Aalto University
School of Science
Degree programme in Engineering Physics and Mathematics

# A Study on Decision Making Using Fuzzy Decision Trees

Bachelor's thesis
22.4.2015

Eero Lehtonen

Aalto University
School of Science

Aalto University, P.O. BOX 11000, 00076 AALTO
www.aalto.fi

**Abstract of bachelor's thesis**

| | |
|---|---|
| **Author** Eero Lehtonen | |
| **Title of thesis** A Study on Decision Making Using Fuzzy Decision Trees | |
| **Degree programme** Degree Programme in Engineering Physics and Mathematics | |
| **Major** System Sciences | **Code of major** F3010 |
| **Supervisor** Prof. Raimo P. Hämäläinen | |
| **Thesis advisor(s)** Ph. D. Matteo Brunelli | |
| **Date** 22.4.2015 | **Number of pages** 3+25 **Language** English |

**Abstract**

In operations research one often faces scenarios and phenomena exhibiting imprecision and uncertainty. Stochastic aspects of these scenarios are often accounted for by means of probabilistic models. In addition to these models, fuzzy augmentation to traditional decision analysis has been advocated as a possible method that takes into account imprecision in quantities available to the decision maker. Fuzzy methodology models these imprecise quantities as fuzzy numbers. Rules exist for computation of such quantities allowing for augmenting existing decision-making models.

The objective of this thesis is to explore augmenting traditional decision trees with such fuzzy methods. Relevant definitions in the literature are given and an example decision tree is used to illustrate the augmentation process and further the results given by such fuzzy decision trees. Results given by the tree are then analyzed and compared to results given by traditional decision tree analysis. The contributions of the process are also discussed and further study is recommended.

**Keywords** Fuzzy Set Theory, Fuzzy Arithmetics, Alpha Cuts, Fuzzy Ordering, Fuzzy Decision Analysis, Decision Trees

# Contents

# 1 Introduction

The field of operations research frequently concerns itself with phenomena exhibiting uncertainty and randomness. Often these stochastic phenomena are accounted for by means of probabilistic models. Namely, randomness is modeled using random variables following some possibly multivariate probability distribution. However, decision makers face situations where such models do not adequately represent uncertainty. Fuzzy sets, and more particularly fuzzy numbers, have been proposed as a means to capture subjective uncertainty a decision maker might face. In face of problems with weak data Kaufmann [1986] advocated fuzzy sets as alternative representations of ambiguous quantities. This approach is expected to yield more interpretable and intelligible results for the decision maker. More recently Dubois [2011] also provided a critical point of view on fuzzy decision analysis questioning whether simply adding a fuzzy component to existing decision making models constitutes a compelling contribution.

To explore the topic, this thesis concentrates on decision tree models, which are often used to illustrate stochastic discrete-time scenarios, where the decision maker specifies a conditional strategy maximizing her expected utility. Traditional decision tree analysis requires the elicitation of exact quantities of utilities and probability distributions associated with outcomes in the tree. The objective of this thesis is to present and discuss a fuzzy augmentation of traditional decision tree analysis in order to allow for ambiguity in model parameters. Existing fuzzy methods presented in the literature are explored and their use in decision tree analysis is evaluated.

The thesis is divided into two principal segments. First, required definitions for fuzzy arithmetics and decision trees are given in Section 2. Decision analysis using fuzzified decision trees is discussed in detail in Section 3. The augmentation process is illustrated using an example decision tree model of a gambling scenario. The model is first presented in Section 2.4 and later augmented in Section 3. In conclusion the utilized fuzzy methods and their results are investigated. Shortcomings of this approach are also considered and further steps are suggested.

# 2 Definitions

## 2.1 Fuzzy Sets

Sets, as described by elementary set theory, are omnipresent in mathematics and a foundation from which most theory is derived from. Still, classes and sets we construct by observing real-world phenomena are often vague or ambiguous, and set theory is limited in terms of describing set membership: a set is described by a *characteristic function* mapping an object to $\{0, 1\}$ indicating whether it belongs to said set. Rather, we might prefer to express a *degree* of membership in those sets. Fuzzy set theory provides a theoretical framework to reason about these kinds of sets. It extends the definition of the characteristic function to account for imprecise knowledge. Hereafter we refer to concepts introduced by elementary set theory as *crisp* in contrast to their fuzzy counterparts.

The motivation for such theory rises from its ability to describe essentially imprecise or ambiguous knowledge. In operations research and decision analysis we might argue most sets and quantities could be interpreted as fuzzy objects, and forming a theory to approach such objects is clearly useful.

A fuzzy set as defined by Zadeh [1965] and Klir and Yuan [1995] is an extension of crisp sets as defined by elementary set theory. A fuzzy set $A$ associates a degree of membership for each object belonging to the universal set $X$. Usually a fuzzy set is characterized by a *membership function* on a crisp set. Often denoted with the same notation as the corresponding fuzzy set, the membership function $A$ maps each $x \in X$ to its degree of membership in the fuzzy set $A$:

$$A : X \to [0, 1] \tag{1}$$

Here membership of degree 0 is interpreted as $x$ not belonging to $A$ and membership of degree 1 is interpreted as $x$ fully belonging to $A$. Intermediate values are interpreted as $x$ having in turn intermediate membership in $A$.

Clearly from this definition it follows that crisp sets are a special case of fuzzy sets, since a characteristic function can be interpreted as a special case of a membership function with a range of $\{0, 1\}$. Such a characteristic function maps $x$ as either belonging to $A$ or not. In this case we can write in crisp terms $A \subset X$.

We next introduce the notation necessary to further discuss fuzzy concepts in the literature.

A fuzzy set $A$ satisfying $\sup_{x\in X} A(x) = 1$ is called *normal.*

Another often used description of fuzzy sets are *alpha cuts* or *$\alpha$-cuts* denoted in this thesis as ${}^{\alpha}A$ and defined as

$$ {}^{\alpha}A = \{x|A(x) \geq \alpha\}. \tag{2} $$

As in Zadeh [1965], we also call a fuzzy set $A : X \to [0, 1]$ with $X \in \mathbb{R}$ *convex* if all of its alpha cuts are convex intervals.

When discussing fuzzy sets, we also find it useful to define extensions to both intersections and unions on fuzzy sets. These operations are called the *standard intersection* and *standard union* defined as

$$ (A \cap B)(x) = \min\{A(x), B(x)\} \tag{3} $$
$$ (A \cup B)(x) = \max\{A(x), B(x)\} \tag{4} $$

Also discussed later, we call $cl\{x \in X|A(x) > 0\}$ the *support* of $A$, where $cl$ is the topological closure operator.

In order to construct fuzzy sets using its alpha cuts we introduce the first decomposition theorem. First define a special fuzzy set ${}_{\alpha}A(x) = \alpha \cdot^{\alpha} A(x)$.

**Theorem 1 (First Decomposition Theorem, Klir and Yuan [1995])** *Given an arbitrary membership function $A : X \to [0, 1]$ it follows that*

$$ A = \bigcup_{\alpha\in[0,1]} {}_{\alpha}A(x) \tag{5} $$

Importantly, these results indicate that all fuzzy sets can fully and uniquely be described by their alpha cuts.

## 2.2 Fuzzy Numbers

Fuzzy numbers as defined by Klir and Yuan [1995] are a special case of fuzzy sets defined on $\mathbb{R}$. Hence, these sets are defined by membership functions of the form

$$ A : \mathbb{R} \to [0, 1] \tag{6} $$

Intuitively a fuzzy number could be interpreted as representing a quantity involving a degree of uncertainty. The characterization by Zadeh [1978] is that the value of the membership function describes the degree of possibility that $x$ is the "true" value of the variable. We could for example represent the statement "a number close to two" using a fuzzy number $A$ with a membership function $A$ for which $A(2) = 1$ and $A(x)$ decreases with $x$ further away from 2. Figure 1 shows an example $A(x) = \max\{0, 1 - |2 - x|\}$.

