Department of Mathematics and Systems Analysis

The principle of least action and stochastic dynamic optimal control

- Applications to economic, financial and physical systems

Jussi Lindgren

Aalto University

DOCTORAL DISSERTATIONS

Aalto University publication series
DOCTORAL DISSERTATIONS 136/2021

The principle of least action and stochastic dynamic optimal control

- Applications to economic, financial and physical systems

Jussi Lindgren

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Science, at a public examination held at the lecture hall H304 of the school on 17.12.2021 at 12 noon.

The public defence will be organized also via remote technology.
Link: https://aalto.zoom.us/j/63597221776
Zoom quick guide: https://www.aalto.fi/en/services/zoom-quick-guide

Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervising professor
Professor Ahti Salo, Aalto University School of Science, Finland

Thesis advisors
PhD Jukka Liukkonen, Radiation and Nuclear Safety Authority, Finland
Professor Ahti Salo, Aalto University School of Science, Finland

Preliminary examiners
Professor Abel Cadenillas, University of Alberta, Canada
Professor Gregorios A. Pavliotis, Imperial College London, United Kingdom

Opponent
Professor Gregorios A. Pavliotis, Imperial College London, United Kingdom

Aalto University publication series
DOCTORAL DISSERTATIONS 136/2021

ISBN 978-952-64-0537-7 (printed)
ISBN 978-952-64-0538-4 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-64-0538-4

Unigrafia Oy
Helsinki 2021

Finland

Printed matter
4041-0619

Aalto University

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

Abstract

Author
Jussi Lindgren

Name of the doctoral dissertation
The principle of least action and stochastic dynamic optimal control - Applications to economic, financial and physical systems

Publisher School of Science

Unit Department of Mathematics and Systems Analysis

Series Aalto University publication series DOCTORAL DISSERTATIONS 136/2021

Field of research Systems and Operations Research

Manuscript submitted 19 May 2021 **Date of the defence** 17 December 2021

Permission for public defence granted (date) 9 September 2021 **Language** English

☐ **Monograph** ☒ **Article dissertation** ☐ **Essay dissertation**

Abstract

Economic and financial systems as well as the physical laws of nature can be studied within a common mathematical framework. In particular, the principle of least action and stochastic optimal control can be applied both to resource allocation problems within the society, as well as to derive physical laws. In economic and financial systems, optimal performance is vital, given that economic policies affect all citizens and general welfare. It is also paramount to try to understand the mathematical structure of efficient financial markets. Both these issues are discussed in this Dissertation.

First, a stochastic optimal control model is developed to model the dynamics of public debt. In such a dynamical model of public debt, the variance of the debt to GDP ratio is determined in order to assess the risk of insolvency. The model demonstrates also the risks stemming from various feedback mechanisms due to hidden fiscal multipliers and hidden credit risk premia. The model is potentially useful for finance ministries and national debt managers and investors alike.

Second, stochastic optimal control is used to derive the key pricing equation from finance theory as an optimality condition for the financial market to be informationally efficient. With such assumptions a nonlinear transport equation is derived for the market instantaneous returns. The model could be used to predict average returns on various assets. Thus the model could be useful for asset managers and investment professionals.

Third, it is shown how the key equations of quantum mechanics can also be derived as an optimality condition, when there is background noise stemming from the spacetime fluctuations at small scales. Furthermore, the Heisenberg uncertainty principle is derived from the stochastic optimal control model.

Finally, the field equations of electromagnetism are derived from a least action principle and it is shown how Maxwell's equations relate to the Einstein field equation. In particular, the link of electromagnetism and spacetime curvature could be tested empirically in principle and the results could facilitate further engineering applications.

The results indicate that strive for efficiency is abundant in natural as well as in economic and financial systems and that the principle of least action is even more omnipresent and important than previously has been known.

Keywords principle of least action, stochastic optimal control, public finance, derivatives pricing, quantum mechanics, general relativity

ISBN (printed) 978-952-64-0537-7 **ISBN (pdf)** 978-952-64-0538-4

ISSN (printed) 1799-4934 **ISSN (pdf)** 1799-4942

Location of publisher Helsinki **Location of printing** Helsinki **Year** 2021

Pages 105 **urn** http://urn.fi/URN:ISBN:978-952-64-0538-4

Aalto-yliopisto

Aalto-yliopisto, PL 11000, 00076 Aalto www.aalto.fi

Tiivistelmä

Tekijä
Jussi Lindgren

Väitöskirjan nimi
Pienimmän vaikutuksen periaate ja stokastinen dynaaminen optimisäätö - Sovelluksia talous- ja finanssijärjestelmiin ja fysikaalisiin systeemeihin

