| - 1\}$. We show (i) that there exists a rank-ordering $r \in R(I, J)$, and (ii) that $v(x^i) \geq v(x^j)$ if $r(x^i) < r(x^j)$.

Partition X into four sets $X(1,1), X(0,1), X(1,0), X(0,0)$ such that $x^i \in X(a,b) \iff y_{\underline{j}-1}(x^i) = a, y_{\overline{j}}(x^i) = b$ for $a, b \in \{0,1\}$. Strict inequality $z_{\overline{j}} > z_{\underline{j}-1}$ cannot hold, because then (6)–(7) would yield to $X(0,1) = \emptyset$. This would contradict the assumption $\overline{j} > \underline{j} - 1$ through (8). With $z_{\overline{j}} \leq z_{\underline{j}-1}$, straightforward application of (6)–(7) leads to $v(x^{11}) \geq z_{\underline{j}-1} \geq v(x^{01,10}) \geq z_{\overline{j}} \geq v(x^{00})$ $\forall x^{11} \in X(1,1), x^{01,10} \in X(0,1) \cup X(1,0), x^{00} \in X(0,0)$.

Constraint (8) with $j = \underline{j} - 1$ gives

$$|X(1,1)| = \underline{j} - 1 - |X(1,0)| \tag{12}$$

and (8) with $j = \overline{j}$ yields

$$|X(1,1)| + |X(0,1)| = \overline{j}. \tag{13}$$

Inserting (13) into condition $|X| = |X(1,1)| + |X(0,1)| + |X(1,0)| + |X(0,0)|$ gives

$$|X(0,0)| = |X| - \overline{j} - |X(1,0)|. \tag{14}$$

If $X(1,1) \neq \emptyset$, assign rankings $1, \ldots, \underline{j} - 1 - |X(1,0)|$ to alternatives in $X(1,1)$, and rankings $\overline{j} + |X(1,0)| + 1, \ldots, |X|$ to those in $X(0,0)$ (if non-empty) such that $r(x^i) < r(x^k) \Rightarrow v(x^i) \geq v(x^k)$.

If $|I| \geq |J|$, constraint (9) implies $\overline{I} \subseteq X(1,1) \cup X(0,0)$ and therefore $X(0,1) \cup X(1,0) \subseteq I$. Now, assign rankings $|X(1,1)|+1, \ldots, |X|-|X(0,0)|$ to alternatives in $X(0,1) \cup X(1,0) \subseteq I$ such that $r(x^i) < r(x^k) \Rightarrow v(x^i) \geq v(x^k)$. Because $|X(1,1)|+1 = \underline{j} - |X(1,0)| \leq \underline{j}$ by (12) and $|X|-|X(0,0)| = \overline{j} + |X(1,0)| \geq \overline{j}$ by (14), the rankings in J are attained by alternatives in I and hence $r \in R(I,J)$.

If $|I| < |J|$, $I \subseteq X(0,1) \cup X(1,0)$ by constraint (9). If $X(1,0) \neq \emptyset$, then $z_{\underline{j}-1} = z_{\overline{j}}$, because $X(0,1) \neq \emptyset$. Then all alternatives in $X(0,1) \cup X(1,0)$ have equal values, and rankings $\underline{j}, \ldots, \underline{j}+|I| < \overline{j}$ can be assigned to alternatives in I without violating condition $r(x^i) < r(x^k) \Rightarrow v(x^i) \geq v(x^k)$. If $X(1,0) = \emptyset$, the cardinalities in (12) and (14) are $|X(1,1)| = \underline{j}-1$, $|X(0,0)| = |X| - \overline{j}$, wherefore the rankings that have not been assigned to $X(1,1)$ and $X(0,0)$ are exactly J. Thus each alternative in $X(0,1) \supseteq I$ can be assigned ranking from J such that $r(x^i) < r(x^k) \Rightarrow v(x^i) \geq v(x^k)$. $\square$

Proof of Theorem 1: $\subseteq$: Assume $K > 1$, because case $K = 1$ is equal to Lemma 1. Take any $s \in S(I,J)$. Then $\exists\, r \in R(I,J)$ such that $s \in S(r)$, i.e., $v(x^i) \geq v(x^j)$ whenever $r(x^i) < r(x^j)$. Let $z_j = v(r^{-1}(j))$, and $y_j(r^{-1}(p)) = 1 \iff p \leq j$ for all $j \in j_\cup$. We show that $v, z_j, y_j(x^i)$ satisfy constraints (6)–(8) and (10).

Showing that (6)–(8) hold can be done as in the proof of Lemma 1 and is therefore omitted. To show that (10) holds, assume first that $|I| \geq |J|$. Take any $k \in \{1, \ldots, K\}, j \in [\underline{j_k}, \overline{j_k}]$ and $x^i \in \overline{I}$. Following the arguments of the proof of Lemma 1, $r^{-1}(j) \notin \overline{I}$. Thus, if $r(x^i) \leq \overline{j_k}$ then $r(x^i) \leq \underline{j_k} - 1$, whereby $y_{\overline{j}}(x^i) = y_{\underline{j}-1}(x^i)$, which also holds if $r(x^i) > \overline{j_k}$. Thus (10) is fulfilled. If $|I| < |J|$, take any $x^i \in I$. Because $r \in R(I,J)$, $r(x^i) \in [\underline{j_l}, \overline{j_l}]$ for exactly one $l \in L = \{1, \ldots, K\}$. By construction, $y_{\overline{j_l}}(x^i) = 1, y_{\underline{j_l}-1} = 0$ for this l. For $k \in L \setminus \{l\}$, $y_{\overline{j_k}}(x^i) - y_{\underline{j_l}-1} = 0$ and thus (10) holds.

$\supseteq$: Assume $|I| \geq |J|$. Then also $|I| \geq |J_k|$ for every $k = 1, \ldots, K$. Because (6)–(8) and (9) hold for all $\underline{j_k} - 1, \overline{j_k}$, $k = 1, \ldots, K$, consecutive application of Lemma 1 yields that $\exists r \in \cap_{k=1}^K R(I, J_k)$ such that $s \in S(r)$. By Definition 3, $r \in R(I, J_k)$ iff $r^{-1}(k) \in I \ \forall j \in J_k$. Thus, $r \in \bigcap_{k=1}^K R(I, J_k) = \bigcap_{k=1}^K \{r \mid r^{-1}(j_k) \in I \ \forall j_k \in J_k\} = \{r \mid r^{-1}(j_k) \in I \ \forall j_k \in J_k, k = 1, \ldots, K\} = \left\{r \mid r^{-1}(j) \in I \ \forall j \in \cup_{k=1}^K J_k\right\} = \{r \mid r^{-1}(j) \in I \ \forall j \in J\} = R(I,J)$.

Case $|I| < |J|$: Recall from the proof of Lemma 1 that strict inequality $z_{\overline{j_k}} > z_{\underline{j_k}-1}$ cannot hold for any k. Thus, if $y_{\overline{j_k}}(x^i) = 0, y_{\underline{j_k}-1}(x^i) = 1$ for some $k \in \{1, \ldots, K\}, x^i \in X$, (6)–(7) give $v(x^i) = z_{\overline{j_k}} = z_{\underline{j_k}-1}$. Thus, for x^j with $y_{\overline{j_k}}(x^j) = 1, y_{\underline{j_k}-1}(x^j) = 0$, $v(x^j) =$

$z_{\overline{j_k}} = z_{\underline{j_k}-1}$. There are $\overline{j_k} - \underline{j_k} + 1 + |\{x^i \in X \mid y_{\overline{j_k}}(x^i) = 0, y_{\underline{j_k}-1}(x^i) = 1\}|$ such alternatives, because (8) holds for both $\overline{j_k}$ and $\underline{j_k} - 1$.

If $|I| < |J|$, then $|\overline{I}| > |\overline{J}|$. According to Salo and Punkka (2005), $R(I, J) = R(\overline{I}, \overline{J})$. We show that (6)–(8) and (10) imply that $s \in S(\overline{I}, \overline{J}) = S(I, J)$.

Set

$$J = \{\underline{j_1}, \dots, \overline{j_1}\} \cup \{\underline{j_2}, \dots, \overline{j_2}\} \cup \dots \cup \{\underline{j_{K-1}}, \dots, \overline{j_{K-1}}\} \cup \{\underline{j_K}, \dots, \overline{j_K}\}$$

and thus its complement

$$\overline{J} = \{j \mid 1 \le j \le \underline{j_1}-1\} \cup \{\overline{j_1}+1, \dots, \underline{j_2}-1\} \cup \dots \cup \{\overline{j_{K-1}}+1, \dots, \underline{j_K}-1\} \cup \{j \mid \overline{j_K}+1 \le j \le |X|\},$$

in which each of the subsets consists of consecutive integers. If $1 \in J$ (and/or $|X| \in J$), then $\underline{j_1} = 1$ ($\overline{j_K} = |X|$), and the first (last) set is empty. Constraints (10) imply $y_{\overline{j_0}}(x^i) = 0 = y_{\underline{j_1}-1}(x^i)$, $y_{\overline{j_{k-1}}}(x^i) = y_{\underline{j_k}-1}(x^i)\ \forall k = 2, \dots, K$ and $y_{\overline{j_K}}(x^i) = y_{\underline{j_{K+1}}-1}(x^i) = 1$. Now $s \in S(\overline{I}, \overline{J})$ based on the proof of case $|I| \ge |J|$. □

Proof of Lemma 2:

Take any $s \in \bigcap_{l=1}^{L} S(I^l, J^l)$. Then there exists a rank-ordering $r \in \bigcap_{l=1}^{L} R(I^l, J^l)$ such that $s \in S(r) \subseteq \bigcap_{l=1}^{L} S(I^l, J^l)$. Choose $s' \in S(r)$ such that $s' \notin S(r')$ for $r' \neq r$ (e.g., s' such that $v(x^i, A') = (1 - \frac{r(x^i)-1}{|X|-1}) \sum_{\{j|a_j \in A'\}} w_j\ \forall\ x^i \in X$). We show that any solution $z_j^l, y_j^l(x^i), l = 1, \dots, L, x^i \in X$ such that (6)–(8) and (10) hold for s', fulfills the conditions $y_j^l(x^i) = y_j^p(x^i)$, whenever $j \in j_\cup(J^l) \cap j_\cup(J^p)$ for some $\{l, p\} \subseteq \{1, \dots, L\}$ and $j < j' \Rightarrow y_j^l(x^i) \le y_{j'}^p(x^i)$ for any $p, l \in \{1, \dots, L\}$. Furthermore, we show that $z_j^p = z_j^l$ holds for some of these solutions and that such a solutions exists for all $s \in S(r)$.

Because $s' \notin S(r')$ for $r' \neq r$, the values $v(x^i), v(x^k)$ are unequal whenever $k \neq i$. Let $j \in j_\cup(J^l) \cap j_\cup(J^p)$ for some statements l, p and assume contrary to the claim that $y_j^l(x^i) = 1$, $y_j^p(x^i) = 0$ for some $x^i \in X$. Then, because (8) holds for both y_j^l and y_j^p, there exists x^k such that $y_j^l(x^k) = 0$, $y_j^p(x^k) = 1$. Inserting these conditions to (6)–(7) leads to $v(x^i) = v(x^k)$, which is a contradiction with the assumption of unequal values. Thus, $y_j^l(x^i) = y_j^p(x^i)$ and we continue with the notational convention $y_j(x^i) := y_j^l(x^i) = y_j^p(x^i)$.

Assume contrary to the claim that $y_j(x^i) > y_{j'}(x^i)$, $j < j'$, for some $x^i \in X$ (indices $j, j' \in J_{\text{ind}}$). Then, $z_j^p \le v(x^i) \le z_{j'}^l$ for some statements p, l, based on (6)–(7). Because

(8) holds for j and j', there also exist at least two other alternatives x^k for which $z_j^p \geq v(x^k) \geq z_{j'}^l$, leading to $z_j^p = v(x^k) = v(x^i) = z_{j'}^l$ and thus violating the assumption $v(x^i) \neq v(x^k)$. Hence, $j < j' \Rightarrow y_j(x^i) \leq y_{j'}(x^i)$ for all $x^i \in X$.

Because $v(x^k) \neq v(x^i)$ and $s' \in S(r)$, the solution to the binary variables $y_j(x^i)$ is unique.

Introduce further constraints $z_j^l = v(r^{-1}(j))$ for all $j \in j_\cup(J^l)$, for all $l = 1, \ldots, L$. This requirement is tighter than $z_j^p = z_j^l$, but it does not violate constraints (6)–(7), (8) and (10). Acknowledging that the solution to the binary variables is unique, constraint $z_j^l = v(r^{-1}(j))$ simplifies the constraints (6)–(7), (8) and (10) to

$$v(r^{-1}(j)) \leq v(x^i) + (1 - y_j(x^i)M)$$
$$v(x^i) \leq v(r^{-1}(j)) + y_j(x^i)M$$

for all $x^i \in X, j \in J_{\text{ind}}$. Inserting the values of $y_j(x^i)$ further simplifies the constraints to

$$v(r^{-1}(j)) \geq v(r^{-1}(k(j))), \quad \forall\, j \in J_{\text{ind}}, j \neq \max J_{\text{ind}}, k(j) = \min_k\{k \in J_{\text{ind}} \mid j < k\}. \quad (15)$$

The scores that fulfill (15) are a superset of $S(r)$, because the constraints in the definition of $S(r)$ in (4) are equivalent to (15) with $J_{\text{ind}} = \{1, \ldots, |X| - 1\}$. Thus, even if s is such that $v(x^i) = v(x^k)$ for some alternatives, it belongs to set of scores characterized by the (6)–(8) for $v(x^i), y_j(x^i), z_j \; \forall x^i \in X, j \in J_{\text{ind}}$, (10) on the statements (I^l, J^l) and (11).

Any $s \in S^L$ belongs to $\bigcap_{l=1}^L S(I^l, J^l)$, because the constraints that define S^L are a superset of those that characterize $\bigcap_{l=1}^L S(I^l, J^l)$. □

Proof of Lemma 3: If $|I| \geq |J|$: because $y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i) \geq 0$, in $\sum_{x^i \in \bar{I}} \sum_{k=1}^K [y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i)] = 0$ each of the summed terms must be zero.

If $|I| < |J|$: Due to the monotonicity of the binary variables, $\sum_{k=1}^K [y_{\overline{j_k}}(x^i) - y_{\underline{j_k}-1}(x^i)] \in \{0, 1\}$. Because summation of $|I|$ such terms equals $|I|$, each of them must equal 1. Therefore, there exists $k^* \in \{1, \ldots, K\}$ such that $y_{\overline{j_k^*}}(x^i) = 1, y_{\underline{j_k^*}-1}(x^i) = 0$. But this implies that for any $k \leq k^*$, $y_{\overline{j_{k-1}}}(x^i) = 0, y_{\underline{j_k}-1}(x^i) = 0$ and for any $k > k^*$, $y_{\overline{j_{k-1}}}(x^i) = 1, y_{\underline{j_k}-1}(x^i) = 1$. □

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Paper [III]

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Ranking intervals in additive value models with incomplete preference information

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Abstract:

Decision rules can be employed to compare alternatives' additive multi-attribute values under incomplete preference information. We show that the recommendations of decision rules that compare magnitudes of value differences across value functions that represent different preferences can depend on how the value function is normalized. To overcome this problem, we propose *ranking intervals* which provide information on how alternatives compare with *all* other alternatives for *all* value functions that are consistent with the incomplete preference information. These intervals can be computed with mixed integer linear programming models which admit several forms of preference statements about the parameters of the value function. We demonstrate uses of ranking intervals in the sensitivity analysis of university rankings and discuss their uses in project portfolio selection.

Keywords: Multiple criteria decision analysis, ranking, incomplete information, sensitivity analysis

1 Introduction

The specification of an additive multi-attribute value function (Keeney and Raiffa 1976, Dyer and Sarin 1979) is often based on the decision-maker's (DM's) subjective preference statements. Difficulties in the exact specification of preferences, among other reasons, have motivated the development of methods that allow the DM to provide incomplete preference information (e.g., White et al. 1982, see also Weber 1987). The *Preference Programming* methods compare pairs of alternatives with *all* additive value functions that are consistent with the incomplete preference information. Specifically, if one alternative is preferred to another for all these value functions, then it *dominates* the other alternative (e.g., White et al. 1982, Salo and Hämäläinen 1992).

Because there can be several non-dominated alternatives under incomplete preference specification, *decision rules* have been proposed to provide insights into how 'good' the alternatives are. Some of these rules and related solution concepts compare the magnitudes of alternatives' numerical value differences within a class of *representative value functions*, which include all value functions that are consistent with the stated preference information but still fulfill predefined *normalization* conditions (e.g., Park and Kim 1997, Dias and Climaco 2000, Salo and Hämäläinen 2001, among others). In this paper, we show that the recommendations provided by these decision rules and, for example, the results of sensitivity analyses that are based on distances between different value functions (Rios Insua and French 1991) can depend on the normalization of value functions.

As a remedy to this problem, we determine all the rankings that the alternatives can attain with those value functions that are consistent with the DM's preference statements. The resulting *ranking intervals* do not depend on the chosen normalization of the value functions. They provide ordinal information about how the alternatives perform *among all* alternatives and they can therefore be used to complement dominance relations for the purpose of screening a large set of alternatives or to examine sensitivity of *full rank-orderings*, such as priority listings of project proposals.

Our work is partly inspired by the work of Butler et al. (1997) who propose exploring the attainable rankings to study the robustness of the alternatives' rankings. Yet, earlier models have focused on one specific form of preference information (Kämpke 1996), applied Monte Carlo simulation to compute a subset of these rankings (Butler et al. 1997), or computed only the best possible rankings of the alternatives over a set of feasible weights (Köksalan et al. 2010). We extend these approaches by developing mixed integer linear programming (MILP) models for computing all the rankings that the alternatives can attain for all the additive value functions that are consistent with (i) the DM's incomplete preference statements and (ii) the attribute-specific values that fulfill the incomplete characterization of the alternatives.

The remainder of this paper is organized as follows. In Section 2 we show that the recommendations provided by several decision rules and other related solution concepts depend on what representative value functions are chosen to represent the preference information. Section 3 develops models for the computation of the ranking intervals. Section 4 gives an illustrative example on the sensitivity analysis of university rankings. Section 5 provides some insights into how multi-criteria project portfolio selection under incomplete preference information (e.g., Liesiö et al. 2007, 2008) can benefit from the ranking intervals by revisiting a case study by Könnölä et al. (2007). Section 6 concludes.

2 Decision recommendations with incompletely defined additive value functions

2.1 Additive value function and positive affine transformations

We assume that there are n attributes. Alternative x is represented by its achievement levels x_i which belong to (measurement scale) sets X_i that are not singleton sets: $(x_1, \ldots, x_n) \in X = \times_{i=1}^{n} X_i$.

The additive multi-attribute value of alternative x is

$$V(x) = \sum_{i=1}^{n} v_i(x_i), \tag{1}$$

in which $v_i : X_i \to \mathbb{R}$ is the attribute-specific value function of attribute a_i (Keeney and Raiffa 1976). We assume that V is a measurable value function. It then measures the strength of preference so that $V(x^b) - V(x^a) > V(x^d) - V(x^c)$ if and only if x^a is more preferred to x^b than x^c is to x^d (Dyer and Sarin 1979). Measurable value functions are unique up to positive affine transformations. Thus, any $V'(x) = \alpha V(x) + \beta$ (in which $\alpha > 0$) represents the same preferences as $V(x)$. Specifically, the preferences captured by (1) can therefore be represented by the "weighted additive value function" $V^N(x) = \sum_{i=1}^n w_i v_i^N(x_i)$, as discussed later in Subsection 2.5.

If the DM's preferences are captured by an additive value function, attribute-specific value functions v_i can be elicited independently of other attributes. Many elicitation methods are based on indifference judgements in which achievement levels $x_i^a, x_i^b, x_i^c, x_i^d \in X_i$ are specified so that $v_i(x_i^b) - v_i(x_i^a) = v_i(x_i^d) - v_i(x_i^c)$ holds (see, e.g., von Winterfeldt and Edwards 1986). We assume that all attributes are meaningful in the sense that $v_i(x_i) > v_i(x_i')$ for some $x_i, x_i' \in X_i$, for all $i = 1, \ldots, n$.

Similarly, the elicitation of preferences between attributes can be based on specifying alternatives $x^a, x^b, x^c, x^d \in X$ so that $V(x^b) - V(x^a) = V(x^d) - V(x^c)$ holds. In such preference elicitation, often all but one achievement level are fixed and the DM is requested to specify this one level so that indifference holds. These methods based on indifference judgements have been complemented by ratio-based methods, in which $x^a, x^b, x^c, x^d \in X$ are all fixed, and the DM specifies the ratio $\kappa(x^a, x^b, x^c, x^d)$ so that

$$V(x^b) - V(x^a) = \kappa(x^a, x^b, x^c, x^d)[V(x^d) - V(x^c)] \tag{2}$$

holds. Statements (2) are used for example in the SWING weighting method (von Winterfeldt and Edwards 1986). In SWING, the DM estimates the ratio $[v_j(x_j^*) - v_j(x_j^\circ)]/[v_i(x_i^*) - v_i(x_i^\circ)]$, in which x° and x^* are alternatives represented by the "least" and "most" preferred levels on all attributes, respectively.

Once the attribute-specific value functions have been elicited and sufficiently many consistent statements (2) have been obtained, the ratio $\kappa(x^a, x^b, x^c, x^d)$ is known for any $x^a, x^b, x^c, x^d \in X$ $(V(x^d) \neq V(x^c))$. This ratio is invariant under positive affine transformations of V. Consequently, there are infinitely many additive value functions (1) that are consistent with the preference information, but which are all positive affine transformations of each other. Therefore, to compute a numerical value for $V(x^b)$, $x^b \in X$, one of these value functions is taken to be a *representative value function*. This choice can be done by selecting any two alternatives $x^+, x^- \in X$ so that x^+ is preferred to x^-, and $V(x^+)$ and $V(x^-)$ are any real numbers so that $V(x^+) > V(x^-)$. The value of x^b is then $V(x^b) = \kappa(x^-, x^b, x^-, x^+)[V(x^+) - V(x^-)] + V(x^-)$. The selection of this representative value function is often referred to as *normalization*. Specifically, one popular way to normalize is to choose $x^- = x^\circ$ and $x^+ = x^*$, and fix the overall values of these alternatives to 0 and 1, respectively. Then the values of the alternatives whose achievement levels x_i fulfill $v_i(x_i^*) \geq v_i(x_i) \geq v_i(x_i^\circ)$ are in the range $[0, 1]$.

For example, assume that a DM is choosing a fruit basket of apples (x_1) and oranges (x_2), $(x_1, x_2) \in X = \mathbb{R}^2_+ \cup (0,0)$. His preference statements are: (i) v_1 and v_2 are linear and increasing and (ii) $V(1,0) - V(0,0) = V(0,1) - V(0,0)$. To compute a numerical value for the fruit basket $x^b = (3,0)$, it is necessary to choose one of the infinitely many value functions that are consistent with these preferences. One possibility is to set $x^+ = (2,0), x^- = (0,1)$ and $V(x^+) = 2, V(x^-) = -1$. Then, the value of $x^b = (3,0)$ is

$$V(3,0) = \underbrace{\frac{V(3,0) - V(0,1)}{V(2,0) - V(0,1)}}_{=[V(3,0)-V(1,0)]/[V(2,0)-V(1,0)]=2} \cdot \underbrace{[V(2,0) - V(0,1)]}_{=2+1=3} + \underbrace{V(0,1)}_{=-1} = 5.$$

If another representative value function were to be chosen, the same preferences would be represented by the value function $V' = \alpha V + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}$. Yet, value difference ratios, such as the above $\kappa((0,1),(3,0),(0,1),(2,0))$

$$= \frac{V'(3,0) - V'(0,1)}{V'(2,0) - V'(0,1)} = \frac{\alpha V(3,0) + \beta - \alpha V(0,1) - \beta}{\alpha V(2,0) + \beta - \alpha V(0,1) - \beta} = \frac{V(3,0) - V(0,1)}{V(2,0) - V(0,1)} = 2,$$

remain unchanged. The numerical value $V'(3,0) = 2 \cdot [\alpha V(2,0) + \beta - \alpha V(0,1) - \beta] + \alpha V(0,1) + \beta = \alpha[2 \cdot V(2,0) - V(0,1)] + \beta$, however, depends on the selection of the representative value function.