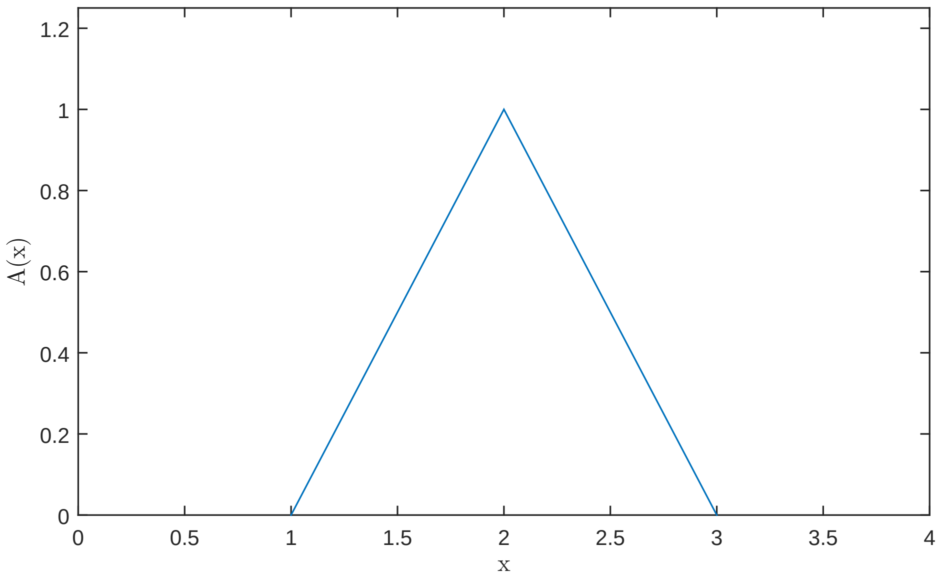


Figure 1: An example membership function representing a "number close to two"

According to the formal definition by Klir and Yuan [1995], a fuzzy number $A$ needs to satisfy the following conditions

- $A$ is normal, i.e. $\sup_{x\in\mathbb{R}} A(x) = 1$
- Alpha cuts of $A$ are closed intervals
- The support of $A$ is bounded

The argument for setting these requirements lies in defining meaningful arithmetic operations on fuzzy numbers. The example in Figure 1 satisfies these additional requirements. Conversely, a fuzzy set $B$ with $B(x) = \frac{1}{|2-x|}$ does not have a bounded support and thus does not qualify as a fuzzy number.

## 2.3 Fuzzy Arithmetics

To further discuss fuzzy calculations we define arithmetic operations on fuzzy sets. Literature presents two equivalent approaches to defining fuzzy arithmetic operations. Both approaches could be interpreted as extensions to crisp sets. First we introduce the extension principle as an elegant means to extend a crisp function $f$ to a fuzzy range and domain. However, due to its weighty computational burden we also introduce a more computationally accommodating method employing alpha cuts and the decomposition theorem.

### 2.3.1 Fuzzy arithmetics by employing the Extension Principle

First, as fuzzy sets could be considered to be an extension of crisp sets it is reasonable to approach fuzzy operations and functions as extensions of their crisp equivalents. In the literature, extending a function $f : X \to Y$ of the crisp set $X$ to $Y$ to a mapping of membership functions is called *fuzzification*. The principle of fuzzifying crisp functions is referred to as *extension principle* by Klir and Yuan [1995] and *fuzzification principle* by Dubois and Prade [1978]. In this thesis we refer to this approach as the extension principle.

**Extension Principle** For arbitrary membership functions $A$ on $X$ and $B$ on $Y$, any given $f : X \to Y$ induces fuzzy functions $F$ and $F^{-1}$ as follows

$$[F(A)](y) = \sup_{x|y=f(x)} A(x) \tag{7}$$

$$[F^{-1}(B)](x) = B(f(x)) \tag{8}$$

Using this principle we may develop fuzzy arithmetics. This method is discussed in detail by both Klir and Yuan [1995] as well as Dubois and Prade [1978].

Denote any of the four arithmetic operations by $*$. For fuzzy numbers $A$ and $B$ we write

$$(A * B)(z) = \sup_{z=x*y} \{A \cap B\}. \tag{9}$$

By (3) it further follows that

$$(A * B)(z) = \sup_{z=x*y} \min\{A(x), B(y)\}. \tag{10}$$

From this definition we have

$$\begin{aligned}(A+B)(z) &= \sup_{z=x+y} \min\{A(x), B(y)\} \\ (A-B)(z) &= \sup_{z=x-y} \min\{A(x), B(y)\} \\ (A\cdot B)(z) &= \sup_{z=x\cdot y} \min\{A(x), B(y)\} \\ (A/B)(z) &= \sup_{z=x/y} \min\{A(x), B(y)\}\end{aligned}$$

for each arithmetic operation.

### 2.3.2 Fuzzy arithmetics by applying alpha cuts

Alternatively, in order to fuzzify crisp arithmetic operations, we define arithmetic operations on closed intervals. Defining an arithmetic operation $*$ on closed intervals as

$$[a,b]*[d,e] = \{f*g | a \leq f \leq b, d \leq g \leq e\} \tag{11}$$

yields

$$[a,b]+[d,e] = [a+d, b+e] \tag{12}$$

$$[a,b]-[d,e] = [a-e, b-d] \tag{13}$$

$$[a,b]\cdot[d,e] = [\min\{ad,ae,bd,be\}, \max\{ad,ae,bd,be\}] \tag{14}$$

$$[a,b]/[d,e] = [a,b]\cdot[1/e, 1/d] \tag{15}$$

Having defined arithmetic operations on crisp intervals we further discuss arithmetic operations on fuzzy numbers. Defining a basic arithmetic operation on alpha cuts of two fuzzy numbers $A$ and $B$ as

$$^{\alpha}(A*B) =^{\alpha} A *^{\alpha} B \tag{16}$$

we can apply the decomposition theorem given in (5):

$$A*B = \bigcup_{\alpha\in[0,1]} {}_{\alpha}(A*B) \tag{17}$$

For the sake of illustration, consider two fuzzy numbers $A$ and $B$ describing numbers "close to two" and "close to three", respectively. As before, denote their membership functions by $A(x) = \max\{0, 1-|2-x|\}$ and $B(x) =$

$\max\{0, 1 - |3 - x|\}$. In order to consider their respective alpha cuts we write these functions in their piecewise linear forms

$$A(x) = \begin{cases} x - 1 & \text{when } 1 \le x < 2 \\ -x + 3 & \text{when } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} x - 2 & \text{when } 2 \le x < 3 \\ -x + 4 & \text{when } 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

From these forms we construct their respective alpha cuts

$$^{\alpha}A = [1 + \alpha, 3 - \alpha]$$
$$^{\alpha}B = [2 + \alpha, 4 - \alpha]$$

Applying (12)–(15) we find

$$^{\alpha}(A + B) = [3 + 2\alpha, 7 - 2\alpha]$$
$$^{\alpha}(A - B) = [-3 + 2\alpha, 1 - 2\alpha]$$
$$^{\alpha}(A \cdot B) = [\alpha^2 + 3\alpha + 2, \alpha^2 - 8\alpha + 12]$$
$$^{\alpha}(A/B) = \left[\frac{\alpha + 1}{4 - \alpha}, \frac{3 - \alpha}{\alpha + 2}\right]$$

The corresponding membership functions are

$$A + B = \begin{cases} \frac{x-3}{2} & \text{when } 3 \le x < 5 \\ \frac{7-x}{2} & \text{when } 5 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$$A - B = \begin{cases} \frac{x+3}{2} & \text{when } -3 \le x < -1 \\ \frac{1-x}{2} & \text{when } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A \cdot B = \begin{cases} \frac{\sqrt{4x+1}-3}{2} & \text{when } 2 \le x < 6 \\ \frac{7-\sqrt{4x+1}}{2} & \text{when } 6 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

$$A/B = \begin{cases} \frac{4x-1}{x+1} & \text{when } \frac{1}{4} \le x < \frac{2}{3} \\ \frac{3-2x}{x+1} & \text{when } \frac{2}{3} \le x \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy numbers yielded by these example calculations are shown in Figure 2.

