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Multistage investment under two explicit sources of uncertainty - a real options approach

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AALTO UNIVERSITY SCHOOL OF SCIENCE PO Box 11000, FI-00076 AALTO http://www.aalto.fi	ABSTRACT OF THE BACHELOR'S THESIS

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Abstract:

Real options valuation provides a method to analyze investment situations under uncertainty. Especially, real options valuation overcomes many of the shortfalls of traditional net present value analysis regarding the assessment of uncertainty and optionality in an investment situation.

The aim of this study is to introduce a real option model that captures sequential investment decisions under two sources of uncertainty. The sequentiality of the investment process is induced by assuming that the investor can decide the investment rate at which the initial investment is made continuously in time. Furthermore, the investment rate is assumed to be bounded between zero and a positive maximum investment rate implying that there exists always a minimum time-to-build. The two explicit sources of uncertainty are modeled by assuming the variables that determine the payoff of the investment program to follow continuous-time stochastic processes. We interpret the two variables to represent the discounted cash inflows and outflows of the completed investment.

The investor's problem is shown to have a bang-bang solution, i.e., when it is optimal invest, it is optimal to invest at the maximum rate. The model is then solved numerically using an explicit finite difference scheme yielding both the option value and the investment threshold that determines whether it is optimal to invest or wait.

The results are analyzed using the method of comparative statics. In particular, we show how the expected evolution of the second stochastic variable, representing the costs of the finished investment, affects the optimal investment rule. We also show how these effects rely on the value of the maximum investment rate.

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Contents

1 Introduction

The valuation of investment opportunities is a common problem in allocating capital and planning future actions. For example, private firms need to assess the profitability of project possibilities in the process of capital budgeting. In order to maximize the firm value, executives need estimates of the project proposals' values to be able to choose the most profitable ones and to discard the rest.

Net present value (NPV) analysis provides a simple way to calculate the profitability of investment opportunities by discounting the expected cash flows related to the investment decision net of the initial investment cost [Brealey et al., 2006]. NPV analysis makes it possible to calculate the explicit value of the investment opportunity and leads to a simple investment rule: invest only if the NPV is positive.

However, NPV analysis has its shortcomings. First, NPV analysis deals only with the expected future cash flows instead of taking into account the joint probability distribution of the future cash flows. Second, in simple NPV models, the whole investment process is irreversible. By irreversibility we mean that once a positive investment decision is made, the investor cannot withdraw even if the NPV of the investment program becomes negative in the future.[1] Third, in NPV analysis the investment opportunities that are not profitable at the moment have no value. If there is uncertainty about the future cash flows of the investment, then the investment opportunity may become profitable in the future, and, therefore, the investment possibility has value even if is not yet profitable.

Real options valuation (ROV) is an investment valuation method that overcomes the shortcomings of NPV analysis presented above [Dixit and Pindyck, 1994]. ROV utilizes the methods of valuing financial options in calculating the value of an option to make a real investment, e.g., a possibility to build a nuclear power plant or to exploit a discovered oil reserve. ROV also yields the optimal investment rule informing the investor when it is optimal to invest and when not.

Majd and Pindyck [1987] propose a real options model in which an investor faces an opportunity to invest in a project, the payoff of which follows a

[1] NPV analysis can be applied to build more complicated investment models in which there is optionality. For example, Hespos and Strassmann [1965] propose an investment model where NPV analysis is applied in a decision tree framework to evaluate investment possibilities which include optionality.

continuous-time stochastic process. The investor will start receiving cash inflows only when the initial capital investment is made and the project is finished. In their model, the initial investment cannot be done at once, but the investor can invest sequentially in continuous time at an investment rate that is bounded between zero and a positive constant. Therefore, at all values of remaining initial investment there exists a minimum time-to-build, i.e., the time it takes to complete the initial investment assuming that the remaining investment is made as quickly as possible. Their model yields results that explain the behaviour of rational investors in situations where the outcome of the project is uncertain, the investment decision is sequential, and there is a significant time-to-build. An R&D project that could ultimately lead to the creation of a new product is an example of such a situation.

The aim of this study is to extend the model of Majd and Pindyck by introducing a second variable that follows a continuous-time stochastic process and affects the payoff of the completed project. In some situations, the factors behind the revenues and the costs of the finished project are fundamentally different. Thus, in these situations there is a need to model the revenues and costs explicitly with separate variables following separate processes. Then, the inclusion of the second variable allows us to model the expected cash inflows and outflows of the finished project separately and to isolate the effects that these variables have on the optimal investment behaviour. We show that the inclusion of the second stochastic variable reverses some of the earlier results. In particular, we show that for some parameter values, the investment threshold, i.e., the value of the revenue variable above which it is optimal to invest instead of waiting, decreases in the amount of initial investment left, in contrast to the results that are implied by the model of Majd and Pindyck.

This study is constructed as follows. In Section 2, we discuss the background theory further by examining the fundamental differences between NPV analysis and ROV. We present an example of both valuation methods and compare the results. We also review earlier research on which our model is built. In Section 3, we build our model and discuss its assumptions in more detail. Section 4 presents the general results of the model and shows how the inclusion of the second stochastic variable yields informative results, which we seek to explain. Section 5 concludes by discussing the implications of our results and outlining possible directions for future research.

2 Theory and background

2.1 Net present value analysis

NPV analysis is widely used in evaluating the value of investment opportunities. For example, in a survey sent to the chief financial officers of Fortune 1000 companies, Ryan and Ryan [2002] found out that 96% of the responders use NPV analysis in capital budgeting at least sometimes. In other words, only 4% of the responders never use NPV analysis. Also, almost all of the other valuation methods mentioned in the survey are modified versions of traditional NPV analysis. In effect, many other valuation methods, such as the internal rate of return method, are built on the idea of basic NPV analysis.

One reason for the popularity of NPV analysis is its simplicity. To illustrate the idea of NPV analysis, let us consider an opportunity to invest in a factory that costs K amount of capital to build. Let us assume that the capital investment can be done at once resulting in a finished factory of market value V_t that evolves according to the geometric Brownian motion (GBM)

$$dV_t = \alpha V_t dt + \sigma V_t dz_t, \tag{1}$$

where α and σ are the capital appreciation rate and the volatility of V, respectively.[2] By capital appreciation rate, we mean that the total rate of return, which equals to the capital appreciation rate plus the payout or dividend rate, is generally greater than the capital appreciation rate. This assumption implies that the factory continues production forever once it is built and that the sum of the discounted expected future cash flows that the project yields equals to V at all times.[3] Therefore, the NPV of the now-or-never investment opportunity at time $t = 0$ is

$$NPV = V_0 - K. \tag{2}$$

The investment rule implied by Eq. (2) is that it is optimal to invest if NPV is non-negative, i.e., $V \geq K$, and not to invest if NPV is negative, i.e.,

[2] We will denote the value of V at time t with V in the following discussion if it is not necessary to use V_t for clarity.

[3] Let us note the required rate of return for holding the factory with r and assume that $r > \alpha$. By the properties of GBM, the sum of discounted expected future cash flows is $\int_0^t (r-\alpha)V_0 e^{\alpha s} e^{-rs} ds = V_0(1 - e^{-(r-\alpha)t})$ assuming that the factory is scrapped without costs after time t [Luenberger, 1998]. This with Eq. (1) implies that the factory is operational forever since otherwise the factory would be either under- or overvalued.

$V < K$. This implies that by NPV analysis the value of the opportunity to invest is $\max(V - K, 0)$.

This example was presented in continuous time. If the situation were that the cash flows would occur discretely in time, then we would sum instead of integrating, and the NPV would be given by the formula

$$NPV = \sum_{t=0}^{\infty} \frac{CF_t}{(1+r)^t}, \tag{3}$$

where CF_t is the cash flow at time t and r is the total required rate of return for holding the factory.

2.2 Real options valuation

To motivate ROV, we revisit the assumptions of the NPV results in Eq. (1). Since V evolves stochastically in time, there is a finite possibility that, even though $V < K$ at the moment, V will exceed K in some point in the future, implying that the investment opportunity may become profitable by the NPV investment rule. Therefore, the possibility to build the factory has at least some positive value to the investor.

The situation here is similar to the investor holding a perpetual call option with a strike price of K on the factory. It is known that although a financial call option might be out of the money today, it has value to investors as the price of the underlying asset might increase in the future providing a positive payoff [Hull, 2010].[4] This suggests that the investment opportunity is not correctly valued by traditional NPV analysis if the investor has the option to invest in the future instead of now.

The theory of financial options also proposes that even if $V > K$, it may be optimal not to exercise the option yet. This is because there is a possibility that the value of the underlying asset increases further creating an incentive not to invest yet. This effect is often called as the value of waiting. Hence, in this framework also the investment rule resulting from the NPV analysis above is incorrect.

Motivated by the shortfalls of NPV analysis, we consider the investor's problem solved in the previous chapter using the ROV framework. The investment

[4]The fact that one does not need to exercise an option if the payoff is negative is crucial to this result.

opportunity can be valued using ROV in two different ways: by using dynamic programming or contingent claims analysis [Dixit and Pindyck, 1994]. The former approach utilizes the Bellman equation in continuous time. The latter assumes that the risk of the completed project can be replicated by a dynamic portfolio formed of assets sold in the financial markets.

The main difference between the two approaches is in determining the required rate of return for the option. In the former approach, the discount rate determining the required rate of return must be chosen in most cases without capital market equilibrium models such as the capital asset pricing model (CAPM).[5] In the latter approach, one does not need to know the required rate of return for the option as the partial differential equation governing the option value is formed by delta hedging the uncertainty of the option by using the portfolio that spans the risk of the underlying investment opportunity. However, the qualitative results yielded by the approaches are similar.

We use the dynamic programming approach because then we do not need to assume that the risk of the underlying investment opportunity can be replicated by other financial assets. This is often the case in projects that lead to the adoption of novel technologies. Also, our focus is on gaining insight into how the investor's behaviour is qualitatively affected by different parameters in our model and not on capital market considerations.

Let us return to modeling the investor's problem using ROV. We will denote the value of the option to build the factory with $F(V) \equiv F$ where V is the value of the completed factory following the GBM given by Eq. (1). Note that the option value is not explicitly a function of time as the investment opportunity is taken to be perpetual. The initial capital investment is K. Let us denote the required rate of return for holding the option with μ. We should note that μ does not necessarily need to equal to the required rate of return for the completed factory. We will also have to assume that $\mu > \alpha$ because otherwise it would be optimal never to exercise the option. Let us further assume that there exists an unique constant V^*, for which applies that it is optimal not to exercise the option if $V < V^*$ and optimal to exercise the option if $V = V^*$.[6]

By the assumptions made above we obtain the option value in the no-exercise

[5] For example, McDonald and Siegel [1986] determine the correct required rate of return for the option implied by CAPM in a certain real option model.