2.2 Incomplete preference information

Because the exact specification of preferences can be difficult (e.g., Weber 1987), methods for deriving decision recommendations from an incomplete preference specification have been developed. In Preference Programming (e.g., White et al. 1982, Salo and Hämäläinen 1992, 2001, 2010), the DM gives a series of incomplete preference statements about the relative importance of the attributes. These statements lead to a finite number of inequalities

$$V(x^b) - V(x^a) \geq \kappa^-(x^a, x^b, x^c, x^d)[V(x^d) - V(x^c)], \quad x^a, x^b, x^c, x^d \in X. \tag{3}$$

We assume that attribute-specific value functions are known so that the ratio $[v_i(x_i^b) - v_i(x_i^a)]/[v_i(x_i^d) - v_i(x_i^c)]$ can be computed for all alternatives that appear in the stated preference statements (3). These statements can thus be employed to capture incompletely defined ratios of value differences in different attributes $[v_i(x_i^b) - v_i(x_i^a)]/[v_j(x_j^d) - v_j(x_j^c)] \geq \kappa^-(x^a, x^b, x^c, x^d)$ (Salo and Hämäläinen 1992), as in the interval versions of SWING and SMARTS methods (Mustajoki et al. 2005). They can also capture ordinal statements $[v_i(x_i^*) - v_i(x_i^\circ)] \geq [v_j(x_j^*) - v_j(x_j^\circ)]$ (e.g., Kirkwood and Sarin 1985) or holistic comparisons $V(x^b) \geq V(x^d)$ (e.g., Salo and Hämäläinen 2001). Statements (3) can result in a complete preference specification, because statement (2) is equal to statements $V(x^b) - V(x^a) \geq \kappa(x^a, x^b, x^c, x^d)[V(x^d) - V(x^c)]$ and $V(x^d) - V(x^c) \geq \kappa(x^a, x^b, x^c, x^d)^{-1}[V(x^b) - V(x^a)]$ of form (3).

The additive value functions V that are consistent with the a set of preference statements (3) are denoted by $\mathcal{V}$. In general, these functions are not all positive affine transformations of each other. Formally, the set $\mathcal{V}$ is defined as follows.

Definition 1 *Let attribute-specific value functions $v_i'(\cdot)$, $i = 1, \ldots, n$, be defined. Then $V(x) = \sum_{i=1}^n v_i(x_i) \in \mathcal{V}$ if (i) $V(\cdot)$ fulfills the set of stated preference statements (3), and (ii) $v_i = \alpha_i v_i' + \beta_i$ for all $i = 1, \ldots, n$ for some $\alpha_i > 0$, $\beta_i \in \mathbb{R}$.*

Pairwise dominance (White et al. 1982) can be used to compare alternatives with all value functions in $\mathcal{V}$, and, more specifically, to examine whether the overall value of an alternative is at least as high as that of another for all value functions and strictly higher for some (see e.g., Kirkwood and Sarin 1985, Hazen 1986, Rios Insua and French 1991, Salo and Hämäläinen 1992, 2001, Salo and Punkka 2005).

Definition 2 *Alternative $x^j \in X$ dominates $x^k \in X$ among $\mathcal{V}$, if $V(x^j) \geq V(x^k)$ for all $V \in \mathcal{V}$ and $V(x^j) > V(x^k)$ for some $V \in \mathcal{V}$.*

2.3 Representative value functions and value comparisons under incomplete preference information

With an incomplete preference specification, there can be several non-dominated alternatives. As a result, proposals to provide recommendations about which non-dominated alternative is the most preferred have been made. Many of these compare the magnitudes of value differences $V(x^j) - V(x^k)$ among the alternatives, across the value functions $\mathcal{V}$, and then seek to answer questions like "can alternative x^j be more preferred to x^k than x^k to x^j?"

Because $\mathcal{V}$ includes all positive affine transformations of the value functions in it ($V \in \mathcal{V}$, then $\alpha V + \beta \in \mathcal{V}$, $\alpha > 0$, $\beta \in \mathbb{R}$), the value differences $V(x^j) - V(x^k)$ are not bounded. For example, if the preference statement of form (2) is replaced by incomplete preference statements $V(2,0) \geq V(0,1)$ and $V(0,2) \geq V(1,0)$ in the fruit basket example introduced in Subsection 2.1, then the value functions that are consistent with the preference statements are

$$\mathcal{V} = \{V(x_1, x_2) = \alpha(x_1 + rx_2) + \beta \mid \alpha > 0, \beta \in \mathbb{R}, r \in [0.5, 2]\}. \tag{4}$$

The value difference of fruit baskets $(1,2)$ and $(2,1)$ is $V(1,2) - V(2,1) = V(0,1) - V(1,0) = \alpha(r-1)$. Because this difference can be both positive and negative, the fruit baskets do not dominate each other. Furthermore, this difference is not bounded by the preference statements; it can be arbitrarily small or large depending on the choice of α. As a result, decision recommendations based on the magnitude of this value difference would be, in a sense, arbitrary.

The lack of unequivocal numerical value differences has been eluded by considering only a subset of value functions in $\mathcal{V}$ (e.g., Park and Kim 1997, Dias and Climaco 2000, Salo and Hämäläinen 2001). In particular, value difference comparisons have been applied to what we call a set of *representative value functions* $\mathcal{V}' \subset \mathcal{V}$, defined so that any value function in $\mathcal{V}$ is represented in $\mathcal{V}'$ by one of its positive affine transformations. Conversely, any value function in $\mathcal{V}$ is a positive affine transformation of a value function in $\mathcal{V}'$. Following the normalization approach in Subsection 2.1, the set $\mathcal{V}'$ is typically defined by choosing two reference alternatives x^- and x^+ whose values are taken to be constant throughout $\mathcal{V}'$. Due to this condition, $\mathcal{V}'$ does not include two different value functions that are positive affine transformations of each other and consequently alternatives in X have unequivocal overall values for any $V \in \mathcal{V}'$.

Definition 3 *$\mathcal{V}'$ represents $\mathcal{V}$ if*

(i) $\forall\ V \in \mathcal{V}\ \exists \alpha > 0, \beta \in \mathbb{R}$ *so that* $\alpha V + \beta \in \mathcal{V}'$

(ii) there exist $x^+, x^- \in X$, $V^+, V^- \in \mathbb{R}$ *such that* $V(x^+) = V^+ > V(x^-) = V^- \; \forall \; V \in \mathcal{V}'$.

Again, *one* possibility to choose the representative value functions is to assign values 0 and 1 to the hypothetical alternatives x° and x^*. Representative value functions $\mathcal{V}'$ exist for any non-empty $\mathcal{V}$, because if $V \in \mathcal{V}$, then $\alpha V(x^+) + \beta = V^+$ and $\alpha V(x^-) + \beta = V^-$ by choosing $\alpha = [V^+ - V^-]/[V(x^+) - V(x^-)]$ (which is by definition positive), and $\beta = V^+ - \alpha V(x^+)$.

In the fruit basket example, if we choose $\mathcal{V}'$ so that $V(x^+) = V(1,0) = 1$ and $V(x^-) = V(0,0) = 0$, we *measure value differences in apples*. This choice is fulfilled by value functions (4) for which $\alpha = 1$ and $\beta = 0$. For any $\mathcal{V}'$ that represents (4), value difference $V(1,2) - V(2,1)$ is maximized with $V^* \in \mathcal{V}'$ so that $r = 2$. With the above choice of $\mathcal{V}'$, the biggest value difference $V(1,2) - V(2,1) = \alpha(r-1)$ is thus equal to 1 (apple), while the biggest value difference in favor of $(2,1)$ is $V(2,1) - V(1,2) = \alpha(1-r) = 0.5$ (with $r = 0.5$). Thus, $(1,2)$ can be more preferred to $(2,1)$ than vice versa in the sense that $\max_{V \in \mathcal{V}'}[V(1,2) - V(2,1)] > \max_{V \in \mathcal{V}'}[V(2,1) - V(1,2)]$.

The opposite conclusion would be reached if value differences were measured *in oranges* by choosing $\mathcal{V}' = \tilde{\mathcal{V}}'$ so that $\tilde{V}(0,1) = 1$, $\tilde{V}(0,0) = 0$ for all $\tilde{V} \in \tilde{\mathcal{V}}'$ (leading to $\alpha = 1/r$, $\beta = 0$ for value functions of form (4)). Because ratios of value differences are invariant to the selection of $\mathcal{V}'$ and $\tilde{V}(0,1) - \tilde{V}(0,0) = 1$ for all $\tilde{V} \in \tilde{\mathcal{V}}'$, the value differences $\tilde{V}(1,2) - \tilde{V}(2,1)$ can be expressed with value functions $V \in \mathcal{V}'$ as follows:

$$\tilde{V}(1,2) - \tilde{V}(2,1) = \frac{\tilde{V}(1,2) - \tilde{V}(2,1)}{\tilde{V}(0,1) - \tilde{V}(0,0)} = \frac{V(1,2) - V(2,1)}{V(0,1) - V(0,0)} = \frac{V(1,2) - V(2,1)}{r}.$$

Each value function in $\mathcal{V}'$ is indeed a positive affine transformation of a value function in $\tilde{\mathcal{V}}'$, but this transformation depends on the parameter r.

Value difference $\tilde{V}(1,2) - \tilde{V}(2,1)$ is maximized when $r = 2$; with this value of r, $V(1,2) - V(2,1) = 1$. Inserting these into the above ratio gives $\max_{\tilde{V} \in \tilde{\mathcal{V}}'}[\tilde{V}(1,2) - \tilde{V}(2,1)] = 0.5$. If $r = 0.5$, then $V(2,1) - V(1,2) = 0.5$ and thus $\tilde{V}(2,1) - \tilde{V}(1,2) = 1$ indicating that $(2,1)$ can be more preferred to $(1,2)$ than vice versa.

Figure 1 presents the values of $(1,2)$ and $(2,1)$ with two choices of $\mathcal{V}'$ that follow the convention of assigning values 0 and 1 to the "least" and "most" preferred alternatives.

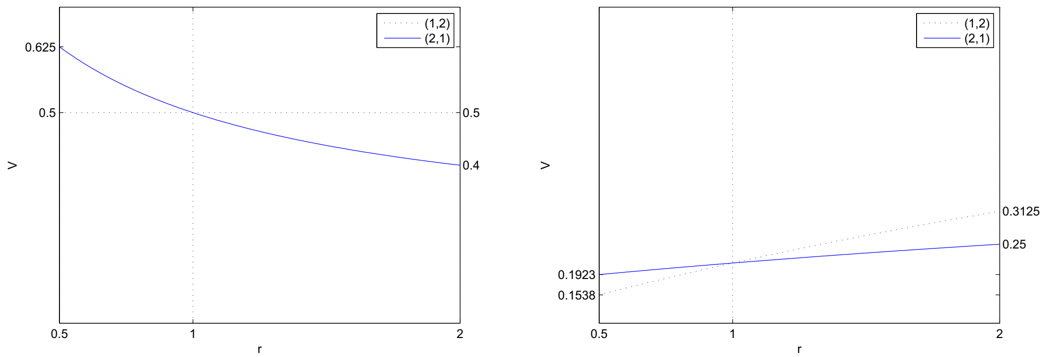

Figure 1: Values of $(1,2)$ and $(2,1)$ as a function of r with two representations $\mathcal{V}'$ of preference information $\mathcal{V}$ in (4) so that $V(x^+) = 1, V(x^-) = 0$.

2.4 Decision rules and rank reversals

Comparisons of values and value differences are employed in many decision rules which recommend a non-dominated alternative from the set $\tilde{X} = \{x^1, \ldots, x^m\} \subseteq X$. For example, the *weak dominance* rule (Park and Kim 1997) or the equivalent *minimax regret* rule (Dias and Climaco 2000, Salo and Hämäläinen 2001) recommends $x^k \in \tilde{X}$ for which $\max_{V \in \mathcal{V}', x^l \in \tilde{X}}[V(x^l) - V(x^k)]$ is the smallest. The *maximax* rule (Salo and Hämäläinen 2001) recommends $x^k \in \tilde{X}$ for which $\max_{V \in \mathcal{V}'} V(x^k)$ is the highest. The *maximin* rule (Salo and Hämäläinen 2001) recommends $x^k \in \tilde{X}$ for which $\min_{V \in \mathcal{V}'} V(x^k)$ is the highest. The *central values* rule (Salo and Hämäläinen 2001) recommends $x^k \in \tilde{X}$ for which $\min_{V \in \mathcal{V}'} V(x^k) + \max_{V \in \mathcal{V}'} V(x^k)$ is the highest. In the example of Figure 1, all these decision rules provide different recommendations with $x^+ = (12, 2)$ than with $x^+ = (2, 4)$. Thus, the recommendations depend on which $\mathcal{V}'$ is chosen to represent $\mathcal{V}$. *Quasi-dominance* (Dias and Climaco 2000) and the related decision rules (Sarabando and Dias 2009), *absolute dominance* (Salo and Hämäläinen 2001), the *domain criterion* (Eiselt and Laporte 1992; and the closely related measure *acceptability index* by Lahdelma et al. 1998) as well as sensitivity analyses based on computing the *closest competitor* (Rios Insua and French 1991) are further examples of concepts that provide results which depend on the choice of $\mathcal{V}'$, as illustrated in Appendix A.

Indeed, decision rules based on comparing the magnitudes of these value differences can exhibit *rank reversals* (Belton and Gear 1983) in the sense that changing x^+ and x^- can change the recommendations. The next theorem provides sufficient conditions under which the recommendations of weak dominance, maximin and maximax rules can be altered by a suitable choice of x^+ and x^-. All proofs are in Appendix B.

Theorem 1 *Let $v_i(x_i): X_i = [l_i, u_i] \to \mathbb{R}$, $i = 1, \ldots, n$ be continuous and let $x^1, x^2 \in X$ and $\mathcal{V}$ be such that $V^1(x^1) > V^1(x^2)$, $V^2(x^2) > V^2(x^1)$ for some $V^1, V^2 \in \mathcal{V}$.*

a) Let $x \in \{x^j \in X \mid V(x^j) > \max\{V(x^1), V(x^2)\} \vee V(x^j) < \min\{V(x^1), V(x^2)\}\ \forall\ V \in \mathcal{V}\}$. Then there exists $\mathcal{V}'$ that represents $\mathcal{V}$ so that

$$\max_{V \in \mathcal{V}'}[V(x^1) - V(x)] > \max_{V \in \mathcal{V}'}[V(x^2) - V(x)].$$

b) Assume that for any pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, i-1, i+1, \ldots, n\}$, there exist achievement levels $x_i, x_i' \in X_i$, $v_i(x_i) < v_i(x_i')$ and $x_j, x_j' \in X_j$, $v_j(x_j) < v_j(x_j')$, so that $v_i(x_i') + v_j(x_j) \geq v_i(x_i) + v_j(x_j')\ \forall\ V \in \mathcal{V}$. Then there exists $\mathcal{V}'$ that represents $\mathcal{V}$ so that

$$\max_{V \in \mathcal{V}'}[V(x^1) - V(x^2)] > \max_{V \in \mathcal{V}'}[V(x^2) - V(x^1)].$$

Part a) of Theorem 1 states that it is possible to make maximax and maximin rules recommend either one of two non-dominated alternatives through some normalization $\mathcal{V}'$ if the attribute-specific value functions are continuous. Part b) assumes continuous attribute-specific value functions, too, and states that weak dominance can recommend any of the two alternatives, if for any pair of attributes there exist improvements $(x_i \to x_i')$ and $(x_j \to x_j')$ such that the DM prefers the first improvement to the second one, and examples of such improvements can be derived from the preference statements that define $\mathcal{V}$. This condition is not strong; it is implicitly assumed by all methods that elicit complete preference information and for example in interval SMARTS/SWING (Mustajoki et al. 2005).

In view of the above, comparing the magnitudes of values or value differences across different value functions in $\mathcal{V}'$ does not provide meaningful results. However, because the relative ordering of the alternatives' values remains invariant across positive affine transformations of the value function, dominance relations that compare the order of two alternatives' values in $\mathcal{V}$ do not depend on the selection of $\mathcal{V}'$. This idea can be extended to complete rank-orderings of the alternatives: by rank-ordering the alternatives for all value functions that are consistent with the DM's preferences, the intervals of the alternatives' attainable rankings can be obtained. In order to compute these *ranking intervals* also when alternatives' achievement levels are incompletely characterized (e.g., Salo and Hämäläinen 2001), we first define the concept of *feasible region.*

2.5 Incompletely characterized alternatives and feasible regions

In the rest of this paper, we consider a discrete set of alternatives $\tilde{X} = \{x^1, \ldots, x^m\} \subseteq X$. It is customary to choose $x^\circ, x^* \in X$ so that $v_i(x_i^*) > v_i(x_i^\circ)$ and $v_i(x_i^*) \geq v_i(x_i) \geq v_i(x_i^\circ)$ for all $x \in \tilde{X}, i = 1, \ldots, n$. We can then select $\alpha_i > 0, \beta_i \in \mathbb{R}$ so that (1) can be written $V(x) = \sum_{i=1}^n [\alpha_i v_i^N(x_i) + \beta_i]$, in which $v_i^N(x_i^\circ) = 0$ and $v_i^N(x_i^*) = 1$. A positive affine transformation of $V(x)$ to $V^N(x) = [V(x) - \sum_{i=1}^n \beta_i]/\sum_{i=1}^n \alpha_i$ leads to

$$V^N(x) = \sum_{i=1}^n [\alpha_i / \sum_{i=1}^n \alpha_i] v_i^N(x_i) = \sum_{i=1}^n w_i v_i^N(x_i), \tag{5}$$

in which the *attribute weights* w_i sum up to one and reflect the relative importance of the attributes. Statements (2) and (3) then correspond to linear constraints on the weights. All information is contained in matrix s so that $[s]_{ji} = v_i(x_i^j), j = 1, \ldots, m$, $[s]_{(m+1)i} = w_i$ in recognition that the *normalized scores* fulfill $v_i^N(x_i^j) = v_i(x_i^j)/w_i$. Matrices s are in

$$S_0 = \left\{ s \in \mathbb{R}^{(m+1)\times n} \mid 0 \leq v_i(x_i^j) \leq w_i, \sum_{i=1}^n w_i = 1 \right\}. \tag{6}$$

The attribute weights and the alternatives can be characterized by statements that lead to linear inequalities on the scores and weights. These constraints can result from preference statements of form (3), intervals of achievement levels $x_i^b \in [\underline{x_i^b}, \overline{x_i^b}]$ which correspond to constraints $w_i \cdot \min\{v_i^N(\underline{x_i^b}), v_i^N(\overline{x_i^b})\} \leq v_i(x_i^b) \leq w_i \cdot \max\{v_i^N(\underline{x_i^b}), v_i^N(\overline{x_i^b})\}$ and ordinal comparisons of attribute-specific values $v_i(x_i^b) \geq v_i(x_i^c)$ (Salo and Hämäläinen 2001), for example. The constraints define the feasible region $S \subseteq S_0$, which is a convex polyhedron. The dominance relation of Definition 2 is modified so that x^j dominates x^k, if dominance holds for all characterizations of these alternatives.

Definition 4 *Let $S \subseteq S_0$. Alternative $x^j \in \tilde{X}$ dominates $x^k \in \tilde{X}$ with information S, denoted by $x^j \succ_S x^k$, if $V^N(x^j) \geq V^N(x^k)\ \forall s \in S$ and $V^N(x^j) > V^N(x^k)$ for some $s \in S$.*

3 Ranking intervals

For any $s \in S$, the alternatives can be ranked based on their overall values so that the alternative with the highest value is assigned ranking one, the one with the second highest value is assigned ranking two, and so on. By rank-ordering the alternatives for all $s \in S$, we obtain the *rank-orderings* (Salo and Punkka 2005) that are *compatible* with S. We work with the representation (5) of V but omit the superscript 'N' from V^N to highlight that the results do not depend on the selection of $\mathcal{V}'$.

Definition 5 *Let $\tilde{X} = \{x^1, \ldots, x^m\} \subseteq X$, $\emptyset \neq S \subseteq S_0$ and assume that $r(\cdot; \tilde{X}) : \tilde{X} \to \{1, \ldots, m\}$ a bijection. The rank-ordering r is compatible with S (denoted by $r \in R(S)$), if $r(x^j) < r(x^k) \Rightarrow V(x^j) \geq V(x^k)$ for some $s \in S$. Rank-ordering r respects dominance in S (denoted by $r \in R^{\succ}(S)$), if $r \in R(S)$ and $r(x^j) < r(x^k) \Rightarrow \neg\left(x^k \succ_S x^j\right)$.*

The sets $R(S)$ and $R^{\succ}(S)$ treat ties differently. In $R(S)$, if the value of x^j is equal to that of x^k for some $s \in S$, then x^j can have a smaller (that is, better) ranking than x^k. In $R^{\succ}(S)$, x^j can have a smaller ranking than x^k only if its value is higher than that of x^k for some $s \in S$ or equals the value of x^k throughout S.

These two sets of rank-orderings are suitable for different settings. In decision support, it is not rational to prefer a dominated alternative to the one it is dominated by. Consequently, the DM could focus on the rank-orderings in $R^{\succ}(S)$. But if the objective is to examine what rankings an alternative can attain with different, complete preferences $s^* \in S$, it can be meaningful to report ties also in favor of the dominated alternative.

As an example, consider alternatives x^1, x^2, x^3 and x^4 with normalized scores $v^N(x^1) = (1/2, 1/2), v^N(x^2) = (1, 0), v^N(x^3) = (2/5, 1), v^N(x^4) = (5/8, 5/8)$, and assume the weights to be constrained by $w_2 \leq w_1 \leq 2w_2$. The overall values in Figure 2 show that x^2, x^3 and x^4 are non-dominated, and they all dominate x^1. The rank-ordering $r = (r(x^1), r(x^2), r(x^3), r(x^4)) = (3, 4, 1, 2)$ is compatible with S, because $V(x^3) = 7/10 > V(x^4) = 5/8 > V(x^1) = 1/2 = V(x^2)$ if $w = (w_1, w_2) = (1/2, 1/2)$, but this rank-ordering does not respect the dominance relation $x^2 \succ_S x^1$. Among the other rank-orderings, $(4, 3, 1, 2)$ is the only compatible one for weights such that $w_1 < 5/8$ and $(4, 1, 3, 2)$ for weights such that $w_1 > 5/8$. If $w_1 = 5/8$, the values of x^2, x^3 and x^4 are all $5/8$, and thus also $(4, 1, 2, 3)$, $(4, 2, 1, 3)$, $(4, 2, 3, 1)$ and $(4, 3, 2, 1)$ are in $R^{\succ}(S)$.