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

Sarja Aalto University publication series DOCTORAL DISSERTATIONS 136/2021

Tutkimusala Systeemi- ja operaatiotutkimus

Käsikirjoituksen pvm 19.05.2021 **Väitöspäivä** 17.12.2021

Väittelyluvan myöntämispäivä 09.09.2021 **Kieli** Englanti

☐ **Monografia** ☒ **Artikkeliväitöskirja** ☐ **Esseeväitöskirja**

Tiivistelmä
Talous- ja finanssijärjestelmiä sekä luonnon fysikaalisia lakeja voidaan tarkastella samassa matemaattisessa viitekehyksessä. Erityisesti pienimmän vaikutuksen periaatetta ja stokastista optimisäätöä voidaan soveltaa niin resurssien jakamiseen yhteiskunnassa kuin fysikaalisten lakien johtamiseen. Talous- ja finanssijärjestelmissä optimaalinen suorituskyky on tärkeää, koska talouspolitiikat vaikuttavat kansalaisiin ja yleiseen hyvinvointiin. On myös erityisen tärkeää yrittää ymmärtää tehokkaiden markkinoiden matemaattista rakennetta. Tässä väitöskirjassa käsitellään näitä molempia asiakokonaisuuksia.
Ensimmäiseksi, stokastista optimisäätömallia hyödynnetään julkisen velan dynamiikan mallintamisessa. Saadun dynamiikan perusteella määritetään julkisen velan BKT-suhteelle varianssi valtion maksukyvyttömyysriskin arvioimiseksi. Malli osoittaa myös, kuinka piilotetut luottoriskipreemiot ja piilotetut finanssipolitiikan kertoimet vaikuttavat riskeihin takaisinkytkentämekanismien kautta. Malli on mahdollisesti hyödyllinen valtiovarainministeriöille ja velkatoimistoille, sekä sijoittajille.
Toiseksi stokastisella optimisäädöllä johdetaan rahoitusteorian keskeinen hinnoitteluyhtälö optimaalisuuskriteerinä sille, että rahoitusmarkkina on informaatiomielessä tehokas. Mallin oletuksista johdetaan epälineaarinen kuljetusyhtälö markkinan hetkellisille tuotoille. Mallia voitaisiin hyödyntää markkinan keskimääräisten tuottojen ennustamisessa erilaisille rahoitusvaateille. Se voisi olla hyödyllinen varainhoitajille ja sijoitusammattilaisille.
Kolmanneksi, kvanttimekaniikan keskeiset yhtälöt johdetaan optimaalisuusehtona, kun oletetaan taustakohina aika-avaruuden heilahteluista pienillä mittakaavoilla. Lisäksi Heisenbergin epätarkkuusperiaate johdetaan tästä stokastisesta optimisäätömallista.
Lopuksi, sähkömagnetismin kenttäyhtälöt johdetaan pienimmän vaikutuksen periaatteesta ja näytetään, miten Maxwellin yhtälöt liittyvät Einsteinin kenttäyhtälöön. Erityisesti sähkömagnetismin ja aika-avaruuden kaarevuuden välinen yhteys voitaisiin testata empiirisessä koeasetelmassa ja tuloksilla voi olla merkitystä uusien teknisten sovellutusten kehittämiseksi.
Tulokset viittaavat siihen, että pyrkimys tehokkuuteen on hyvin yleinen niin luonnollisissa, kuin talous- ja finanssijärjestelmissä ja että pienimmän vaikutuksen periaate on jopa vielä enemmän kaikkialla läsnä olevampi ja tärkeämpi, kuin aiemmin on ymmärretty.

Avainsanat pienimmän vaikutuksen periaate, stokastinen optimisäätö, julkistalous, rahoitusvaateiden hinnoittelu, kvanttimekaniikka, yleinen suhteellisuusteoria

ISBN (painettu) 978-952-64-0537-7 **ISBN (pdf)** 978-952-64-0538-4

ISSN (painettu) 1799-4934 **ISSN (pdf)** 1799-4942

Julkaisupaikka Helsinki **Painopaikka** Helsinki **Vuosi** 2021

Sivumäärä 105 **urn** http://urn.fi/URN:ISBN:978-952-64-0538-4

Preface

Doing scientific research over these last 5 years has been a remarkable adventure. I would like to take this opportunity to express my greatest gratitude to professor Ahti Salo, as this work was realized due to his belief that the author would deserve a chance to complete this journey. I also thank prof. Salo for his numerous suggestions and support during the process. Warmest thanks also to Dr. Jukka Liukkonen for advising me during the process. I also thank Dr. Kimmo Soramäki, who induced me to consider doing doctoral studies at Aalto University - coming back home in a sense, since I started my intellectual journey as a student in the department of Engineering Physics and Mathematics in 2001. Thanks to Dr. Antti Savinainen and MSc, MDiv Rauno Perälä for their excellent didactical skills and support at Kuopion Lyseon lukio. Ultimately doing my master's in economics, it was a fruitful route to be inspired from both physics and economics. I was introduced to optimal control in 2009, when I attended the Moscow State University Summer School. Numerous discussions I have had on mathematics with MSc Vili Aapro have been great in terms of inspiration. I would like to also thank Dr. Elena Rovenskaya (International Institute for Applied Systems Analysis) for her valuable comments. I am also in infinite debt to my dear spouse Leena, who has endured me during these not always easy years. Writing a Dissertation has been something very interesting during the COVID-19 pandemic. As the present Dissertation argues for the ingeniousness of nature, I hope that there is some teleological meaning as regards the pandemic as well, even though we may not realize it now. I believe there is always a certain duality in things - maybe something good will come out of this pandemic as well. I believe that growing human knowledge is paramount in order to progress and move forward - science has intrinsic as well as extrinsic value. I dedicate this Dissertation to my dear Leena and to my father and my mother, Rauno and Iris.

Helsinki, September 30, 2021,

Jussi Lindgren

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Lindgren J. Examination of Interest-Growth Differentials and the Risk of Sovereign Insolvency. *Risks*, 9(4),75, April 2021.

II Lindgren J. Efficient Markets and Contingent Claims Valuation: An Information Theoretic Approach. *Entropy*, 22(11),1283, November 2020.

III Lindgren J, Liukkonen J. Quantum Mechanics can be Understood Through Stochastic Optimization on Spacetimes. *Scientific Reports*, 9, 19984, December 2019.

IV Lindgren J, Liukkonen J. The Heisenberg Uncertainty Principle as an Endogenous Equilibrium Property of Stochastic Optimal Control Systems in Quantum Mechanics. *Symmetry*, 12(9),1533, September 2020.

V Lindgren J, Liukkonen J. Maxwell's Equations from Spacetime Geometry and the Role of Weyl Curvature. *Journal of Physics: Conference Series*, 1956, July 2021.

Author’s Contribution

Publication I: “Examination of Interest-Growth Differentials and the Risk of Sovereign Insolvency”

Lindgren is the sole author.

Publication II: “Efficient Markets and Contingent Claims Valuation: An Information Theoretic Approach”

Lindgren is the sole author.

Publication III: “Quantum Mechanics can be Understood Through Stochastic Optimization on Spacetimes”

Lindgren is the primary author. Lindgren initiated this project and developed the concepts related to the invariance and spacetime diffusion. Lindgren and Liukkonen jointly wrote and revised the entire paper. Lindgren and Liukkonen read and approved the final paper.