[6] In our case the previous assumptions imply that this assumption holds [Dixit and Pindyck, 1994].

region $V \in (0, V^*)$ from the Bellman equation:

$$\mu F = \frac{\mathbb{E}[dF]}{dt}. \tag{4}$$

By extending $\mathbb{E}[dF]$ using Itō's lemma, we obtain the following ordinary differential equation (ODE) that the option value must solve in $V \in [0, V^*]$ [Hull, 2010]:

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + \alpha V F_V - \mu F = 0.^7 \tag{5}$$

If $V \geq V^*$, then the option is as valuable as the payoff $V - K$ since for these values of V the option will be exercised.

The appropriate boundary conditions in this situation are

$$F(0) = 0, \tag{6a}$$
$$F(V^*) = V^* - K, \tag{6b}$$
$$F_V(V^*) = 1. \tag{6c}$$

Eq. (6a) states that if the market value of the factory becomes zero, then the option loses its value because zero is an absorbing barrier for the GBM given by Eq. (1). Eq. (6b) simply states that at the moment of exercise the option is as valuable as the payoff it yields. This condition is usually called as the value-matching condition. Eq. (6c) demands that the partial derivative of the option value with respect to the stochastic variable is continuous at the point of exercise.[8] This condition is often referred to as the smooth-pasting condition.

Without going into further detail, we state that the solution for the problem above is [Dixit and Pindyck, 1994]

$$F(V) = \begin{cases} AV^\gamma & \text{if } V \in [0, V^*] \\ V - K & \text{if } V \in (V^*, \infty) \end{cases}, \tag{7}$$

[7] Here, we adopted a shorthand notation for partial derivatives. For example, F_V is the partial derivative of F with respect to V. We will continue to use this notation for the rest of the thesis.

[8] See Dixit and Pindyck [1994] for an intuitive explanation to this condition.

where γ, V^*, and A satisfy equations

$$\gamma = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\mu}{\sigma^2}} \tag{8}$$

$$V^* = \left(\frac{\gamma}{\gamma - 1}\right) K \tag{9}$$

$$A = \frac{V^* - K}{V^{*\gamma}}. \tag{10}$$

.

Note that by Eq. (8), $\gamma > 1$. Therefore, Eq. (9) implies that $V^* > K$. In other words, the optimal investment rule of the real options model is different than the rule that NPV analysis yielded: the result of ROV proposes that it is suboptimal to exercise the option when $V = K$ because there is more value in waiting for the market value of the project to increase.

By using Eqs. (8)-(10), one can show that both $F(V)$ in the waiting region given any value of V and V^* are increasing in σ. This means that an increase in the uncertainty of the future project values increases the option value since the possibility that the project value becomes larger in the future increases. We should note that an increase in uncertainty increases also the possibility that the project value becomes smaller. As the option does not need to be exercised, the investor benefits of increases in the project value more than she loses if the value decreases. Therefore, V^* increases in σ since uncertainty increases the value of waiting.

One can also show that both the option value in the waiting region and the investment threshold are increasing in α. The reason for this is that since the difference between μ and α represents the cost of waiting, an increase in α decreases the cost of waiting and, thus, postpones investment. The conclusions above are inherent in many real option models. However, the inclusion of new variables or an investment lag may reverse these results, as we shall soon discover.

2.3 Literature review

One of the restricting assumptions above is that the factory can be built instantly if the investor wishes to exercise the option. This is an unrealistic approximation of most of real investment situations because there is usually a considerable lag between the investment decision and the moment that the finished investment starts to yield cash inflows.

Majd and Pindyck [1987] extend the model discussed above by limiting the rate by which the investor can invest the initial cost. Let us use the same notation as above. In their model the remaining capital investment evolves according to the differential equation

$$\frac{dK}{dt} = -I, \tag{11}$$

where $I \in [0, k]$, and k presents the maximum investment rate implying that the minimum time-to-build is K/k. Consequently, I can be seen as a control variable that the investor manages during the investment process. From the investor's perspective, the problem is then to solve for the optimal control law that tells when and at which rate it is optimal to invest.

Majd and Pindyck show that the optimal control problem has a bang-bang solution: it is optimal to either invest at the maximum rate k or not at all. The problem is solved numerically yielding both the option value $F(V, K)$ and the investment threshold $V^*(K)$. Then, the optimal behaviour is to invest at rate k if $V \geq V^*(K)$ and to wait if $V < V^*(K)$. $V^*(K)$ is found to be increasing in K reflecting the facts that the longer is the minimum time-to-build, the more uncertain is the value of the completed project, and the larger is the value of K, the larger is also the discounted initial investment remaining.

Majd and Pindyck also performed comparative statics with respect to parameters σ, δ, and k to gain further insight into the investor's behaviour.[9] They found out that $V^*(K)$ is increasing in σ. This is explained by the fact that in this model the investor has a compound option. By compound option, we mean that by investing an infinitesimal amount of capital, the investor receives a new option on the project, only with less initial investment to go. Therefore, the payoff of an incremental investment is a convex function of V. This implies that the investor benefits from increased uncertainty as she can profit from increases in V but can wait if V decreases. In contrast, the cost of waiting is presented by δ that is constant with respect to all other parameters. To conclude, the benefits of waiting increase in σ, whereas the costs of waiting remain the same.

The effects of δ and k are not as straightforward. We first consider the case of δ. As δ represents the opportunity cost of waiting, an increase in its value has an effect of hastening investment. On the contrary, by Eq. (1), a

[9]Here, $\delta = \mu - \alpha$ is the difference between the total required rate of return for holding asset V and the capital appreciation rate of V. Therefore, for example, a decrease in δ implies an increase in the growth rate of V.

larger δ means that the growth rate of V is smaller when holding r constant. Therefore, an increase in δ also discourages investment as in some cases the value of V is expected to decrease during the investment period. Majd and Pindyck show that for low values of δ the former effect dominates, while for larger values the latter one takes over. The overall result is that when increasing δ from value of zero, $V^*(K)$ first decreases and then increases after some point.

Because the value of k determines the minimum time-to-build, the effect of δ on $V^*(K)$ is dependent on k. Since a large value of k means shorter construction times and vice versa, a decrease in k should amplify both of the effects discussed above. Indeed, Majd and Pindyck show that this happens. The growth rate effect above is especially sensitive to the value of k as for large values of k the effect that an increase in δ can discourage investment nearly vanishes.

As discussed above, since k determines how fast the project can be finished, its value also has a substantial effect on the results. The effect of k is that the faster the project can be finished, the less is the discounted expected cash flow foregone because of the time-to-build. Because of this, Majd and Pindyck find that $V^*(K)$ is decreasing in k as larger values of k mean that the project can be finished faster. The effect of k on $V^*(K)$ is also dependent on δ as the cash flows foregone are proportional to δ. Because higher values of δ mean that the cash flow foregone is larger, the effect of k on $V^*(K)$ is more substantial if δ is relatively large.

To conclude the discussion of the model of Majd and Pindyck, we restate their key results. The introduction of a time-to-build increases the investment threshold in comparison with the earlier model, in which the factory could be built instantly. Also, the time-to-build can reverse the earlier result that an increase in δ always decreases the investment threshold as the investor must consider how V is expected to evolve during the construction period. Finally, Majd and Pindyck showed that the lower is the maximum rate of construction, the higher is the investment threshold.

In the Majd and Pindyck model, there is a good deal of optionality in the investment process because the investor can decide whether to invest or not continuously in time. This might be a good approximation of multistage investments in which the investment program can be stalled without cost between the stages if the situation turns unfavourable. Also, the more consecutive stages there are in the investment process, the more realistic this assumption becomes.

However, in many investment situations the initial investment decision is completely irreversible and only the finished project can be scrapped after the construction period. This changes the situation fundamentally as once the investment decision is done, the investor cannot act on bad news until the construction period is over.

Bar-Ilan and Strange [1996] propose a model, in which the investor has an opportunity to invest in a project, the value of which is a function of a revenue variable evolving according to a GBM and a constant cost parameter. In their model the investment decision is made at once and there is an investment lag. Once the investment decision is made, the investor can abandon the project at a cost only after the construction period. If the finished project is abandoned, then the investor holds again an opportunity to invest in the same project from scratch having to bear the initial investment cost and construction period again.

By assuming a more irreversible investment process, Bar-Ilan and Strange reverse some of the results of Majd and Pindyck. Their main outcome is that if the abandonment of the finished project is relatively costless and an investment lag is present, then a larger uncertainty over the future values of V can actually lower the investment threshold. The logic behind this is that if the payoff of the investment is a convex function of V, Jensen's inequality implies that the expected payoff is increasing in the volatility of V. For some values this effect might overcome the effect that increasing uncertainty also increases the benefits of waiting. In other words, by making a positive investment decision, the investor receives an European call option on the project with a slightly modified payoff because of the option to abandon the project after completion. Then, because the value of the call option is increasing in the volatility, the marginal cost of waiting is also increasing in the volatility of the underlying project.

Bar-Ilan and Strange also discovered that for the parameters they used, a longer investment lag leads to a lower investment threshold, contradicting the results of Majd and Pindyck [1987] yet again. The reason for this is similar to the logic above. A longer investment lag means that the variance of the future payoff is larger. This leads to two opposite effects. First, because of the larger uncertainty over the future payoff, the benefits of waiting increase due to the argument above. Second, the larger uncertainty yet again implies that the expected payoff is larger. For the parameters used in their study, Bar-Ilan and Strange found that the latter effect dominates and greater investment lags encourage investment.

It should be noted that the results of Bar-Ilan and Strange are not contradictory with those of Majd and Pindyck because of the different investment processes assumed. In Majd and Pindyck model, the investor has more optionality within the investment process. Therefore, in their model the costs of waiting are smaller in general. Moreover, the cost of waiting is captured entirely by the term δ. In the Bar-Ilan and Strange model, the investment period, during which no actions can be made, leads to the cost of waiting being a function of volatility. Hence, the differences in the investment processes explain the differences in the results.

The models discussed above consist of only one stochastic variable. In some investment situations, there is a need to incorporate other sources of uncertainty explicitly as well by adding more stochastic variables into the model. This serves as a key motivation to this thesis.

McDonald and Siegel [1986] study a model in which both the market value of the project and the investment cost follow a GBM. In their model, the option is perpetual, too. They note that the investment cost variable can also be seen as the discounted sum of expected future costs incurred after the project is launched. In this case, however, one cannot model the initial investment cost explicitly anymore. Also, the model of McDonald and Siegel is based on the assumption that the option can be exercised instantly.