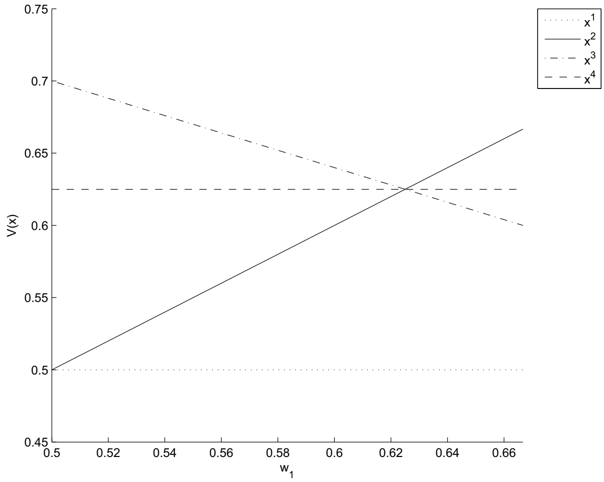

Figure 2: Overall values

The number of compatible rank-orderings can be very large. The rankings that the alternatives can attain provide a more concise way to communicate rank-based results. Let the minimum and maximum rankings of x^k among $R(S)$ be denoted by $\underline{r}_S(x^k)$ and $\overline{r}_S(x^k)$, respectively. Similarly, $\underline{r}^{\succ}_S(x^k)$ and $\overline{r}^{\succ}_S(x^k)$ refer to the minimum and maximum rankings among $R^{\succ}(S)$. The next lemma shows that attainable rankings are sets of consecutive integers, *ranking intervals.*

Lemma 1 *Let $S \subseteq S_0$ be a convex set. Then*

a) $\bigcup_{r \in R(S)} r(x^k) = \{\underline{r}_S(x^k), \ldots, \overline{r}_S(x^k)\}$, *b)* $\bigcup_{r \in R^{\succ}(S)} r(x^k) = \{\underline{r}^{\succ}_S(x^k), \ldots, \overline{r}^{\succ}_S(x^k)\}$.

By Lemma 1, the attainable rankings of an alternative can be determined from its extremum rankings. Among $R(S)$, these can be described in terms of V and they are solutions to MILP models (cf. Kämpke 1996, Köksalan et al. 2010).

Theorem 2 *Let $S \subseteq S_0$ be a convex polyhedron, $x^k \in \{x^1, \ldots, x^m\} = \tilde{X} \subseteq X$, $y = (y_1, \ldots, y_m)$ and constant $M > 1$. Then,*

a) $\underline{r}_S(x^k) = 1 + \min_{s \in S} \left|\{j \in \{1, \ldots, m\} \mid V(x^j) > V(x^k)\}\right|$, *which is equal to the optimum of*

$$\min_{\substack{s\in S,\\ y\in\{0,1\}^m}} \left\{ \sum_{j=1}^{m} y_j \mid V(x^j) \le V(x^k) + y_j M \ \forall j \in \{1,\ldots,m\},\ y_k = 1 \right\}, \tag{7}$$

b) $\overline{r}_S(x^k) = \max_{s\in S} \left|\{j \in \{1,\ldots,m\} \mid V(x^j) \ge V(x^k)\}\right|$, *which is equal to the optimum of*

$$\max_{\substack{s\in S,\\ y\in\{0,1\}^m}} \left\{ \sum_{j=1}^{m} y_j \mid V(x^k) \le V(x^j) + (1-y_j) M \ \forall j \in \{1,\ldots,m\},\ y_k = 1 \right\}. \tag{8}$$

To compute ranking intervals in $R^{\succ}(S)$, dominance relations among alternatives $\tilde{X}$ are determined first. If the optimum of the linear program $\min_{s\in S}[V(x^j) - V(x^k)]$ is positive, then $x^j \succ_S x^k$. If it is negative, then $x^j \not\succ_S x^k$. If it is zero, then $x^j \succ_S x^k$ if and only if $\max_{s\in S}[V(x^j) - V(x^k)] > 0$.

The interpretation of the binary variables in Theorem 2 is that $y_j = 1$ if and only if $r(x^j) < r(x^k)$, for $k \neq j$. In view of this interpretation, the constraints

$$y_l = 1 \quad \forall\, l \in \{j \in \{1,\ldots,m\} \mid x^j \succ_S x^k\} \tag{9}$$

guarantee that an alternative that dominates x^k has a smaller ranking than x^k and

$$y_l = 0 \quad \forall\, l \in \{j \in \{1,\ldots,m\} \mid x^k \succ_S x^j\} \tag{10}$$

impose the requirement that x^k has a smaller ranking than the alternatives it dominates.

Lemma 2 *Let* $S \subseteq S_0$, $x^k \in \tilde{X}$, $y = (y_1,\ldots,y_m)$ *and constant* $M > 1$. *Then*

a) $\underline{r}^{\succ}_S(x^k)$ *is equal to the optimum of (7) subject to (9),*

b) $\overline{r}^{\succ}_S(x^k)$ *is equal to the optimum of (8) subject to (10).*

4 Sensitivity analysis of university rankings

In the Academic Ranking of World Universities (ARWU; http://www.arwu.org/), over 1000 universities are evaluated with an additive model. Table 1 presents the attributes and the weights $\overline{w}_i$ applied in ARWU model (see Liu and Cheng 2005 for details). For each attribute a_i, the universities $x^j = (x_1^j, \ldots, x_6^j)$ have non-negative normalized scores $v_i^N(x_i^j) = 100x_i^j/x_i^*$ in which x_i^* is the achievement level of the university that performs best with regard to attribute a_i. The variability in the size of the universities is accounted for by attribute a_6. Attribute a_6 is normalized so that the best-performing university has a score of 100 and $v_6^N = 0$, if the scores for other attributes a_1–a_5 are zero.

Table 1: Attributes, measurement scales and weights. SSCI=Social Science Citation Index, SCIE=Science Citation Index-expanded

	attribute	measure x_i^j	weight $\overline{w}_i$
a_1	Quality of education	Alumni of an institution winning Nobel Prizes and Fields Medals	0.10
a_2	Quality of faculty 1	Staff of an institution winning Nobel Prizes and Fields Medals	0.20
a_3	Quality of faculty 2	Highly cited researchers in 21 broad subject categories	0.20
a_4	Research output 1	Articles published in Nature and Science	0.20
a_5	Research output 2	# of Articles in SCIE + 2 times # of Articles in SSCI	0.20
a_6	Size	$\sum_{l=1}^{5} \overline{w}_l v_l^N(x_l^j)/M(x^j)$, $M(x^j)$ = # of full-time equivalent academic staff	0.10

The ARWU web site gives complete data on 498 of the 500 best universities in 2010. Using this data, we perform sensitivity analysis on the ARWU weights $\overline{w}$ to get insights in the following questions: How much can the universities' rankings change, if weights other than $\overline{w}_i$ are used? For which universities are these weights optimal, in the sense that their rankings can only become worse as a result of 'small' weight perturbation in these weights? Which universities perform well across a wide range of weights?

The attribute weights $\overline{w}_i$ have stayed the same for many years and they do not reflect how the attributes' scales $[0, x_i^*]$ have changed. Thus, for example, the value of a single *article published in Nature* compared to the value of a *Nobel prize winning staff member* can differ from one year to another. We do not seek to capture the sensitivity of such valuations across the attributes, but merely treat weights as numerical parameters. Technically, we consider subsets of $S_w^0 = \{s \in S_0 \mid s_{ji} = v_i(x_i^j) = w_i v_i^N(x_i^j)/100 \ \forall \ i = 1, \ldots, 6, j = 1, \ldots, 498\}$ and, more specifically, feasible weight regions S_1–S_5 that are subsets of each other so that $S_k \subset S_{k-1}$:

- No weight information: $S_1 = S_w^0$.

- Weights are bounded from below by 0.02: $S_2 = \{s \in S_w^0 \mid w_i \geq 0.02 \; \forall \; i = 1, \ldots, 6\}$.
- Weights are bounded from below by 0.02 and the weights of attributes a_1 and a_6 are smaller than or equal to those of a_2–a_5:

$$S_3 = \left\{s \in S_w^0 \mid w_i \geq w_j \geq 0.02 \; \forall \; i = 2, 3, 4, 5, \forall \; j = 1, 6\right\}.$$

- Weights are allowed to vary within 30 % and 10% intervals around the ARWU weights $\overline{w}_i$:

$$\begin{aligned} S_4 &= \left\{s \in S_w^0 \mid 0.7\overline{w}_i \leq w_i \leq 1.3\overline{w}_i \; \forall \; i = 1, \ldots, 6\right\}, \\ S_5 &= \left\{s \in S_w^0 \mid 0.9\overline{w}_i \leq w_i \leq 1.1\overline{w}_i \; \forall \; i = 1, \ldots, 6\right\}. \end{aligned}$$

We focus on rank-orderings which respect dominance. The universities are indexed so that x^j is ranked j-th in the ARWU. Table 2 presents the data and the ranking intervals of universities x^1–x^{50} among all 498 universities. The university x^{334} illustrates how much the choice of weights can influence the rankings; with no weight information, all rankings from 10 to 498 are in its ranking interval, as seen in Figure 3.

The results show that:

- Some universities' rankings are more robust to the weights than others'. For example, x^1 is among the top two no matter what the weights are. The rankings of x^{10} and x^{21} can drop only by 10 to 20 and by 15 to 36, respectively, while x^7 can attain ranking 126 and x^{19} ranking 210 for some non-negative weights (set S_1). The ranking of x^{34} is quite sensitive even to 'small' weight perturbation (sets S_4–S_5), while the ranking of x^{14} is sensitive to 'large' (S_1–S_3), but not 'small' weight perturbation. The rankings of the top ten universities are at most 12 with weights from S_4; but with weights from S_3, universities x^6, x^7 and x^9 can attain rankings above 50, while the other top ten universities' maximum rankings are below 20. Pairwise comparisons are possible, too: x^{44} is ranked behind x^{43} in ARWU, but its maximum rankings are smaller than those of x^{43} for all weight sets S_1–S_5.
- The rankings of some universities do not drop, if 'slightly' different weights are used. For example, the ranking interval of x^{13} is $[11, 13]$ with weights from S_4. On the contrary, a 'small' change in weights would not improve the ranking of x^{17} as it is in

Table 2: Data and ranking intervals for 51 universities

x^j	$v_1^N(x_1^j)$	$v_2^N(x_2^j)$	$v_3^N(x_3^j)$	$v_4^N(x_4^j)$	$v_5^N(x_5^j)$	$v_6^N(x_6^j)$	S_1	S_2	S_3	S_4	S_5
1	100	100	100	100	100	69.2	[1, 2]	[1, 2]	[1, 1]	[1, 1]	[1, 1]
2	67.6	79.3	69	70.9	70.6	54.2	[2, 9]	[2, 7]	[2, 7]	[2, 4]	[2, 3]
3	40.2	78.4	87.6	68.4	69.7	50.1	[2, 15]	[2, 14]	[2, 8]	[2, 5]	[2, 4]
4	70.5	80.3	66.8	70.1	61.4	64.5	[2, 31]	[2, 19]	[2, 19]	[2, 5]	[3, 4]
5	88.5	92.6	53.9	54.3	65.7	53.1	[2, 18]	[2, 13]	[2, 13]	[2, 6]	[5, 5]
6	50.3	68.8	56.7	64.8	46.9	100	[1, 108]	[1, 66]	[5, 66]	[5, 7]	[6, 6]
7	56.4	84.8	61.1	43.3	44.3	65.5	[2, 126]	[3, 91]	[3, 91]	[6, 9]	[7, 8]
8	70.7	67.4	56.2	47.6	69.9	32.1	[3, 28]	[3, 20]	[5, 15]	[6, 8]	[7, 8]
9	65.5	83.9	50.9	39.8	50.5	40	[3, 82]	[3, 57]	[4, 57]	[9, 12]	[9, 10]
10	56.2	57.6	48.8	49.8	68.5	41.1	[5, 20]	[6, 19]	[7, 19]	[8, 11]	[9, 10]
11	48.6	44.9	58.5	56.3	62	37	[5, 28]	[5, 22]	[6, 22]	[9, 11]	[11, 11]
12	42.3	51.1	54.3	49.9	59.5	38.1	[9, 41]	[9, 30]	[9, 30]	[12, 13]	[12, 13]
13	27.2	42.6	56.9	49.2	75.1	31.2	[4, 38]	[4, 37]	[4, 16]	[11, 13]	[12, 13]
14	15.1	35.8	60.2	54.6	65.1	37.9	[5, 114]	[5, 89]	[5, 21]	[14, 16]	[14, 14]
15	32.9	34.3	57.1	46.9	68.6	28.5	[8, 48]	[9, 36]	[9, 22]	[14, 17]	[15, 16]
16	24.4	31.7	53.9	51.6	72.5	28.1	[6, 53]	[8, 41]	[8, 28]	[14, 16]	[15, 16]
17	36.5	35.4	51.9	40.2	66.1	25.7	[13, 81]	[13, 53]	[14, 26]	[17, 21]	[17, 20]
18	43.6	32.1	42	49.4	64	27.2	[10, 61]	[11, 44]	[15, 28]	[17, 21]	[17, 20]
19	0	40.1	53.4	51.8	60.7	33.6	[9, 210]	[10, 204]	[10, 31]	[17, 24]	[17, 21]
20	33.3	14.1	42	52	80.4	34.5	[2, 107]	[2, 88]	[2, 88]	[16, 24]	[17, 20]
21	32.9	32.1	39.4	44.6	67	31.6	[13, 36]	[14, 35]	[14, 35]	[19, 23]	[21, 22]
22	36.5	0	59.8	43.4	79.8	26.3	[2, 155]	[2, 138]	[2, 138]	[17, 28]	[20, 24]
23	34.1	36.1	36.3	43.6	53.6	47.1	[8, 60]	[8, 53]	[17, 53]	[19, 28]	[22, 24]
24	33.7	34.7	38.1	36	67.6	31	[13, 42]	[14, 40]	[14, 40]	[21, 25]	[23, 24]
25	35.4	36.5	42.6	37.1	58.6	27.8	[15, 57]	[16, 43]	[18, 37]	[21, 28]	[25, 26]
26	17.7	37.2	41.4	36.9	62.3	33	[16, 91]	[17, 73]	[17, 33]	[23, 28]	[26, 27]
27	23.8	19.2	38.8	38.3	80.3	27.9	[2, 71]	[3, 60]	[3, 60]	[23, 29]	[25, 27]
28	30.6	16.2	50.4	36.1	66.6	23.9	[13, 107]	[16, 82]	[16, 80]	[25, 32]	[28, 28]
29	18.5	18.9	48.3	35.9	59.7	28.4	[20, 84]	[21, 68]	[21, 64]	[29, 34]	[29, 31]
30	21.3	25.9	38.8	41	54.8	26.7	[23, 68]	[24, 55]	[24, 52]	[29, 33]	[29, 31]
31	32.4	24.4	40.7	36.2	54.4	22.4	[22, 131]	[24, 109]	[28, 55]	[29, 34]	[30, 34]
32	14.1	30.7	38.8	41.7	44.7	33.5	[20, 134]	[21, 115]	[24, 115]	[29, 37]	[32, 34]
33	16	35.1	42	33.3	42.6	37.3	[15, 148]	[16, 129]	[22, 129]	[29, 36]	[31, 34]
34	19.2	58.4	28.8	42.3	21	35.6	[10, 477]	[11, 450]	[11, 450]	[26, 52]	[29, 38]
35	17.7	0	45.8	42.2	62	24.4	[20, 190]	[22, 147]	[22, 139]	[31, 50]	[34, 38]
36	17.7	18.9	32.2	30.8	65.7	23.7	[18, 111]	[20, 95]	[20, 67]	[33, 42]	[35, 38]
37	22	19.9	41.4	29	53.6	26.2	[29, 74]	[30, 64]	[31, 61]	[34, 40]	[35, 38]
38	18.5	16.6	46.1	28.4	54.4	24.7	[22, 95]	[22, 84]	[22, 82]	[33, 43]	[36, 39]
39	34.8	23.5	24.9	28.8	59.9	21.9	[20, 144]	[21, 113]	[29, 82]	[32, 46]	[36, 39]
40	26.1	24.1	26	26	56.4	32.3	[23, 83]	[25, 76]	[34, 76]	[36, 47]	[40, 43]
41	10.7	16.2	39.4	27.7	60.6	23.9	[30, 183]	[31, 167]	[31, 86]	[38, 53]	[40, 43]
42	26.1	27.2	31.4	20.5	49.9	38.1	[14, 102]	[15, 87]	[33, 87]	[35, 49]	[40, 44]
43	11.9	0	46.6	37.4	56.1	23.2	[22, 239]	[22, 167]	[22, 149]	[36, 64]	[40, 50]
44	23.2	18.9	27.9	28	59.1	23.1	[31, 118]	[33, 110]	[35, 71]	[40, 53]	[43, 48]
45	31.7	46	12.5	20.8	49.9	23.6	[13, 237]	[14, 206]	[14, 206]	[34, 66]	[41, 52]
46	0	29.3	36.7	26.3	49.3	26.9	[28, 240]	[28, 209]	[28, 89]	[38, 59]	[43, 51]
47	0	0	47.2	31.7	63	26	[20, 247]	[22, 232]	[22, 143]	[37, 70]	[42, 56]
48	0	26.7	38.8	26.3	53.1	20	[29, 278]	[32, 225]	[32, 64]	[39, 59]	[43, 51]
49	20.6	33.1	30.5	29.9	38.4	23.5	[25, 208]	[28, 182]	[28, 182]	[39, 62]	[44, 52]
50	26.1	20.9	27.9	30.4	48.2	26.1	[35, 98]	[35, 92]	[39, 92]	[43, 55]	[46, 50]
334	0	0	14.4	5.7	16.4	44.7	[10, 498]	[12, 498]	[212, 498]	[252, 450]	[306, 371]

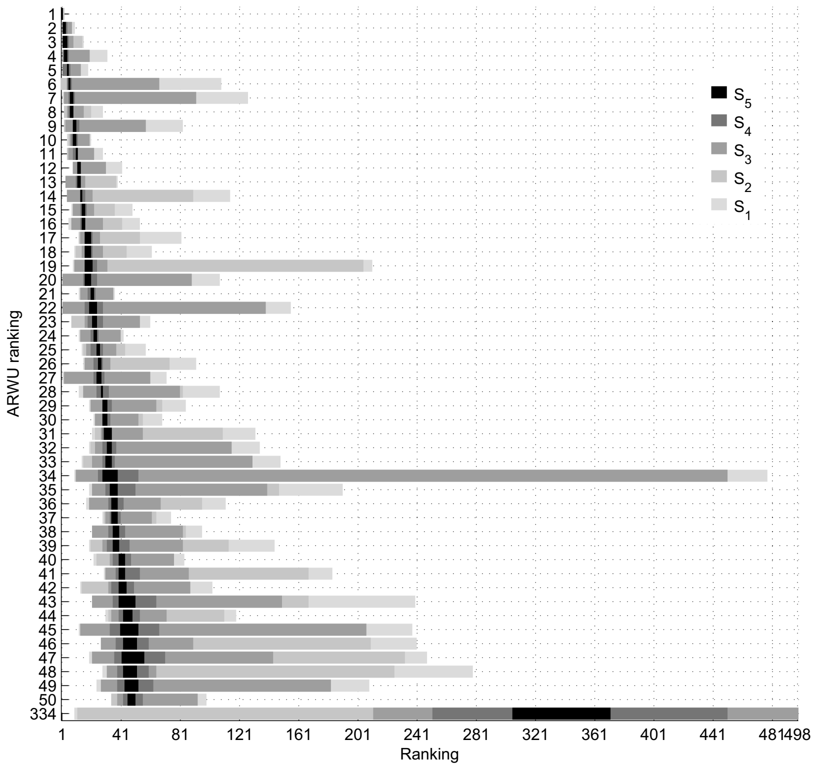

Figure 3: Ranking intervals for 51 universities

[17, 21] for all weights in S_4. The ranking intervals also show whether the ranking can improve or drop more as a result of perturbing weights: The ranking of x^{37} can drop by 24 to 61, but improve only by 6, if weights in S_3 are considered. The largest corresponding drop for x^{26} is 7, but the biggest possible improvement is 9.

- Incomplete ordinal information and lower bounds (set S_3) can narrow the ranking intervals effectively. For example, the maximum ranking of x^{19} improves from 210 to 31 as a result of introducing this preference information. Focusing on rank-orderings that respect dominance can narrow the intervals, too. For example, the value of x^{19} is zero with weights $(1, 0, 0, 0, 0, 0)$, but $498 \notin R^{\succ}(x^{19}, S_1)$, because x^{19} dominates

288 universities.

5 Ranking intervals in project portfolio selection

5.1 Additive-linear portfolio value

Multi-criteria methods are extensively employed to select subsets of projects from a large set of proposals $\tilde{X}$. Several applications use an additive value function to model the positive value that would be gained if the project was selected, and define the value of the *portfolio* as the sum of its constituent projects' values (e.g., Golabi et al. 1981, Ewing et al. 2006, Kleinmuntz and Kleinmuntz 1999). Optimization models can then be applied to solve the portfolio $p \subseteq \tilde{X}$ that maximizes this additive-linear value

$$V(p) = \sum_{x^j \in p} V(x^j) = \sum_{x^j \in p} \sum_{i=1}^{n} w_i v_i^N(x_i^j) \tag{11}$$

subject to resource and other possible portfolio-level constraints as well as possible interactions among the projects (e.g., Stummer and Heidenberger 2003). These constraints are typically modeled with linear inequalities which characterize *feasible* portfolios $P_F \subseteq 2^{\tilde{X}}$.

We first revisit a case study by Könnölä et al. (2007) and show how ranking intervals can be used to support multi-criteria portfolio selection when feasible portfolios are constrained only by the maximum number of projects they can contain and there are no project interactions. Second, we discuss how ranking intervals can be used in the early stages of project portfolio selection.

5.2 Ranking intervals in screening of innovation ideas

Könnölä et al. (2007) use an additive-linear model of portfolio value (11) to support the screening of innovation ideas. Their model accommodates numerical evaluations on a scale from 1 to 7, provided by dozens of experts with regard to novelty (a_1), feasibility (a_2), and relevance (a_3) of each innovation idea. Of the three approaches for using these numerical evaluations, we consider the one in which the normalized score of an innovation idea is essentially the average of the participants' evaluations. Feasible portfolios are constrained by the 'budget', defined by the number of projects to be included in the portfolio.

Könnölä et al. (2007) used the RPM method (Liesiö et al. 2007), which accommodates incomplete preference information and uses dominance (see Definition 4) to compare feasible portfolios. Project-level decision recommendations are based on the analysis of the *non-dominated portfolios*. A project (an innovation idea in this context) that is included in all of them is a *core project* and should be selected; *exterior projects* do not belong to any of them and should be discarded; the rest are *borderline projects*. Figure 4 presents this categorization and the ranking intervals for the 28 innovation ideas in 'Health care and societal services' without any information about the attribute weights. The 'budget' is ten ideas.