Publication IV: “The Heisenberg Uncertainty Principle as an Endogenous Equilibrium Property of Stochastic Optimal Control Systems in Quantum Mechanics”

Lindgren is the primary author. Lindgren invented the link between the stationary distribution and the uncertainty principle through the linear–quadratic optimization program; Liukkonen edited the paper and wrote the discussion and conclusions. Both authors reviewed and approved the paper.

Publication V: "Maxwell's Equations from Spacetime Geometry and the Role of Weyl Curvature"

Lindgren is the primary author. Lindgren developed the link between the electromagnetic four-potential and metric tensor and derived the field equations. Liukkonen drew the figures and tables and wrote the discussion. Both authors reviewed the manuscript.

List of Figures

List of Tables

Abbreviations

QM Quantum Mechanics

PDE Partial Differential Equation

HJB Hamilton-Jacobi-Bellman

SDE Stochastic Differential Equation

GR General Relativity

EM Electromagnetism

1. Introduction

"Nihil est sine ratione." - Gottfried Wilhelm Leibniz

Physical systems seem to aim to reach some optimal configuration in order to be as efficient as possible. Virtually all laws of classical physics can be formulated from this variational principle, see [41] and [4]. On the other hand, in modern economics and finance theory, dynamic programming is one of the key tools of contemporary modeling. Constrained optimization is widely employed in our society and efficiency considerations are central both in natural, as well as in man-made systems. In natural science, the *principle of least action* is concerned with energy minimization, whereas in economics, usually some utility functional is maximized. In physics, one main benefit of variational principles is that one can embed coordinate-invariance naturally in such optimization models, [27]. Physics can be seen as nature's economics, and vice versa [30], [31].

1.1 Origins of the principle of least action, calculus of variations, optimal control theory and dynamic programming

Calculus of variations, optimal control theory and dynamic programming can be traced back to antiquity. Already around 300 BC, Euclid considers the minimum distance from a point to a line. Archimedes (287-212 BC) demonstrated the law of the lever and set up axioms for equilibrium for floating bodies. He can be called the first classical mechanic. The mathematician Heron of Alexandria (c. 10-70 AD) considers in Catoptrica the minimum distances traveled by light between two mirrors. Johannes Kepler (1571-1630) considered the secretary problem, which is a classical problem in dynamic programming. Galileo Galilei (1564-1642) tried to determine the shape of hanging chains (1638), but failed. René Descartes (1596-1650), who considered nature as a machine, can be seen as an early advocate of cybernetics. Cybernetics as a discipline was later proposed by Norbert Wiener in the 20th century, see [47].

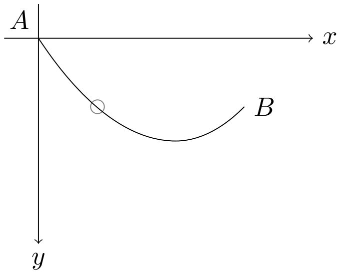

Figure 1.1. What kind of a curve would minimize the time for a bead to reach point B from point A?

G.W. Leibniz (1646-1716) and Pierre de Fermat (1601-1665) were key figures in the intellectual development of calculus of variations. Leibniz's concept of *best of all possible worlds* is immensely relevant in its development, as calculus of variations has a natural link with the concept, although Leibniz considered it primarily in a theological-philosophical context, not solely in the context of mathematical programming. His concept of *vis viva* was the first pre-modern conceptualization of kinetic energy and the law of conservation of energy.

In 1657, Pierre de Fermat showed that light travels between two points in minimal time or more precisely that in a medium with variable velocity, light prefers the path that guarantees the minimal time. This, together with Johann Bernoulli's (1667-1748) brachistochrone problem (1696) and Isaac Newton's (1642-1726) problem of minimal resistance (1687) were prime movers for further developments. The actual birth of calculus of variations can be naturally linked with the brachistochrone problem, see Fig.1.1. Christopher Huygens' (1629-1695) principle of wave diffraction had repercussions as well, especially on Hamilton-Jacobi theory and Pontryagin's maximum principle. Although Isaac Newton is the giant of classical mechanics with his three famous laws, the most interesting parallel developments of calculus of variations and classical mechanics took place only in the 18th century.

The *principle of least action* has been a cornerstone of classical mechanics since the publication of *Les Lois du movement et du repos d 'eduites d'un principe metaphysique* by Pierre Louis Moreau de Maupertuis (1698-1759) in 1746. This principle says that a particle will take the route which will minimize a certain functional, called the action functional. As is well documented, Maupertuis was not the first scholar to consider this principle; it is manifested already in the teleological thinking of Aristotle and later in the works of Fermat and in particular the thinking of Leibniz. For the history of the principle of least action and variational principles, see [41]

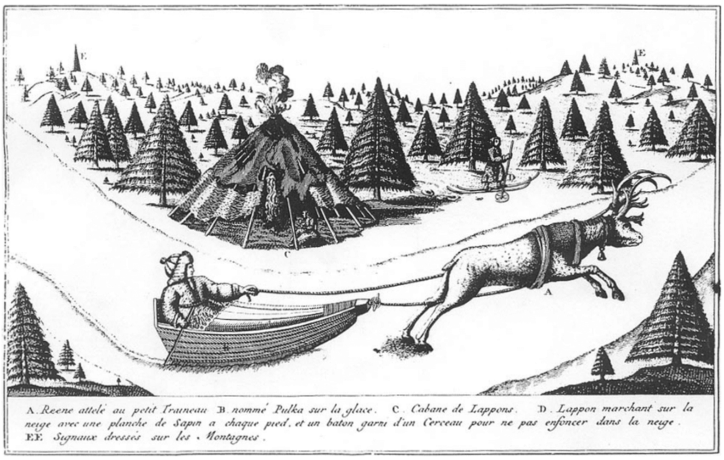

Figure 1.2. Maupertuis on his expedition in Tornio valley

and [4]. There is an ongoing debate on the metaphysics and the possible ontological meaning of the principle, see for example [45].