McDonald and Siegel discover that in this situation the optimal investment rule is to exercise the option when $V/K \geq D$ where $D > 1$ is a constant with respect to V, K, and time. Furthermore, D is increasing in the volatilities of the GBMs that the stochastic variables follow. In other words, uncertainty discourages investment as the benefit of waiting increases in uncertainty in this model as well. Also, an increase in the growth rate of V and a decrease in the growth rate of K or the required rate of return for the option increase the value of D since then the growth rate of the payoff increases.

The McDonald and Siegel model is linear homogenous and can be hence solved as a one-factor model after a variable transform. Adkins and Paxson [2011] study real asset renewing decisions in a two-factor real options framework. Unlike the McDonald and Siegel model, their model cannot be transformed into a one-factor model. To solve the model, Adkins and Paxson propose a quasi-analytical solution that serves as a reference on how to solve two-factor real options models built on similar assumptions in general.

3 Analytical model

3.1 Assumptions

In this study we extend the models proposed in both Majd and Pindyck [1987] and McDonald and Siegel [1986] by considering a situation, where there is an investment lag, initial capital cost, and two stochastic variables that determine the payoff of the project. Especially, our aim is to study how introducing a stochastic variable that represents the costs that the finished project incurs affects the investor's optimal investment policy.

Let us denote the discounted expected cash inflows of the finished project with V and the discounted expected cash outflows with C. Then, the payoff of the finished project is $\max(V-C, 0)$. Note that we implicitly assume that the finished investment can be scrapped without any costs. This assumption is not restricting as a rational investor will never finish the investment program if the payoff will be negative.

We shall continue to assume that both the option to invest in the project and the project itself are perpetual.[10] This is not true in reality, but because the cash flows are discounted, the effect of distant cash flows is negligible. The gain of assuming perpetuality is that it relieves us from assuming that the option value is an explicit function of time.

We shall assume that V and C follow the following GBMs

$$dV_t = \alpha_V V_t dt + \sigma_V V_t dz_t \tag{12a}$$
$$dC_t = \alpha_C C_t dt + \sigma_C C_t dw_t, \tag{12b}$$

where dz_t and dw_t are increments of uncorrelated Wiener processes. We assume that the increments of the GBMs are uncorrelated as there are often no reliable reasons for why one should assume them to be correlated. Also, there is some discussion on whether the assumption that the value increments of financial assets are correlated is reasonable [Wilmott, 2007]. Finally, the induction of a correlation between the increments would be rather easy to implement.

Note that the assumption that the drift and volatility parameters of GBMs given by Eqs. (12a) and (12b) are constant implies that the decisions of the

[10]The assumption that the project is perpetual could be easily relaxed by modifying the payoff of the investment option. We will not do so as it would not have interesting effects on the results.

investor do not affect the evolution of V and C. In the case of V, this is essentially a perfect market assumption, i.e., the investor takes the market value of the output of the firm as given. In the case of C, the interpretation depends on the situation. If $\alpha_C = 0$, then the interpretation is simply that the evolution of the cost variable is stochastic yet not mean reverting. When $\alpha_C > 0$, the interpretation might be that the costs are expected to increase in the long run. For example, if the main cost determinant of the finished project is a diminishing natural resource, then the interpretation might be that because the price of this resource will increase in the future due to decreasing supply; consequently, the costs of production will rise.

By contrast, if $\alpha_C < 0$, then we can form an interesting interpretation. Consider the case of new technology adoption, e.g., the adoption of electric vehicles (EVs) and a charging infrastructure. If the adoption of EVs is in line with the goals of governmental organizations, then they might invest in the R&D involved to initiate private sector investment and further accelerate the adoption process. In this case, it is also feasible for the public organizations to make their information and progress available to the public so that the private sector can capitalise on the evolving technology, thereby fulfilling the goals behind the public investments.[11] This implies that the private investor in our case experiences an exogenous experience curve effect that decreases the costs of the finished project over time. Hence, by considering the case $\alpha_C < 0$, we can study how an exogenous learning curve effect described above affects the actions of rational investors.

We shall model the investment process following the lead of Majd and Pindyck [1987]. Let us denote the initial capital investment left with K, the investment rate with I, and the maximum investment rate with k. Then the dynamics of K are captured by Eq. (11).

As seen above, the assumptions made about the investment process have great influence over the results. Therefore, we shall consider the situations in which our assumptions hold in greater depth. The key attribute of the investment dynamics assumed here is that the investor can continuously adapt the rate at which she invests as new information about the expected profitability of the finished project arrives. This implies that our framework is most relevant in modeling situations in which the investment is made in mul-

[11] According to the Joint Research Centre of the European Commission [2013], circa 65% of the outstanding total European EV R,D&D budget of 1.9 B€ is from public funding. The report also finds that an increased exchange of information between the projects would result in a better societal return for the investments due to the exogenous learning effects described above.

tiple stages and the investor can halt the investment between the stages. If the investor has an opportunity to halt the process during the stages, then our model is even more relevant.

The more irreversible the investment process becomes, the less appropriate our model turns in describing the optimal investment behaviour. In these cases a model, in which the investment process is similar to the one in Bar-Ilan and Strange [1996], should be employed. The difference between the approaches matters in that they lead to different results.

As we will use the dynamic programming approach to value the investment option, we denote the required rate of return for the option with μ. As noted before, we have generally no way of using a capital market model to determine the appropriate value of μ. One way to determine μ would be to use Itō's lemma to determine the stochastic process by which the analytical solution for the option value evolves and then solve μ as a function of the other parameters by using some financial market model. However, since we will have to use a numerical program to calculate the option value, it is clear that this is not feasible in our situation. Therefore, we will interpret μ as a parameter that represents the cost of maintaining the investment possibility in general.

We need to place one restriction on the value of μ, i.e., $\mu > \alpha_V$. Otherwise, for some values of α_C it would be never optimal to exercise the option as the capital gain of V would exceed the required rate of return.

3.2 Derivation of the model

Let us again denote the option value with $F(V, C, K) \equiv F$. Then, the option value in $(V, C, K) \in X \equiv (0, \infty) \times (0, \infty) \times (0, \infty)$ given the investment policy $I(V, C, K) \equiv I$ can be obtained from the Bellman equation

$$\mu F = \max_{I \in [0,k]} \left(\frac{\mathbb{E}[dF]}{dt} - I \right). \tag{13}$$

Note that dF is a function of I. Here dt in the denominator means that the expression in the nominator is divided by the increment of time, not differentiated with respect to time. By expanding dF using Itō's lemma and taking the expected value, we get

$$\mu F = \max_{I \in [0,k]} \left(\frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - I F_K - I \right). \tag{14}$$

By noting that the expression to be maximized with respect to I is linear in I, we conclude that if it is optimal to invest at all, then it is also optimal to invest at the maximum rate k. Therefore, the optimal investment policy is a bang-bang control as in the model of Majd and Pindyck [1987].

Let us assume that there exists a unique continuous surface $V^*(C,K)$ in X so that it is optimal to invest if $V \geq V^*(C,K)$ and wait otherwise. This assumption is based on the intuition that the option value will be increasing in V. Let us denote the option value in the investment region $R \equiv X \cap \{V \geq V^*(C,K)\}$ with F and in the waiting region $W \equiv X \setminus R$ with f.

Under the assumption above the option value functions in the two regions are given by PDEs

$$\frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - kF_K - \mu F - k = 0 \text{ in } R, \tag{15a}$$

$$\frac{1}{2}\sigma_V^2 V^2 f_{VV} + \frac{1}{2}\sigma_C^2 C^2 f_{CC} + \alpha_V V f_V + \alpha_C C f_C - \mu f = 0 \text{ in } W. \tag{15b}$$

Note that only Eq. (15a) contains partial derivatives with respect to K as no investment occurs in W.

The appropriate boundary conditions to the problem are

$$F(V,C,0) = \max(V - C, 0) \tag{16a}$$

$$\lim_{V \to 0} f(V,C,K) = 0 \tag{16b}$$

$$\lim_{C \to \infty} f(V,C,K) = 0 \tag{16c}$$

$$F(V^*(C,K),C,K) = f(V^*(C,K),C,K) \tag{16d}$$

$$F_V(V^*(C,K),C,K) = f_V(V^*(C,K),C,K) \tag{16e}$$

$$F_C(V^*(C,K),C,K) = f_C(V^*(C,K),C,K) \tag{16f}$$

Eq. (16a) is simply the payoff of the option, whereas Eq. (16b) states that when V reaches zero, the option becomes worthless. This is because zero is an absorbing barrier to the GBM given by Eq. (12a). Eq. (16c) means that the option value converges to zero as the expected costs of the finished project grow arbitrarily large. Eq. (16d) is the value-matching condition discussed above. Eqs. (16e) and (16f) are the smooth-pasting conditions. Note that now there are two smooth-pasting conditions as there are two stochastic variables.

A general solution to Eq. (15b) is of the form

$$f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)}, \tag{17}$$

where coefficients $\beta(K)$ and $\eta(K)$ must satisfy condition

$$\frac{1}{2}\sigma_V^2\beta(\beta-1) + \frac{1}{2}\sigma_C^2\eta(\eta-1) + \alpha_V\beta + \alpha_C\eta - \mu = 0 \tag{18}$$

for each value of K. We use short-hand notation for $\beta(K)$ and $\eta(K)$ here. By general solution, we mean that any linear combination of functions of the form given by Eq. (17) satisfies the PDE given by Eq. (15b).

Eq. (18) has solutions in all four quadrants of the (β, η)-plane [Adkins and Paxson, 2011]. However, we can rule out three of the four quadrants by using the boundary conditions given by Eqs. (16b) and (16c). Doing so we obtain that for a solution of the option value in W it must hold that $\beta > 0$ and $\eta < 0$.

From now on, we will assume that the solution to PDE (15b) is $f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)}$ where $(\beta(K), \eta(K)) \in (0, \infty) \times (-\infty, 0)\ \forall\ K \in (0, \infty)$ so that Eq. (18) holds. $A(K)$ must be solved using the other boundary conditions and the option value in R.