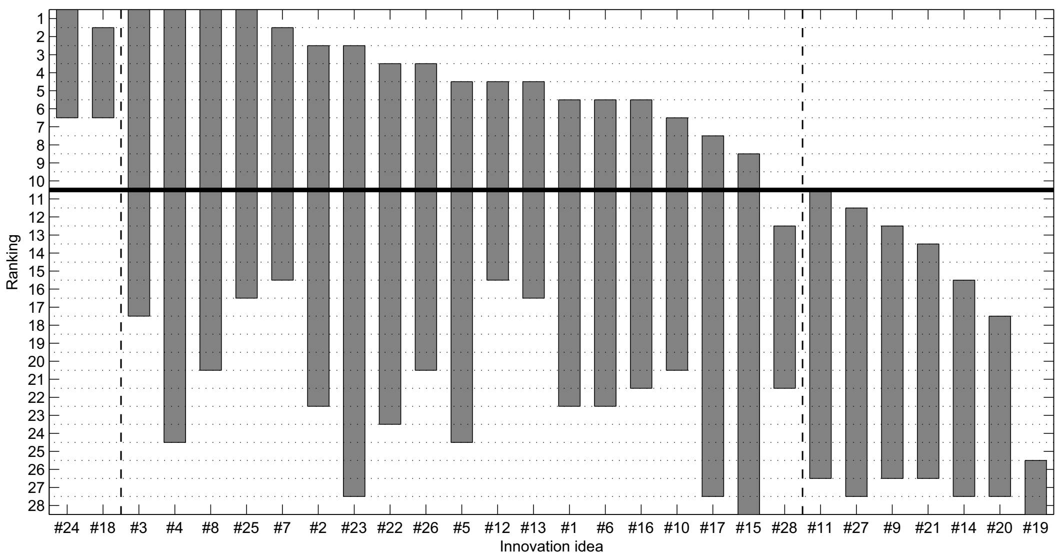

Figure 4: Ranking intervals of the innovation ideas. The horizontal line highlights the budget of ten. The vertical dashed lines separate core, borderline and exterior projects.

The results show that (i) only the core projects have maximum rankings that are at most ten, (ii) the minimum rankings of the exterior projects are greater than ten, and (iii) there is one borderline project, #28, whose minimum ranking exceeds ten. Indeed, the ranking intervals identify core, borderline and exterior projects among *potentially optimal* portfolios

$$P_{PO}(S) = \{p \in P_F \mid V(p) \geq V(p') \;\forall\; p' \in P_F \text{ for some } s \in S\}$$

simultaneously for all budgets $K = 1, \ldots, m-1$, when there are no other portfolio-level constraints or project interactions.

Theorem 3 *Let $S \subseteq S_0$ be convex, $K \in \{1, \ldots, m-1\}$, $V(p)$ as in (11), and $P_F = \{p \subseteq X \mid |p| \leq K\}$. Then $p \in P_{PO}(S)$ if and only if there exists $r \in R(S)$ so that $r(x^j) \leq K$ for all $x^j \in p$.*

A non-dominated portfolio is not necessarily potentially optimal. For example, project #28 belongs to a non-dominated portfolio which is not potentially optimal. But, as a rule, a potentially optimal portfolio p is non-dominated: it is dominated by p' only, if the values of these portfolios are equal for all those scores with which p maximizes V in addition to the requirement that the dominance conditions – $V(p') \geq V(p)$ for all feasible scores and $V(p') > V(p)$ for some feasible scores – hold (for more detailed discussion, see Hazen 1986 and Rios Insua and French 1991).

Theorem 3 states that if feasible portfolios are constrained only by the number of projects, then choosing the projects with the highest values leads to a value-maximizing portfolio. Furthermore, the ranking intervals identify the projects which are among the K most high-valued ones, no matter how the weights and scores are selected from the feasible region. This is useful, because the budget can be a 'soft constraint' (Liesiö et al. 2008, see also Phillips and Bana e Costa 2007), and the ranking intervals can thus be used to analyze how the recommendations change as a function of the budget. Specifically, if $r_S^-(x^k) > K$, then x^k is an exterior project, if $r_S^-(x^k) \leq K < r_S^+(x^k)$, then x^k is a borderline project, and if $r_S^+(x^k) \leq K$, then x^k is a core project among the potentially optimal portfolios. In this way, ranking intervals provide decision support even if the budget is incompletely defined as an interval $K^- \leq K \leq K^+$.

Könnölä et al. (2007) set the budget to approximately one third of the total number of innovation ideas. The ranking intervals in Figure 4 suggest that the decision recommendations are in this case study robust to budget selection: because there are no innovation ideas with maximum rankings between 7 and 14, ideas #18 and #24 are the only core projects among the potentially optimal portfolios for any budget between 6 and 14. A budget reduction of one innovation idea would not change the set of exterior projects either, but increasing the budget by one to 11 would change the status of #11 from exterior to borderline project.

5.3 Ranking intervals in project proposal screening

Often, the task is to select a portfolio of projects with *different costs* such that the total cost of the portfolio is limited by a budget. One widely used approach to support such portfolio decisions is to rank the proposals in terms of their value-to-cost ratios $V(x^j)/c^j$ (e.g., Kirkwood 1997). In comparison with many methods for portfolio decision analysis, the benefit of these ratios is that several budgets can be considered simultaneously (Kleinmuntz and Kleinmuntz 1999) so that cost differences are transparently accounted for. However, if the preferences are incompletely defined, then the proposals' value-to-cost ratios generally take values over intervals. Comparisons of these intervals involve the same difficulties as comparing alternatives' value intervals (see Section 2), which suggests that it maybe better to focus on the rankings that these value-to-cost ratios can have. The project proposals' value-to-cost ratio ranking intervals can be computed with the linear models of Theorem 2 and Lemma 2 by replacing the overall values $V(x^j)$ by these ratios, if the cost estimates c^j are crisp numbers.

The ranking intervals can be especially useful in the early phases of portfolio decision analysis. Even if additive value function represented the DM's preferences between project proposals, the additive-linear portfolio value (11) requires additional preference independence conditions to hold (Golabi et al. 1981). If some other value function than (11) is used, linear models to compute value-maximizing or non-dominated portfolios (Liesiö et al. 2007, 2008) cannot be used. Indeed, for example, Stummer and Heidenberger (2003) and Grushka-Cockayne et al. (2008) enumerate all 2^m possible portfolios before portfolio-level analysis. Because this is a tractable approach only if the number of proposals is small, Stummer and Heidenberger (2003) propose that a screening phase should be conducted first to compare individual projects and to discard 'inferior' ones so that some 20–30 proposals qualify for the portfolio-level analysis. Similarly, Linton et al. (2002) divide projects into categories *discard*, *consider further*, and also *accept* before modeling interdependencies among the projects and imposing portfolio-level constraints.

The ranking intervals help such screening, because they identify ‘inferior’ and ‘very good’ project proposals also under incomplete preference information. In addition to making it possible to enumerate all project portfolios, decreasing the number of proposals reduces the effort needed (i) to evaluate interdependencies among the proposals, and (ii) to evaluate the proposals in more detail, if the data or the evaluations that describe the proposals is inaccurate or incomplete at the outset: it can be useless to evaluate proposals that perform well (poorly) even if the most pessimistic (optimistic) characterization of the proposal is applied, as they would be selected (discarded) even if more accurate information was available. The RPM method, too, can benefit from such a screening phase, because the computation of the non-dominated portfolios becomes computationally demanding if the number of proposals is very large. In addition, the algorithms of the RPM method assume that the projects are evaluated through score intervals $\underline{v_{ij}} \leq v_i^N(x_i^j) \leq \overline{v_{ij}}$, which are not necessarily readily available at the early stages of the analysis.

6 Conclusion

We have shown that several decision rules and other solution concepts for the comparison of alternatives with incompletely defined preferences provide results which depend on the normalization of the additive value function. Towards this end, we propose ranking intervals which show what rankings the alternatives can attain under incomplete preference information, to complement dominance relations as a means of characterizing which alternatives are better than others. These ranking intervals are easy to understand, take all alternatives into account simultaneously and can be visualized in an understandable format no matter how many criteria there are. The ranking intervals can be efficiently computed by solving MILP problems, even when there are hundreds of alternatives.

Ranking intervals can be particularly useful when conducting sensitivity analyses of ranking lists or rank-orderings, and when screening alternatives. Screening can be useful especially in multi-criteria project portfolio selection, where there may be hundreds of project proposals. In such settings, ranking intervals can be used to discard weak project proposals or to select very good ones already in the early phases of the decision process. This leaves fewer projects for further consideration, so that less resources are needed for more detailed project evaluation before good portfolios are computed in view of the available preference information.

A Examples of results that depend on normalization

Alternative x^1 **quasi-dominates** x^2 if $\max_{V\in\mathcal{V}'}[V(x^2)-V(x^1)] \leq \mu$, in which μ is a "small" predefined parameter (Dias and Climaco 2000). If $\mu = 0.05$, then $(1,2)$ quasi-dominates $(2,1)$ but not vice versa in the example illustrated in Figure 1 with choice $x^+ = (12,2)$. However, if $x^+ = (2,12)$, then $(2,1)$ quasi-dominates $(1,2)$ but not vice versa. Decision rules **quasi-optimality** and **quasi-dominance** by Sarabando and Dias (2009) are based on the concept of quasi-dominance and thus exhibit rank reversals in the same way.

Absolute dominance between alternatives x^j and x^k holds, if $\min_{V\in\mathcal{V}'} V(x^j) > \max_{V\in\mathcal{V}'} V(x^k)$ (Salo and Hämäläinen 2001). If value functions (4), and the two choices of $\mathcal{V}'$ illustrated in Figure 1 are considered, then $V(1,2) = 0.5$ and $V(0.99,1.98) = 0.495$ for all $V \in \mathcal{V}'$ with $x^+ = (2,4)$ and thereby x^1 dominates $(0.99,1.98)$ in the sense of absolute dominance. But if $x^+ = (12,2)$, then $\min_{V\in\mathcal{V}'} V(1,2) \approx 0.1538 < \max_{V\in\mathcal{V}'} V(0.99,1.98) \approx 0.31$ and absolute dominance does not hold.

The **domain criterion** (Eiselt and Laporte 1992) and the **acceptability index** (Lahdelma et al. 1998) measure the "share of attribute weights" for which a particular alternative is recommended. Value functions of form (5) that represent (4) can be written

$$V^N(x) = \underbrace{\frac{x_1^* - x_1^\circ}{x_1^* - x_1^0 + r(x_2^* - x_2^0)}}_{=w_1}\underbrace{\frac{x_1 - x_1^\circ}{x_1^* - x_1^\circ}}_{=v_1^N(x_1)} + \underbrace{\frac{r(x_2^* - x_2^\circ)}{x_1^* - x_1^0 + r(x_2^* - x_2^0)}}_{=w_2}\underbrace{\frac{x_2 - x_2^\circ}{x_2^* - x_2^\circ}}_{=v_2^N(x_2)}, \qquad (12)$$

in which $r \in [0.5, 2]$. Now $V^N(1,2) \geq V^N(2,1) \iff 2 \geq r \geq 1 \iff$

$$[x_1^* - x_1^\circ]/[x_1^* - x_1^\circ + 2(x_2^* - x_2^\circ)] \leq w_1 \leq [x_1^* - x_1^\circ]/[x_1^* - x_1^\circ + x_2^* - x_2^\circ],$$

$$V^N(1,2) \leq V^N(2,1) \iff 1 \geq r \geq 0.5 \iff$$

$$[x_1^* - x_1^\circ]/[x_1^* - x_1^\circ + x_2^* - x_2^\circ] \leq w_1 \leq [x_1^* - x_1^\circ]/[x_1^* - x_1^\circ + 0.5(x_2^* - x_2^\circ)].$$

If $x^* = (2,4), x^\circ = (0,0)$, these conditions are $V^N(1,2) \geq V^N(2,1) \iff 1/5 \leq w_1 \leq 1/3$, $V^N(2,1) \geq V^N(1,2) \iff 1/3 \leq w_1 \leq 1/2$. The acceptability index of $(1,2)$ is thus

$(1/3-1/5)/(1/2-1/5) = 4/9$ which is smaller than $5/9$, the acceptability index of $(2,1)$. If $x^* = (12,2), x^\circ = (0,0)$, the acceptability index of $(1,2)$ is $13/21 > 8/21$.

Rios Insua and French (1991) consider completely defined (5) and propose sensitivity analysis to be based on computing the **closest competitor** of the alternative that maximizes (5) among $\tilde{X}$. This closest competitor (x^{closest}) is the one for which the 'smallest change' in attribute weights – measured for example in Euclidean distance – is required so that $V^N(x^{\text{closest}}) \geq \max_{x^k \in \tilde{X}}\{V^N(x^k)\}$. Let the DM's value function be (12) with $r = 1$, and consider $\tilde{X} = \{x^1, x^2, x^3\} = \{(1,1), (0.9, 1.05), (1.05, 0.8)\}$. Then $V^N(x^1) > \max\{V^N(x^2), V^N(x^3)\}$.

Choice $x^\circ = (0,0), x^* = (1.05, 1.05)$ leads to attribute weights $w = (w_1, w_2) = (0.5, 0.5)$. Then, $V^N(x^2) \geq \max\{V^N(x^1), V^N(x^3)\} \iff w_2 \geq 2/3$ (i.e., $r \geq 2$), and $V^N(x^3) \geq \max\{V^N(x^1), V^N(x^2)\} \iff w_2 \leq 1/5$ (i.e., $r \leq 1/4$). In terms of Euclidean distance, x^2 is the closest competitor of x^1, because

$$\sqrt{(1/2-1/3)^2 + (1/2-2/3)^2} = \sqrt{2}/6 < 3\sqrt{2}/10 = \sqrt{(1/2-4/5)^2 + (1/2-1/5)^2}.$$

But if $x^\circ = (0,0), x^* = (10.5, 1.05)$, then $w = (10/11, 1/11)$ and x^3 is the closest competitor because $V^N(x^3) \geq \max\{V^N(x^1), V^N(x^2)\} \iff w_2 \leq 1/41$ (i.e., $r \leq 1/4$), $V^N(x^2) \geq \max\{V^N(x^1), V^N(x^3)\} \iff w_2 \geq 1/6$ ($r \geq 2$), and $\sqrt{(10/11-40/41)^2 + (1/11-1/41)^2} < \sqrt{(10/11-5/6)^2 + (1/11-1/6)^2}$.

B Proofs

Proof of Theorem 1: a) Let $x_i^{\min} = \text{argmin}_{x \in [l_i, u_i]} v_i(x)$ and $x_i^{\max} = \text{argmax}_{x \in [l_i, u_i]} v_i(x)$. Because V is increasing in $v_i, i = 1, \ldots, n$, for example $(x_1^{\min}, \ldots, x_n^{\min}), (x_1^{\max}, \ldots, x_n^{\max}) \in \{x \in X \mid V(x) > \max\{V(x^1), V(x^2)\} \vee V(x) < \min\{V(x^1), V(x^2)\} \;\forall\; V \in \mathcal{V}\}$ so this set is not empty. Choose $C \neq 0$ so that it is positive, if $V(x) > V(x^1)$ and negative otherwise and $\mathcal{V}'$ so that $V(x) = C$, $V(x^2) = 0$ for all $V \in \mathcal{V}'$. By assumption there exists $V \in \mathcal{V}'$ so that $V(x^1) > V(x^2) = 0$, whereby $\max_{V \in \mathcal{V}'}[V(x^1) - V(x)] = \max_{V \in \mathcal{V}'}[V(x^1)] - C > -C$.

b) Because for any pair $(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, i-1, i+1, \ldots, n\}$ there exist $x_i, x_i' \in X_i$, $v_i(x_i) < v_i(x_i')$ and $x_j, x_j' \in X_j$, $v_j(x_j) < v_j(x_j')$, so that $V(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \geq$

$V(x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_n)\ \forall\ V \in \mathcal{V}$, and v_i are continuous, ratio

$$\frac{\sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)]}{\sum_{i:\ x_i^2 \succ^i x_i^1}[v_i(x_i^2) - v_i(x_i^1)]} \tag{13}$$

has extrema values over $\mathcal{V}$. Let the minimum and maximum of (13) be K^- and K^+, respectively. Maximum K^+ is greater than one, because otherwise $V(x^2) \geq V(x^1)\ \forall\ V \in \mathcal{V}$. Similarly $K^- < 1$. The maximum (minimum) of ratio (13) over $V \in \mathcal{V}$ is equal to its maximum (minimum) over $V \in \mathcal{V}'$, for any $\mathcal{V}'$ that represents $\mathcal{V}$.

If $K^+ + K^- < 2$, choose μ_a, μ_b so that $\min\{1, \mu_b K^+ + (K^+ - K^-)(K^+ - 1)/(2 - K^- - K^+)\} > \mu_a > \mu_b K^+ > 0$. If $K^+ + K^- = 2$, choose them so that $1 > \mu_a > K^+ \mu_b > 0$. Otherwise, set $\mu_b = 0$, and choose $\mu_a \in (0, 1)$.

For all i so that $x_i^1 \succeq^i x_i^2$, set $x_i^+ = x_i^2$ and x_i^b so that $v_i(x_i^b) = \mu_b[v_i(x_i^1) - v_i(x_i^2)] + v_i(x_i^2)$. Such x_i^b exists, because there exists $x_i^b \in [x_i^2, x_i^1] \subseteq X_i$, for any $v_i(x_i^b)$ so that $v_i(x_i^2) \leq v_i(x_i^b) \leq v_i(x_i^1)$, because v_i is continuous. For all i so that $x_i^1 \prec^i x_i^2$, set $x_i^- = x_i^1$ and x_i^a so that $v_i(x_i^a) = \mu_a[v_i(x_i^2) - v_i(x_i^1)] + v_i(x_i^1)$. Now

$$V(x^+) - V(x^-) = -\mu_b \sum_{i:\ x_i^1 \succ^i x_i^2} [v_i(x_i^1) - v_i(x_i^2)] + \mu_a \sum_{i:\ x_i^2 \succ^i x_i^1} [v_i(x_i^2) - v_i(x_i^1)] \tag{14}$$

is (strictly) positive for all $V \in \mathcal{V}$, because the attribute-specific value differences in (14) are positive and $\mu_a > K^+ \mu_b \geq 0$.

$$\max_{V \in \mathcal{V}} \frac{V(x^1) - V(x^2)}{V(x^+) - V(x^-)} = \max_{V \in \mathcal{V}} \frac{\sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)] - \sum_{i:\ x_i^2 \succ^i x_i^1}[v_i(x_i^2) - v_i(x_i^1)]}{-\mu_b \sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)] + \mu_a \sum_{i:\ x_i^2 \succ^i x_i^1}[v_i(x_i^2) - v_i(x_i^1)]}$$

$$= \max_{V \in \mathcal{V}} \frac{\frac{\sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)]}{\sum_{i:\ x_i^2 \succ^i x_i^1}[v_i(x_i^2) - v_i(x_i^1)]} - 1}{-\mu_b \frac{\sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)]}{\sum_{i:\ x_i^2 \succ^i x_i^1}[v_i(x_i^2) - v_i(x_i^1)]} + \mu_a} = \frac{K^+ - 1}{-\mu_b K^+ + \mu_a}.$$

Define $\mathcal{V}'$ so that $V(x^+) = 1, V(x^-) = 0$. Then $\max_{V \in \mathcal{V}'}[V(x^1) - V(x^2)] = \frac{K^+ - 1}{-\mu_b K^+ + \mu_a}$.

Value difference (14) is now equal to one. Thereby

$$\sum_{i:\ x_i^2 \succ^i x_i^1} [v_i(x_i^2) - v_i(x_i^1)] = \frac{1}{\mu_a} + \frac{\mu_b}{\mu_b} \sum_{i:\ x_i^1 \succ^i x_i^2} [v_i(x_i^1) - v_i(x_i^2)]. \tag{15}$$

Inserting (15) to minimal and maximal (13) leads to constraints

$$\sum_{i:\ x_i^1 \succ^i x_i^2} [v_i(x_i^1) - v_i(x_i^2)] \geq \frac{K^-}{\mu_a - \mu_b K^-} \tag{16}$$

and

$$\sum_{i:\ x_i^1 \succ^i x_i^2} [v_i(x_i^1) - v_i(x_i^2)] \leq \frac{K^+}{\mu_a - \mu_b K^+}.$$

Inserting (15) to $\max_{V \in \mathcal{V}'}[V(x^2) - V(x^1)]$ gives

$$\max \left[\frac{1}{\mu_a} + \left(\frac{\mu_b}{\mu_a} - 1 \right) \sum_{i:\ x_i^1 \succ^i x_i^2} [v_i(x_i^1) - v_i(x_i^2)] \right] \tag{17}$$

Because $\mu_b < \mu_a$, the objective function of (17) is decreasing in $\sum_{i:\ x_i^1 \succ^i x_i^2}[v_i(x_i^1) - v_i(x_i^2)]$, and the maximum is obtained when (16) is binding. Thus, $\max_{V \in \mathcal{V}'}[V(x^2) - V(x^1)]$ is

$$\frac{1}{\mu_a} + \left(\frac{\mu_b}{\mu_a} - 1 \right) \frac{K^-}{\mu_a - \mu_b K^-} = \frac{1 - K^-}{\mu_a - \mu_b K^-}.$$

We can now compare the two maximal value differences:

$$\max_{V \in \mathcal{V}'}[V(x^1) - V(x^2)] > \max_{V \in \mathcal{V}'}[V(x^2) - V(x^1)] \iff \frac{K^+ - 1}{-\mu_b K^+ + \mu_a} > \frac{1 - K^-}{\mu_a - \mu_b K^-}.$$

Because $\mu_a > \mu_b K^+$, $(-\mu_b K^+ + \mu_a)(\mu_a - \mu_b K^-)$ is positive. Multiplying both sides of the above inequality by this term leads to

$$\mu_a(K^+ + K^- - 2) + \mu_b(K^+ + K^- - 2K^+K^-) > 0.$$

If $K^+ + K^- > 2$, $\mu_b = 0$ and the inequality holds. If $K^+ + K^- < 2$, then $\mu_a = K^+\mu_b + \epsilon$ for ϵ so that $(K^+ - K^-)(K^+ - 1)/(2 - K^- - K^+) > \epsilon > 0$. Replacing μ_a by $K^+\mu_b + \epsilon$ and dividing both sides by $\mu_b > 0$ leads to

$$(K^+ + \epsilon)(K^+ + K^- - 2) + (K^+ + K^- - 2K^+K^-) =$$

$$(K^+ - K^-)(K^+ - 1) - \epsilon(2 - K^- - K^+) >$$

$$(K^+ - K^-)(K^+ - 1) - \frac{(K^+ - K^-)(K^+ - 1)(2 - K^- - K^+)}{2 - K^- - K^+} = 0.$$

If $K^+ + K^- = 2$, then it suffices to show that $K^+ + K^- - 2K^+K^- > 0$. Inserting $K^+ = 2 - K^-$ to this inequality gives

$$2 - 2(K^-(2 - K^-)) > 0 \iff 2(1 - K^-)^2 > 0,$$

which holds because $K^- < 1$. □

Proof of Lemma 1: Let $R(x^k, S) = \cup_{r \in R(S)} \{r(x^k)\}$, $R^{\succ}(x^k, S) = \cup_{r \in R^{\succ}(S)} \{r(x^k)\}$,

$$R^{\succ}(s, S) = R^{\succ}(S) \cap R(\{s\}) \tag{18}$$

and $R(x^k, s, S) = \{r(x^k) \mid r \in R^{\succ}(s, S)\}$. We first show that $R(x^k, s, S)$ is non-empty and both $R(x^k, s, S)$ and $R(x^k, \{s\})$ consist of consecutive integers.