Constrained optimization is used pervasively. In modern economics, from game theory to dynamic stochastic general equilibrium modeling, optimization is the fundamental assumption of methodological individualism, [30] and [31]. Nature also seems to be rational in this sense. The principle of least action was put onto a solid foundation with the development of calculus of variations and analytical mechanics. Calculus of variations is concerned with the task of finding a minimum or a maximum for some bounded functional over a set of functions satisfying some constraints. The well-known necessary stationarity condition for such problem is the celebrated *Euler-Lagrange equation*. In the physical context, calculus of variations stipulates, that given a Lagrangian, which is the difference of kinetic and potential energies, the necessary stationarity condition requires that a test particle will follow Newton's second law of motion. In contemporary physics education, a course in analytical mechanics highlights usually three approaches to mechanics; Lagrangian mechanics, Hamiltonian mechanics and the Hamilton-Jacobi approach. The first approach is the approach considering the Euler-Lagrange necessary conditions, the second one is a dual approach to Lagrangian mechanics (the Hamiltonian is the Legendre dual of the Lagrangian) and the third one is the most interesting and fruitful according to the author of the Dissertation, as it has the most natural link to dynamic programming and optimal control theory. In classical field theories, calculus of variations can be utilized to derive Maxwell's equations in electromagnetism and Einstein equations in General Relativity. Furthermore, Richard Feynman's approach to quantum mechanics makes use of the principle of least action as well [17].

While optimization and calculus of variations were developed already during the 18th century, control theory and systems engineering began to develop in the latter part of the 19th century thanks to the works of J.C. Maxwell (1831-1879), E.J. Routh (1831-1907) and A. Hurwitz (1859-1919). Control systems engineering is generally about trying to affect a given system with through some means to achieve desirable effects, over time. Therefore, control engineering is a rather general paradigm of intentional behavior. Control engineering is an important enabler of the Industrial Revolution itself, as for example J. Watt's (1736–1819) steam engine is a control system through the operation of the engine's valves and thus its pressure regulation. Indeed, control theory and control engineering are omnipresent in a modern society, be it aerospace engineering, heating, air-conditioning or industrial process control.

From the perspective of this Dissertation, especially relevant is the concept of feedback control. In feedback control systems, the control of the system depends in particular on the current state of the system. Feedback is an essential element of cybernetics, which is a research strand developed by Norbert Wiener (1894-1964). Cybernetics was especially popular in the former Soviet Union, where control engineering and systems analysis was applied to virtually everything within the society, see [22]. Optimal control theory is an engineering approach, where there is an explicit objective to be minimized, subject to the dynamical system at hand.

Optimal control can be given by an explicit feedback rule. In the late 1950's the Russians and the Americans advanced optimal control theory simultaneously. In the USSR, Lev S. Pontryagin (1908-1988) and his group developed the celebrated Pontryagin's maximum principle by 1956, whereas Richard Bellman (1920-1984) was developing the paradigm of dynamic programming in the USA.

In optimal control theory and dynamic programming, the controller is trying to control some dynamical system in order to minimize some cost functional. The research motivations were driven by the need to understand how to control some technical or mechanical systems, such as weapons systems, missiles, aircraft or space stations. The Cold War was probably a key motivating factor in this respect [38]. Stochastic optimal control theory or stochastic dynamic programming is another extension to the tradition, enabled by the celebrated Itô's lemma and the invention of stochastic calculus and rigorous treatment of Wiener processes. Dynamic programming and control theory have been utilized also more and more in artificial intelligence and machine learning (reinforcement learning) research during the last thirty years, [6].

The relationship of Richard Bellman's dynamic programming [5] and Lev Pontryagin's maximum principle [8] is analogous to Hamilton-Jacobi theory and Hamilton's equations. In dynamic programming, the optimal value for the objective functional obeys the Hamilton-Jacobi-Bellman (HJB) partial differential equation, and Pontryagin's maximum principle essentially describes the characteristics of this PDE. The Hamilton-Jacobi-Bellman PDE is close to the adjoint equation in Pontryagin's approach, because if the value function is smooth, the (negative) gradient of the value function is the adjoint variable or the costate. Pontryagin's approach is more efficient in deterministic problems, whereas Bellman's approach can be more easily generalized into a stochastic setting where one tries to optimally control some stochastic differential equation. The functional to be minimized then becomes a random variable so that the decision-making problem transforms into a problem, where the conditional expectation is to be maximized or minimized. Stochastic optimal control theory is nowadays well established with many applications especially in for example finance, see the seminal paper [29], and in macroeconomics, [9]. Moreover, (stochastic) optimal control theory has been applied in differential game theory, see the book [13]. On the relationship of dynamic programming and the maximum principle, see the book [49].

1.2 Objectives of the Dissertation

The main objective of this Dissertation is to illustrate the potential of calculus of variations and principle of least action and dynamic optimization more generally as a tool to model various natural and man-made systems. This Dissertation thus consists of a collection of applications of the principle of least action and dynamic programming in physics, economics and finance.

The usefulness of stochastic optimal control theory in macroeconomics is demonstrated in Publication I, where public debt sustainability is analyzed using the tools of linear-quadratic control theory. Publication I highlights how, with minimal assumptions, the debt dynamics for a sovereign can become nonlinear and therefore potentially even chaotic.

The results in Publication II show the power of stochastic optimal control in finance theory. Publication II shows how the celebrated Black-Scholes equation can be seen as an optimality condition for the financial market when the market as a system tries to be as efficient as possible. A systemic approach to financial theory is therefore advocated.

In Publications III-IV stochastic optimal control is utilized to derive the fundamental equations of quantum mechanics from an alternative perspective. In particular, the approach demonstrates the links between stochastic analytical mechanics, statistical mechanics, and relativity. The aim of these Publications is to show how the imaginary structure of quantum mechanics can be understood by considering stochastic mechanics in a coordinate invariant fashion. The key hypothesis is that the randomness in the movement of the test particle can be understood as if the spacetime was randomly fluctuating at small scales. Furthermore in Publication IV, the Heisenberg uncertainty principle is explored and analyzed in the context of the model presented in Publication III.