Since the PDE in the investment region has no analytical solutions, we will use a numerical approach based on an explicit finite difference method to solve the rest of the investor's problem. However, now that we know the form of the analytical solution in the waiting region, we can write boundary conditions (16d)-(16f) in a more convenient form. By inserting the quasi-analytical solution given by Eq. (17) in the conditions mentioned above, we obtain that at the investment threshold the following conditions must be met:

$$\frac{F(V^*(C,K),C,K)}{F_V(V^*(C,K),C,K)} = \frac{V^*(C,K)}{\beta(K)}, \tag{19a}$$

$$\frac{F(V^*(C,K),C,K)}{F_C(V^*(C,K),C,K)} = \frac{C}{\eta(K)}, \tag{19b}$$

where $\beta(K)$ and $\eta(K)$ satisfy Eq. (18). We will utilize conditions (19a) and (19b) in the numerical program to find the placement of the investment threshold. Once the investment threshold is solved, we can solve the values of $A(K)$, $\beta(K)$, and $\eta(K)$ for each discrete value of K. The numerical program is discussed in further detail in Appendix A.

4 Results

4.1 Base case

We will present the results of the model in two parts. First, we will consider a base case and provide a discussion of the results in general. Then, we will present the most interesting results by using the method of comparative statics, i.e., examine how the results change when one parameter is changed while the others remain the same. This allows us to isolate the effects of single parameters on the investor's optimal investment policy.

Let us consider a situation where the investor has an investment opportunity of the kind discussed in the previous section. Let the total investment required to finish the investment program be $K = 6$ (M€) and the maximum investment rate be $k = 1$ (M€/year). This implies that the minimum time to complete the investment program is six years and that the unit of time is years. We shall assume that the drift of V is $\alpha_V = 0.04$ and the volatility of V is $\sigma_V = 0.14$. Let us consider at first a case where the drift of C is the same as the drift of V ($\alpha_C = 0.04$) and the volatility of C is equal to the volatility of V as well ($\sigma_C = 0.14$). We will assume that the required rate of return of the option is $\mu = 0.08$. In further discussion, we will refer to this set of parameter values as the base case.

If we were considering an all-equity firm that consisted only of the investment opportunity studied here, then the base case values would imply that the volatility of the firm's stock is approximately $\sqrt{0.14^2 + 0.14^2} = 19.8\%$. Considering that the implied volatility of the S&P 500 index options sold on the Chicago Board Options Exchange is usually around 20%, the assumptions made on the volatilities of the processes are fairly realistic [Yahoo Finance].[12]

The assumption that the drift rates of V and C are equal in the base case implies that the instantaneous growth rate of $V - C$ is linear in $V - C$ so that it is negative if $V - C < 0$, positive if $V - C > 0$, and zero if $V - C = 0$. Moreover, the capital rate of return for $V - C$ equals to $\alpha_V = \alpha_C = 0.04$. This assumption is reasonably realistic as well considering that the total rate of return for $V - C$ equals to the capital rate of return plus the payout rate. If the payout rate were for example 0.04 for both V and C, then the total rate of return would be 0.08, which is realistic considering the historical returns

[12] One should note that the S&P 500 represents a well diversified portfolio of leveraged assets. Therefore, using its volatility to represent the volatility of a typical firm is a rough approximation.

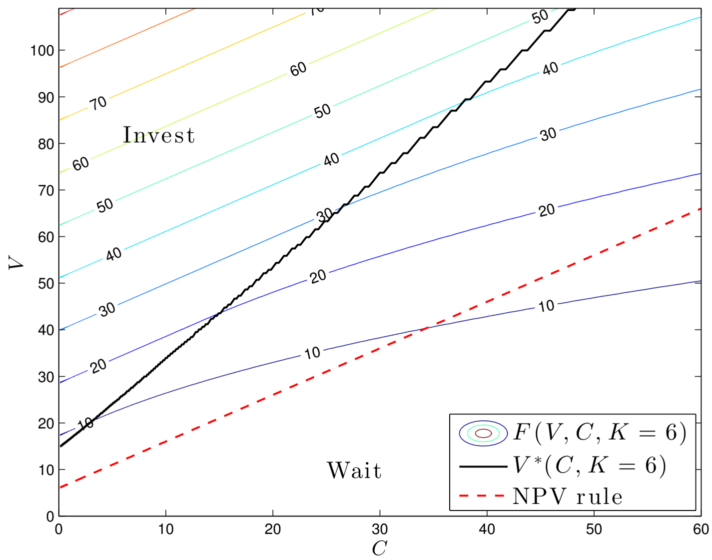

Figure 1: The option value, investment threshold and NPV rule when $K = 6$

on equity investments.

Figure 1 shows the option value and the investment threshold in the base case when $K = 6$. We can see that the option value is increasing in V and decreasing in C as intuition suggests. This is the case for other values of K as well. The black line in the figure shows the position of the investment threshold $V^*(C, K = 6)$.[13] As we can see, the threshold is rising in C, which is also intuitively explained by the fact C represents the costs of the finished project. Note also that the investment threshold is not an isocurve of the option value. Therefore, we cannot in general draw a straight connection between the option value and the location of the investment threshold.

The red dashed line in Figure 1 shows the NPV investment rule in this situation assuming that the whole investment is finished at the full rate if it is optimal to invest.[14] We can see that the NPV rule is to invest in cases where it is optimal to wait according to the ROV rule. Also, in the waiting area of the NPV rule, the option to invest has no value contradicting the

[13] The threshold curve is not smooth because of the numerical finite difference scheme used to solve the problem.

[14] In this case the NPV rule is to invest only if $e^{-\mu\frac{K}{k}}\left(Ve^{\alpha_V\frac{K}{k}} - Ce^{\alpha_C\frac{K}{k}}\right) - \frac{k}{\mu}\left(1 - e^{-\mu\frac{K}{k}}\right) \geq 0$.

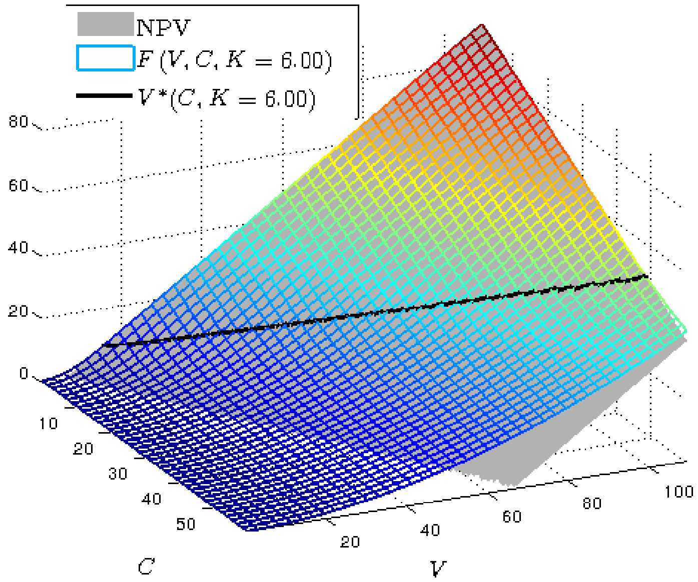

Figure 2: The option value surface, now-or-never NPV, and investment threshold when $K = 6.00$

ROV results.

The NPV rule is by its definition obtained by calculating the expected cash flows of the project net of the initial investment costs. Therefore, there must be other reasons than the initial investment cost for the investment threshold $V^*(C, K = 6)$ to be above the NPV rule. The reason is twofold. First, since both V and C evolve stochastically in time, there is a chance that the investment opportunity might increase in value over time. This implies that there are benefits to waiting that are not present in the NPV analysis. Second, as there is uncertainty on the value of the finished project due to the time-to-build, it is optimal to wait longer than the NPV rule suggests in order to cover this uncertainty by waiting for the expected value of the finished project to rise well above the NPV rule.

One should also note that the slope of $V^*(C, K = 6)$ is greater than that of the NPV rule. This is explained by the fact that because of the assumption that V and C follow GBMs, the volatility of the process that $V - C$ follows increases in both V and C. Therefore, at larger values of V and C, the benefits of waiting are larger than at lower values. We should note that even when K decreases, the slopes of the curves remain different due to this reason.

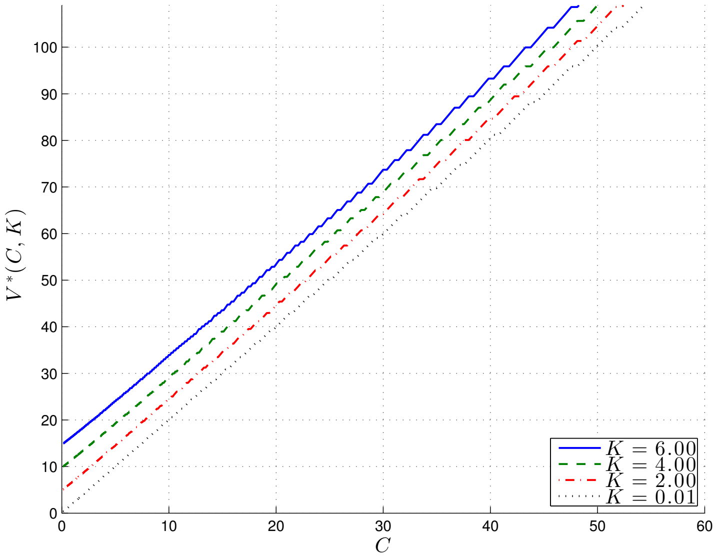

Figure 3: The investment threshold of the base case for different values of K

Figure 2 shows the option value surface, the now-or-never NPV, and the projection of the investment threshold on the option value surface when $K = 6.00$. We can see that the value-matching and smooth-pasting conditions are met by the numerical solution: the option values in the investing and waiting regions meet on the investment threshold, and the surface is smooth on the threshold. Also, we notice that the option value is non-negative for all values of (V, C). Furthermore, by comparing the option value and the NPV, we observe that the option value is greater than the NPV for all values of (V, C) reflecting the fact that unlike NPV analysis, ROV considers also the value of waiting and the possibility to vary the investment rate. We also notice that the difference between the option value and the NPV converges to zero as V increases and C decreases. This happens since if $V >> 1$ and $C << 1$, then the investment program will be completed almost certainly at full pace without pauses yielding on average a total payoff that equals to the NPV.

Figure 3 shows the investment thresholds for various values of K in the base case. We can see that the threshold curves are increasing in C for each value of K as they should by the argument that the value of the finished project is decreasing in C. Also, in the base case the investment thresholds increase in K. This is due to two reasons. First, the remaining initial investment increases in K. Second, the uncertainty over the value of the payoff when the investment program is completed is increasing in K since a large value of K indicates that the minimum time-to-build is large as well.

Note that Figure 3 can be used as a decision rule: since we have implicitly assumed that the investor can observe V, C, and K at each point in time, the investor can use the investment thresholds at different values of K as a guide on how to proceed optimally with the investment program.