Take any $s \in S$ and define sets

$$I^{>}(s) = \{j \in \{1, \ldots, m\} \mid V(x^j) > V(x^k)\}, \tag{19}$$

$$I^{=}(s) = \{j \in \{1, \ldots, m\} \mid V(x^j) = V(x^k)\}, \tag{20}$$

$$I^{<}(s) = \{j \in \{1, \ldots, m\} \mid V(x^j) < V(x^k)\}. \tag{21}$$

Assign rankings $1, \ldots, |I^{>}(s)|$ to alternatives $\{x^j \in \tilde{X} \mid j \in I^{>}(s)\}$ and rankings $|I^{>}(s)| + |I^{=}(s)| + 1, \ldots, m$ to alternatives $\{x^j \in \tilde{X} \mid j \in I^{<}(s)\}$ so that $V(x^j) > V(x^l) \Rightarrow r(x^j) < r(x^l), x^j \succ_S x^l \Rightarrow r(x^j) < r(x^l)$. Such assignment is possible, because (i) $V(x^j) > V(x^l) \Rightarrow x^l \not\succ_S x^j$ and (ii) alternatives that have equal values (denoted by index set X') can be assigned rankings in the following way: let X_p contain the indices of alternatives that are non-dominated among $X' \setminus \cup_{i=0}^{p-1} X_i$, $X_0 = \emptyset$. Assign rankings $|\{j \in \{1, \ldots, m\} \mid V(x^j) > V(x'), x' \in X'\}| + 1 + \sum_{0 \le i < p} |X_i|, \ldots, |\{j \in \{1, \ldots, m\} \mid V(x^j) > V(x'), x' \in X'\}| + \sum_{1 \le i \le p} |X_i|$ to x^l so that $l \in X_p, p = 1, \ldots, P$ in which $P = \min\{i \in \{1, \ldots, m\} \mid X_{i+1} = \emptyset\}$.

As there are $|I^{>}(s)|$ alternatives with strictly higher value than x^k and $|I^{<}(s)|$ with strictly lower value than x^k, set

$$B(s) = \{b \in \mathbb{Z} \mid |I^{>}(s)| + 1 \le b \le |I^{>}(s)| + |I^{=}(s)|\} \tag{22}$$

is equal to $R(x^k, \{s\})$.

Partition $I^{=}(s)$ so that $I^{=}_{\succ}(s)$ includes indices of the alternatives that dominate x^k, $I^{=}_{\prec}(s)$ those that x^k dominates and $I^{=}_{\sim}(s) = I^{=}(s) \setminus (I^{=}_{\succ}(s) \cup I^{=}_{\prec}(s))$. Assign rankings $|I^{>}(s)| + 1, \ldots, |I^{>}(s)| + |I^{=}_{\succ}(s)|$ to alternatives $\{x^j \mid j \in I^{=}_{\succ}(s)\}$, and rankings $|I^{>}(s)| + |I^{=}_{\succ}(s)| + |I^{=}_{\sim}(s)| + 1, \ldots, |I^{>}(s)| + |I^{=}(s)|$ to alternatives $\{x^j \mid j \in X^{=}_{\prec}(s)\}$ so that $x^j \succ_S x^l \Rightarrow r(x^j) < r(x^l)$.

Among $R^{\succ}(s,S)$, the ranking of alternative x^k is not smaller than $|I^{>}(s)|+|I^{=}_{\succ}(s)|+1$ because there exist $|I^{>}(s)|$ alternatives whose value is higher than that of x^k, and additional $|I^{=}_{\succ}(s)|$ that dominate x^k. Similarly, it is at most $|I^{>}(s)|+|I^{=}_{\succ}(s)|+|I^{=}_{\sim}(s)|$.

No alternative x^i, $i \in I^{=}_{\sim}(s)$ dominates x^j, $j \in X^{=}_{\succ}(s)$, because $x^i \succ_S x^j \succ_S x^k$ would lead to $x^i \succ_S x^k$ which violates $i \in I^{=}_{\sim}(s)$. Similarly, no alternative x^l, $l \in I^{=}_{\prec}(s)$ dominates x^i, because then $x^k \succ_S x^l \succ_S x^i$, violating $i \in I^{=}_{\sim}(s)$. Take any z in

$$B^{\succ}(s) = \{b \in \mathbb{Z} \mid |I^{>}(s)|+|I^{=}_{\succ}(s)|+1 \leq b \leq |I^{>}(s)|+|I^{=}_{\succ}(s)|+|I^{=}_{\sim}(s)|\} \tag{23}$$

and assign it to x^k, and other rankings $B^{\succ}(s)\setminus\{z\}$ to the alternatives $I^{=}_{\sim}(s)\setminus\{k\}$ so that $x^j \succ_S x^l \Rightarrow r(x^j) < r(x^l)$. Thus, $R(x^k,s,S)$ consists of consecutive integers $B^{\succ}(s)$.

If $s^-, s^+ \in S$ and $\lambda \in [0,1]$, then $s^\lambda = \lambda s^+ + (1-\lambda)s^- \in S$, because S is convex. Because V is linear, (i) if $j \in I^{\equiv}(s^-,s^+) = \{j \in \{1,\ldots,m\} \mid V(x^k) = V(x^j)$ for both s^- and $s^+\}$, then $V(x^k)=V(x^j)$ for any $s=s^\lambda$ and (ii) if $j \in \{1,\ldots,m\}\setminus I^{\equiv}(s^-,s^+)$, then $\Lambda_j(s^-,s^+) = \{\lambda \in [0,1] \mid V(x^j)=V(x^k)$ for $s=s^\lambda\}$ is either empty or a singleton. Let $\Lambda(s^-,s^+) = \cup_{j\in\{1,\ldots,m\}\setminus I^{\equiv}(s^-,s^+)}\Lambda_j(s^-,s^+) = \{\lambda \in [0,1] \mid V(x^k) = V(x^j)$ for some $j \in \{1,\ldots,m\}\setminus I^{\equiv}(s^-,s^+)\}$. By construction, $0 \leq |\Lambda(s^-,s^+)| \leq m-1$.

If $\Lambda(s^-,s^+)=\emptyset$, then the sets $I^{>}(s^\lambda)$, $I^{<}(s^\lambda)$, $I^{=}_{\succ}(s^\lambda), X^{=}_{\sim}(s^\lambda)$ and $I^{=}_{\prec}(s^\lambda)$ do not depend on $\lambda \in [0,1]$. Consequently $R(x^k,\{s^\lambda\})$ and $R(x^k,s^\lambda,S)$ do not depend on λ. If $\Lambda(s^-,s^+)\neq\emptyset$, index the items of $\Lambda(s^-,s^+)$ in an ascending order, i.e., $\lambda_i<\lambda_j \iff i<j$, for any $\lambda_i,\lambda_j \in \Lambda(s^-,s^+)$. Take $\lambda' \in [0,1]$ so that $\lambda' \notin \Lambda(s^-,s^+)$. Then, $I^{=}(s^{\lambda'}) = I^{\equiv}(s^-,s^+)$ by construction. If $\lambda'<\lambda_1$, set $i=0$. If $\lambda'>\lambda_{|\Lambda|}$, set $i=|\Lambda(s^-,s^+)|$. Otherwise, set i so that $\lambda_i<\lambda'<\lambda_{i+1}$. Take $l \in \{i,i+1\}$. Then,

$$j \in I^{>}(s^{\lambda'}) \Rightarrow j \notin (I^{<}(s^{\lambda_l}) \cup I^{=}_{\prec}(s^{\lambda_l})) \iff j \in (I^{>}(s^{\lambda_l}) \cup I^{=}_{\succ}(s^{\lambda_l}) \cup I^{=}_{\sim}(s^{\lambda_l})),$$

$$j \in I^{<}(s^{\lambda'}) \Rightarrow j \notin (I^{>}(s^{\lambda_l}) \cup I^{=}_{\succ}(s^{\lambda_l})) \iff j \in (I^{<}(s^{\lambda_l}) \cup I^{=}_{\prec}(s^{\lambda_l}) \cup I^{=}_{\sim}(s^{\lambda_l})),$$

$$j \in I^{=}_{\sim}(s^{\lambda'}) \Rightarrow j \in I^{=}_{\sim}(s^{\lambda_l}), \quad j \in I^{=}_{\succ}(s^{\lambda'}) \Rightarrow j \in I^{=}_{\succ}(s^{\lambda_l}), \quad j \in I^{=}_{\prec}(s^{\lambda'}) \Rightarrow j \in I^{=}_{\prec}(s^{\lambda_l}).$$

Hence, $|I^{>}(s^{\lambda_l})| \leq |I^{>}(s^{\lambda'})|$ and $|I^{<}(s^{\lambda_l})| \leq |I^{<}(s^{\lambda'})|$. The cardinalities fulfill $|I^{>}(s)|+|I^{=}(s)| = m-|I^{<}(s)|$, whereby using the notation of (22) gives

$$B(s^{\lambda'}) \subseteq B(s^{\lambda_l}). \tag{24}$$

Similarly, $|I^{>}(s^{\lambda_l})|+|I^{=}_{\succ}(s^{\lambda_l})| \le |I^{>}(s^{\lambda'})|+|I^{=}_{\succ}(s^{\lambda'})|$ and $|I^{<}(s^{\lambda_l})|+|I^{=}_{\prec}(s^{\lambda_l})| \le |I^{<}(s^{\lambda'})|+|I^{=}_{\prec}(s^{\lambda})|$. Using the notation of (23) gives

$$B^{\succ}(s^{\lambda_l}) \supseteq B^{\succ}(s^{\lambda'}). \tag{25}$$

a) Take $s^- \in S$ so that $\underline{r}_S(x^k) \in R(x^k,\{s^-\})$ and $s^+ \in S$ so that $\overline{r}_S(x^k) \in R(x^k,\{s^+\})$. If $\Lambda(s^-,s^+)=\emptyset$, then $R(x^k,S)=R(x^k,\{s^\lambda\})$, a set of consecutive integers.

Assume that $\lambda_1 > 0$. Then if $\lambda = 0$, (24) becomes $B(s^-) \subseteq B(s^{\lambda_1})$, whereby $R(x^k,\{s^-\}) \cap R(x^k,\{s^{\lambda_1}\}) \neq \emptyset$ (if $\lambda_1 = 0$, then $s^- = s^{\lambda_1}$, and the above holds). For any $i \in \{1,\ldots,|\Lambda(s^-,s^+)|-1\}$, it holds that $\lambda = (\lambda_i+\lambda_{i+1})/2 \notin \Lambda(s^-,s^+)$. Thereby, $B(s^{\lambda_{i+1}}) \supseteq B(s^{(\lambda_i+\lambda_{i+1})/2}) \subseteq B(s^{\lambda_i})$, and consequently $R(x^k,\{s^{\lambda_i}\}) \cap R(x^k,\{s^{\lambda_{i+1}}\}) \neq \emptyset$. Finally, setting $\lambda = 1, i = |\Lambda(s^-,s^+)|$ gives $R(x^k,\{s^+\}) \cap R(x^k,\{s^{\lambda_{|\Lambda(s^-,s^+)|}}\}) \neq \emptyset$.

Thus, $R(x^k,S) = \cup_{i=1}^{|\Lambda(s^-,s^+)|} R(x^k,\{s^{\lambda_i}\})$, a union of sets $B(s^{\lambda_i})$. As a union of two intersecting sets of consecutive integers, any $B(s^{\lambda_i}) \cup B(s^{\lambda_{i+1}})$ also consists of consecutive integers. By induction, $\cup_{i=1}^{|\Lambda(s^-,s^+)|} R(x^k,\{s^{\lambda_i}\})$ consists of consecutive integers, too.

b) Choose $s^-, s^+ \in S$ so that $r(x^k,s^-,S) = \underline{r}^{\succ}_S(x^k)$ and $r(x^k,s^+,S) = \overline{r}^{\succ}_S(x^k)$. With the same selection of parameters λ as above and the application of (25), $R(x^k,s^-,S) \cap R(x^k,s^{\lambda_1},S) \neq \emptyset$, $R(x^k,s^{\lambda_i},S) \cap R(x^k,s^{\lambda_{i+1}},S) \neq \emptyset$ and $R(x^k,s^+,S) \cap R(x^k,s^{\lambda_{|\Lambda(s^-,s^+)|}},S) \neq \emptyset$. Because $R^{\succ}(S) = \cup_{s\in S} R^{\succ}(S) \cap R(\{s\})$, then $R^{\succ}(x^k,S) = \cup_{i=1}^{|\Lambda(s^-,s^+)|} R(x^k,s^{\lambda_i},S)$. As above in a), this set consists of consecutive integers. □

Proof of Theorem 2: a) Using the partition (19)–(21), $\underline{r}_S(x^k)$

$$= \min_{r(x^k)\in R(x^k,S)} r(x^k) = \min_{\substack{s\in S,\\ r\in R(\{s\})}} r(x^k) = 1 + \min_{\substack{s\in S,\\ r\in R(\{s\})}} \left|\left\{j \in \{1,\ldots,m\} \mid r(x^j) < r(x^k)\right\}\right|$$

$$= 1 + \min_{\substack{s\in S\\ r\in R(\{s\})}} \left(|I^{>}(s)| + \left|\left\{j \in I^{=}(s) \mid r(x^j) < r(x^k))\right\}\right|\right) = 1 + \min_{s\in S} |I^{>}(s)|.$$

Let constant $M > 1$, greater than value difference of any two alternatives. Then, $\underline{r}_S(x^k)$

$$\begin{aligned} &= 1 + \min_{s\in S} |I^{>}(s)| = 1 + \min_{s\in S} \left|\{j \in \{1,\ldots,m\} \mid V(x^j) > V(x^k)\}\right| \\ &= \min_{\substack{s\in S,\\ y\in\{0,1\}^m}} \left\{ \sum_{j=1}^m y_j \mid V(x^j) \le V(x^k) + y_j M \ \forall\ j \in \{1,\ldots,m\} \right\} + 1 \qquad (26) \\ &= \min_{\substack{s\in S,\\ y\in\{0,1\}^m}} \left\{ \sum_{j=1}^m y_j \mid V(x^j) \le V(x^k) + y_j M \ \forall\ j \in \{1,\ldots,m\}, y_k = 1 \right\}. \qquad (27) \end{aligned}$$

In (26), $V(x^j) > V(x^k) \Rightarrow y_j = 1$. If $V(x^j) \leq V(x^k)$, both $y_j \in \{0,1\}$ are feasible, but the minimization of $\sum y_j$ guarantees that such binary variables, including y_k, are zero at the optimum. Thus, the number of indices j so that $V(x^j) > V(x^k)$ is $\sum_{j=1}^m y_j$. Because the selection of $y_k \in \{0,1\}$ does not affect the feasiblity of the solution (the respective constraint is $0 \leq y_k M$), constant 1 in (26) can be replaced by setting $y_k = 1$ to get representation (27).

b) like in a), $\overline{r}_S(x^k) = \max_{r(x^k)\in R(x^k,S)} r(x^k) =$

$$1 + \max_{\substack{s\in S\\ r\in R(\{s\})}} \left(|I^>(s)| + |\{j \in X^=(s) \mid r(x^j) < r(x^k)\}|\right) = \max_{s\in S} \left(|I^>(s)| + |I^=(s)|\right).$$

For each rank-ordering r, define an inverse rank-ordering r^{inv} so that $r(x^j) = z \iff r^{\text{inv}}(x^j) = m - z + 1$. By Definition 5, $r^{\text{inv}}(x^k) > r^{\text{inv}}(x^l) \iff r(x^k) < r(x^l) \Rightarrow V(x^k) \geq V(x^j) \iff -V(x^k) \leq -V(x^j)$. Then, the maximum of $r(x^k)$, $r \in R(S)$ can be obtained by applying the result of part a) to $r^{\text{inv}}(x^k)$:

$$\begin{aligned}
&\overline{r}_S(x^k) = \max_r \left\{r(x^k) \mid r \in R(S)\right\} = \min_{r^{\text{inv}}} \left\{m + 1 - r^{\text{inv}}(x^k) \mid r \in R(S)\right\} \\
= \;\; & m + 1 - \min_{\substack{s\in S,\\ y\in\{0,1\}^m}} \left\{\sum_{j=1}^m y_j \mid -V(x^j) \leq -V(x^k) + y_j M \;\forall\, j \in \{1,\ldots,m\}, y_k = 1\right\}.
\end{aligned}$$

By defining $y'_j = 1 - y_j$, applying the property $\min f = -\max(-f)$, and rearranging terms, the above problem becomes

$$\begin{aligned}
&\overline{r}_S(x^k) \\
= \;\; & m + 1 + \max_{\substack{s\in S,\\ y'\in\{0,1\}^m}} \left\{-m + \sum_{j=1}^m y'_j \mid V(x^k) \leq V(x^j) + y'_j M \;\forall\, j \in \{1,\ldots,m\}, y'_k = 0\right\} \\
= \;\; & \max_{\substack{s\in S,\\ y'\in\{0,1\}^m}} \left\{\underbrace{1 + y'_k}_{=1} + \sum_{\substack{j=1\\ j\neq k}}^m y'_j \mid V(x^k) \leq V(x^j) + y'_j M \;\forall\, j \in \{1,\ldots,m\}, y'_k = 0\right\} \\
= \;\; & \max_{\substack{s\in S,\\ y'\in\{0,1\}^m}} \left\{\underbrace{0 + y'_k}_{=1} + \sum_{\substack{j=1\\ j\neq k}}^m y'_j \mid V(x^k) \leq V(x^j) + y'_j M \;\forall\, j \in \{1,\ldots,m\}, y'_k = 1\right\} \\
= \;\; & \max_{\substack{s\in S,\\ y'\in\{0,1\}^m}} \left\{\sum_{j=1}^m y'_j \mid V(x^k) \leq V(x^j) + y'_j M \;\forall\, j \in \{1,\ldots,m\}, y'_k = 1\right\}.
\end{aligned}$$

Above, it has been acknowledged that with $j = k$, the respective constraint $0 \leq (1 - y'_k)M$ does not affect the solution independently of $y'_k \in \{0,1\}$.$\square$

Proof of Lemma 2: a) Let $R^{\succ}(s,S)$ and $I_{\succ}^{=}(s)$ like in the proof of Lemma 1. Then,

$$\min_{r\in R^{\succ}(S)} r(x^k) = 1+ \min_{\substack{s\in S,\\ r\in R^{\succ}(s,S)}} \left|\left\{j\in\{1,\ldots,m\} \mid r(x^j)<r(x^k)\right\}\right| = 1+\min_{s\in S}\left|I^{>}(s)\cup I_{\succ}^{=}(s)\right|.$$

Divide $I^{>}(s)$ into $I_{\succ}^{>}(s)$ and $I_{\not\succ}^{>}(s)$, indices of alternatives that dominate and do not dominate x^k in S, respectively. Set $I^{\succ} = I_{\succ}^{>}(s)\cup I_{\succ}^{=}(s)$ contains the indices of the alternatives that dominate x^k in S. Then,

$$\min_{r\in R^{\succ}(S)} r(x^k) = 1+\min_{s\in S}|I_{\not\succ}^{>}(s)\cup I_{\succ}^{>}(s)\cup I_{\succ}^{=}(s)| = |I^{\succ}|+1+\min_{s\in S}|I_{\not\succ}^{>}(s)|. \tag{28}$$

By Theorem 2, the term $1+\min_{s\in S}|I_{\not\succ}^{>}(s)|$ in (28) is the minimum ranking of x^k among $X^{\not\succ} = \tilde{X}\backslash\{x^j \mid j\in I^{\succ}\}$ and thus equal to

$$1+\min_{\substack{s\in S\\ y''\in\{0,1\}^m}}\{\sum_{j=1}^{m} y_j'' \mid V(x^j)\le V(x^k)+y_j''M \;\forall\, x^j\in X^{\not\succ}, y_j''=0 \;\forall\, j\in I^{\succ}\}. \tag{29}$$

Set $I^{\succ}$ is the same for any $s\in S$, and thus we can write

$$|I^{\succ}| = \min_{\substack{s\in S\\ y'\in\{0,1\}^m}}\{\sum_{j=1}^{m} y_j' \mid y_j'=1 \;\forall\, j\in I^{\succ}, y_j'=0 \;\forall\, j\in I^{\not\succ}\}, \tag{30}$$

in which $I^{\not\succ}=\{j \mid x^j\in X^{\not\succ}\}$. In (29) and (30), the variables y' and y'' are independent of each other. Inserting (29) and (30) into (28) gives $\min_{r\in R^{\succ}(S)} r(x^k) = 1+$

$$\min_{\substack{s\in S\\ y',y''\in\{0,1\}^m}}\{\sum_{j=1}^{m}[y_j''+y_j'] \mid y_j'=1, y_j''=0 \;\forall\, j\in I^{\succ}, y_j'=0, V(x^j)\le V(x^k)+y_j''M \;\forall\, j\in I^{\not\succ}\}.$$

Substitution $y_j=(y_j''+y_j')\in\{0,1\}$ simplifies this expression to

$$1+\min_{\substack{s\in S\\ y\in\{0,1\}^m}}\{\sum_{j=1}^{m} y_j \mid y_j=1 \;\forall\, j\in I^{\succ}, V(x^j)\le V(x^k)+(y-\underbrace{y_j'}_{=0})M \;\forall\, j\in I^{\not\succ}\}.$$

b) The proof is similar to that of a) and therefore omitted. □

Proof of Theorem 3: Take any $s\in S$ and $p\in P_{PO}(\{s\})$ such that $|p|=K$. Then, $\nexists x^j\in\tilde{X}\setminus p$ such that $V(x^j)>V(x^k), x^k\in p$, because otherwise $V((p\cup\{x^j\})\setminus\{x^k\})>V(p)$ and p would not be potentially optimal in $\{s\}$. Because no such x^j exists, there exists a rank-ordering $r\in R(\{s\})$ such that $r(x^k)<r(x^j)$, $x^k\in p, x^j\notin p$.

Take any $s\in S, r\in R(\{s\})$. Let $p=\{x^k \mid r(x^k)\le K\}$. Then, $\sum_{x^k\in p}V(x^k)\ge\sum_{x^j\in p'}V(x^j), p\neq p'\in P_F$, because $V(x^j)\le V(x^k)$ for any $x^j\notin p$, $x^k\in p$. □

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Paper [IV]

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Ranking Intervals and Dominance Relations for Ratio-Based Efficiency Analysis

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We develop comparative results for ratio-based efficiency analysis (REA) based on the decision-making units' (DMUs') relative efficiencies over sets of feasible weights that characterize preferences for input and output variables. Specifically, we determine (i) *ranking intervals*, which indicate the best and worst efficiency rankings that a DMU can attain relative to other DMUs; (ii) *dominance relations*, which show what other DMUs a given DMU dominates in pairwise efficiency comparisons; and (iii) *efficiency bounds*, which show how much more efficient a given DMU can be relative to some other DMU or a subset of other DMUs. Unlike conventional efficiency scores, these results are insensitive to outlier DMUs. They also show how the DMUs' efficiency ratios relate to each other for *all* feasible weights, rather than for those weights only for which the data envelopment analysis (DEA) efficiency score of *some* DMU is maximized. We illustrate the usefulness of these results by revisiting reported DEA studies and by describing a recent case study on the efficiency comparison of university departments.

Key words: efficiency analysis; data envelopment analysis; preference modeling
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1. Introduction

The seminal paper of Charnes et al. (1978) has spawned a growing literature on data envelopment analysis (DEA) which offers numerous methods for examining the efficiency of decision-making units (DMUs) (see, e.g., Cooper et al. 2007). These methods are often employed in contexts where information about the unit prices of input and output variables is not readily available, but where it is still possible to elicit subjective information about how valuable these variables are relative to each other (Thompson et al. 1986, Allen et al. 1997; cf. Thanassoulis et al. 2004). This is the case in contexts such as higher education, health care, and technology management, among others (see, e.g., Sarrico and Dyson 2000).