Finally, in Publication V, the principle of least action is utilized in deriving the field equations of electromagnetism purely from the geometric properties of the spacetime. The approach makes use of the Lagrangian formulation of General Relativity. In Publication V, it is shown how two of Maxwell's equation are hidden within the Einstein field equation, when one identifies the metric of the spacetime with a simple coupling of the electromagnetic four-potential. Publication V therefore tries to offer more insight on electromagnetism and especially on the nature of the electromagnetic four-potential, from which the Faraday tensor derives.

2. Methodological Foundations

"All stable processes we shall predict. All unstable processes we shall control." - John von Neumann (1948)

2.1 The principle of least action and calculus of variations

The overarching theme of this Dissertation is the principle of least action and dynamic optimization. Generally speaking, the principle of least action suggests that a system acts in a way that makes a certain functional stationary. Such a simple action functional could be for example of the form

$$\int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})dt, \tag{2.1}$$

with some fixed $x(t_1)$ and $x(t_2)$. The problem is to then find a function $x(t)$ in such a way that the integral is minimal. Formally, in the simple situation, for example we may require as a necessary condition that the variation of the action integral vanishes

$$\delta \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})dt = 0. \tag{2.2}$$

The notation δ means that the functional variation of the action should vanish. Formally, the necessary condition stipulates that the Fréchet differential is to vanish at optimal x, see [28]. Thus in the abstract formulation, we want to find a function $x(t)$ in such a way that the integral is stationary (usually we seek a minimum or a maximum). In classical mechanics this requirement is called *Hamilton's principle*. The integrand may depend on the derivatives of arbitrary order of the function $x(t)$ as well as on the independent variables, such as t here. The principle of least action is therefore mathematically a problem in calculus of variations. The theory of dynamic programming and optimal control can be seen as an extension to calculus of variations.

For a simple problem, where the integrand $\mathcal{L}$ depends only on the time derivative $\dot{x}(t)$ and the function $x(t)$ itself, the stationarity condition implies, using functional variation, that such an optimal function $x(t)$ must satisfy the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right). \tag{2.3}$$

Calculus of variations was developed during the 18th century by Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813), but variational principles were developed already during the 17th century especially by Pierre de Fermat (1601-1665). In 1745, Pierre-Louis Moreau de Maupertuis (1698-1759) stated the principle of least action in mechanics. Classical or analytical mechanics is thus based on the *principle of least action*, where the Lagrangian $\mathcal{L}$ in the problem of calculus of variations above is the difference of kinetic and potential energies of the test particle. Equation 2.3 is then just Newton's II law of motion. The principle of least action and calculus of variations can be extended to field theories (field theories involve multiple integrals and volume elements), see [42].

In order to appreciate the link of classical mechanics and optimal control theory, we may consider the Hamiltonian function $\mathcal{H}$, which is the Legendre transformation [49] of the Lagrangian:

$$\mathcal{H} =< \dot{x}, p > - \mathcal{L}, \tag{2.4}$$

with the generalized momenta or conjugate variable $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$. The brackets notation $<,>$ refers to an inner product (dual pairing) of vectors. In Hamiltonian mechanics the laws of motion are then:

$$\begin{cases} \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} \\ \dot{x} = \frac{\partial \mathcal{H}}{\partial p}. \end{cases} \tag{2.5}$$

The pair of differential equations above 2.5 are called *Hamilton's canonical equations*. These equations are equivalent to the Euler-Lagrange equations and in many cases they are easier to solve than the corresponding Euler-Lagrange equations.

William Rowan Hamilton (1805-1865) and Carl Gustav Jacobi (1804-1851) showed that the problem of classical mechanics can be transformed to solving a first-order nonlinear partial differential equation. This Hamilton-Jacobi equation is of the following form, a first-order PDE

$$\frac{\partial S}{\partial t} + \mathcal{H}(t, x, \frac{\partial S}{\partial x}) = 0. \tag{2.6}$$

The function S is called Hamilton's principal function and it relates directly to the action functional 2.1, see [49]. In dynamic programming, the analogue of the Hamilton-Jacobi PDE is the Hamilton-Jacobi-Bellman equation and Pontryagin's maximum principle is analogous to Hamilton's equations, see [49]. In view of the above considerations, the Hamiltonian function can be understood as the rate at which the action integral changes. This intuition helps to understand why in optimal control theory we must make the Hamiltonian stationary with respect to the control function. In a stochastic setting, the Hamilton-Jacobi-Bellman PDE is a second-order nonlinear PDE due to Itô calculus.

2.2 Riemannian geometry

In Publication V, the mathematical framework is that of differential geometry using tensor calculus. Differential geometry is also more generally a useful tool in control theory and classical mechanics, see [1] and [2].

Tensors are objects, which transform in a certain way under coordinate transformations, take inputs of *vectors and covectors* and return real numbers in a linear manner. A tensor is therefore a multi-linear map. A simple example of a tensor can be understood through quadratic forms in linear algebra. In tensor calculus, upper and lower indices of an object have different meanings. The lower indices represent covariant components and upper indices represent contravariant components of a tensor. Contravariant components and covariant components transform in a different manner. The key idea of tensor calculus is coordinate-invariance, for example, if a quadratic form pairs a vector and a covector, and if one is to change coordinates, the matrix must change as well, to cater for the different representation of the vectors and covectors in a new coordinate system.

The basic object of Riemannian geometry is the metric tensor $g_{\mu\nu}$, computing inner products on a general manifold M. The idea is that if there are for example two contravariant vectors A^{μ} and A^{ν} (are also tensors), the object $g_{\mu\nu}A^{\mu}A^{\nu}$ is a scalar, the inner product of the two vectors. The metric tensor thus defines the length of vectors. The Einstein summation convention means that upper and lower indices are to be summed over, if they have the same symbol, as above. In a Euclidean space, the metric tensor would be represented by a matrix with 1s on the diagonal. On a general curved manifold, the entries of the metric tensor depend on the position on the manifold. A Lorentzian spacetime is a four-dimensional (pseudo-)Riemannian manifold, which looks locally like a Minkowski spacetime. A Minkowski spacetime is a flat spacetime, whose metric can be represented as a four-by-four matrix with the diagonal $-1, 1, 1, 1$.