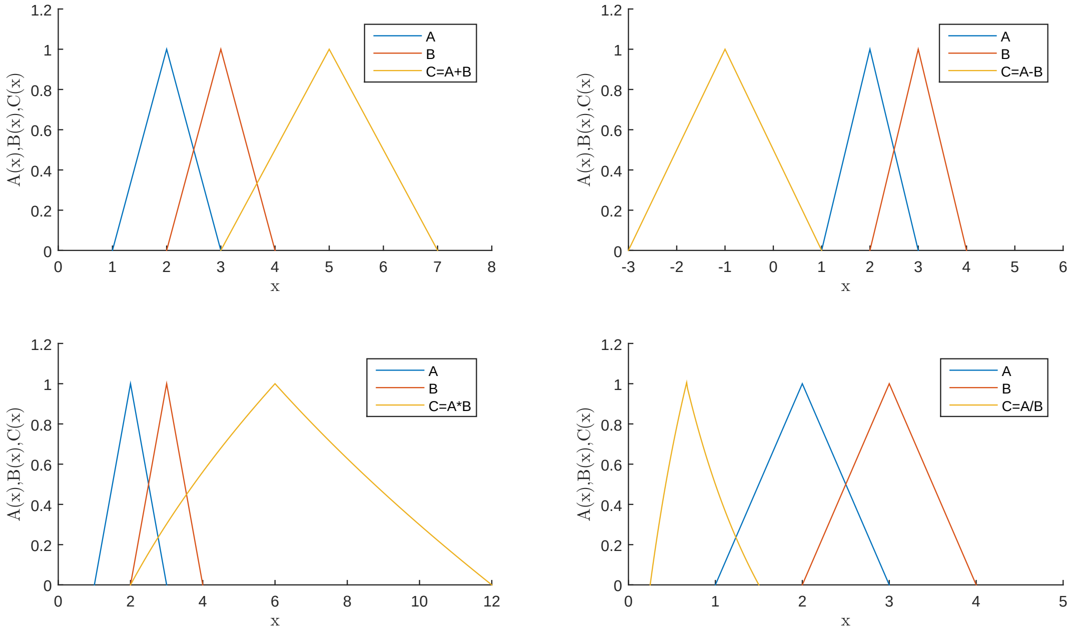


Figure 2: Arithmetic operations on triangular fuzzy numbers

## 2.4 Decision Trees

Often a decision maker (DM for short) faces a scenario, where she needs to make a series of decisions during a period of time, and where these decisions affect some final outcome. Outcomes and options for future choices might also be affected by changes in the *state of the world* during this period. Clearly, the decision maker should then adjust her strategy based on information available at each time she makes a decision. Scenarios such as these could be considered stochastic discrete-time dynamic optimization problems.

A natural representation for such a scenario is a tree with leaf nodes representing outcomes reached by a sequence of decisions and stochastic events. Such trees are referred to as decision-flow-diagrams or *decision trees* by Raiffa and Schlaifer [1961], Raiffa [1968] and others.

A decision tree is a simple, flowchart-like graph used to analyze decision making as well as to determine optimal decision paths for a given scenario. The tree represents a multi-step decision making process where a chain of discrete decisions have discrete outcomes yielding some stochastic utility.

The graph usually contains three kinds of nodes to represent these properties:

**Decision nodes** Decision nodes that represent options available to the decision maker at each point.

**Chance nodes** Chance nodes represent possible outcomes of a single decision.

**Outcome nodes or leaves** Final outcome nodes are leaves that represent possible outcomes of the chain of decisions. Each outcome is associated with some non-stochastic utility.

As an illustration, a simple decision tree describing a betting process is presented in Figure 3.

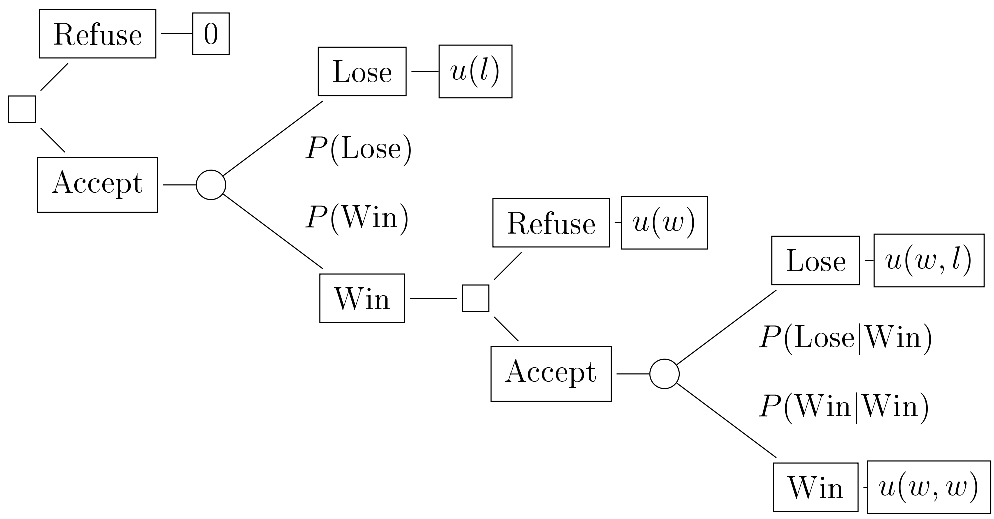


Figure 3: A simple decision tree

In order to discuss optimal decision rules for such a scenario we first introduce the Principle of Optimality:

**Principle of Optimality** An optimal decision rule has the property that regardless of previous state, the remaining decisions must constitute an optimal decision rule in regard to the current state resulting from the previous decisions. [Bellman, 1957, Chap. III.3.]

Most often decision making over choices in the tree is directed to maximizing expected utility. Interpreting the scenario as a dynamic optimization problem, a natural approach to solving an optimal strategy is by dynamic programming. Following Bellman's Principle of Optimality we analyze the tree from leaf to root so that each decision is associated with expected utility from future nodes. Expected utility for earlier decisions is then calculated assuming that later decisions are made optimally. It should be noted, that in a stochastic scenario this approach yields an optimal strategy for each decision node. In comparison to deterministic scenarios, once initial decisions are made and events take place, the optimal strategy for remaining nodes may change. Therefore, the optimal decision rule at time $t$ is dependent on outcomes of previous stochastic events, and thus the decision maker needs to re-evaluate her strategy at each decision node.

As represented by the tree shown in Figure 3. The decision maker is presented with a scenario in which she needs to decide whether to accept a simple gamble, and in case of a win, whether to accept a second gamble. In order to maximize expected utility, the DM first evaluates the *second* gamble by comparing expected utilities of her choices in that subproblem. Once the optimal decision for the subproblem for the second gamble is known, the DM is able to compare expected utility for the first gamble. Denoting the first and second decisions as $d_1$ and $d_2$ respectively the DM first calculates

$$d_2^* = \underset{d_2 \in \{\text{Accept,Refuse}\}}{\arg\max} \mathbb{E}\left[u(d_2)\right]$$

and having found the optimal decision $d_2^*$ for the second gamble she calculates

$$d_1^* = \underset{d_1 \in \{\text{Accept,Refuse}\}}{\arg\max} \mathbb{E}\left[u(d_1) \mid d_2 = d_2^*\right].$$

# 3 Fuzzy Decision Trees

Decision Tree models as discussed above require for the decision maker to specify precise quantities for each parameter in the tree. This approach might result in loss of information for parameters that are only known approximately or which reflect subjective knowledge of the decision maker. As proposed by Watson et al. [1979] especially changes in multiple parameters might alter the recommended decision, while sensitivity analysis on individual parameters might not indicate variation.