4.2 Comparative statics

4.2.1 Sensitivity with respect to α_C

We shall begin this section by considering how the value of α_C affects the investment threshold. Figure 4 shows the investment thresholds at different values of K for various values of α_C while holding the other parameters the same as in the base case. We can see that for the smaller values of K the effect of α_C on the results is monotonic: a decrease in α_C shifts the investment threshold up and, thus, increases the incentive to wait. However, for $K = 6$ the effect is more subtle: when α_C decreases from 0.08 to -0.10 the investment threshold shifts up, but as α_C decreases further the investment threshold $V^*(C, K = 6)$ shifts downwards. As the main motivation for this study is to gain insight on how the inclusion of C affects the investor's choices, we need to discuss the mechanics behind the effects of α_C on the investor's optimal behaviour in detail.

The effect of α_C on the results can be understood by considering the expected evolution of $V - C$. By using Eqs. (12a) and (12b) we get

$$\mathbb{E}[(V - C)_{t+s}|\mathscr{F}_t] = V_t e^{\alpha_V s} - C_t e^{\alpha_C s}, \tag{20}$$

where $\mathscr{F}_t$ is a set containing all information on the evolution of V and C up to time t. Now we can see that the expected evolution of the payoff depends on the value of α_C. The expected increase of $V - C$ is increasing in V and α_V and decreasing in C and α_C.

Let us consider first how α_C affects the investment boundary when $K = 0.01$, i.e, the capital investment is nearly finished and the payoff can be obtained almost instantaneously. This will help us understand the effect on the investment thresholds for larger values of K. The plots in Figure 4 show that the investment threshold $V^*(C, K = 0.01)$ increases monotonically as α_C decreases. In other words, when α_C decreases, the future probability distribution of the payoff shifts to a desirable direction creating incentives for waiting since the gap between the capital appreciation of $V - C$ and

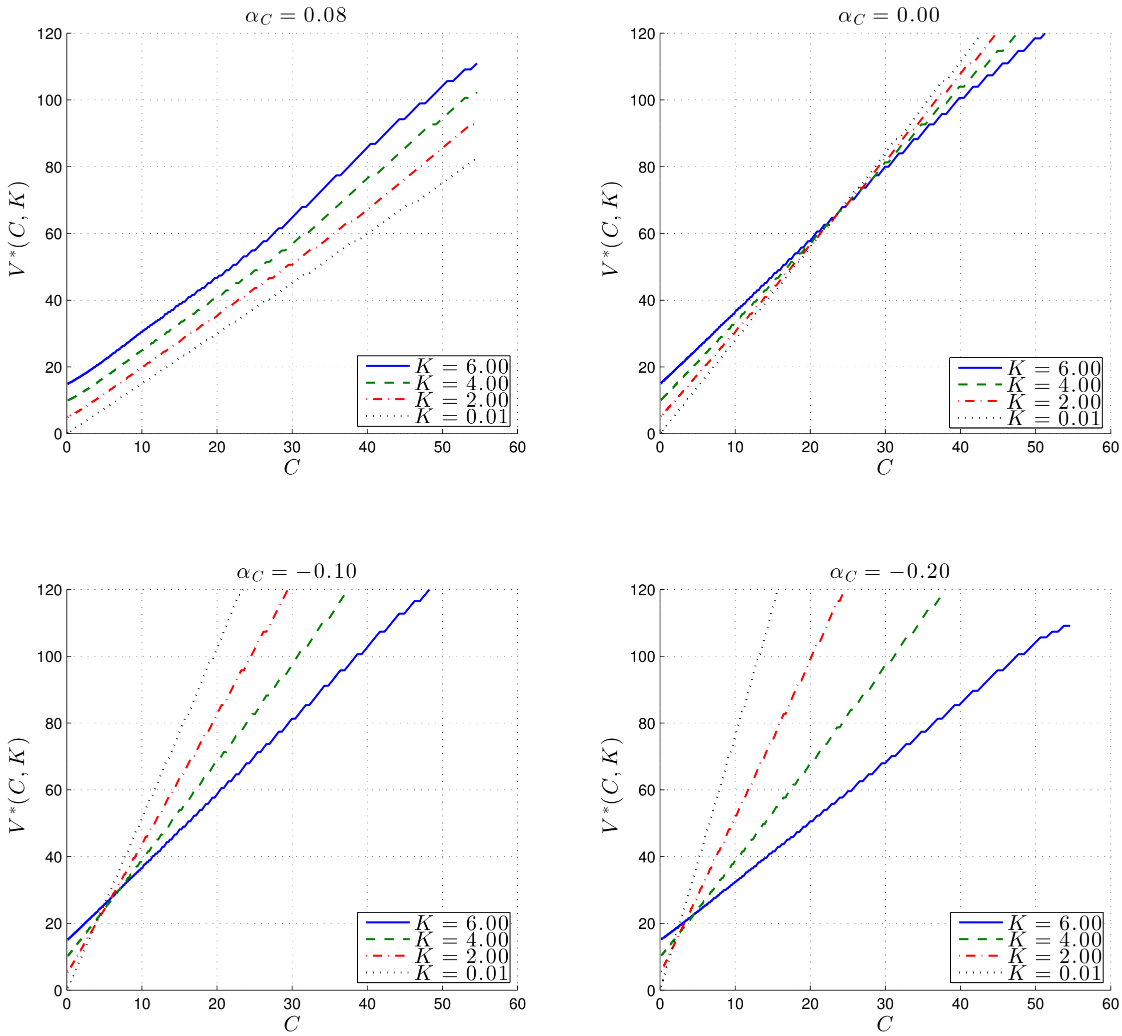

Figure 4: The sensitivity of the investment threshold with respect to α_C

the required rate of return μ narrows. This is also shown by McDonald and Siegel [1986] in the case $K \to 0$.[15]

Now we can consider the effect of α_C on the investment thresholds when $K > 0.01$ and the payoff cannot be obtained instantaneously on demand. Recall that the underlying idea behind the dynamic programming approach used to solve the investor's problem is that at each state (V, C, K) the optimal decision is derived assuming that the subsequent decisions are optimal as well. In our case, this means that the investor holding the option to invest with K amount of initial investment remaining knows the optimal investment rule for smaller values of K as well. Due to the fact that the cash flows are discounted, this implies that for larger values of K it is optimal to wait for V and C to reach such values that the remaining initial investment will be done with minimal pauses on average assuming that the investor follows the optimal investment rule.[16] In this way, the initial investment costs will be paid as late as it is reasonable while still allowing the investor to obtain the payoff as soon as it is optimal to do so in most cases. The drivers behind this logic are that, first, the discount factor implies that cash outflows paid in the future are less valuable than if they were paid now, and second, because of the discount rate, it is better to obtain the payoff now than in the future assuming that it would actually be optimal to obtain the payoff now. Therefore, the placement of the investment thresholds at larger values of K depends on both the placement of the investment threshold when $K \to 0$ and the stochastic evolution of the payoff.

To fully understand the logic above, we shall first consider the situation in the upper left plot of Figure 4, where $\alpha_V < \alpha_C$. By Eq. (20) this implies that the payoff is expected to increase in the future less than in the base case displayed by Figure 3. Therefore, as α_C increases from the base case value $\alpha_C = 0.04$, the investment threshold for $K = 0.01$ decreases for each value of C since the incentive to wait diminishes. Following the threshold $V^*(C, K = 0.01)$, the thresholds for larger values of K shift down as well since otherwise the investor would wait for too long to begin investing and the expected discounted payoff at the end of the investment program would decrease. Notice also that the spread between the thresholds for different values of K increases in C. This is explained by Eq. (20): since the expected increase of C is linear in C, the investor will wait for V to increase further

[15] It can be shown numerically that $V^*(C, K)$ converges to the analytical results of McDonald and Siegel [1986] when $K \to 0$.

[16] It can be shown numerically that if $V - C$ evolves according to Eq. (20), the investor will receive the payoff after the minimum time-to-build once it is optimal to invest at $K = 6$ by investing at full rate up to the completion of the investment program.

for larger values of C to offset the larger expected increase of C. This argument applies generally as the effects of α_C on the investment thresholds are amplified at large values of C.

Let us then consider what happens when α_C decreases from 0.08 to 0.00 by examining the upper plots in Figure 4. We notice that $V^*(C, K = 0.01)$ shifts upwards for each value of C in comparison to the same threshold curve for $\alpha_C = 0.08$ as α_C decreases. This reflects the increased incentive to wait since for $\alpha_C = 0.00$ C is not expected to increase at all, whereas the stochastic process of V remains the same as before. In effect, when α_C decreases, the gap between μ and the growth rate of the payoff decreases and, thus, the cost of waiting decreases as discussed above.

However, we notice that the investment thresholds for large values of K shift up less than the thresholds for smaller values of K for each value of C as α_C decreases. The explanation for this is that the decrease of α_C from 0.08 to 0.00 increases the growth rate of the payoff. Therefore, since after the decrease of α_C the payoff is expected to increase more during the minimum time-to-build than in the case of $\alpha_C = 0.08$, it is optimal to start investing at lower values of V with respect to the investment threshold at $K \to 0$ given a value of C than in the case $\alpha_C = 0.08$. As explained above, this enables the investor to obtain the payoff as soon as it is optimal to do so on average.

Moreover, we observe that for small (large) values of C, the investment threshold is increasing (decreasing) in K. This is explained by the effect of the initial investment cost on the threshold. Since the investor needs to pay K amount of capital to obtain the payoff, it is not optimal to start investing if the expected payoff of the investment program exercised by the optimal policy does not at least exceed the discounted initial investment left. This is depicted by the fact that the intercept of the investment threshold and the vertical axis is positive in all cases where $K > 0$. Also, as K decreases, this intercept converges to zero as the initial investment left decreases and its effect on the investment threshold vanishes. Recall that the initial investment outflows are completely irreversible. Therefore, the initial investment outflows that are already paid are not taken into consideration in the subsequent investment decisions. This explains why for small values of C, the investor will in the end settle for a payoff that is smaller than it is on the investment thresholds for larger values of K, as can be seen in Figure 4.

This threshold-increasing effect of K is present for all values of C and its magnitude does not depend on the value of C. For small values of C, first, the expected payoff of the optimally completed investment program is not large in comparison to K for values of V that are near the investment threshold, and second, the absolute change in the value of the payoff during the time-to-build is on average small according to Eq. (20). Therefore, for small values of C, the investment threshold is increasing in K since the need to wait for V to reach such values that the expected payoff overcomes the initial investment dominates the investment-hastening effect of the expected growth of the payoff during the investment period.

In contrast, for larger values of C, the expected payoff of an optimally executed investment program is significantly larger than the initial investment and the expected absolute increase of the payoff during the investment program is substantial. Therefore, the effect of the expected growth of the payoff during the investment process dominates the effect of the initial investment and, thus, the investment threshold is actually decreasing in K for large values of C.