As on a general curved manifold the metric varies depending on the position, a *covariant derivative* ∇ is needed for derivatives to transform as tensors on the manifold. A tensor itself is a geometrical object, which transforms under coordinate transformations in a certain manner. Scalars are invariant under coordinate transformations. The covariant derivative is the partial derivative of a tensor, accompanied with a correction term to cater for the locally varying metric, this correction term is called the Christoffel connection $\Gamma^{\lambda}_{\mu\nu}$, but is not a tensor, merely a collection of coefficients or symbols, [44]. The Christoffel symbols are obtained from the partial derivatives of the metric tensor. In the present analysis, we consider only what is called the Levi-Civita connections, that is connections which are torsionless (a symmetry property of the Christoffel symbols) and metric compatible (the covariant derivative of the metric vanishes). In a physical setting, the Christoffel symbols are analogous to forces and the metric is analogous to potentials.

It is essential to understand from Riemannian geometry in the context of General Relativity that a *curvature tensor* can be constructed from the second derivatives of the metric tensor. A curvature tensor $R^{\lambda}_{\nu\sigma\mu}$ tells how parallel transport of vectors around a closed loop fails locally. In essence, the definition of the Riemann-Christoffel curvature tensor is the anticommutativity of covariant derivatives. Suppose that one has a covariant vector A_λ. Then the implicit definition for the Riemann-Christoffel curvature tensor is

$$R^{\lambda}_{\nu\sigma\mu}A_\lambda = \nabla_\sigma\nabla_\mu A_\nu - \nabla_\mu\nabla_\sigma A_\nu. \tag{2.7}$$

The Ricci tensor is the contraction of the curvature tensor $R_{\nu\mu} = R^{\sigma}_{\nu\sigma\mu}$. Contracting a tensor is analogous of taking the trace of a matrix. Furthermore, the Ricci scalar is the trace of the Ricci tensor $R = g^{\nu\mu}R_{\nu\mu}$. For an introduction to tensor calculus, see [44]. Good sources on General Relativity are for example [11] and [32].

In General Relativity, the principle of least action can be used in a very elegant way to derive the Einstein field equations in vacuum. In General Relativity, the principle of least action is

$$\min_{g^{\mu\nu}} \int R\sqrt{-g}d^4x, \tag{2.8}$$

where we seek a metric $g^{\mu\nu}$ for the spacetime in such a way that the invariant scalar curvature over the spacetime is minimized. The scalar curvature R is a simple nontrivial curvature invariant of the Riemann-Christoffel curvature tensor. It depends on the second derivatives of the metric tensor. The metric determinant g ensures that the integral is coordinate-invariant. The stationarity condition is then the Einstein field equation in vacuum

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.9}$$

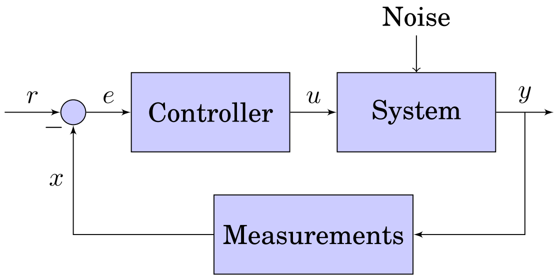

Figure 2.1. A simple conceptualization of a state-feedback control system.

2.3 Dynamic programming and optimal control theory

In optimal control theory, one has a dynamical system, be it an economic or financial or a physical system. As the systems concerned are dynamical, i.e. evolving with respect to time, (stochastic) differential equations are used to model the system dynamics. Once the system model is chosen, an optimal control is applied to the model to produce results that are in principle measurable and testable. In particular, closed-loop controls or feedback controls are considered, where the controller can in principle observe the state of the system fully at each time instant. The state then gives feedback to the controlling unit or the controller and this information then is mapped to the control applied. In particular, optimal control theory is control theory, for problems in which it is possible to precisely define the cost criterion or the performance criterion, usually as a bounded functional (sum or integral). In stochastic optimal control, the performance criterion is a mathematical expectation, when the state dynamics is governed by an Itô process.

In systems theory, a general state-feedback control system may be viewed as a system with vector inputs $u(s, x(s))$ and vector outputs $y(s)$. The state-space form of such a system without external noise is

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s, x(s))) \\ \dot{y}(s) = g(s, x(s)). \end{cases} \tag{2.10}$$

The vectoral state $x(s)$ evolves given the input and on the other hand the state gives feedback to the controller $u(s, x(s))$. We assume that the control system is fully controllable and reachable (can be steered to any state in finite time) and fully observable (the state is fully observable for all times). For a good overview of systems theory, see [36].

A general optimal control problem in continuous time is given by the following: suppose we have the dynamical system

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)) \\ x(t) = x, \end{cases} \tag{2.11}$$

where $x(s)$ is the state of the system at time s and $u(s) \in U$ is a (measurable) control variable. In this Dissertation it is assumed that the state is fully observable and therefore state-feedback control is in principle possible. The (bounded) functional to be minimized is of the form

$$J = \int_t^T \mathcal{L}(s, x(s), u(s))ds + h(x(T)). \tag{2.12}$$

2.3.1 Pontryagin's maximum principle

Optimal control problems can be solved primarily either with the Pontryagin's maximum principle, or dynamic programming. While this Dissertation utilizes predominantly the latter approach, it is important the present the basics of Pontryagin's approach. Suppose that we want to minimize 2.12 subject to the dynamical system 2.11. Then, for an optimal trajectory $x(s)$, adjoint variable $p(s)$ and optimal control $u(s)$ the following necessary conditions must hold

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} \\ x(t) = x \\ p(T) = -\frac{\partial h(x(T))}{\partial x} \\ \mathcal{H}(s, x(s), p(s)) = \sup_{u \in U} \left(< f, p > -\mathcal{L} \right). \end{cases} \tag{2.13}$$

The necessary conditions above become sufficient, if the Hamiltonian is concave and with some other technical conditions. The optimal control obtained by solving the system 2.13 gives an open-loop control. Moreover, if the value function $V = \inf_{u \in U} J$ is smooth enough, the costate variable p is the negative gradient of the value function, so that $p = -\frac{\partial V}{\partial x}$, see [49].