Technically, the Charnes-Cooper-Rhodes-DEA (CCR-DEA; Charnes et al. 1978) computes efficiency scores for the DMUs relative to an efficient frontier, characterized by the DMUs that have the highest efficiency ratio between the aggregate value of their outputs and aggregate value of their inputs for *some* feasible input/output weights. By definition, an efficient DMU will have a score of one. For inefficient DMUs, the score is typically less than one and serves as a measure of how close to the efficient frontier a DMU can be when its inputs and outputs are aggregated with weights that are *most favorable* to this DMU. However, a concern with these scores is that they do not convey information about how the efficiency ratio of the DMU compares with the efficiency ratios of other DMUs for *other* input/output weights even though other weights reflect relevant preference information. This recognition has motivated the development of cross-efficiency (CE) methods where the efficiency score for every DMU is computed based on the weights for which the efficiency of *some* DMU is maximized (see Sexton et al. 1986, Doyle and Green 1994). Yet the consideration of these weights *only* does not show how the DMUs' efficiency ratios change relative to each other for *all* feasible weights.

A second concern with conventional efficiency scores is that they can be sensitive to which DMUs are included in or excluded from the analysis: for instance, the introduction or removal of a single outlier may shift the efficient frontier drastically and thus disrupt efficiency scores, which may be perplexing to users (see, e.g., Seiford and Zhu 1998a, b; Zhu 1996). A third concern with conventional scores is that they call for returns-to-scale assumptions, which may be difficult to justify. In effect, these three concerns can be addressed by focusing on pairwise one-on-one comparisons of efficiency ratios among DMUs because such comparisons (i) account for all feasible input/output weights, (ii) are less sensitive to the presence of outlier DMUs, and (iii) do not necessitate assumptions about what the set of production possibilities is beyond the DMUs that are included in the analysis (see, e.g., Galagedera and Silvapulle 2003, Dyson et al. 2001).

Motivated by the above considerations, we develop efficiency results in response to the following questions:

- What are the best and worst rankings that a given DMU can attain in comparison with other DMUs based on the comparison of DMUs' efficiency ratios for all feasible weights?
- Given a pair of DMUs, does the first DMU dominate the second one (in the sense that the efficiency ratio of the first DMU is higher than or equal to that of the second for all feasible weights and strictly higher for some weights)?
- How much more/less efficient can a given DMU be relative to some other DMU or, more generally, relative to the most and least efficient DMU in some subset of DMUs?

The first question is partly motivated by the popularity of ranking lists, as exemplified by the ranking of "best" universities by the Shanghai Jiao Tong University (cf. Liu and Cheng 2005, see also Köksalan et al. 2010). The resulting *ranking intervals*—defined by the DMUs' best/worst rankings over all feasible input/output weights—are robust, because the integer-valued bounds of these intervals can change at most by one when a single DMU is introduced or removed. The second question establishes *dominance relations* based on pairwise comparisons between two DMUs at a time. The third question (which is related to superefficiency; see, e.g., Andersen and Petersen 1993) yields *efficiency bounds* that provide information about the relative efficiency differences among the DMUs. All of these results can be employed in the specification of performance targets. With more preference information, the results become usually more conclusive in terms of narrower ranking intervals, additional dominance relations, and tighter efficiency bounds. Furthermore, ratio-based results can be presented even when the number of DMUs is small, because the results are not computed relative to an efficient frontier for the reliable estimation of which the number of DMUs would have to large compared with the number of input and output variables.

The rest of this paper is organized as follows. Section 2 discusses earlier methods for ratio-based efficiency analysis and their applications in selected application domains. Section 3 formulates ratio-based efficiency results, considers their uses in target setting, and contrasts them with cross-efficiency analysis. Section 4 illustrates these results in the context of reported DEA studies and describes a case study where they were employed in the comparison of university departments. Section 5 concludes.

2. DEA Methods and Their Applications

In the DEA literature, there are numerous methods for analyzing the relative efficiencies of DMUs that transform multiple inputs into multiple outputs (see, e.g., Cooper et al. 2007). Early approaches for incorporating preference information in these methods include the specification of assurance regions (Thompson et al. 1990) and cone ratios (Charnes et al. 1990). Subsequently, relationships between DEA models and multicriteria decision-making methods have been explored extensively (Stewart 1996, Joro et al. 1998, Bouyssou 1999). These relationships also underpin the value efficiency analysis method (Halme et al. 1999, Halme and Korhonen 2000, Korhonen et al. 2002), which makes inferences about the DMUs' value efficiencies with the help of an implicit value function. Recent advances include approaches based on the explicit construction of the decision maker's (DM's) value function (Gouveia et al. 2008) and the specification of context-sensitive assurance regions for input/output weights (Cook and Zhu 2008).

Instead of seeking to survey DEA applications (see, e.g., Cooper et al. 2007, Emrouznejada et al. 2008, Avkiran and Parker 2010), we only provide some pointers to selected DEA models in the three decision contexts—higher education, technology management, and health care—for which we provide numerical efficiency results in §4.

First, higher education is an attractive domain for DEA, because universities consume many inputs and produce multiple outputs to which prices may be difficult to attach. As a result, DEA has been employed extensively in higher education by treating universities, departments, research units, or even students as DMUs. For instance, Ahn et al. (1988) analyze the production behavior of higher education institutions and compare the relative efficiencies of public and private doctoral-granting universities in the United States. Johnes (2006) discusses the role of DEA in higher education and analyzes more than 100 higher educational institutions in the United Kingdom. Tauer et al. (2007) examine the efficiencies of the 26 academic departments at Cornell University when specifying performance targets. Korhonen et al. (2001) establish efficiency scores for research units at the Helsinki School of Economics and present an approach for allocating resources to support the attainment of higher aggregate efficiency. Sarrico and Dyson (2000) describe a DEA-based planning tool for the formulation of strategic options at the University of Warwick.

Second, comparative analyses in technology management involve subjective preferences about the inputs that are needed to develop and deploy technologies with the aim of generating desired outputs.

For example, Shafer and Bradford (1995) compare alternative machine group solutions based on DEA efficiencies. Baker and Talluri (1997) provide decision support for screening robots based on cross-efficiency analysis. Talluri and Yoon (2000) evaluate robots using an extended cone-ratio DEA approach. Eilat et al. (2008) integrate DEA models with a balanced scorecard approach and evaluate research and development projects in different stages of their life cycle. Farzipoor (2009) supports technology selection decision by developing a framework that captures preferences through assurance regions and accommodates both cardinal and ordinal information about the DMUs.

Third, DEA models in health care give insights into which DMUs are more efficient than others when health indicators are viewed as outputs and when inputs consist of health-care investments and possibly contextual factors as well. For instance, Garcia et al. (2002) analyze the efficiency of primary health units and explore how sensitive the DEA results are to the selection of output variables. Hollingsworth et al. (1999) review DEA applications in health care with a particular emphasis on the efficiency evaluation of hospitals. In his comprehensive book, Ozcan (2008) discusses uses of DEA models across a broad range of health-care planning problems.

3. Comparative Results for Ratio-Based Efficiency Analysis

3.1. Efficiency Ratios

Assume that there are K DMUs that consume M types of inputs and produce N types of outputs. The kth DMU (DMU$_k$ for short) consumes $x_{mk} \geq 0$ units of the mth input and produces $y_{nk} \geq 0$ units of the nth output. The input consumption and output production vectors are $x_k = (x_{1k}, \ldots, x_{Mk})^T$ and $y_k = (y_{1k}, \ldots, y_{Nk})^T$, respectively.

Preference information about the relative values of inputs and outputs is captured by nonnegative weights $v = (v_1, \ldots, v_M)^T$ and $u = (u_1, \ldots, u_N)^T$, respectively. These weights are assumed to satisfy homogeneous linear constraints (cf. Podinovski 2001, 2005)

$$S_v = \{v = (v_1, \ldots, v_M)^T \neq 0 \mid v \geq 0, A_v v \leq 0\}, \quad (1)$$

$$S_u = \{u = (u_1, \ldots, u_N)^T \neq 0 \mid u \geq 0, A_u u \leq 0\}, \quad (2)$$

where A_v and A_u are coefficient matrices derived from the DM's preference statements about how valuable different amounts of inputs and outputs are. These statements can be elicited with well-known techniques for the specification of assurance regions (see, e.g., Thompson et al. 1986, 1990); for instance, if the DM states that one unit of output 1 is at least as valuable as a unit of output 2 but not more valuable than two units of output 2, then the constraints $u_2 \leq u_1 \leq 2u_2$ must hold. If such statements are elicited from several DMs, a group preference representation for these DMs can be built by forming convex combinations of those weights that satisfy the constraints of some DM (Salo 1995).

For any feasible input weights $v \in S_v$, the virtual input of DMU$_k$ is $v^T x_k = \sum_{m=1}^{M} v_m x_{mk}$. Similarly, the virtual output for $u \in S_u$ is $u^T y_k = \sum_{n=1}^{N} u_n y_{nk}$. We assume that the virtual inputs and the virtual outputs are strictly positive for all feasible weights (i.e., $\sum_m v_m x_{mk} > 0$, $\forall\, v \in S_v$, and $\sum_n u_n y_{nk} > 0$, $\forall\, u \in S_u$, for all $k = 1, \ldots, K$). This assumption holds, for example, if all inputs and outputs have strictly positive weights and if there is at least one input (output) that is consumed (produced) by every DMU. It also holds if all DMUs consume/produce some positive amounts of all inputs/outputs. The assumption of positive virtual inputs/outputs implies that the (absolute) efficiency ratio (cf. Podinovski 2001) of DMU$_k$, defined as

$$E_k(u, v) = \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}}, \quad (3)$$

is well defined for any $u \in S_u$, $v \in S_v$ (see also Dyson et al. 2001).

3.2. Ranking Intervals

For any feasible input/output weights, the DMUs can be ranked based on their efficiency ratios (3). The resulting rankings can change relative to each other for different weights. We first determine what is the best (highest) efficiency ranking that a DMU can attain relative to other DMUs over the set of input/output weights (1) and (2). For instance, this ranking is three for a DMU if the least number of other DMUs with a strictly higher efficiency ratio is two. Similarly, we compute the worst (lowest) ranking for a DMU. These two bounds establish a *ranking interval*, which conveys information about the relative efficiencies of the DMUs.

Toward this end, we define the sets

$$R_k^{>}(u, v) = \{l \in \{1, \ldots, K\} \mid E_l(u, v) > E_k(u, v)\},$$

$$R_k^{\geq}(u, v) = \{l \in \{1, \ldots, K\} \backslash \{k\} \mid E_l(u, v) \geq E_k(u, v)\},$$

which contain the indexes of those other DMUs whose efficiency ratios are either strictly higher than that of DMU$_k$ (for $R_k^{>}(u, v)$) or at least as high as that of DMU$_k$ (for $R_k^{\geq}(u, v)$). By construction, $R_k^{>}(u, v) \subseteq R_k^{\geq}(u, v)$.

The corresponding efficiency rankings are defined as $r_k^{>}(u, v) = 1 + |R_k^{>}(u, v)|$ and $r_k^{\geq}(u, v) = 1 + |R_k^{\geq}(u, v)|$ (here, $|R|$ denotes the cardinality of the set R). For example, if the efficiency ratio of DMU$_k$ is strictly higher than the efficiency ratios of all other DMUs for some $(u, v) \in (S_u, S_v)$, then $r_k^>(u, v)$ and $r_k^\geq(u, v)$ equal one, because $R_k^>(u, v) = R_k^\geq(u, v) = \varnothing$. Yet these rankings treat ties differently: if exactly two DMUs have same highest efficiency ratio at $(u', v') \in (S_u, S_v)$, then $r^>(u', v')$ ranks them both as first, but $r^\geq(u', v')$ ranks them as second.

The *ranking interval* for DMU_k is now defined as $[r_k^{\min}, r_k^{\max}]$, where the best and worst rankings for DMU_k are given by

$$r_k^{\min} = \min_{u,v} r_k^>(u, v),$$

$$r_k^{\max} = \max_{u,v} r_k^\geq(u, v),$$

and where the optimization problems are solved over $(u, v) \in (S_u, S_v)$. Both optimum solutions exist, because $r_k^>(u, v)$ and $r_k^\geq(u, v)$ assume values in the set $\{1, \ldots, K\}$.

Based on Theorems 1 and 2, the ranking interval $[r_k^{\min}, r_k^{\max}]$ can be determined from mixed integer linear programming problems where the weight sets are closed and bounded by constraints (5) and (7), respectively. In these and also later theorems, C denotes a large positive constant. The proofs are in the appendix.

If DMU_k is CCR-DEA efficient, then for some feasible weights its efficiency ratio is higher than or equal to the efficiency ratio of any other DMU, and thus its best ranking in Theorem 1 will be one.

Theorem 1. *The optimum of the minimization problem*

$$\min_{u,v,z} \; 1 + \sum_{l \neq k} z_l$$

$$\text{subject to } \sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} + C z_l, \quad l \in \{1, \ldots, K\}, \quad l \neq k, \quad (4)$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1, \quad (5)$$

$$z_l \in \{0, 1\}, \quad l \neq k,$$

$$(u, v) \in (S_u, S_v)$$

is $r_k^{\min}$, the best (highest) efficiency ranking of DMU_k.

Theorem 2. *The optimum of the maximization problem*

$$\max_{u,v,z} \; 1 + \sum_{l \neq k} z_l$$

$$\text{subject to } \sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1 - z_l), \quad l \in \{1, \ldots, K\}, \quad l \neq k, \quad (6)$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk} = 1, \quad (7)$$

$$z_l \in \{0, 1\}, \quad l \neq k,$$

$$(u, v) \in (S_u, S_v)$$

is the $r_k^{\max}$, the worst (lowest) efficiency ranking of DMU_k.

With the introduction of additional preference information, the constraints on the feasible input/output weights become tighter. In view of Theorems 1 and 2, such information may lead to narrower (but not wider) ranking intervals.

In general, DMUs that are outliers in the sense that their input/output profiles differ considerably from what is consumed/produced by most DMUs are likely to have wider ranking intervals. This is because these outlier DMUs can have either good (high) or bad (low) rankings at the extreme points of S_u and S_v. Conversely, DMUs whose profiles are more typical are likely to have narrower ranking intervals.

3.3. Efficiency Dominance

Although ranking intervals provide information about the relative efficiencies of the DMUs, they are not well suited for the comparison of *pairs* of DMUs. Specifically, even if two DMUs have overlapping ranking intervals, it is possible that one of them has a higher efficiency ratio (3) for all feasible input/output weights.

To compare the efficiency ratios of DMUs on a one-on-one basis, we build on concepts from preference programming (see, e.g., Salo and Hämäläinen 1992, 2001) and define *efficiency dominance* between DMUs as follows.

Definition 1. DMU_k dominates DMU_l (denoted by $\text{DMU}_k \succ \text{DMU}_l$) if and only if

$$E_k(u, v) \geq E_l(u, v) \quad \text{for all } (u, v) \in (S_u, S_v), \quad (8)$$

$$E_k(u, v) > E_l(u, v) \quad \text{for some } (u, v) \in (S_u, S_v). \quad (9)$$

If $\text{DMU}_k \succ \text{DMU}_l$, the efficiency ratio of DMU_k is at least as high as that of DMU_l for all feasible weights, and moreover, there exist some weights for which its efficiency is strictly higher. By construction, Definition 1 establishes a strict partial order that is an irreflexive, asymmetric, and transitive binary relation among the DMUs. This relation, however, may not be total (i.e., it may be that neither $\text{DMU}_k \succ \text{DMU}_l$ nor $\text{DMU}_l \succ \text{DMU}_k$).

The dominance relation in Definition 1 can be determined based on the pairwise efficiency ratio

$$D_{k,l}(u, v) = \frac{E_k(u, v)}{E_l(u, v)}. \quad (10)$$

By Lemma 1, this ratio is invariant subject to multiplication of input/output weights by positive constants.

Lemma 1. *Take any $(u, v) \in (S_u, S_v)$, and let (u', v') be vectors that are obtained from (u, v) by multiplying them componentwise so that $u' = c_u u$, $v' = c_v v$ for some $c_u > 0$, $c_v > 0$. Then, $(u', v') \in (S_u, S_v)$ and $D_{k,l}(u, v) = D_{k,l}(u', v')$.*

In view of Lemma 1, the ratio (10) remains invariant even if weights are normalized through constraints such as $\sum_n u_n = 1$ and $\sum_m v_m = 1$. After the introduction of such constraints, the feasible sets S_u and S_v become closed and bounded. Because the ratio $D_{k,l}(u, v)$ is continuous in input/output weights, it therefore achieves its maximum and minimum values, denoted by $\overline{D}_{k,l}$ and $\underline{D}_{k,l}$, respectively.

The relative efficiency ratio (10) is nonlinear in weights (u, v). Yet, by Theorem 3, this ratio can be maximized and minimized through linear programming.

Theorem 3. *The optimum of the maximization (minimization) problem*

$$\max_{u,v} \ (\min_{u,v}) \ \sum_n u_n y_{nk} \tag{11}$$

$$\text{subject to} \ \sum_n u_n y_{nl} = \sum_m v_m x_{ml}, \tag{12}$$

$$\sum_m v_m x_{mk} = 1, \tag{13}$$

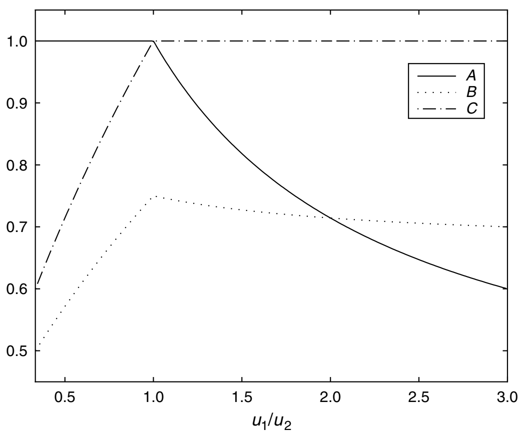
$$(u, v) \in (S_u, S_v) \tag{14}$$

is the maximum (minimum) of $D_{k,l}(u, v)$ over (S_u, S_v).

The optimization problems in Theorem 3 provide upper and lower bounds on how efficient DMU_k can be relative to DMU_l across feasible weights. For example, if $\overline{D}_{k,l} = 1.42$, the efficiency ratio of DMU_k can be *at most* 42% greater than that of DMU_l. Conversely, if $\underline{D}_{k,l} = 1.10$, the efficiency ratio of DMU_k is *at least* 10% higher than that of DMU_l. Thanks to Theorem 3, the dominance structure can be computed efficiently with linear programming. First, if the minimum $\underline{D}_{k,l}$ is greater than one, DMU_k dominates DMU_l. Second, if it is less than one, (8) is violated and dominance does not hold. Third, if the minimum is exactly one, the sufficiency condition (9) can be checked by maximizing (11) subject to (12)–(14). If the resulting maximum $\overline{D}_{k,l}$ exceeds one, dominance does hold; but if not, then DMU_k and DMU_l have the same efficiency ratio (3) for all feasible weights, and there is no dominance. Also, the transitivity and asymmetricity properties of $\succ$ can be exploited to further reduce the number of pairs for which the dominance relation must be explicitly computed.

A DMU is not necessarily dominated by another DMU that has a higher CCR-DEA efficiency score. For example, consider three DMUs, A, B, and C, that all consume one unit of a single input and produce two outputs so that $A = (1, 3)$, $B = (2, 1)$, and $C = (3, 1)$. For the weight information $1/3u_1 \le u_2 \le 3u_1$, there are two CCR-DEA-efficient DMUs, A and C. Yet, for feasible output weights such that $u_1/u_2 > 2$, the virtual output of DMU_B is higher than that of DMU_A so that DMU_A does not dominate DMU_B. Figure 1

Figure 1 The Efficiency Ratios for DMUs A, B, and C



shows how the DMUs' efficiency ratios change relative to each other for feasible output weights when the highest efficiency ratio is normalized to one. For example, the efficiency ratio of DMU_C is higher than that of DMU_B for all weights so that DMU_C dominates DMU_B. The ranking intervals are $[1, 3]$ for A, $[2, 3]$ for B, and $[1, 2]$ for C.

With the introduction of additional preference information, new dominance relations are often established. Furthermore, existing dominance relations are preserved, except in the unlikely case where both DMUs have the same efficiency ratio for all weights in the revised feasible set.

3.4. Efficiency Bounds

The analysis of relative efficiencies can be extended to situations where the efficiency DMU_k is benchmarked simultaneously with a group $DMU_L = \{DMU_l \mid l \in L \subseteq \{1, \ldots, K\}\}$ consisting of several DMUs. In this case, the ratios

$$D_{k,\bar{L}}(u, v) = \frac{E_k(u, v)}{\max_{l \in L} E_l(u, v)} = \min_{l \in L} \frac{E_k(u, v)}{E_l(u, v)}, \tag{15}$$

$$D_{k,\underline{L}}(u, v) = \frac{E_k(u, v)}{\min_{l \in L} E_l(u, v)} = \max_{l \in L} \frac{E_k(u, v)}{E_l(u, v)} \tag{16}$$

indicate how efficient DMU_k is relative to the most and least efficient DMUs in the benchmark group for different input/output weights. The maximum and minimum values of (15) over feasible weights are denoted by $\overline{D}_{k,\bar{L}}$ and $\underline{D}_{k,\bar{L}}$, whereas $\overline{D}_{k,\underline{L}}$ and $\underline{D}_{k,\underline{L}}$ are the corresponding maximum and minimum values of (16). By Theorems 4 and 5, these bounds can be solved with linear programming. It is worth noting that if the benchmark set L contains all DMUs, then $\overline{D}_{k,\bar{L}}$ is equal to the CCR-DEA score. If DMU_k is not contained in the benchmark set L, the maximum $\overline{D}_{k,\bar{L}}$

gives the superefficiency of DMU_k relative to this set of DMUs (see, e.g., Zhu 1996).

Theorem 4. $\underline{D}_{k,\bar{L}} = \min_{l\in L} \underline{D}_{k,l}$. *$\bar{D}_{k,\bar{L}}$ is the optimum of the maximization problem*

$$\max_{u,v} \sum_n u_n y_{nk} \tag{17}$$

$$\text{subject to } \sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \quad l \in L, \tag{18}$$

$$\sum_n v_m x_{mk} = 1,$$

$$(u, v) \in (S_u, S_v).$$

Theorem 5. $\bar{D}_{k,\underline{L}} = \max_{l\in L} \bar{D}_{k,l}$. *$\underline{D}_{k,\underline{L}}$ is the optimum of the minimization problem*

$$\min_{u,v} \sum_n u_n y_{nk} \tag{19}$$

$$\text{subject to } \sum_n u_n y_{nl} \ge \sum_m v_m x_{ml}, \quad l \in L, \tag{20}$$

$$\sum_n v_m x_{mk} = 1,$$

$$(u, v) \in (S_u, S_v).$$

3.5. Specification of Performance Targets

All of the above results can be employed to specify performance targets. For example, one can introduce targets such that DMU_k will be among (i) the R_k^* most efficient DMUs for *some* feasible weights or (ii) the R_k° most efficient DMUs for *all* feasible weights. These two cases are addressed by Theorems 6 and 7 for the case where efficiency improvements are sought through radial increases in output production.

Theorem 6. *Assume that $R_k^* < r_k^{\min}$. Then, the maximization problem*

$$\max_{u,v,z} \sum_n u_n y_{nk}$$

$$\text{subject to } 1 + \sum_{l\ne k} z_l \le R_k^*, \tag{21}$$

$$\sum_n u_n y_{nl} \le \sum_m v_m x_{ml} + C z_l, \quad l \ne k, \tag{22}$$

$$\sum_m v_m x_{mk} = 1,$$

$$z_l \in \{0, 1\}, \quad l \ne k,$$

$$(u, v) \in (S_u, S_v)$$

has an optimum $\zeta^* < 1$ such that $\rho^* = 1/\zeta^*$ gives the least radial output increase for which the best ranking of DMU_k is R_k^* or better.