One approach suggested by Watson et al. [1979] to address this imprecision in decision making processes is to extend known quantitative decision making models to a fuzzy domain and analyze these models using fuzzy arithmetics as discussed before.

In this thesis we limit our approach to fuzzy extensions to decision trees and study possible models resulting from applying fuzzy methodology to simple decision trees. Such fuzzification methods were discussed by Janikow [1998] and Olaru and Wehenkel [2003] in a machine learning context, where such an extension has proved useful in reducing model sensitivity to imprecise knowledge. In this thesis we again limit ourselves to discuss a model based on the extension principle as described by Watson et al. [1979]. This approach applies the extension principle in fuzzifying expected value computation for fuzzy probabilities and utilities.

## 3.1 Fuzzy expectation

For a simple case of a discrete random variable $X$ with possible outcomes $x_1, x_2, \ldots, x_n$ and with respective utilities $u_i \in \mathbb{R}\ \forall i$ and probabilities $p_i \in [0,1]\ \forall i$ so that $\sum_i p_i = 1$, we have according to well-known probabilistic rules the expected value

$$\mathbb{E}\left[X\right] = \sum_{i=1}^{n} p_i u_i. \tag{18}$$

For such a discrete random variable a crisp expected value is then a mapping of

$$\mathbb{E}\left[\cdot\right] : \mathbb{R}^n \times [0,1]^n \to \mathbb{R}. \tag{19}$$

For a fuzzy technique we need to extend this definition into a fuzzy domain and range. We now consider a discrete random variable $X$ whose possible outcomes yield fuzzy utilities corresponding to membership functions $U_i$ and take place according to fuzzy probabilities corresponding to membership functions $P_i$. Namely fuzzy numbers are used to compute the expectation $\mathbb{E}\left[X\right]$, which due to $U_i$ and $P_i$ being fuzzy is also fuzzy. We denote this expectation as $\mathbb{E}\left[X\right](u)$, with $u \in \mathbb{R}$. According to the extension principle and (3) we now have

$$\begin{aligned}
\mathbb{E}\left[X\right](u) &= \sup_{u=\sum_{i=1}^{n} p_i u_i} \{U_1 \cap P_1 \cap \ldots \cap U_n \cap P_n\} && (20)\\
&= \sup_{u=\sum_{i=1}^{n} p_i u_i} \min\{U_1(u_1), P_1(p_1), \ldots, U_n(u_n), P_n(p_n)\}. && (21)
\end{aligned}$$

Note that we may also decompose $U_i$ and $P_i$ into their respective alpha cuts and employ (5) in order to arrive at the same outcome with regards to $\mathbb{E}[X]$.

When fuzzifying probabilities, the issue of normalization needs to be addressed. For a discrete probability distribution we require $\sum_i p_i = 1$. However, in a fuzzy context verifying this identity is not straightforward. Pavlačka [2014] discussed several approaches to fuzzy normalization. In this instance we adopt an approach based on the extension principle. We only require fuzzy probabilites to satisfy the following property

$$\left(\sum_i P_i(p_i)\right)(p) > 0 \Rightarrow \sum_i p_i = 1. \tag{22}$$

In essence, we only consider normalized probability distributions to be possible. From this it follows, that projecting $\left(\sum_i P_i(p_i)\right)(p)$ onto the axis $p_i$ yields $P_i(p_i)$.

Consider for example triangular fuzzy numbers $A_i$ defined by triplets $(a_i, b_i, c_i)$ so that

$$A_i(x) = \begin{cases} \frac{x-a}{b-a} & \text{when } a_i \leq x < b_i \\ \frac{x-c}{b-c} & \text{when } b_i \leq x \leq c_i \\ 0 & \text{otherwise} \end{cases}. \tag{23}$$

Modeling two possible outcomes with such fuzzy numbers we then require

$$a_1 + c_2 = 1$$
$$c_1 + a_2 = 1$$
$$b_1 + b_2 = 1$$

in order to satisfy (22). Such a case is visualized in Figure 4. From this representation the aspect of projection is clear: projecting $\sum_i P_i$ onto any of its respective components yields $P_i$.

For the sake of an example, let us consider a scenario as presented in Figure 5. Here two fuzzy utilities $U_1(u_1)$ and $U_2(u_2)$ as well as fuzzy probabilities $P_1(p_1)$ and $P_2(p_2)$ satisfying (22) correspond to a random variable $X$ whose expectation we compute using the rule presented above.

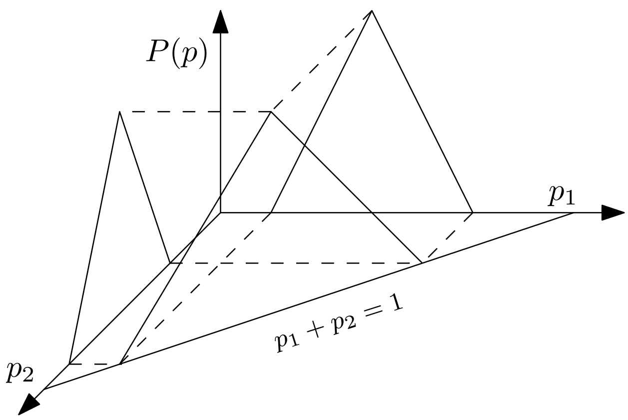


Figure 4: Fuzzy probabilities $P_1$ and $P_2$ satisfying equation (22)

Notably this procedure allows us to sequentially compute expected utilities for trees such as in Figure 3.

## 3.2 Fuzzy preference

In addition to a method for computing fuzzy expectations, one still needs a method to solve preferable alternatives based on fuzzy quantities available. In essence, the preference relation $\preceq$ needs to be defined for membership functions.

More generally we consider this problem as one of ordering. For fuzzy quantities such a problem has no unambiguous approach and several ranking methods have been proposed. Such methods can be divided into those reducing fuzzy quantities in question to crisp numbers allowing for straightforward comparison as presented in Wang and Kerre [2001a], and allowing for fuzzy relations expressing the degree of belief in the statement "The fuzzy quantity $A$ is larger than the fuzzy quantity $B$.", as respectively presented in Wang and Kerre [2001b].

Both papers introduced a set of reasonable properties for ordering fuzzy quantities and test their rationality. Brunelli and Mezei [2013] also conducted a numerical study comparing such rankings in order to find similarities among a set of ordering methods. Watson et al. [1979] also used a method later developed by Dubois and Prade [1983] and discussed by Wang and Kerre [2001b] for deriving the belief for a binary statement on preference order. The method satisfies Wang and Kerres' reasonable properties with the ex-