2.3.2 Dynamic programming and the Hamilton-Jacobi-Bellman equation

In this Dissertation, the feedback controls are given by dynamic programming considerations. Dynamic programming concerns with sequential decision-making in which the decisions affect the system. The essence of dynamic programming is the principle of optimality, which requires that:

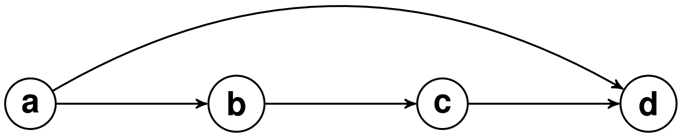

Figure 2.2. Principle of optimality

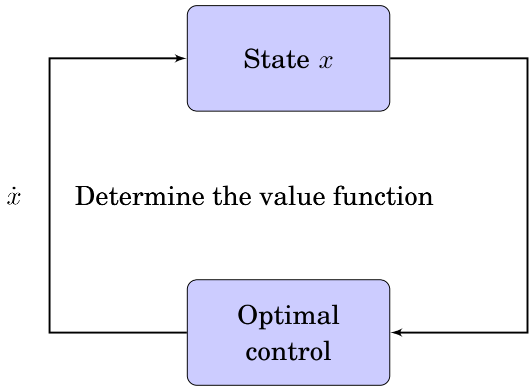

Figure 2.3. An optimal control feedback-loop

"An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

This principle is illustrated in Fig.2.2.: If the path from a to d via b and c is optimal, then each of the sub-paths must be optimal as well. The principle of optimality then is manifested in the Hamilton-Jacobi-Bellman (HJB) equation, which is a nonlinear PDE for the value function $V = \inf_{u \in U} J$ of the dynamic optimization problem 2.12. The HJB PDE is of the form

$$\frac{\partial V}{\partial t} = \mathcal{H}, \tag{2.14}$$

where the function on the right side of the equation is the Hamiltonian function (assuming the value function is smooth enough)

$$\mathcal{H} = \sup_{u \in U} \left(- < f(t,x,u), \nabla V > - \mathcal{L}(t,x,u) \right), \tag{2.15}$$

and $V(x,t)$ is the value function for the dynamic optimization program. Solving the HJB equation is under certain technical conditions sufficient in order to obtain closed-loop optimal controls. Generally the solutions of this PDE are not however smooth; fortunately there is a mature theory of viscosity solutions for the HJB PDE, [12]. The Hamilton-Jacobi PDE from classical mechanics is a special case of the Hamilton-Jacobi-Bellman partial differential equation. For details on the relation between Hamiltonian systems, Hamilton-Jacobi theory and HJB PDEs, see [49].

In this setting, Pontryagin's maximum principle can be thought of as a set of ordinary differential equations (Hamilton's equations) describing the characteristic curves of the Hamilton-Jacobi-Bellman partial differential equation. In classical mechanics an analogous relationship can be found between the Hamilton-Jacobi equation and Hamilton's equations.

If the value function is smooth, the adjoint variable or the costate variable is the (negative) gradient of the value function. In economics the adjoint variable is known as the shadow price. It describes at what marginal rate the optimal performance of the system increases if the state constraint is infinitesimally relaxed.

2.4 Stochastic calculus

In this Dissertation, stochastic calculus is to account for the "noise" in the state dynamics. Specifically, noisy state dynamics can be modeled using stochastic differential equations (SDE)

$$\begin{cases} dX = f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dW \\ x(t) = x, \end{cases} \tag{2.16}$$

where W is a standard Brownian motion. Much of stochastic analysis is concerned with the integration of random processes. Such stochastic integrals can be defined and a chain rule for stochastic functions exists (Itô's lemma).

Markovian stochastic processes are the relevant class of processes in this Dissertation, as the Markovian structure is the natural assumption that the information relevant to the process is amalgamated in the current state of the process. The relevant history of the stochastic process is therefore accumulated in the current state. This property is especially useful from the point of view of feedback control. Given an adapted process $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ on some probability space $(\Sigma, \mathcal{F}, P^\mu)$ it is Markov with initial distribution μ, if $P^\mu[X_{t+s} \in B|\mathcal{F}_t] = P^\mu[X_{t+s} \in B|X_t]$. In other words, in terms of information, only the current state of the process matters. In this sense, a Markovian process does not have any memory other than what is accumulated in the current state.

Given an Itô process of the form 2.16, two PDEs can be associated with such Markovian processes: the Kolmogorov forward equation (also called the Fokker-Planck equation) and the backward equation. Assuming that the transition probability density $\rho(s; x, y)dy = P^x[X_s \in dy]$ exists, we can

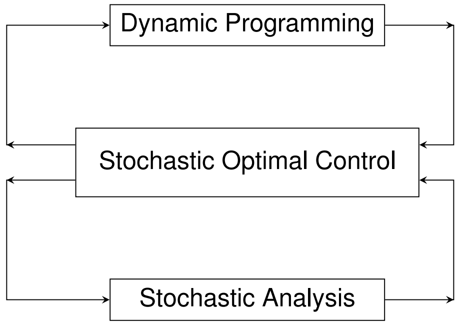

Figure 2.4. Stochastic optimal control as a fusion of dynamic programming and stochastic analysis

consider the forward equation

$$\frac{\partial \rho}{\partial s} = \mathcal{A}^* \rho, \tag{2.17}$$

where $\mathcal{A}^*$ is the formal adjoint of the infinitesimal generator of the Markovian process. The generator of the Itô process 2.16 is determined by using Itô's lemma. This tells how the process moves forward in time locally. It is given by

$$\mathcal{A}f(x) := \lim_{s \to 0} \frac{E^x[f(X_s)] - f(X)}{s}. \tag{2.18}$$

The reason why the generator is a second order differential operator, can be seen through Itô's lemma, which effectively shows that Taylor's expansion must include second-order terms due to the quadratic variation of martingale processes. For a thorough introduction to stochastic calculus, see the book [26].