Theorem 7. *Assume that $R_k^\circ < r_k^{\max}$. Then, the minimization problem*

$$\min_{u,v,z} \sum_n u_n y_{nk}$$

$$\text{subject to } 1 + \sum_{l\ne k} z_l \le K - R_k^\circ, \tag{23}$$

$$\sum_m v_m x_{ml} \le \sum_n u_n y_{nl} + C z_l, \quad l \ne k, \tag{24}$$

$$\sum_m v_m x_{mk} = 1,$$

$$z_l \in \{0, 1\}, \quad l \ne k,$$

$$(u, v) \in (S_u, S_v)$$

has an optimum $\zeta^* \le 1$ such that $\rho^* = 1/\zeta^*$ is the infimum of those radial output increases for which the worst ranking of DMU_k is R_k° or better.

Differences in Theorems 6 and 7 reflect asymmetric discontinuities when rankings improve. For instance, let there be three DMUs, A, B, and C, which all consume one unit of a single input and produce $y_A = 1$ and $y_B = y_C = 2$ units of a single output. Then, when A doubles its production, its best possible ranking jumps to one, but its worst possible ranking remains three until its production is strictly greater than two. Neither the best nor the worst ranking of A will be *exactly* two, no matter how much it increases its production.

An increase in the production of outputs by a factor of $\rho > 1$ corresponds to a decrease in the use of inputs by a factor of $1/\rho < 1$, because

$$\frac{\sum_n u_n[\rho y_{nk}]}{\sum_m v_m x_{mk}} = \frac{\sum_n u_n y_{nk}}{\sum_m v_m[1/\rho] x_{mk}}.$$

Thus, radial output targets can be translated into corresponding requirements on the input side. Similarly, the overall target ρ^* can be factored into a radial output target ρ_u and a radial input target ρ_v such that $\rho_u\rho_v = \rho^*$, $y'_{nk} = \rho_u y_{nk}$, and $x'_{mk} = [1/\rho_v] x_{mk}$.

Dominance relations, too, can be employed in target setting. For example, one may ask by how much a DMU_k that *does not* dominate DMU_l should increase its output to reach the threshold level beyond which it starts to dominate DMU_l. Based on $\underline{D}_{k,l} \le 1$ in Theorem 3, DMU_k achieves the efficiency level of DMU_l for all weights when it increases its production by $\rho_l^* = 1/\underline{D}_{k,l}$. If the target is to ensure that DMU_k begins to dominate several DMUs contained in the index set L, the threshold level for the required increase is $\rho^* = \max_{l\in L} \rho_l^*$. One may also ask by how much DMU_k that *is* dominated by DMU_l needs to increase its production so as not to be dominated. In this case, $1 \le \underline{D}_{l,k}$, and the threshold level for the required increase is $\rho_l^* = \underline{D}_{l,k}$. Even bounds for benchmark sets in §3.4 can be used in target specification.

3.6. Comparisons with Cross-Efficiency Analysis

In cross-efficiency analysis, every DMU is assigned a single CE score using those weights for which the efficiency of some DMU is maximized. By design, this approach recognizes that different weights are relevant in efficiency evaluation, in contrast to the standard CCR-DEA approach where the efficiency score for a DMU is determined using only those weights that are most favorable to it (see, e.g., Doyle and Green 1994).

Specifically, the DMUs' cross-efficiencies are computed from a square matrix $\theta_{l,k}$, $k, l = 1, \ldots, K$, whose lth row $\theta_{l,k} = [E_1(u^l, v^l), \ldots, E_K(u^l, v^l)]$ contains the efficiencies of DMUs with weights (u^l, v^l), which maximize the efficiency of DMU_l subject to the constraint that the maximum efficiency of any DMU is one. If there are multiple optima, alternative rules may be applied in weight selection. In the aggressive formulation, for example, weights are chosen by minimizing the relative efficiency of the aggregate DMU, which is formed by summing the inputs and outputs of all the other DMUs. In the benevolent formulation, the relative efficiency of the same aggregate DMU is maximized. Once the weights (u^l, v^l), $l = 1, \ldots, K$ have been chosen, the cross-efficiency of DMU_k is computed as

$$CE_k = \frac{1}{K}\sum_{l=1}^{K}\theta_{l,k} = \frac{1}{K}\sum_{l=1}^{K}E_k(u^l, v^l). \qquad (25)$$

DMUs can be ranked based on their cross-efficiencies. If there are no ties, DMU_k has a unique CE ranking that is equal to one plus the number of those DMUs that have a strictly higher cross-efficiency, i.e., $r_k^{CE,>} = 1 + |\{CE_l \mid CE_l > CE_k\}|$. In the case of ties, the CE ranking can drop to $r_k^{CE,\geq} = 1 + |\{CE_l \mid CE_l \geq CE_k, l \neq k\}|$ if DMU_k is assigned the worst ranking among all the DMUs that have the same cross-efficiency. Except for the possibility of ties, a major difference between CE rankings and ranking intervals in §3.2 is that CE rankings typically assign a single ranking to each DMU. In contrast, ranking intervals show *all* the rankings that DMUs can attain across the full set of feasible input/output weights.

We draw attention to three concerns with cross-efficiency analysis. First, the CE rankings of any two DMUs may depend on what *other* DMUs are included in the analysis. Indeed, Theorem 8 shows that whenever there are two DMUs that do not dominate each other and whose efficiency ratios differ for some input/output weights, then it is possible to introduce additional DMUs so that the CE ranking of the first DMU becomes better than that of the second. By Theorem 9, the nondominance assumption is necessary so that a DMU that dominates some other DMU will have a higher CE ranking than the DMU that it dominates.

Theorem 8. *Assume that* $\mathrm{DMU}_k \nsucc \mathrm{DMU}_l$ *and* $\mathrm{DMU}_l \nsucc \mathrm{DMU}_k$ *and* $\exists(u, v)$ *such that* $E_l(u, v) \neq E_k(u, v)$. *There then exist* DMU_i, $i = K+1, \ldots, K+K'$ *such that* $CE_k > CE_l$ *in the augmented set* $\{\mathrm{DMU}_i \mid i = 1, \ldots, K+K'\}$.

Theorem 9. *If* $\mathrm{DMU}_k \succ \mathrm{DMU}_l$, *then* $CE_k \geq CE_l$.

The phenomenon addressed by Theorem 8 is problematic because it means that cross-efficiency analyses are, in principle, susceptible to purposeful manipulation where the relative CE ranking of a nondominated DMU is altered by introducing appropriately chosen DMUs. Here, there are parallels to the contested rank reversal phenomenon where the introduction of a new alternative to a multicriteria decision problem changes the relative rankings of previously analyzed alternatives. Rank reversals have aroused plenty of controversy, and, for instance, they have been widely regarded as a shortcoming of the analytic hierarchy process (Belton and Gear 1983; see also Dyer 1990, Salo and Hämäläinen 1997).

A second concern is that the inequality in Theorem 9 may not be strict, meaning that a dominated DMU may have as high a CE score as the DMU it is dominated by. For example, consider three DMUs, A, B, and C, that consume one unit of a single input and produce three outputs according to profiles $A = (3, 3, 2)$, $B = (3, 2, 2)$, and $C = (0, 2, 3)$. Clearly, A dominates B. If there are no constraints on output weights, the efficiencies of DMUs B and C are maximized for weights $u^B = (1/3, 0, 0)$ and $u^C = (0, 0, 1/3)$, whereas DMU_A achieves its maximum efficiency for all convex combinations of weights $u^{A,1} = (1/3, 0, 0)$ and $u^{A,2} = (0, 1/3, 0)$. If the selection among the alternative optima is based on the aggressive formulation, the value of the aggregate output vector $u^{B+C} = (3+0, 2+2, 2+3) = (3, 4, 5)$ is minimized using output weights $u^{A,1} = (1/3, 0, 0)$. In this case, the case the cross-efficiency matrix becomes

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2/3 & 2/3 & 1 \end{pmatrix}.$$

Here, the three rows contain the DMUs' efficiencies for weights $u^{A,1}$, u^B, and u^C, respectively. The DMUs' cross-efficiencies (25) are now obtained as column averages $[1+1+2/3]/3 = 8/9 = CE_A = CE_B$ and $CE_C = [0+0+1]/3 = 1/3 = CE_C$, which show that DMUs A and B have the same cross-efficiency although DMU_A dominates DMU_B. An analogous conclusion can be reached for the benevolent formulation by replacing the output vector of DMU_C by $C' = (0, 0, 3)$ and by choosing the output weights $u^{A,1} = (1/3, 0, 0)$ to maximize the virtual value of the aggregate output vector $u^{B+C'} = (3, 2, 5)$.

A third concern is that the CE ranking of a DMU may lie outside the ranking interval $[r_k^{\min}, r_k^{\max}]$. For example, consider three DMUs, A, B, and C, that produce a single output, $y_A = 26$, $y_B = 19$, and $y_C = 16$, and consume two inputs, $x_A = (13, 9)$, $x_B = (7, 9)$, and $x_C = (16, 1)$. If there are no weight constraints, all three DMUs are efficient and achieve a DEA efficiency of one for weights $u^k = 1/y_k, k = A, B, C$ and $v^A = (1/26, 1/18)$, $v^B = (1/7, 0)$, and $v^C = (0, 1)$, respectively. With these weights, the cross-efficiency matrix becomes

$$\begin{pmatrix} 1 & \frac{19}{20} & \frac{144}{157} \\ \frac{14}{19} & 1 & \frac{7}{19} \\ \frac{13}{72} & \frac{19}{144} & 1 \end{pmatrix},$$

which yields the cross-efficiencies $CE_A \approx 0.639 < CE_B \approx 0.694 < CE_C \approx 0.762$. Thus, DMU_A has the worst CE ranking.

However, DMU_A has the smallest efficiency ratio only for weights (u, v) such that

$$E_A(u,v) \le E_C(u,v) \iff \frac{26u}{13v_1 + 9v_2} \le \frac{16u}{16v_1 + v_2}$$
$$\iff -\frac{1}{2}v_1 + \frac{59}{208}v_2 \ge 0, \quad \text{and}$$
$$E_A(u,v) \le E_B(u,v) \iff \frac{26u}{13v_1 + 9v_2} \le \frac{19u}{7v_1 + 9v_2}$$
$$\iff \frac{5}{38}v_1 - \frac{63}{494}v_2 \ge 0.$$

Multiplying the first inequality by 5/19 and summing up the inequalities gives $-(11/208)v_2 \ge 0$, which implies that v_2 must be zero. But then the first inequality gives $v_1 \le 0$, violating the assumption that $v_1 + v_2 > 0$. This proves that DMU_A has the worst CE ranking among the three DMUs although it is either the most or the second most efficient DMU for *all* feasible weights.

4. Applications of Ratio-Based Efficiency Measures

We next illustrate uses of ratio-based efficiency measures by revisiting two reported studies and also by describing a real case study where they supplied useful insights.

4.1. Ratio-Based Efficiency Results for the Evaluation of Robots

Baker and Talluri (1997) present an efficiency model for screening 27 robots using velocity and load capacity as outputs and cost and repeatability as inputs. They do not elicit preference information about the relative values of these input/output variables.

In Table 1, the CCR-DEA score of a robot is in the second column, followed by its best and worst efficiency rankings, a list of those robots it is dominated by, and lower and upper bounds for the robot's efficiency ratio relative to the highest efficiency ratio among all *other* robots over the set of feasible weights. Here, the worst efficiency ranking $r^{\max}$

Table 1 Efficiency Results for the Comparison of Robots

Robot	Eff.	$r^{\min}$	$r^{\max}$	Dominated by	$[\underline{D}_{k,\bar{l}}, \overline{D}_{k,\bar{l}}]$	CE	FPI(%)
1	1	1	21	—	[0.038, 1.012]	0.58	72.41
2	0.90	3	24	14	[0.024, 0.904]	0.48	88.28
3	0.53	7	23	11, 15, 19	[0.038, 0.529]	0.30	76.28
4	1	1	27	—	[0.004, 1.100]	0.31	222.58
5	0.59	3	27	1, 14, 19	[0.001, 0.592]	0.19	211.76
6	0.48	11	25	7, 8, 10, 13, 14, 19, 23, 24	[0.017, 0.482]	0.28	72.28
7	1	1	17	—	[0.055, 1.322]	0.70	42.86
8	0.78	5	15	—	[0.063, 0.783]	0.56	39.74
9	0.38	11	25	1, 7, 8, 10, 13, 14, 19	[0.029, 0.378]	0.27	40.14
10	1	1	17	—	[0.049, 1.043]	0.70	42.86
11	0.67	3	19	19	[0.063, 0.671]	0.42	59.84
12	0.10	18	27	1, 3, 7, 8, 10, 11, 13, 14, 15, 16, 19, 23, 25, 26, 27	[0.004, 0.102]	0.06	70.61
13	1	1	15	—	[0.061, 1.091]	0.73	36.99
14	1	1	13	—	[0.060, 1.769]	0.82	21.95
15	0.61	3	22	—	[0.038, 0.613]	0.36	70.14
16	0.60	3	24	—	[0.029, 0.604]	0.34	77.50
17	0.40	17	26	3, 7, 8, 10, 11, 13, 14, 15, 19, 23, 25	[0.013, 0.405]	0.19	112.92
18	0.37	12	25	1, 7, 8, 10, 13, 14, 19, 25	[0.031, 0.365]	0.26	40.47
19	1	1	10	—	[0.064, 1.021]	0.66	51.52
20	1	1	27	—	[0.001, 8.265]	0.34	194.12
21	0.85	2	25	—	[0.023, 0.852]	0.34	150.45
22	0.83	4	26	10, 13, 14	[0.005, 0.829]	0.46	80.19
23	0.69	3	22	7, 10	[0.039, 0.694]	0.44	57.79
24	0.64	5	22	7, 10, 13, 23	[0.036, 0.636]	0.41	55.15
25	0.55	10	18	7, 8, 13, 14, 19	[0.054, 0.553]	0.38	45.62
26	0.58	2	22	—	[0.037, 0.581]	0.36	61.40
27	1	1	25	—	[0.014, 3.880]	0.59	69.49

and the bounds $\underline{D}_{k,\bar{L}}$ show that for some weights the efficiency ratios of even CCR-DEA-efficient robots are quite low relative to the other robots. Moreover, the bounds $\overline{D}_{k,\bar{L}}$ show that CCR-DEA-efficient robots are superefficient, meaning that for any one of them it is possible to find feasible weights such that its efficiency ratio is strictly higher than that of all other robots. For example, the efficiency ratio of robot 4 can be 1.1 times as high as the maximum efficiency ratio of other robots.

The last two columns show the robots' cross-efficiency and so-called false positive index (FPI). The FPI index (Baker and Talluri 1997) is an indicator of how much the efficiency of the robot improves when its efficiency ratio is evaluated using weights that are most favorable to it rather than using also weights that favor other robots. Thus, the smaller the FPI, the less sensitive the efficiency of a robot is to the selection of weights.

There are 13 dominated robots that can be eliminated. Among the remaining 14 nondominated robots, 4 and 20 can be the least efficient of all for some weights, although they are efficient in the CCR-DEA sense. In the same vein, robots 1, 15, 16, 21, 26, and 27—which have large FPI values in excess of 60%—can be among the seven least efficient robots for some weights. Robots 7, 8, 10, 13, 14, and 19, in contrast, are more robust and belong to the 17 most efficient robots for all weights; they also have low FPI values below 50%. Robots 14 and 19 have the best ranking intervals. Robot 14 has a higher superefficiency value 1.769, whereas the ranking of 19 is never below 10. In this way, dominance structures and ranking intervals help identify nondominated DMUs like robots 14 and 19 that are more efficient than others across a broad range of weights. These results complement cross-efficiencies and FPI indices, yet they are based on a rigorous dominance concept and, in particular, the consideration of *all* feasible weights instead of only those weights for which the efficiency of some DMU is maximized.

4.2. Efficiency Comparison of Hospitals

Here, we revisit the example of Cooper et al. (2007, p. 155) with 14 hospitals whose inputs consist of nurses (x_1) and doctors (x_2), and whose outputs are outpatients (y_1) and inpatients (y_2). In the first phase, there is no preference information about the relative values of these variables. In the second phase, assurance regions for weights are introduced by stating that (i) neither input can be more than five times as valuable as the other and that (ii) neither output can be more than five times as valuable as the other. These statements correspond to the constraints $0.2v_1 \leq v_2 \leq 5v_1$ and $0.2u_1 \leq u_2 \leq 5u_2$.

Table 2 shows how the efficiency results change because of this preference information. Initially, hospitals H2, H3, H6, H8, and H10 are efficient in view of their CCR-DEA scores. DMU H8 becomes dominated when preference information is introduced. In view of Table 2, DMU H10 appears more efficient than others on several accounts, because (i) it is among the three most efficient hospitals for all feasible weights, (ii) all the dominated DMUs are dominated by it, (iii) it has the highest superefficiency ($\overline{D}_{k,\bar{L}} = 1.04$, i.e., for some weights it is up to 4% more efficient than the next most efficient DMU), and (iv) the bound $\underline{D}_{k,\bar{L}} = 0.98$ means its efficiency ratio is for all weights at least 98% of the highest efficiency ratio among all DMUs.

4.3. A Case Study on the Comparison of University Departments

This case study was carried out at a large technical university consisting of 12 departments responsible for research activities and educational degree programmes. The impetus for the study came from the board, which asked the resource committee of the university to consider alternative models for efficiency analysis and resource allocation.

The outputs consisted of three-year departmental averages in the university's reporting system that contained 44 outputs, structured under seven classes (degrees and credits awarded, international publications, domestic publications, international mobility of staff, other international scientific activities, other domestic scientific activities, and student exchanges). Statements about the relative values of these outputs were elicited from 10 members of the resource committee using a spreadsheet tool. First, in each output class, 10 points were associated with a reference output (e.g., an MS degree), whereafter each respondent was asked to assign points to the other outputs in the same class. For instance, by giving 80 points to a PhD degree the respondent could state that a single PhD degree is as valuable as eight MS degrees. Second, the respondent was asked to provide statements about the values of these seven reference outputs through similar point allocations. From these statements, the corresponding vector of normalized weights was derived for every respondent. The feasible output weights consisted of convex combinations of these weights, and thus contained the viewpoints of all respondents.[1]

The two input variables were basic funding, which is provided by the government and allocated to the departments by the rector, and external funding, which is acquired by research groups from external sources. Only these two inputs were chosen because most other inputs (e.g., annual person-years, office space) are ultimately financed through these two

[1] The original data are available from the authors upon request.

Table 2 **Results for Hospitals H1–H14 Without Preference Information (First Row) and With Preference Information** $0.2 \leq u_1/u_2, v_1/v_2 \leq 5$ **(Second Row)**

DMU	Eff.	r^{min}	r^{max}	Dominated by	$[\underline{D}_{k,\bar{L}}, \overline{D}_{k,\bar{L}}]$	$[\underline{D}_{k,\underline{L}}, \overline{D}_{k,\underline{L}}]$
H1	0.95	3	13	6	[0.75, 0.95]	[1.20, 2.56]
	0.93	6	11	2, 3, 6, 9, 10	[0.78, 0.93]	[1.32, 2.15]
H2	1	1	10	—	[0.80, 1.06]	[1.21, 2.91]
	1	1	7	—	[0.83, 1.02]	[1.36, 2.41]
H3	1	1	5	—	[0.91, 1.02]	[1.45, 2.72]
	1	1	4	—	[0.95, 1.01]	[1.58, 2.36]
H4	0.7	11	14	1, 2, 3, 6, 7, 8, 9, 10, 11, 12	[0.34, 0.70]	[0.71, 1.27]
	0.63	14	14	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	[0.40, 0.63]	[0.74, 0.92]
H5	0.83	7	13	3, 6, 9, 10	[0.59, 0.83]	[1.01, 2.37]
	0.82	7	12	2, 3, 6, 9, 10	[0.73, 0.82]	[1.16, 2.03]
H6	1	1	6	—	[0.91, 1.08]	[1.44, 2.67]
	1	1	5	—	[0.91, 1.00]	[1.55, 2.30]
H7	0.84	7	12	3, 6, 10, 12	[0.56, 0.84]	[1.22, 1.64]
	0.8	10	12	1, 2, 3, 6, 8, 9, 10, 11, 12	[0.61, 0.80]	[1.27, 1.53]
H8	1	1	12	—	[0.66, 1.00]	[1.25, 2.21]
	0.87	6	11	3, 6, 9, 10	[0.69, 0.87]	[1.31, 1.81]
H9	0.99	2	10	10	[0.79, 0.99]	[1.26, 2.87]
	0.98	2	5	10	[0.90, 0.98]	[1.42, 2.44]
H10	1	1	6	—	[0.88, 1.04]	[1.41, 2.88]
	1	1	3	—	[0.98, 1.04]	[1.56, 2.48]
H11	0.91	5	11	3, 6	[0.69, 0.91]	[1.26, 2.08]
	0.85	8	10	2, 3, 6, 9, 10, 12	[0.71, 0.85]	[1.32, 1.79]
H12	0.97	3	10	3	[0.70, 0.97]	[1.40, 2.03]
	0.93	4	9	3, 6, 10	[0.75, 0.93]	[1.47, 1.86]
H13	0.79	10	14	2, 3, 6, 7, 9, 10, 12	[0.38, 0.79]	[0.95, 1.40]
	0.74	13	13	1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14	[0.50, 0.74]	[1.09, 1.35]
H14	0.97	3	14	3, 10	[0.43, 0.97]	[0.90, 2.26]
	0.93	4	12	3, 9, 10	[0.63, 0.93]	[1.16, 2.00]

sources. Because the management of external funding involves more work, and because such funding places constraints on its use, the respondents were asked to state how much more "valuable" basic funding is compared with external funding. Most respondents noted that basic funding is 1.25–2.00 times as valuable as funding from external sources (e.g., the value of $100,000 of basic funding would be the same as that of $125,000–$200,000 of external funding).

Based on Theorem 4, the efficiency bounds in Figure 2 indicate the ranges within which the departments' efficiency ratios vary relative to the highest

Figure 2 **Efficiency Intervals for the 12 Departments**

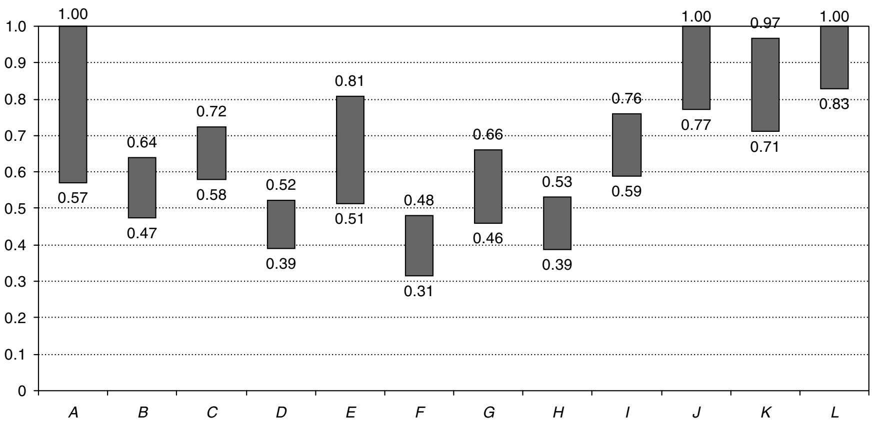
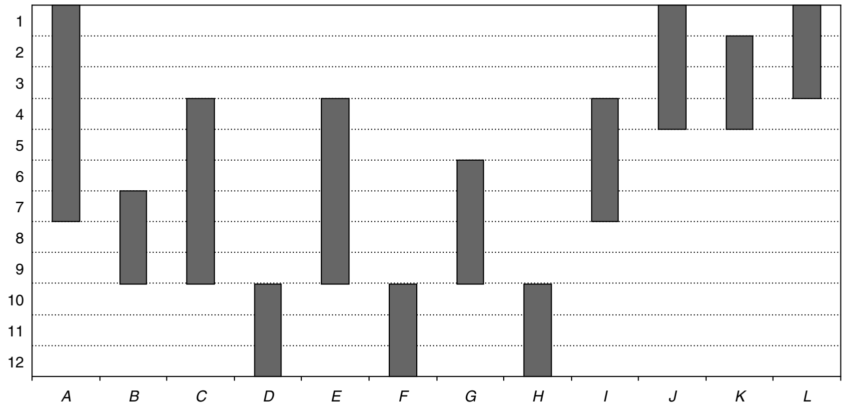


Figure 3 **Best and Worst Efficiency Rankings for the 12 Departments**



efficiency ratio among all departments. Specifically, the upper bounds are the usual CCR-DEA efficiency scores according to which there are three efficient departments (A, J, and L), followed by the "nearly" efficient department K (with an efficiency score of 0.97), then five departments with efficiency scores in the range 0.60–0.90, and, finally, three inefficient departments with scores less than 0.60. The lower bounds show how low the departments' efficiency ratios can be relative to the highest efficiency ratio over the set of feasible weights. Thus, for instance, the efficiency ratio of department L is for all weights at least 83% of the efficiency ratio of the most efficient department.