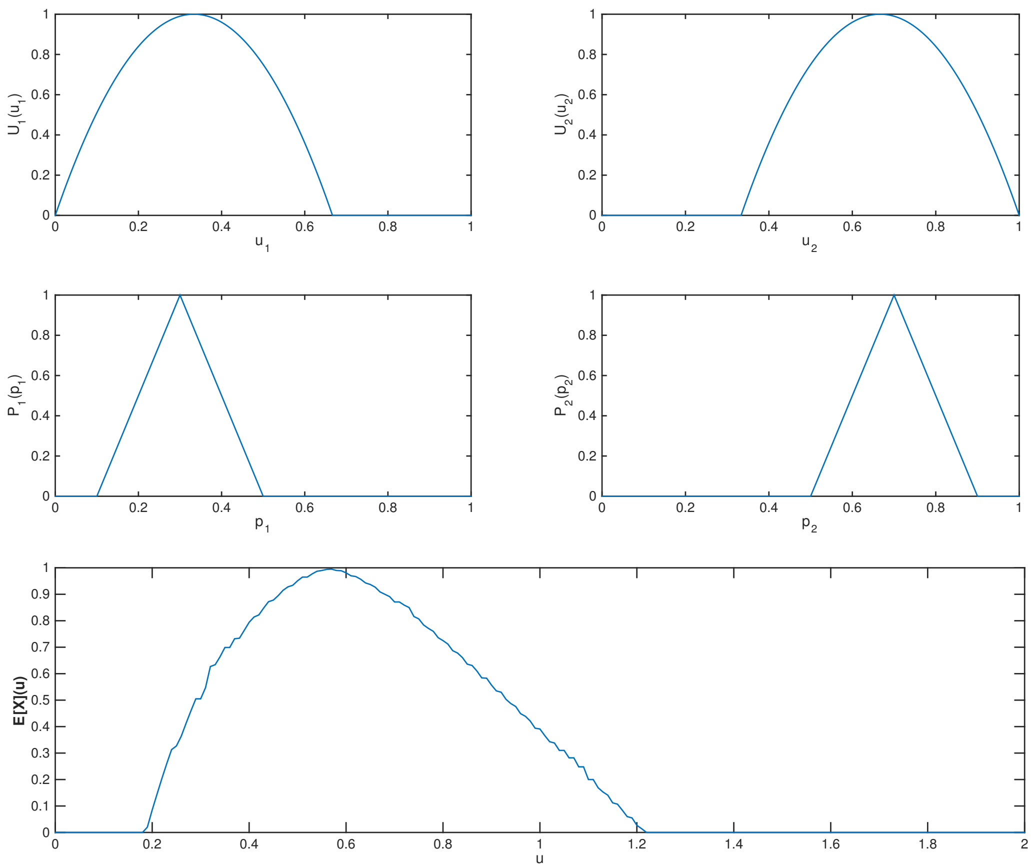


Figure 5: Fuzzy expected utility $\mathbb{E}[X](u)$ corresponding to the fuzzy utilities $U_1(u_1)$, $U_2(u_2)$ and the fuzzy probability $P_1(p_1)$

ception of $\mathbf{A_5}$ and $\mathbf{A'_6}$. In essence the ranking is dependent on the set being ranked, namely, that with arbitrary sets of fuzzy numbers $S$ and $S'$ and fuzzy quantities $A, B \in S \cap S'$ $A \prec B$ on $S$ does not imply $A \prec B$ on $S'$. In addition for fuzzy quantities $A, B, A+C, B+C \in S, C \neq \emptyset$, $A \prec B \Rightarrow A+C \prec B+C$ does not hold. Note that $A \preceq B \Rightarrow A+C \preceq B+C$ in turn does hold.

Following Watson et al. [1979] we define $S_1$ and $S_2$ as fuzzy statements referring to fuzzy relationships between two fuzzy expectations $\mathbb{E}[X](u_x)$ and $\mathbb{E}[Y](u_y)$ corresponding to membership functions $\mu_1(u_x, u_y)$ and $\mu_2(u_x, u_y)$ so that $\mu_1$ describes the intersection of membership functions $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ and $\mu_2$ describes the preference of $\mathbb{E}[X]$ over $\mathbb{E}[Y]$.

We also define the fuzzy implication as

$$\mu(S_1 \to S_2) = \min_{u_x,u_y} \max\{1 - \mu_1(u_x, u_y), \mu_2(u_x, u_y)\} \tag{24}$$

and strict fuzzy preference relation as

$$\mathbb{E}[X] \prec \mathbb{E}[Y] : \quad \mu_{\prec}(u_x, u_y) = \begin{cases} 1, & u_x > u_y, \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Hence we may construct the strict fuzzy preference implication $X \succ Y$ as

$$\begin{aligned} \mu(X \succ Y) &= \min_{u_x,u_y} \max\{1 - \mu_1(u_x, u_y), \mu_2(u_x, u_y)\} \\ &= \min_{u_x,u_y} \max\{1 - \min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}, \mu_{\prec}(u_x, u_y)\} \\ &= \min_{u_x \le u_y} \{1 - \min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}\} \\ &= 1 - \max_{u_x \le u_y} \{\min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}\} \end{aligned} \tag{26}$$

Note that $\mu(X \prec Y) \neq 1 - \mu(X \succ Y)$ but rather

$$\begin{aligned} \mu(X \prec Y) &= \mu(Y \succ X) \\ &= \min_{u_x,u_y} \max\{1 - \min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}, \mu_{\prec}(u_y, u_x)\} \\ &= 1 - \max_{u_y \le u_x} \{\min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}\} \end{aligned} \tag{27}$$

In Figure 6 two fuzzy utilities are shown along with a horizontal line corresponding to $\max_{u_y \le u_x}\{\min\{\mathbb{E}[X](u_x), \mathbb{E}[Y](u_y)\}\}$. Following (26) and (27) we then have $\mu(X \succ Y) = 0$ and $\mu(X \prec Y) = 0.25$.

A more straightforward approach to ordering fuzzy quantities is to reduce them to crisp quantities using one of several available approaches. A simple method for such a computation is the center of gravity as defined by Lee [1990]. In a continuous case we calculate the center of gravity $M$ for a fuzzy number $A$ as

$$M(A) = \frac{\int_{supp(A)} A(x)x dx}{\int_{supp(A)} A(x)}. \tag{28}$$

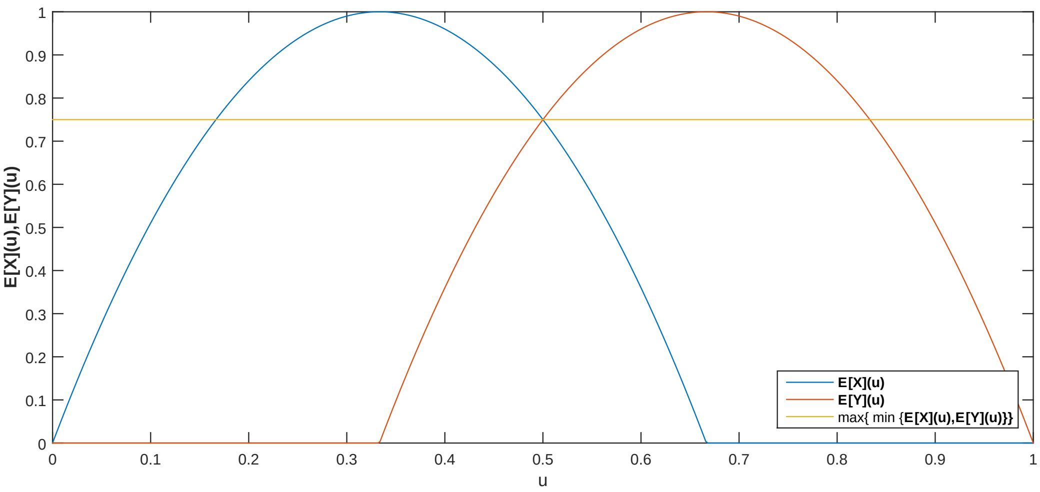


Figure 6: Finding the level of indication for the statements "$X$ is strictly preferred to $Y$" and "$Y$ is strictly preferred to $X$".

In numeric computation we use a naive discretization of this. With $n$ evenly distributed points $x_i$ in the support of $A$ we compute

$$M(A) = \frac{\sum_{supp(A)} A(x_i)x_i}{\sum_{supp(A)} A(x_i)}. \tag{29}$$

Figure 7 shows two expectations corresponding to random variables $X$ and $Y$ with their respective centers of gravity. The clear implication is that taking into account the distributions of both expectations $X \prec Y$. The interpretation for such a defuzzification technique is intuitive and useful.