Finally, we will consider the cases where α_C decreases below zero, i.e., C is expected to decrease in the future. We can see from Figure 4 that $V^*(C, K = 0.01)$ will shift further up as α_C decreases. The effect on the thresholds at higher values of K is not as dramatic, however. We notice that, for example, $V^*(C, K = 6)$ stays the same as α_C changes from 0.00 to -0.10 and actually shifts down when α_C decreases further to -0.20. This happens as now C is expected to decrease in the future, whereas V is expected to increase as before. Therefore, when $K = 6$, it is optimal to start investing even if C is substantially larger than what it should be in order to exercise the option as $K \to 0$.

Our interpretation of the results above is that by starting to invest at larger values of K, the investor buys the right to be able to receive the payoff just as V and C reach such values that it is optimal to do so, rather than having to wait for the minimum time-to-build to receive the payoff once this happens. This interpretation justifies the observation that the investment threshold may be decreasing in K for large values of C since it is optimal to start investing even when the current value of the payoff is suboptimal in comparison to the threshold at $K \to 0$ if the payoff is expected to increase fast enough after the investment process begins. Also, the observation that $V^*(C, K = 6)$ shifts downwards as α_C decreases from -0.10 to -0.20 is then well explained by the fact that since C is expected to decrease at a larger rate when $\alpha_C = -0.20$, it is optimal to start investing at higher values of C given a value of V because the expected decrease of C is greater.

4.2.2 Sensitivity with respect to k

As our explanation for the results above relies on the logic that the investor holding the option considers both the expected evolution of $V - C$ during the investment period and the optimal investment policy at smaller values of K when making decisions on whether to invest or wait, we would assume that the results of the comparative statics above would be amplified for smaller values of k since this would imply a longer investment period. Consider for example the case where $\alpha_C < 0$ and C is expected to decrease while V is expected to increase. Now, if we decrease the maximum investment rate, then we expect that it is optimal to start investing at even higher values of C given a value of V since the minimum time-to-build is longer, thereby implying that the expected decrease of C during the investment period is larger as well. By generalizing the logic above, we would assume that an decrease in k would amplify the results of the comparative statics above. Motivated by this, we will next analyze the results of the same comparative statics as above, but using a smaller maximum investment rate $k = 0.5$. This doubles the minimum time-to-build for every value of K in comparison to the value $k = 1$ used above.

Figure 5 shows the results of the comparative statics with respect to α_C when $k = 0.5$ and the other parameters are the same as in the base case. We notice that the intuition above is correct as the smaller value of k amplifies the effects of α_C on the investment thresholds. Note that the investment thresholds are not affected by the change in the value of k when $K = 0.01$ since then the payoff can be received almost instantly. The reason why the other thresholds react more dramatically to changes in α_C than in the case above is that now the investor needs to look further ahead in time when making decisions for larger values of K as the minimum time-to-build is longer.

An interesting result occurs in the lower right case of Figure 5 where $\alpha_C = -0.20$. For large values of C and K, it is optimal to invest even if $V - C < 0$. However, this is well explained by the expected increase of $V - C$ during the investment program. Also, for each value of C, the NPV rule in this extreme situation is to invest at a smaller value of V than the ROV rule suggests. The observation applies generally: the ROV investment threshold is always larger than the now-or-never NPV threshold. This strengthens our explanation for why it might be optimal to invest even if the current value of the payoff is negative since the fact that the ROV threshold is larger than the NPV threshold in all situations ensures that the average value of the

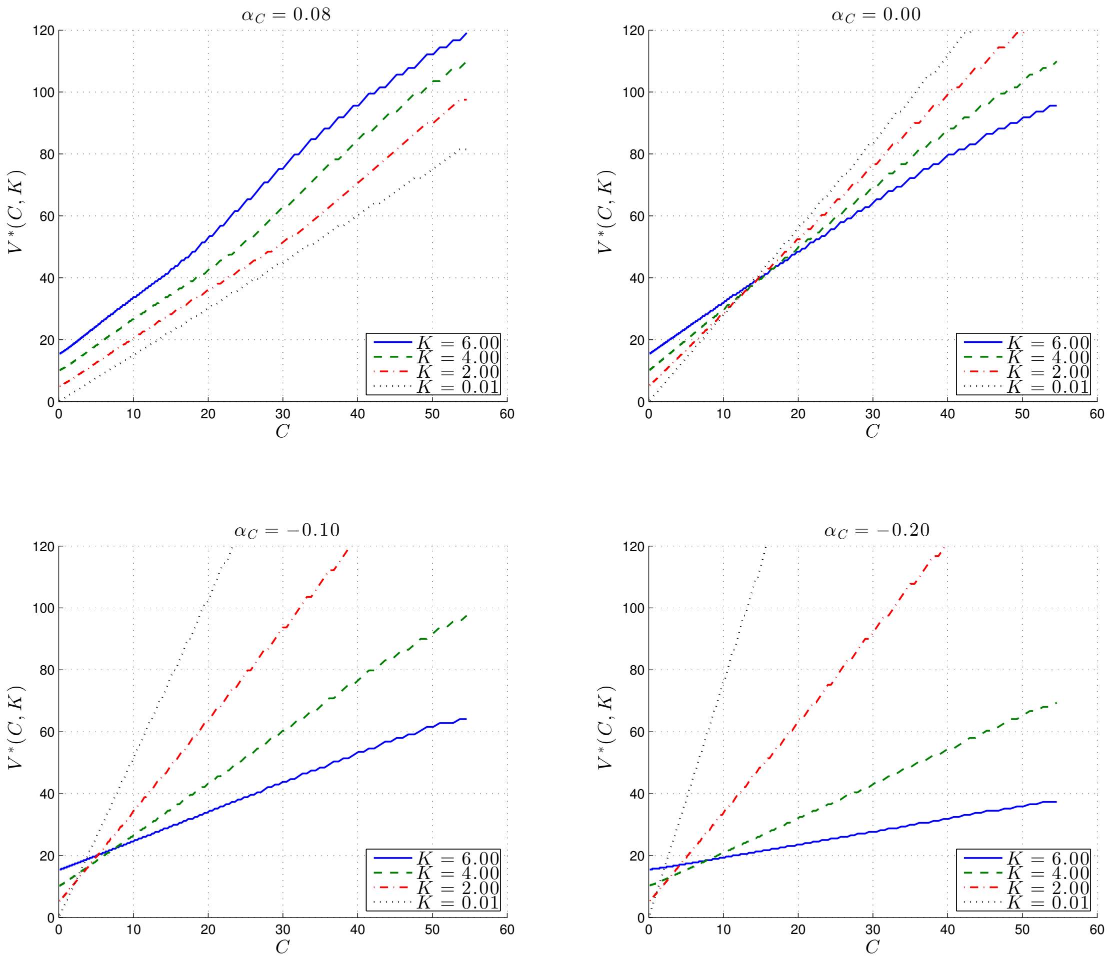

Figure 5: The sensitivity of the investment threshold with respect to α_C when $k = 0.5$

investment program executed by the ROV rule is positive in all cases.

Recall that Majd and Pindyck [1987] found the investment threshold to be increasing in K for all parameter values. On the contrary, in our two-factor model, the investment thresholds may be decreasing in K for certain values of C. What explains this difference? The answer is obvious: our model is built on different assumptions. Particularly, in our model there are two stochastic variables that determine the payoff whereas the model of Majd and Pindyck consists only of one. Therefore, the results are not completely comparable. This is also the explanation for why we find that in our model the investment threshold can be increasing in k, which might seem to contradict the results of Majd and Pindyck. However, this effect is well explained by the evolution of the payoff as seen above.

4.2.3 Sensitivity with respect to other parameters

To conclude the comparative statics, we will briefly discuss the effect of the other parameters as well. We shall start by discussing the parameters that determine the uncertainty in the situation, i.e., σ_V and σ_C. These parameters increase the volatility of the stochastic process that $V - C$ follows. Since our model is similar to the model of Majd and Pindyck [1987] in the sense that the investor is holding a compound option, an increase in the volatility of the payoff only increases the benefits of waiting and shifts the investment thresholds up in all cases. Also, if we assumed that the increments of V and C were correlated, then the effect of the correlation would be to increase or decrease the volatility of the payoff. If the correlation were positive (negative), then uncertainty would decrease (increase) shifting the investment thresholds down (up).[17]

The effect of α_V on the results is similar to that of α_C. An increase in α_V increases the benefits of waiting and shifts the investment threshold $V^*(C, K = 0.01)$ upwards. Again, the investment thresholds at larger values of K are located in a way that once the first initial investment is made, the investor will on average be able to invest continuously at the maximum rate up to the end of the investment program. We should also note that the investment thresholds grow without boundaries as $\alpha_V \to \mu$ (assuming that $\alpha_V > \alpha_C$) since then the long-term capital rate of return of the payoff converges to μ and the cost of waiting diminishes. Finally, as μ represents

[17] By using Eqs. (12a) and (12b), we obtain that in the case of a non-zero correlation, the stochastic part of the increment of the payoff has a variance of $\left(\sigma_V^2 V_t^2 - 2V_t C_t \sigma_V \sigma_C \rho + \sigma_C^2 C_t^2\right) dt$, where ρ is the correlation between dz_t and dw_t.

the cost of waiting in our model, the effect of an increase in μ is to shift the investment threshold down for all values of K and thus hasten investment.

5 Conclusions and discussion

The study proposes a method to compute the option value and the investment thresholds in a situation where the investor can sequentially invest in project opportunity, the payoff of which is a function of two stochastic variables. The sequential nature of the investment process is modeled by allowing the investor to choose at which rate to invest continuously in time. As the investment rate is assumed to be bounded between zero and a positive constant, the investor cannot obtain the payoff instantly but has to wait for at least a minimum time-to-build.

The results of the model are analyzed using the method of comparative statics. We find that when $K \to 0$, the investment threshold increases in α_V and decreases in α_C. For larger values of K, the placement of the investment thresholds depends on the expected stochastic evolution of the payoff $V - C$ alongside with the minimum time-to-build. As the value of k affects the minimum time-to-build, it affects the investment thresholds as well. We explain the results of the comparative statics by considering the investor's problem in the framework of dynamic programming. Uncertainty is found to postpone investment in all cases. Also, the ROV investment threshold is found to be always larger than the now-or-never NPV threshold given a value of C.

Some of the outcomes that the model yields might seem to be in contradiction with the earlier results of Dixit and Pindyck [1994]. However, the differences are explained by the fact that in our model the stochastic evolution of the payoff is different than in the model of Majd and Pindyck because of the inclusion of the second stochastic variable.