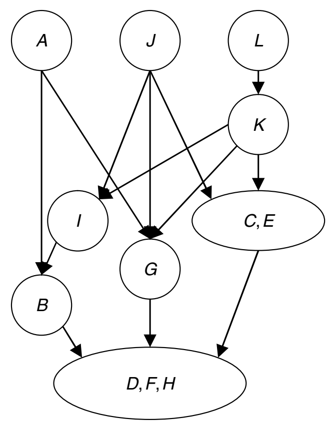
The ranking intervals in Figure 3 complement efficiency bounds. For instance, department L is among the three most efficient departments for all feasible weights, whereas J and the CCR-DEA-inefficient department K are among the four most efficient ones. Department A is efficient, but its ranking drops to 7 for some weights, indicating that its efficiency is sensitive to what input/output weights are employed. Departments D, F, and H are the three least efficient ones. Their ranking intervals show, for instance, that for all weights these three departments are less efficient than department G, although their efficiency intervals overlap with that of G.

Dominance relations are shown in Figure 4, where department X dominates Y if and only if there is a directed path from X to Y. Thus, department L dominates K, but K is not dominated by departments A and J. Also, A does not dominate I, meaning that for some weights the efficiency ratio of I is higher than that of A even though its CCR-DEA efficiency is lower than that of A. Moreover, department A dominates fewer departments (5) than K (8), which also indicates that the relative efficiency of A is more sensitive to the choice of weights. Departments D, F, and H do not dominate each other, but they are dominated by all other departments.

The results in §3.5 can be applied to specify performance targets. First, consider the three "midtier" departments C, E, and I, whose rankings are in the range from the fourth to the ninth most efficient. If department C is challenged to become one of the *three* most efficient departments for some feasible weights, it needs to increase its output by 8.80%; and if it is to be ranked as one of three most efficient departments for all weights, it must increase it output by more than 53.35%. Corresponding targets for departments E and I are 6.80% and 10.72% (for some weights) and 42.65% and 47.97% (for all weights).

Similarly, the least efficient departments D, F, and H could be required to achieve a position among the six most efficient departments. In this case, department D would have to increase its output by 25.97% to achieve such a position for some

Figure 4 **Efficiency Dominance Relations Among the Departments**



weights. Moreover, it would have to increase its output by more than 54.40% to secure this position for all weights. Corresponding results for departments F and H are 32.33% and 31.54% (for some weights) and 94.21% and 62.89% (for all weights).

5. Conclusion

We have developed ratio-based efficiency results (ranking intervals, dominance relations, and efficiency bounds) for comparing the relative efficiencies of DMUs for all feasible input/output weights. Unlike conventional DEA efficiency scores or cross-efficiencies, these results are robust in the sense that they (i) reflect how the DMUs' efficiency ratios change relative to each other over the entire feasible set of weights, (ii) tend to be insensitive to the introduction/removal of outlier DMUs, and (iii) do not necessitate particular assumptions about what production possibilities there are beyond the DMUs that are included in the analysis. Furthermore, these results do not exhibit rank reversals that may arise when ranking DMUs with cross-efficiency analysis. These results can also be employed to specify performance targets for DMUs.

We have illustrated the usefulness of these efficiency results by revisiting reported DEA studies and by describing a case study on the comparison of university departments. The encouraging feedback from this case study, together with the applicability of our efficiency results in many other contexts, leads us to believe that these results are helpful across the full range of decision contexts where ratio-based efficiency comparisons are appropriate.

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Appendix

Proof of Theorem 1. Let the best ranking of DMU_k be attained at $(u, v) \in (S_u, S_v)$. Then there exists $L = R_k^{>}(u, v) \subset \{1, \ldots, K\}$ so that $E_l(u, v) > E_k(u, v), l \in L$ and $E_k(u, v) \geq E_l(u, v), l \notin L$. Let $v'_m = v_m/[\sum_m v_m x_{mk}]$ and $u'_n = u_n/[\sum_n u_n y_{nk}]$. Then $(u', v') \in (S_u, S_v)$ and $\sum_m v'_m x_{mk} = \sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$, and $z_l = 0$ if $l \notin L$. Then, for any $l \notin L$, we have

$$1 \leq \frac{E_k(u, v)}{E_l(u, v)} = \frac{E_k(u', v')}{E_l(u', v')} = \frac{\sum_m v'_m x_{mk}}{\sum_n u'_n y_{nk}} \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}} = \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}},$$

which gives $\sum_n u'_n y_{nl} \leq \sum_m v'_m x_{ml}$. For $l \in L$, multiplying $z_l = 1$ by the large positive constant C implies that the constraint (4) is satisfied for $l \in L$ too. Because $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{>}(u, v)| = r_k^{\min}(u, v)$, the solution to the minimization problem is not larger than the best ranking.

Conversely, let (u, v, z) be a solution to the minimization problem. Let $L = \{l \mid l \neq k, z_l = 1\}$. Then introducing $z_l = 0, l \notin L$ into the first constraint in (4) gives $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml}$ so that

$$\frac{E_k(u, v)}{E_l(u, v)} = \frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}} \geq 1,$$

because $E_k(u, v) = 1$ due to (5). Thus, any $l \notin L$ cannot belong to $R_k^{>}(u, v)$. For $l \in L$, the inequality $\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} \iff E_k(u, v) \geq E_l(u, v)$ cannot hold because z is at optimum (otherwise, any such $z_l = 1$ could be changed to $z_l = 0$ without violating (4) while reducing the value of the objective function); hence $l \in L \subseteq R_k^{>}(u, v)$. It follows that $R_k^{>}(u, v) = L$ and $r_k^{\min} \leq 1 + |R_k^{>}(u, v)| = 1 + |L| = 1 + \sum_{l \neq k} z_l$. □

Proof of Theorem 2. If the worst ranking of DMU_k is attained at $(u, v) \in (S_u, S_v)$, there exists a subset $L = R_k^{\geq}(u, v) \subset \{1, \ldots, K\}, k \notin L$ such that $E_l(u, v) \geq E_k(u, v), l \in L$ and $E_k(u, v) > E_l(u, v), l \notin L$. Let $v'_m = v_m/[\sum_j v_m x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$ and $u'_n = u_n/[\sum_n u_n y_{nk}]$ so that $\sum_n u'_n y_{nk} = 1$.

For any $l \neq k$, let $z_l = 1$ if $l \in L$, and let $z_l = 0$ if $l \notin L$. Then, for any $l \in L$,

$$1 \leq \frac{E_l(u, v)}{E_k(u, v)} = \frac{E_l(u', v')}{E_k(u', v')} = \frac{\sum_m u'_n y_{nl}}{\sum_m v'_m x_{ml}} \Rightarrow \sum_m v'_m x_{ml} \leq \sum_n u'_n y_{nl},$$

and thus (6) holds. For $l \notin L$, multiplying $(1 - z_l) = 1$ by the positive constant C implies that (6) is satisfied in this case too. Now, $1 + \sum_{l \neq k} z_l = 1 + |L| = 1 + |R_k^{\geq}(u, v)| = r_k^{\max}$. Thus, the solution to the maximization problem is at least as large as the worst ranking.

Conversely, assume that (u, v, z) is a solution to the maximization problem, and let $L = \{l \mid l \neq k, z_l = 1\}$. For any $l \in L$ with $z_l = 1$, the constraint $\sum_m v_m x_{ml} \leq \sum_n u_n y_{nl}$ implies

$$\frac{E_l(u, v)}{E_k(u, v)} = \frac{\sum_n u_n y_{nl}}{\sum_m v_m x_{ml}} \geq 1,$$

because u and v satisfy (6)–(7); thus, $L \subseteq R_k^{\geq}(u, v)$. Because z is at optimum, the inequality $E_k(u, v) \leq E_l(u, v)$ cannot hold for $l \notin L$ (otherwise, any such $z_l = 0$ could be changed to $z_l = 1$ without violating constraints while increasing the objective function). Thus, $R_k^{\geq}(u, v)$ does not contain elements that are outside of L. It follows that $L = R_k^{\geq}(u, v)$ and $r_k^{\max} \geq 1 + |R_k^{\geq}(u, v)| = 1 + \sum_{l \neq k} z_l$. □

Proof of Lemma 1. To prove that $u' \in S_u$, note that $u \in S_u$ implies $A_u u \leq 0$ and, hence, $A_u u' = A_u c_u u = c_u(A_u u) \leq 0$; similarly, $v' \in S_v$. The last claim follows from

$$\begin{aligned} D_{k,l}(u', v') &= \frac{E_k(u', v')}{E_l(u', v')} \\ &= \frac{\sum_n u'_n y_{nk}}{\sum_m v'_m x_{mk}} \frac{\sum_m v'_m x_{ml}}{\sum_n u'_n y_{nl}} \\ &= \frac{c_u \sum_n u_n y_{nk}}{c_v \sum_m v_m x_{mk}} \frac{c_v \sum_m v_m x_{ml}}{c_u \sum_n u_n y_{nl}} \\ &= \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}} \frac{\sum_m v_m x_{ml}}{\sum_n u_n y_{nl}} = D_{k,l}(u, v). \quad \square \end{aligned}$$

Proof of Theorem 3. Choose $(u^*, v^*) \in (S_u, S_v)$ such that $D_{k,l}(u^*, v^*) \geq D_{k,l}(u, v)$, $\forall (u, v) \in (S_u, S_v)$. Define v' so that $v'_m = v^*_m/[\sum_i v^*_i x_{ik}]$. By construction, $v' \in S_v$ and $\sum_m v'_m x_{mk} = 1$. Define $u'_n = u^*_n[\sum_m v'_m x_{ml}]/[\sum_j u^*_j y_{jl}]$. Then, $\sum_n u'_n y_{nl} = \sum_m v'_m x_{ml}$. The weights (u', v') satisfy constraints (12)–(14), and the repeated application of Lemma 1 gives $D_{k,l}(u^*, v^*) = D_{k,l}(u^*, v') = D_{k,l}(u', v') = \sum_n u'_n y_{nk}$, proving that the maximum of (11) over (12)–(14) is at least as high as $D_{k,l}(u^*, v^*)$.

Assume that the maximum of (11) is attained at (u^o, v^o). For these weights $(u^o, v^o) \in (S_u, S_v)$, we have

$$D_{k,l}(u^o, v^o) = \frac{E_k(u^o, v^o)}{E_l(u^o, v^o)} = \frac{\sum_n u^o_n y_{nk}}{\sum_m v^o_m x_{mk}} \frac{\sum_m v^o_m x_{ml}}{\sum_n u^o_n y_{nl}} = \sum_n u^o_n y_{nk},$$

because the weights (u^o, v^o) satisfy (12)–(13). Thus, the maximum of $D_{k,l}(u, v)$ over (S_u, S_v) cannot be smaller than the solution to the maximization problem in Theorem 3. The minimization case can be shown analogously. □

Proof of Theorem 4.

$$\begin{aligned}\min_{u,v} D_{k,\bar{L}}(u, v) &= \min_{u,v} \frac{E_k(u, v)}{\max_{l \in L} E_l(u, v)} \\ &= \min_{u,v} \min_{l \in L} \frac{E_k(u, v)}{E_l(u, v)} \\ &= \min_{l \in L} \min_{u,v} D_{k,l}(u, v) \\ &= \min_{l \in L} \underline{D}_{k,l}(u, v).\end{aligned}$$

Let the maximum of (15) be ζ^* so that this optimum is attained at (u^*, v^*). There then exists some $l^* \in L$ such that $E_{l^*}(u^*, v^*) \geq E_l(u^*, v^*)$ $\forall l \in L$. Choose $v' = v^*/[\sum_m v^* x_{mk}]$ so that $\sum_m v'_m x_{mk} = 1$. Also, choose a constant $c_u > 0$ so that $\sum_n u'_n y_{nl^*} = \sum_m v'_m x_{ml^*}$ for $u' = c_u u^*$. For any $l \in L$, we have

$$1 \geq D_{l,l^*}(u^*, v^*) = D_{l,l^*}(u', v') = \frac{E_l(u', v')}{E_{l^*}(u', v')} = \frac{\sum_n u'_n y_{nl}}{\sum_m v'_m x_{ml}}$$

so that the constraint (18) is satisfied by (u', v'). By construction, $\zeta^* = \max_{u,v} D_{k,\bar{L}}(u, v) = D_{k,l^*}(u', v') = \sum_n u'_n y_{nk}$, which shows that the maximum of (17) is at least as high as that of (15).

Conversely, assume that the maximum of (17), ζ', is attained at (u', v'), and choose $l' \in L$ so that the constraint in (18) is binding (such l' exists, for otherwise u' could be increased to improve the value of the objective function, which would be in violation of the optimality assumption). Now,

$$\max_{u,v} D_{k,\bar{L}}(u, v) \geq \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta'$$

so that the maximum (15) must be at least as high as that of (17). □

Proof of Theorem 5.

$$\begin{aligned}\max_{u,v} D_{k,\underline{L}}(u, v) &= \max_{u,v} \frac{E_k(u, v)}{\min_{l \in L} E_l(u, v)} \\ &= \max_{u,v} \max_{l \in L} \frac{E_k(u, v)}{E_l(u, v)} \\ &= \max_{l \in L} \max_{u,v} D_{k,l}(u, v) \\ &= \max_{l \in L} \bar{D}_{k,l}(u, v).\end{aligned}$$

Let the minimum of (16), ζ^*, be attained at (u^*, v^*). There then exists some l^* such that $E_{l^*}(u^*, v^*) \leq E_l(u^*, v^*)$, $\forall l \in L$, and $\zeta^* = \min_{u,v} D_{k,\underline{L}}(u, v) = E_k(u^*, v^*)/E_{l^*}(u^*, v^*)$. As in the proof of Theorem 4, use (u^*, v^*) in defining normalized valuation vectors (u', v') such that $\sum_m v'_m x_{mk} = 1$ and $E_{l^*}(u', v') = 1$. The choice of l^* guarantees that $1 \leq E_l(u', v')$ so that constraint (20) holds for all $l \in L$. Because

$$\zeta^* = \frac{E_k(u^*, v^*)}{E_{l^*}(u^*, v^*)} = \frac{E_k(u', v')}{E_{l^*}(u', v')} = \sum_n u'_n y_{nk},$$

the minimum of (19) is at least as small as the minimum of (16).

Assume that ζ', the minimum of (19), is obtained at (u', v'). Choose l' such that the constraint in (20) is binding (such l' must exist, for otherwise the assumption of optimality would be violated). Then $E_{l'}(u', v') = 1$, whereas constraint (20) implies that $E_l(u', v') \geq 1$ for any other $l \in L$; hence, $E_{l'}(u', v') \leq E_l(u', v')$. It follows that

$$\min_{u,v} D_{k,\underline{L}}(u, v) \leq D_{k,\underline{L}}(u', v') = \frac{E_k(u', v')}{\min_{l \in L} E_l(u', v')} = \frac{E_k(u', v')}{E_{l'}(u', v')} = \zeta',$$

proving that the minimum of (16) is at least as small as the optimum of (19). □

Proof of Theorem 6. Because $u \in S_u \Rightarrow c_u u \in S_u$ for any $c_u > 0$, there exists $u \in S_u$ so that the constraints (21) and (22) are satisfied. The optimum ζ^* is attained at some weights (u^*, v^*), because v fulfills the normalization constraint and assumes values in a compact set $\{v \in S_v \mid \sum_m v_m x_{mk} = 1\}$, and u is maximized but bounded from above by constraint (22). If the optimum ζ^* were equal to one, then according to Theorem 1, DMU_k could only reach ranking $r_k^{\min}$ and constraint (21) would be violated. Thus, $\zeta^* < 1$.

For any feasible (u, v, z) that satisfy the constraints, the constraint (22) gives $z_l = 0 \Rightarrow 1 \geq E_l(u, v)$ so that $E_l(u, v) > 1 > E_k(u, l) \Rightarrow z_l = 1$. By (21), there are therefore at most $\sum_{l \neq k} z_k$ other DMUs whose efficiency is higher than that of DMU_k. By (21), the best ranking of DMU_k is therefore R_k^* or better. By construction, $1/\zeta^*$ is the revised efficiency ratio of DMU_k.

For any $\zeta' > \zeta^*$ and any feasible (u, v), the optimality of ζ^* implies that the constraint (21) will be violated when z_l are chosen by minimizing them so that (22) holds. But then there will be more than $R_k^* - 1$ other DMUs with an efficiency ratio that is strictly higher than that of DMU_k, meaning that the best ranking of DMU_k is worse than R_k^*.

Similarly, if constraint (21) holds and $\zeta' > \zeta^*$, constraint (22) is violated for some $l' \neq k$ such that $z_{l'} = 0$ and $E_{l'}(u, v) > E_k(u, v)$. We can assume that (21) holds with equality, for else the violation of (22) for l' could be eliminated by setting $z_{l'} = 1$. Because $E_l(u, v) > E_k(u, v) \Rightarrow z_l = 1$ for the constraints that are satisfied, there are again more than $R_k^* - 1$ other DMUs with a strictly higher efficiency ratio, and thus DMU_k does not attain the target ranking R_k^*. □

Proof of Theorem 7. Because $u \in S_u \Rightarrow c_u u \in S_u$ for any $c_u > 0$, there exists $u \in S_u$ so that the constraints (23) and (24) are satisfied. The optimum ζ^* is attained at some weights (u^*, v^*), because v fulfills the normalization constraint and thus assumes values in a compact set $\{v \in S_v \mid \sum_m v_m x_{mk} = 1\}$, and u is minimized but bounded from below by constraints (24).

According to Theorem 2, there exists a solution u, v, z' such that $\sum_m v_m x_{mk} = \sum_n u_n y_{nk} = 1$, $\sum_{l\neq k} z'_l = r_k^{\max} - 1$, and $\sum_m v_m x_{ml} \leq \sum_n u_n y_{nl} + C(1 - z'_l)$. Solution u, v, z such that $z_l = 1 - z'_l \ \forall l \neq k$ is feasible because this substitution yields directly the constraints (24) and also the constraint $\sum_{l\neq k} z_l = K - r_k^{\max} \leq K - R_k^\circ - 1$, whereby (23) is fulfilled. Thus, $\zeta^* \leq 1$.

For any feasible (u, v, z), the constraint (24) gives $z_l = 0 \Rightarrow E_l(u, v) \geq E_k(u, v)$ so that $E_k(u, v) > E_l(u, v) \Rightarrow z_l = 1$. By (23), there are at most $\sum_{l\neq k} z_k$ other DMUs whose efficiency is lower than that of DMU_k, and hence the worst possible ranking is $R_k^\circ + 1$ or worse.

We show that $\rho^* = 1/\zeta^*$ is the maximum increase in the outputs of DMU_k such that $R_k^\circ + 1$ belongs to the ranking interval of the revised DMU, DMU_{k^ρ} with $x_{k^\rho} = x_k$, $y_{k^\rho} = \rho y_k$. For any increase greater than ρ^*, only better rankings belong to the interval.

For any $\zeta < \zeta^*$ and feasible (u, v), the optimality of ζ^* implies that constraint (23) must be violated if constraints (24) hold for all $l \neq k$. But then the worst ranking of DMU_{k^ρ} will be R_k° or better.

Conversely, if (23) holds and (24) is violated for $\text{DMU}_{l'}$, then $E_{l'}(u, v) < E_{k^\rho}(u, v)$ and $z_{l'} = 0$. Furthermore, the constraint (23) can be assumed to hold with equality, because otherwise we could set $z_{l'} = 1$, and the constraint would not be violated. This implies that the number of DMUs p for which $E_p(u, v) < E_{k^\rho}(u, v)$ is at least $|\{l \in \{1, \ldots, K\} \mid l \neq k, z_l = 1\}| + 1 = K - R_k^\circ - 1 + 1 = K - R_k^\circ$, and the ranking of DMU_{k^ρ} must be R_k° or better.

Thus, for any $\rho > 1/\zeta^*$, the ranking of DMU_{k^ρ} is R_k° or better for all feasible (u, v). The formulation thus provides the infimum of the radial increases for which the worst ranking is R_k° or better. □

Proof of Theorem 8. By assumption, $E_k(u', v') > E_l(u', v')$ for some $(u', v') \in (S_u, S_v)$. Let constant $M > \max_{i=k,l}[1/\underline{D}_{i,\bar{l}}]$, where $L = \{1, \ldots, K\}$. Define $\text{DMU}_{k'}$ and $\text{DMU}_{l'}$ so that $y_{k'} = M y_k$, $y_{l'} = \bar{D}_{k,l} M y_l$, $x_{k'} = x_k$, $x_{l'} = x_l$. Then, (i) $E_{l'}(u, v) \geq E_{k'}(u, v) > E_i(u, v)$ for all $i \in \{1, \ldots, K\}$, and (ii) there exist $(u', v') \in (S_u, S_v)$ such that $E_{l'}(u', v') = E_{k'}(u', v')$ and for any such weights, $E_k(u', v') > E_l(u', v')$.

Consider DMUs DMU_i, $i \in \{1, \ldots, K\} \cup \{l'\} \cup \{K + 2, \ldots, K + K'\}$ so that DMU_i, $i = K+2, \ldots, K+K'$ are equal to $\text{DMU}_{k'}$. Among these DMUs, $\theta_{k',k} > \theta_{k',l}$. Then, for a sufficiently large K',

$$
\begin{aligned}
&CE_k - CE_l \\
&= \frac{1}{K+K'} \sum_{i=1}^{K+K'} [\theta_{i,k} - \theta_{i,l}] \\
&= \frac{1}{K+K'} \left[\sum_{i=1}^{K} [\theta_{i,k} - \theta_{i,l}] + [\theta_{l',k} - \theta_{l',l}] + (K'-1)[\theta_{k',k} - \theta_{k',l}] \right]
\end{aligned}
$$

is positive, because $\sum_{i=1}^{K} [\theta_{i,k} - \theta_{i,l}] + \theta_{l',k} - \theta_{l',l}$ is bounded from above by $K + 1$. □

Proof of Theorem 9. Let (u^i, v^i), $i = \{1, \ldots, K\}$ be the weights that maximize the efficiency of DMU_i in the specification of the cross-efficiency matrix. Because DMU_k dominates DMU_l, we have $E_k(u^i, v^i) \geq E_l(u^i, v^i)$ so that $\theta_{i,k} \geq \theta_{i,l}$. Summing this inequality over $i = 1, \ldots, K$ gives $CE_k \geq CE_l$. □

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