## 3.3 Fuzzy decision making

We finally consider again the example decision tree in Figure 3. We assign example fuzzy quantities to utilities and probabilities in the tree and evaluate optimal decisions using both fuzzy implication methods discussed above. For simplicity we consider triangular fuzzy probabilities in Figure 8 and utilities in Figure 9 corresponding to quantities in the tree. We also treat refusal to take part in the first gamble to yield a certain, crisp benchmark utility of 0.

Utilities calculated using (5) for the second gamble are presented in Figure 10. We also display centers of gravity for both utilities.

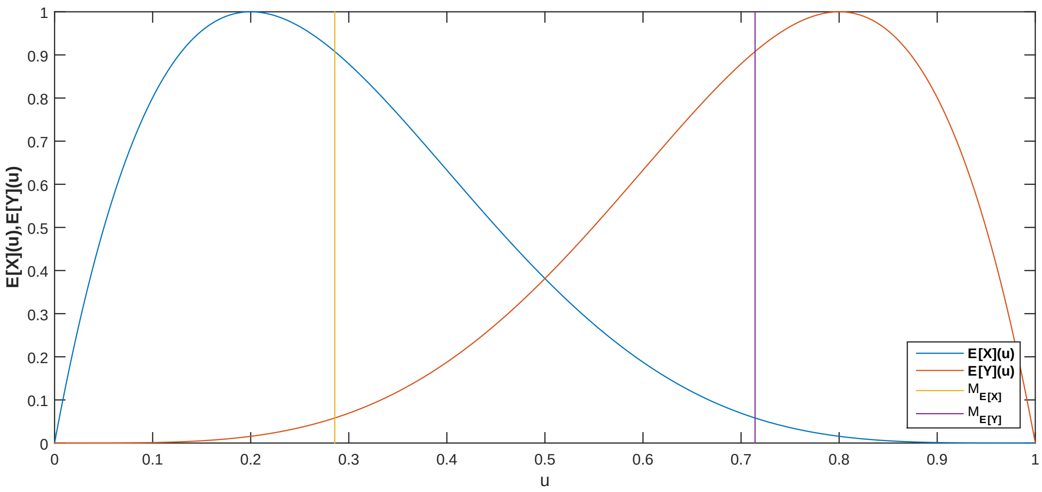


Figure 7: Ordering fuzzy expectations by calculating their respective centers of gravity.

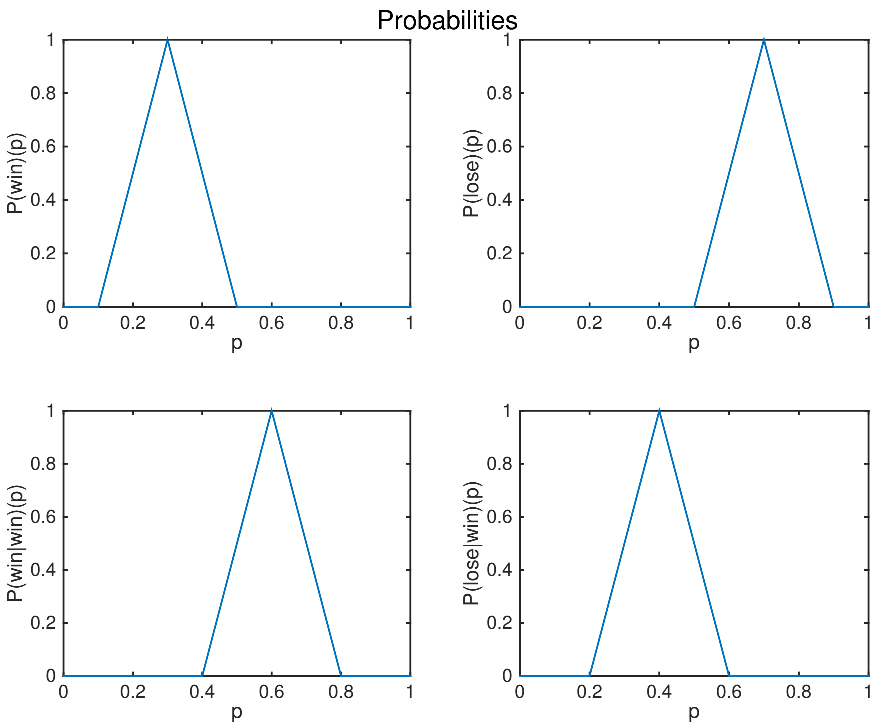


Figure 8: Fuzzy probabilities for the decision tree in Figure 3.

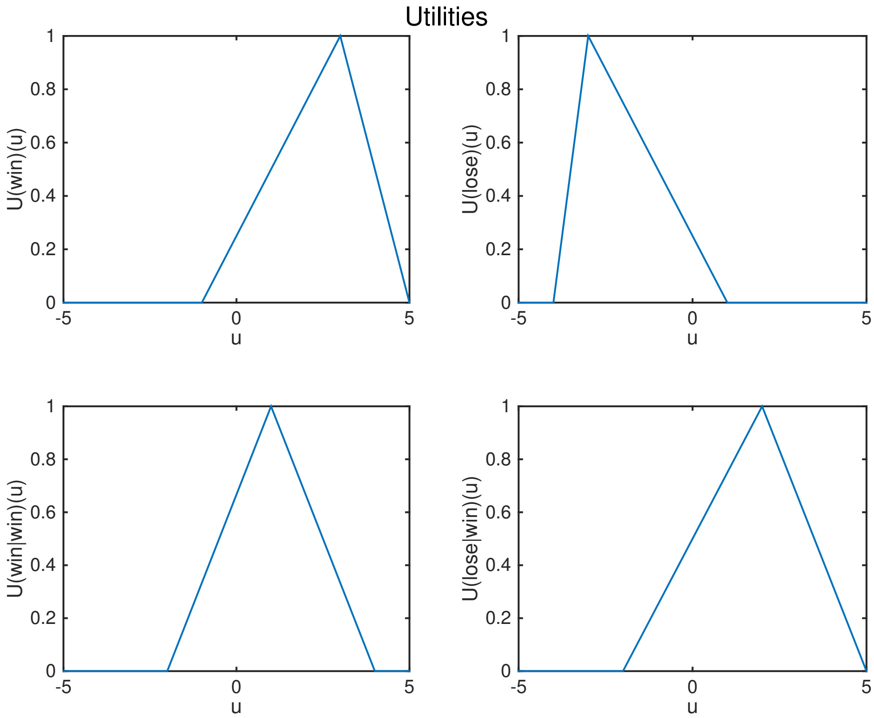


Figure 9: Fuzzy utilities for the decision tree in Figure 3.

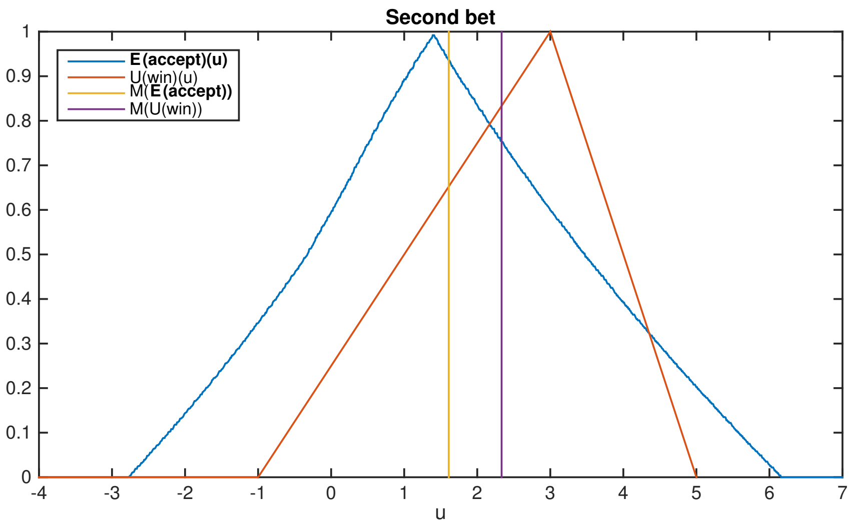


Figure 10: Fuzzy probabilities for the decision tree in Figure 3.