We chose V and C to represent the discounted cash in- and outflows of the completed project. We also assumed that the value of the completed project is $V - C$. However, the stochastic variables could have other interpretations depending on which particular investment situation is at interest. Also, the payoff could be generally any function of V and C in our framework.[18] In this sense, our model is general and can be used to analyze multiple invest-

[18]We do not take a stance on which conditions the payoff function should meet in order for the problem to have a solution.

ment situations that meet the assumptions made about the nature of the investment process and the stochastic variables.

One of the limiting assumptions behind the model is that the investor can decide on whether to invest or wait continuously in time. As discussed above, this assumption might be a rather good approximation in some situations. However, if the initial investment decision is completely irreversible, then our model is irrelevant. For example, the initial decision to build a coal power plant is practically completely irreversible and the construction time of the plant is substantial. Also, the major revenue and cost determinants of the power plant, i.e, the prices of electricity and coal, evolve stochastically in time. Therefore, the exercise of building and solving a two-factor model, in which the investment decision is modeled following the lead of Bar-Ilan and Strange [1996] could be interesting. In this case, the effect of the volatilities of the processes that V and C follow on the results could be the opposite than it is in our model. In addition, it would be interesting to see if the effect of the drift rates would be similar to our model since even if the initial investment decision is made completely irreversible, the investment lag implies that a rational investor considers how the payoff is expected to evolve during the lag when making investment decisions.

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A The numerical method

We first apply the transformation $F(V,C,K) = e^{-\mu\frac{K}{k}}G(X,Y,K)$, where $X = \ln V$ and $Y = \ln C$, to the PDE given by Eq. (15a) to modify the PDE to a simpler form and to ensure numerical stability. After the transformation, the PDE in R is:

$$\frac{1}{2}\sigma_V^2 G_{XX} + \frac{1}{2}\sigma_C^2 G_{YY} + (\alpha_V - \frac{1}{2}\sigma_V^2)G_X + (\alpha_C - \frac{1}{2}\sigma_C^2)G_Y - kG_K - ke^{\mu\frac{K}{k}} = 0. \tag{21}$$

Note that the coefficients of the PDE are now constant. After the transformation, the boundary conditions that solution for Eq. (21) must satisfy are:

$$G(X,Y,0) = e^{XY}, \tag{22a}$$

$$\frac{G(X^*(Y,K),Y,K)}{G_X(X^*(Y,K),Y,K)} = \frac{1}{\beta(K)}, \tag{22b}$$

$$\frac{G(X^*(Y,K),Y,K)}{G_Y(X^*(Y,K),Y,K)} = \frac{1}{\eta(K)}, \tag{22c}$$

where $\beta(K)$ and $\eta(K)$ solve Eq. (18) for each value of K.

Since we will solve the PDE numerically in a cubic grid, we need some additional boundary conditions that apply at the boundaries of the grid. For this purpose, we assume the following second-order boundary conditions:

$$\lim_{X\to\infty} G_{XX} = 0 \tag{23a}$$

$$\lim_{X\to-\infty} G_{XX} = 0 \tag{23b}$$

$$\lim_{Y\to\infty} G_{YY} = 0 \tag{23c}$$

$$\lim_{Y\to-\infty} G_{YY} = 0 \tag{23d}$$

These boundary conditions are chosen since they are known to work well with many financial options [Wilmott, 2007]. Also, these conditions were found to work in the case of our model. We will from now require that these conditions are approximately met at the boundaries of the lattice.

Let us denote $G(i\Delta X, j\Delta Y, l\Delta K) = G^l_{i,j}$, where $i \in \{i_{min}, i_{min}+1, ..., i_{max}\}$, $j \in \{j_{min}, j_{min}+1, ..., j_{max}\}$, and $l \in \{l_{min}, l_{min}+1, ..., l_{max}\}$. ΔX, ΔY, ΔK, and the minimum and maximum indices are predetermined constants that govern the dimensions of the lattice.[19]

[19] i_{min} and j_{min} will be negative in order to obtain option values near the zero-border in the (V,C)-world. The value of l_{min} will be zero.

We will use the following finite difference approximations for the partial derivatives of G:

$$G_X(i\Delta X, j\Delta Y, l\Delta K) = \frac{G^l_{i+1,j} - G^l_{i-1,j}}{2\Delta X}, \tag{24a}$$

$$G_Y(i\Delta X, j\Delta Y, l\Delta K) = \frac{G^l_{i,j+1} - G^l_{i,j-1}}{2\Delta Y}, \tag{24b}$$

$$G_{XX}(i\Delta X, j\Delta Y, l\Delta K) = \frac{G^l_{i+1,j} - 2G^l_{i,j} + G^l_{i-1,j}}{(\Delta X)^2}, \tag{24c}$$

$$G_{YY}(i\Delta X, j\Delta Y, l\Delta K) = \frac{G^l_{i,j+1} - 2G^l_{i,j} + G^l_{i,j-1}}{(\Delta Y)^2}, \tag{24d}$$

$$G_K(i\Delta X, j\Delta Y, l\Delta K) = \frac{G^{l+1}_{i,j} - G^l_{i,j}}{\Delta K}. \tag{24e}$$

By inserting the approximations above in the transformed PDE given by Eq. (21), we obtain the difference equation:

$$G^{l+1}_{i,j} = a_+ G^l_{i+1,j} + a_- G^l_{i-1,j} + b_+ G^l_{i,j+1} + b_- G^l_{i,j-1} + cG^l_{i,j} - n_l, \tag{25}$$

where

$$a_+ = \frac{\Delta K}{2k\Delta X}(\frac{\sigma_V^2}{\Delta X} + \alpha_V - \frac{\sigma_V^2}{2}), \tag{26a}$$

$$a_- = \frac{\Delta K}{2k\Delta X}(\frac{\sigma_V^2}{\Delta X} - \alpha_V + \frac{\sigma_V^2}{2}), \tag{26b}$$

$$b_+ = \frac{\Delta K}{2k\Delta Y}(\frac{\sigma_C^2}{\Delta Y} + \alpha_C - \frac{\sigma_C^2}{2}), \tag{26c}$$

$$b_- = \frac{\Delta K}{2k\Delta Y}(\frac{\sigma_C^2}{\Delta Y} - \alpha_C + \frac{\sigma_C^2}{2}), \tag{26d}$$

$$c = 1 - \frac{\sigma_V^2 \Delta K}{k(\Delta X)^2} - \frac{\sigma_C^2 \Delta K}{k(\Delta Y)^2}, \tag{26e}$$

$$n_l = \Delta K e^{\mu \frac{l\Delta K}{k}}. \tag{26f}$$

If the lattice point considered is on the lattice boundary, then we discretize the boundary conditions given by Eqs. (24a)-(24e). Then, the discretized boundary conditions can be inserted in Eq. (25) to compute the option value at the lattice point.

We shall now illustrate the idea of the computational method. First, we calculate the values of G when $l = 0$ using Eq. (22a). Then we calculate the values of option when $l = 1$ using Eq. (25) and the discretized versions of boundary conditions (24a)-(24e).

Now that we know the preliminary option values at $l = 1$, the next task is to find the investment threshold. For this we use boundary conditions (22b), (22c), and (18). By combining these conditions, the following equation must be met on the investment threshold:

$$\frac{1}{2}\sigma_V^2\frac{G_X}{G}(\frac{G_X}{G}-1)+\frac{1}{2}\sigma_C^2\frac{G_Y}{G}(\frac{G_Y}{G}-1)+\alpha_V\frac{G_X}{G}+\alpha_C\frac{G_Y}{G}-\mu=0 \qquad (27)$$

Our strategy is then to evaluate the left-hand side of the equation at every lattice point for $l = 1$ by using the finite difference approximations in Eqs. (24a) and (24b).[20] The location of the investment threshold given a value of j is then the pair (i, j), for which the absolute value of the left-hand side of Eq. (27) is the smallest in $i \in \{i_{min}+1, i_{min}+1, ..., i_{max}-1\}$.[21] After we have numerically solved the free boundary for $l = 1$, we can solve the values of constants $A(\Delta K)$, $\beta(\Delta K)$, and $\eta(\Delta K)$ using Eqs. (22b) and (22c), the value-matching condition in Eq. (16d), the made functional transformation, and the form of the analytical solution in the waiting region given by Eq. (17). We solve the values of these constants at each investment threshold for $l = 1$ and take the averages of these values to determine the final values.

Now that we have calculated the initial option values, the placement of the investment threshold, and the constants of the analytical solution in the lower region, we should fill the waiting region for $l = 1$ with the values given by the analytical solution before repeating the procedure above for $l = 2$. However, this as this proves to cause numerical instability, we update the option values after the initial option values and the investment thresholds have been determined for all values of l. We note that this is technically wrong as the numerical option values in the waiting region affect the option values in the investment region for larger values of l. However, as the results seem to be realistic, we allow this minor flaw.

Once the iteration above has been completed for all values of l and the option values in the waiting region are updated, the final solution for the investor's problem is obtained by using the functional and variable transformations in the opposite direction than what was initially done.

[20] The locations of the investment threshold at $l = l_{min}$ and $l = l_{max}$ are extrapolated.

[21] Here, we implicitly assume that the investment threshold is not at i_{min} or i_{max} for any value of Y or K. This assumption is met if the lattice dimensions are chosen properly.

B Summary in Finnish

Investointimahdollisuuksien arvon määrittäminen on yleinen ongelma niin yksityisissä kuin myös julkisissa organisaatioissa. Usein ongelmana on se, että päätöksentekijöiden tulee valita optimaaliset sijoitusmahdollisuudet kaikkien mahdollisuuksien joukosta ottaen huomioon budjettirajoite sekä mahdollisesti muita rajoitteita. Koska päätöksenteon kriteerinä on usein ainakin investointikohteiden arvo, on selvää että kohteiden arvon määrittäminen on oleellinen osa kuvailtua päätöksentekoprosessia. Arvon määrittämisen ohella on oleellista tietää myös onko kohteeseen sijoittaminen ylipäätänsä kannattavaa.

Nettonykyarvoanalyysi on eräs yksinkertainen tapa määrittää sijoitusmahdollisuuden arvo. Ideana on määrittää sijoitusmahdollisuuden tuottamien kassavirtojen nykyarvo diskonttaamalla kassavirtojen arvioidut odotusarvot käyttäen diskonttauskertoimena tuottovaatimusta, joka kuvaa sijoituskohteen riskiä. Investointimahdollisuuden nettonykyarvo on täten edellä mainittujen diskontattujen odotusarvoisten kassavirtojen summa. Kun nettonykyarvo on määritetty, saadaan yksinkertainen päätössääntö; investoi vain jos nettonykyarvo on ei-negatiivinen.