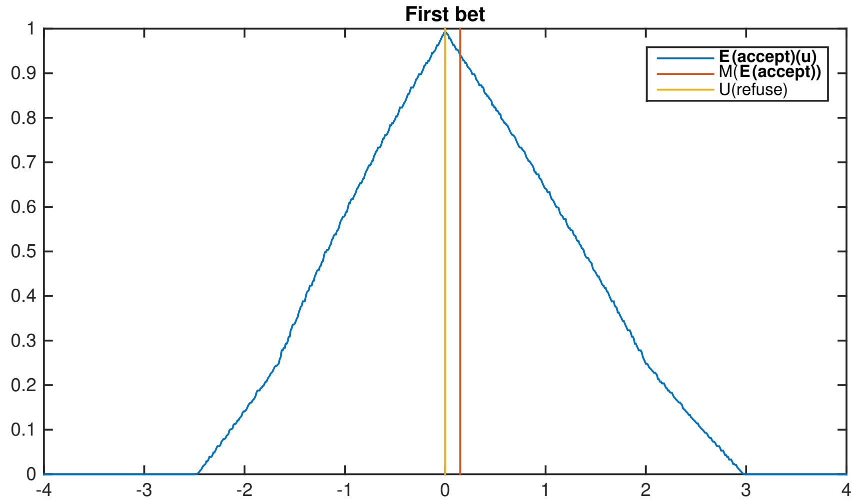


Figure 11: Fuzzy utilities for the decision tree in Figure 3.

Using the center of gravity ordering refusing the second bet and receiving $U(\text{win})$ would be preferable to taking the second gamble corresponding to the utility $\mathbb{E}[U(\text{accept})]$. Knowing the DM would refuse the second gamble, she would then evaluate the first gamble with the assumption that upon winning it she would receive the utility $U(\text{win})$. Using this knowledge she evaluates the expected utilities for the first gamble. The expectations are presented in Figure 11. Again comparing the center of gravity of $\mathbb{E}[U(\text{accept})]$ to the certain utility of 0 she would find accepting the first bet preferable. This elicitation process then yields $(\text{accept}, \text{refuse})$ as the optimal strategy for such a scenario.

Using the fuzzy implication method in (24) and (25) yields a degree of belief of approximately 0.2 for the statement "Refusing the second bet is preferable to accepting it." and conversely 0 for the statement "Accepting the second bet is preferable to refusing it.". Observing this difference in belief and specifically the nonexistent support for the second statement it would be plausible for the DM to refuse the second bet also under the fuzzy implication approach. However, comparing the numerical approximation for the first bet in Figure 11 to a crisp value of zero, the support for the fuzzy implication "Accepting the first bet is preferable to refusing it." is effectively zero. In this sense the method provides a clear opinion for the second bet, but fails to provide meaningful implication for the first bet.

# 4 Conclusion

Methods presented and discussed in the literature allow for fuzzification of crisp arithmetics and ordering methods and thus allow for their application in a decision making context where optimality is determined by maximization of expected utility. The computation of crisp expected value was fuzzified by fuzzifying the underlying arithmetic operations. In a case of decision trees this is adequate since the probability distribution for each event in the scenario is discrete and finite. Thus this approach cannot be applied to computing the expected value of a continuously distributed random variable. What is more, there exists no consensus as to how fuzzy quantities should be ordered, leaving it up to the decision maker to decide how to discern order between utilities.

Along the more elegant extension principle, the approach to fuzzy arithmetics using alpha cuts proved to be a simple means to compute operations on fuzzy numbers. The approach also enabled straightforward discretization of fuzzy numbers and thus simplified numeric computation of fuzzy operations. Compared to computation using the extension principle, computations using discretized set of alpha cuts were fast to execute as well as straight-forward to implement.

Fuzzification of a simple decision tree was used as an example of adding a fuzzy component to traditional decision making processes. The method was found to yield interpretable results with some additional insight to the degree of belief in optimality of given strategies. However, concerns raised by Dubois [2011] are not positively addressed, since in total not much evidence is provided in support of using fuzzy decision trees over traditional models in order to validate their rationale and improved usefulness. To this end, more evidence needs to be gathered of traditional decision tree methods' susceptibility to errors in parameter estimation and the improvement of the fuzzy augmentation in this regard.

# References

R. Bellman. *Dynamic Programming.* Princeton University Press, Princeton, NJ, USA, 1 edition, 1957.

M. Brunelli and J. Mezei. How different are ranking methods for fuzzy numbers? a numerical study. *International Journal of Approximate Reasoning*, 54(5):627–639, 2013.

D. Dubois. The role of fuzzy sets in decision sciences: Old techniques and new directions. *Fuzzy Sets and Systems*, 184(1):3–28, 2011.

D. Dubois and H. Prade. Operations on fuzzy numbers. *International Journal of Systems Science*, 9(6):613–626, 1978.

D. Dubois and H. Prade. Ranking fuzzy numbers in the setting of possibility theory. *Information Sciences*, 30(3):183–224, 1983.

C. Z. Janikow. Fuzzy decision trees: issues and methods. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 28(1):1–14, 1998.

A. Kaufmann. On the relevance of fuzzy sets for operations research. *European Journal of Operational Research*, 25(3):330–335, 1986.

G. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic*, volume 4. Prentice Hall New Jersey, 1995.

C.-C. Lee. Fuzzy logic in control systems: fuzzy logic controller. ii. *IEEE Transactions on Systems, Man and Cybernetics*, 20(2):419–435, 1990.

C. Olaru and L. Wehenkel. A complete fuzzy decision tree technique. *Fuzzy Sets and Systems*, 138(2):221–254, 2003.

O. Pavlačka. On various approaches to normalization of interval and fuzzy weights. *Fuzzy Sets and Systems*, 243:110–130, 2014.

H. Raiffa. *Decision Analysis: Introductory Lectures on Choices Under Uncertainty.* Addison-Wesley, Reading, MA, USA, 1 edition, 1968.

H. Raiffa and R. Schlaifer. *Applied Statistical Decision Theory.* Graduate School of Business Administration, Harvard University, 1961.

X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (i). *Fuzzy Sets and Systems*, 118(3):375–385, 2001a.

X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (ii). *Fuzzy Sets and Systems*, 118(3):387–405, 2001b.

S. R. Watson, J. J. Weiss, and M. Donnell. Fuzzy decision analysis. *IEEE Transactions on Systems, Man and Cybernetics*, 9(1):1–9, 1979.

L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.

# A Summary in Finnish

Päätöksentekoanalyysissä käsitellään tyypillisesti malleja, joissa päätöksentekijä arvioi jonkin asetelman yksittäisistä päätöksistä muodostuvia strategioita. Yleensä näiden strategioiden tarkoitus on maksimoida jokin päätöksentekijän odotettu subjektiivinen hyöty. Tällaisia malleja laadittaessa ja hyödynnettäessä vaaditaan päätöksentekijältä tarkat arviot sekä lopputulemien hyödyistä että hyötyihin vaikuttavien tapahtumien todennäköisyysjakaumista. Monessa reaalimaailman tilanteessa tarkka estimointi voi kuitenkin olla mahdotonta ja päätöksentekijän käsityksen pelkistäminen numeerisiin lukuarvoihin voi vähentää mallin käytössä olevan informaation määrää ja johtaa päätöksentekoprosessin vääristymiin. Perinteisestä joukko-opista yleistetyn sumean joukko-opin menetelmiä on esitetty informaation epätarkkuuden huomioimiseksi. Oleellisesti tämä yleistetty teoria laajentaa perinteisen kuuluvuuskäsitteen sisältämään osittaisen kuulumisen joukkoon. Menettely mahdollistaa epätarkkojen lukuarvojen matemaattisen käsittelyn ja soveltamisen päätöksentekoanalyysiin. Työssä tarkasteltiin sumean joukko-opin menetelmien soveltamista perinteiseen päätöspuuanalyysiin sekä muodostettujen sumeiden päätöspuiden antamia strategiasuosituksia.

Klassisessa joukko-opissa joukko voidaan määritellä täysin sen karakteristisella funktiolla eli funktiolla, joka kuvaa kaikki käsiteltävän universumin alkiot ykköseksi mikäli alkio kuuluu joukkoon, tai nollaksi, mikäli alkio ei kuulu joukkoon. Sumeassa joukko-opissa karakteristinen funktio laajennetaan kuuluvuusfunktioksi, jonka arvojoukko on yksikköintervalli. Toisin sanoen joukolle sallitaan osittainen kuuluvuus. Sumea joukko-oppi käsitetäänkin kirjallisuudessa klassisen joukko-opin laajennuksena, ja siten kaikille klassisille käsitteille ja operaatioille on määriteltävä sumea vastine, joka redusoituu odotetulla lailla klassiseen muotoon. Sumea luku on sumean joukon reaaliluvuille määritelty erityistapaus, joka tulkitaan yleensä epämääräiseksi tai epätarkaksi lukuarvoksi. Kirjallisuudessa käsitellään päätöksentekoanalyysin kontekstissa myös lingvistisiä muuttujia, mutta työssä rajoituttiin käsittelemään vain sumeita lukuja.

Päätösanalyysia varten sumeille luvuille käsiteltiin kirjallisuudessa esitettyjä aritmeettisten laskutoimitusten laajennuksia. Yleisesti reaaliluvuille määritellyt funktiot voidaan niin sanotusti sumeuttaa eli laajentaa käsittämään myös sumeat lukuarvot. Työssä tarkasteltiin kahta eri lähestymistapaa funktioiden ja laskuoperaatioiden sumeuttamiselle: laajennusperiaatetta ja $\alpha$-leikkauksia. Näistä varsinaisessa päätösanalyysissä sovellettiin $\alpha$-leikkausmenetelmää sen soveltuessa paremmin numeeriseen laskentaan. Menetelmässä joukko ilmaistaan klassisten joukkojen, $\alpha$-leikkausten, unionina. Mielivaltainen klassisten joukkojen kuvaus voidaan sumeuttaa laskemalla joukon $\alpha$-leikkausten kuvausten unioni. Sumeiden lukujen $\alpha$-leikkaukset ovat intervalleja, joille määriteltyjen aritmeettisten laskutoimitusten koneellinen laskenta osoittautui yksinkertaiseksi. Laskenta toteutettiin käytännössä poimimalla yksikköintervallista pienen intervallin välein $\alpha$:t, joiden leikkauksille laskutoimitus tehtiin. Lopuksi toimituksen tulos saatiin muodostamalla näistä intervalleista unioni.

Sumean aritmetiikan sovellukset päätösanalyysissä ovat laaja-alaiset. Tässä työssä kuitenkin rajoitetaan käsittely yksinkertaisten päätöspuiden sumeiden laajennusten analyysiin. Päätöspuilla kuvataan stokastisia diskreettiaikaisia päätösprosesseja, joissa päätöksentekijä määrittelee ehdollisen strategian oman hyötynsä maksimoimiseksi. Esimerkiksi monivaiheisessa investointiprojektissa organisaation johto päättää myöhempien vaiheiden investoinnista edellisen vaiheen tuloksien perusteella. Mikäli projektille kriittinen ensimmäinen vaihe ei onnistu tavoitteiden mukaisesti, ei sitä kannata välttämättä jatkaa. Mallin tapahtumien todennäköisyysjakaumat ja lopputulosten hyödyt on määriteltävä tarkkoina lukuarvoina, mikä esimerkin kontekstissa jättää huomiotta päätöksentekoon liittyvän epävarmuuden: projektin eri vaiheiden epäonnistumisten todennäköisyys ei ole välttämättä tarkasti arvioitavissa.

Päätösprosessin numeerinen analyysi pohjattiin työssä kirjallisuuden sumean aritmetiikan menetelmiin. Päätösanalyysissa sumeille lukuarvoille on määriteltävä myös odotusarvon ja suuruusjärjestyksen käsitteet. Kirjallisuudessa on esitetty sumeille todennäköisyysjakaumille ja järjestysrelaatioille lukuisia osittain ristiriitaisia määritelmiä, joita työssä verrattiin. Todennäköisyysjakauman normalisointia varten valittiin käytettäväksi kirjallisuudessa esitetty yksinkertainen malli. Järjestysrelaatiolle verrattiin kirjallisuudessa määriteltyjä vaihtoehtoja ja sovellettiin niistä kahta. Näistä toinen asettaa luvut suuruusjärjestykseen vertaamalla niiden geometrisia painopisteitä, ja toinen muodostaa sumean suuruusjärjestyksen ilmaisemalla luottamuksen väittämään "luku A on suurempi kuin luku B" välillä nollasta yhteen.

Näitä määritelmiä hyödyntäen voitiin työssä muodostaa päätöspuille sumea laajennos, jolla pyrittiin huomioimaan kuvitteellisen päätöksentekijän estimaattien epätarkkuus vedonlyöntiprosessissa. Skenaariossa hän päättää ensin osallistumisestaan vedonlyöntiin, jossa hän joko voittaa tai häviää. Voittaessaan hän päättää vielä osallistumisestaan toiseen vastaavaan vetoon. Näin hän joko jättää osallistumatta kumpaankaan vedonlyöntiin, häviää ensimmäisen, voittaa ensimmäisen ja lopettaa, voittaa ensimmäisen ja häviää toisen tai voittaa kummatkin vedonlyönnit. Sumeassa mallissa sallitaan sekä prosessin eri lopputulosten hyötyjen että vedonlyöntien tulosten todennäköisyyksien epämääräisyys. Jokaiselle skenaarion valinnalle laskettiin hyödyn odotusarvo ja niitä verrattiin kummallakin järjestysrelaatiolla. Näin jokaiselle päätökselle saatiin yksiselitteinen suositus optimista sekä sumea luottamus sen optimaalisuudelle.

Saadut esimerkkipuun strategiat osoittautuivat helposti tulkittaviksi. Erityisesti sumea luottamus vaihtoehtojen paremmuusjärjestyksestä tuotti perinteiseen analyysiin nähden lisäinformaatiota, jonka perusteella päätöksentekijä pystyy mahdollisesti arvioimaan valintojaan paremmin tilanteissa, joissa skenaarion parametrit eivät ole tarkalleen tiedossa tai niihin liittyy muuta epäselvyyttä. Kirjallisuudessa on kuitenkin esitetty kritiikkiä sumeille menetelmille, joissa olemassa olevaan päätösmalliin lisätään sumea ulottuvuus ilman, että tämän menettelyn uskotaan huomattavasti parantavan mallin suorituskykyä. Sumean laajennoksen pitäisi ottaa paremmin huomioon päätöksentekijän käytössään oleva informaatio ja sillä pitäisi olla järjellinen perusta.

Työssä oletettiin sumean mallinnusprosessin ottavan paremmin huomioon päätöksentekijän kokema epävarmuus, mutta prosessin paremmasta suorituskyvystä perinteiseen päätöspuuanalyysin nähden ei esitetty vahvaa näyttöä. Jatkossa olisikin hyödyllistä verrata sumean mallin suorituskykyä todellisissa skenaarioissa, jossa päätöksentekijä arvioi laajennetun mallin tuottaman informaation hyödyllisyyttä. Sumeaa aritmetiikkaa voidaan soveltaa myös muissa malleissa, joissa menettely voi olla työssä esitettyä yksinkertaista tapausta hedelmällisempää.