Nettonykyarvoanalyysillä on kuitenkin puutteensa. Ensiksi, se ei ota lainkaan kantaa investointimahdollisuuden tuottamien kassavirtojen yhteiseen todennäköisyysjakaumaan, sillä nettonykyarvon kaavassa tarvitaan vain tämän yhteisjakauman keskiarvoa. Toiseksi, mikäli kassavirtojen jakauma muuttuu ajassa, ei nettonykyarvoanalyysi ota huomioon sitä mahdollisuutta, että tällä hetkellä kannattamattomasta mahdollisuudesta voi tulla kannattava tulevaisuudessa. Jälkimmäinen seikka tarkoittaa sitä, että nettonykyarvoanalyysi arvioi kannattamattomien investointimahdollisuuksien arvon väärin, sillä yllä esitetyn argumentin mukaan myös näillä on arvoa, mikäli kassavirtojen jakauma kehittyy ajassa.

Reaalioptioanalyysi on vaihtoehtoinen keino mallintaa investointimahdollisuuksia. Reaalioptioanalyysin lähtökohtana on se, että reaalisia investointimahdollisuuksia mallinnetaan finanssioptioiden tavoin. Reaalisuudella tarkoitetaan tässä asiayhteydessä sitä, että sijoituskohde ei ole arvopaperi, vaan esimerkiksi tuotekehitysprojekti. Reaalioptioanalyysin viitekehyksessä esimerkiksi mahdollisuus rakentaa ydinvoimalaitos nähdään osto-optiona ydinvoimalaitokseen investointikustannuksen suuruisella toteutushinnalla. Tästä näkökulman muutoksesta seuraa kaksi asiaa. Ensinnäkin, vaikka sijoitusmahdollisuuden nettonykyarvo olisi negatiivinen, on sijoitusmahdollisuudella arvoa. Toiseksi, myös päätössääntö eroaa nettonykyarvoanalyysin vastaavasta.

Nimittäin, vaikka mahdollisuuden nettonykyarvo olisi positiivinen, ei ole välttämättä vielä optimaalista harjoittaa optiota ja tehdä sijoitusta. Syynä on se, että sijoituskohteesta voi tulla tulevaisuudessa vieläkin kannattavampi. On kuitenkin huomattava, että mikäli investointimahdollisuuden tuottamat kassavirrat ovat täysin deterministiset ja aikainvariantit, ovat reaalioptioanalyysin tulokset identtiset nettonykyarvoanalyysin tulosten kanssa. Reaalioptioanalyysin edut tulevat esiin siis tilanteissa, joissa investointikohteen arvo muuttuu ajassa.

Kandidaatintutkielmassa tarkasteltiin tilannetta, jossa investoijalla on mahdollisuus sijoittaa reaaliseen sijoituskohteeseen, jonka arvo riippuu kahdesta muuttujasta. Muuttujien oletettiin kehittyvän stokastisesti ajassa toisistaan riippumatta. Muuttujien tulkittiin kuvaavan valmiin investointikohteen diskontattuja tuloja ja menoja.

Koska useimmissa käytännön tilanteissa investointikohdetta ei voida rakentaa tai saada valmiiksi hetkessä, mallinnettiin investointiprosessi siten, että investoija voi sijoittaa mahdollisuuteen korkeintaan tietyllä määrällä aikayksikköä kohden. Malli rakennettiin jatkuvassa ajassa, jolloin oletettiin, että investoija voi säätää investointinopeutta niin ikään jatkuva-aikaisesti. Investointinopeuden oletettiin olevan rajoitettu nollan ja positiivisen vakion eli maksimaalisen investointinopeuden välille. Tästä seuraa loogisesti se, että jokaisella jäljellä olevan alkuinvestoinnin arvolla on olemassa minimirakennusaika eli aika, jonka päästä investointikohde olisi valmis olettaen, että jäljellä oleva alkuinvestointi maksetaan mahdollisimman nopeasti.

Tutkielman päämääränä oli rakentaa tilannetta mallintava reaalioptiomalli sekä ratkaista option arvofunktio ja optimaalinen päätössääntö, joka kertoo optimaalisen investointinopeuden kussakin tilanteessa. Tavoitteena oli erityisesti tutkia sitä, kuinka valmiin investoinnin kustannuksia kuvaavan muuttujan odotettu kehityskulku vaikuttaa investoijan optimaaliseen päätössääntöön. Koska saatu malli täytyi ratkaista numeerisin keinoin, oli osatavoitteena myös kehittää toimiva numeerinen ratkaisumenetelmä.

Rakennetun mallin tuloksena saatiin sekä option arvo että optimaalinen päätössääntö numeerisessa muodossa. Tulokset analysoitiin tarkastelemalla ensiksi esimerkkitapausta ja suorittamalla tämän jälkeen herkkyysanalyysi mallin parametrien suhteen.

Saatu päätössääntö on niin kutsuttu "bang-bang"-ohjaus; mikäli on ylipäätänsä optimaalista investoida, kannattaa investoida maksimaalisella nopeudella. Lisäksi päätössääntö on kaikilla jäljellä olevan alkuinvestoinnin arvoilla kasvava funktio kustannusmuuttujasta. Toisin sanottuna, mitä suuremmat ovat projektin arvioidut kulut tällä hetkellä, sitä suurempi tulee olla tulomuuttujan arvo, jotta olisi optimaalista investoida odottamisen sijaan.

Se, kuinka päätössääntö käyttäytyy jäljellä olevan alkuinvestoinnin funktiona, ei ole lainkaan triviaalia. Tämän vuoksi päätössäännön dynamiikka selitettiin kaksivaiheisesti. Ensiksi tarkasteltiin päätössääntöä silloin, kun alkuinvestointia on hyvin vähän jäljellä, jolloin minimirakennusaika on lyhyt. Tämän jälkeen pääteltiin kuinka päätössääntö kehittyy jäljellä olevan investoinnin kasvaessa soveltaen dynaamisen ohjelmoinnin periaatetta ja valmiin investointikohteen arvon odotusarvoista kehittymistä investointijakson aikana.

Herkkyysanalyysissa havaittiin, että mitä nopeammin tulomuuttujan tai mitä hitaammin kustannusmuuttujan odotetaan kasvavan tulevaisuudessa, sitä suurempi tulee tulomuuttujan arvon olla suhteessa kustannusmuuttujaan, jotta optio olisi optimaalista harjoittaa loppuun valmiin investointikohteen saamiseksi. Tulos kuvaa sitä, että mikäli alkuinvestointia on jäljellä vain vähän ja investointiprosessi voidaan saada halutessa nopeasti valmiiksi, ovat odottamisen hyödyt suuret suhteessa haittoihin. Teknisesti ottaen kyse on siitä, että option tuottovaatimuksen ja valmiin projektin odotusarvoisen arvonnousun välinen ero pienenee, jos tulomuuttujan kasvuvauhti kasvaa tai kustannusmuuttujan kasvuvauhti pienenee.

Päätössääntö suuremmilla jäljellä olevan alkuinvestoinnin arvoilla määräytyy sen mukaan, kuinka investointikohteen arvon odotetaan kehittyvän investointiprosessin aikana. Nyrkkisääntö on se, että alkuinvestointi kannattaa lykätä mahdollisimman kauas tulevaisuuteen, muttei kuitenkaan liian kauas ottaen huomioon päätössääntö pienillä jäljellä olevan investoinnin arvoilla. Perusteena tälle on se, että on parempi maksaa investointikulut huomenna kuin tänään. Tämä taas johtuu siitä, että rahavirrat diskontataan nykyhetkeen myös reaalioptioviitekehyksessä. Toisaalta, samasta syystä on epäedullista viivästyttää alkuinvestointeja liian kauan, sillä tällöin investointiprosessin valmistuminen viivästyy liikaa pienentäen investoinnin nykyarvoa.

Yllä mainitun nyrkkisäännön mielenkiintoinen implikaatio on se, että jos kustannusmuuttujan oletetaan pienenevän tulevaisuudessa riittävän nopeasti, on mahdollista, että suurilla jäljellä olevan investoinnin arvoilla on optimaalista investoida tilanteissa, joissa tämänhetkinen arvioitujen tulojen ja menojen erotus on negatiivinen. Tämä tulos selittyy hyvin sillä, että koska kustannusmuuttujan oletetaan laskevan tulevaisuudessa, kannattaa investoijan ennakoida ja alkaa investoimaan ajoissa, jotta investointikohde voidaan saada valmiiksi heti kun optio on optimaalista harjoittaa loppuun.

Mallin muuttujien kasvunopeuksien vaikutuksen ohella tutkittiin sitä, kuinka muuttujien tulevaisuuden arvojen keskihajonta, eli volatiliteetti, ja maksimaalinen investointinopeus vaikuttavat tuloksiin. Herkkyysanalyysin keinoin havaittiin, että muuttujien volatiliteetin kasvattaminen viivästyttää investointeja, sillä volatiliteetin kasvaessa odottamisen hyödyt kasvavat enemmän kuin haitat. Huomattiin myös, että pienemmillä maksimaalisen investointinopeuden arvoilla on optimaalista ennakoida enemmän muuttujien odotettua aikakehitystä. Tulos selittyy sillä, että pienempi investointinopeus johtaa pidempiin investointiaikoihin.

Yksi mallin rajoittavimmista oletuksista on se, että investoijan oletettiin kykenevän päättämään investointinopeudesta kaikkina ajanhetkinä. Tämä johtaa erityisesti siihen, että investoija voi pysäyttää investointiprosessin milloin tahansa. On selvää, että useimmiten tositilanteissa näin ei ole. Mikäli investointiprosessi on melko joustava siinä suhteessa, että investointiprosessin jatkamista voidaan lykätä prosessin eri vaiheiden välissä tai peräti vaiheiden aikana, on rakennettu malli kohtalainen approksimaatio tilanteesta. Kuitenkin, jos investointipäätös on kokonaisuutena peruuttamaton, eli positiivinen päätös johtaa koko investoinnin toteuttamiseen kerrallaan, on malli epärelevantti. Tämän vuoksi olisi mielenkiintoista rakentaa vastaava malli tilanteessa, jossa investointipäätös kokonaisuudessaan on tehtävä kerrallaan. Tällöin esimerkiksi muuttujien volatiliteettien vaikutus investointisääntöön voisi olla päinvastainen. Lisäksi olisi mielenkiintoista nähdä, että olisiko muuttujien kasvunopeuksien ja päätössäännön välinen dynamiikka tutkielman tuloksia vastaava.