2.5 Stochastic optimal control theory

Stochastic optimal control in this Dissertation is applied as dynamic optimization with Markovian structure. In words, the problem is to control an (Markovian) Itô process in order to minimize some conditional expectation. This technique is obtained when optimal control, dynamic programming and stochastic analysis are combined. Given a stochastic differential equation 2.16, we want to minimize the expectation

$$E_{tx}\left(\int_t^T \mathcal{L}(s, x(s), u(s, x(s)))ds + h(x(T))\right) \tag{2.19}$$

for times $t \leq s \leq T$ and with a terminal cost function $h(x(T))$, subject to the Itô process:

$$\begin{cases} dX = f(s, x(s), u(s, x(s)))ds + \sigma(s, x(s), u(s, x(s)))dW \\ x(t) = x. \end{cases} \tag{2.20}$$

With suitable conditions, we generally seek for a Markovian feedback or closed-loop optimal control $u = (s, x(s))$. Given that the state process is a random process, the Markovian control policy is a random variable as well. In this Dissertation the SDE-models are such that the diffusion matrix $\sigma\sigma'$ does not depend on the control and the respective HJB PDE is almost the same as in deterministic case; the only difference is that one has a second-order derivatives of the value function $V(x,t)$. With the above assumptions, the HJB PDE becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} Tr\left(\sigma\sigma' D^2 V\right) = \mathcal{H}, \tag{2.21}$$

where $\mathcal{H} = \sup_{u \in U} (- < f(t,x,u), \nabla V > - \mathcal{L}(t,x,u))$, $D^2 V$ is the Hessian of the value function and $\sigma\sigma'$ is the outer product of the diffusion vector σ with itself. The value function satisfies the boundary condition $V(x,T) = h(x)$. The obvious challenge with stochastic optimal control is that the value function might not be smooth, and that the partial differential equation above is nonlinear and difficult to solve in general. For good basic sources of stochastic optimal control theory, see [18] and [49].

3. Contributions of the Dissertation

This Dissertation illustrates the power and usefulness of the principle of least action and stochastic optimal control by presenting applications in economic, financial and physical systems. The Publications I-V cover both physical applications as well as financial and economic applications. The contributions of the Publications are summarized in Table 3.1.

3.1 Publication I

Public debt sustainability has become an especially relevant macroeconomic policy question due to COVID-19. Governments around the globe are using fiscal policy to support households and firms to overcome the acute crisis. It is very difficult in to assess debt sustainability and solvency of sovereigns in general. In Publication I, a stochastic optimal control model is put forward to model a rational government, who wants to drive down the debt ratio in finite time. The main contribution of Publication I is to further demonstrate the usefulness of stochastic methods in public finance and economic theory more generally. An index of insolvency risk is developed and a nonlinear stochastic differential equation is developed to model the effect of hidden fiscal multipliers and hidden credit risk premiums. Publication I thus adds to the literature, where stochastic optimal control is applied to macroeconomics and public finance theory, [16], [25],[10].

3.2 Publication II

The pricing of financial derivatives was revolutionized in 1973, when the Black–Scholes pricing model was presented [7]. Options, futures and other financial derivatives are being traded nowadays daily with very large nominal volumes. Even though pricing of options is a well-established field of research, less focus has been put on the properties of the financial market itself which support no-arbitrage pricing of financial derivatives.

Table 3.1. Summary of the Publications

Publication	Research objectives	Methodology	Results
I	To demonstrate the nonlinear dynamics of government debt and the risk of sovereign insolvency.	Stochastic Optimal Control, Stochastic Analysis	A nonlinear SDE is derived for the government debt dynamics and an index of insolvency is constructed.
II	To derive the Black-Scholes PDE from a Stochastic Optimal Control Problem, to operationalize the efficient market hypothesis.	Stochastic Optimal Control, Stochastic Analysis	It is shown that the Black-Scholes PDE is a linearized HJB PDE, a Burgers' PDE is derived for the transport of market instantaneous returns.
III	To derive relativistic and nonrelativistic PDEs of Quantum Mechanics from a Stochastic Optimal Control model and to explain the origin of the imaginary structure of QM.	Stochastic Optimal Control, Tensor calculus	The covariant Stueckelberg PDE, Telegrapher's PDE and Schrödinger PDE are derived from a Stochastic Optimal Control framework.
IV	To explore the interpretation of the Heisenberg Uncertainty Principle from a stochastic mechanics viewpoint.	Stochastic Optimal Control, Stochastic Analysis	The Heisenberg Uncertainty Principle is proven and a new interpretation is given.
V	To show how electromagnetism can be understood solely from the properties of the geometry of the space-time. To understand how the electromagnetic four-potential relates to space-time geometry.	Principle of Least Action, Calculus of Variations, Tensor calculus	Maxwell's equations are shown to be a special case of Einstein field equations. The role of Weyl curvature in electromagnetism is explored. The sign invariance of the metric determinant is demonstrated.

Publication II demonstrates the use of stochastic optimal control in finance theory. The main contributions of Publication II are the following: first, it is shown how the Black-Scholes pricing PDE for contingent claims can be recovered as a linearized Hamilton-Jacobi-Bellman PDE. Second, it is shown that such a model for the financial market which supports Black-Scholes pricing, the market drift or the instantaneous return function must obey the backwards Burgers' PDE. Burgers' equation is a simple model in hydrodynamic turbulence and it has been derived earlier in the context of pricing contingent claims, see [24]. The stochastic optimal control model presented in Publication II is also an operationalization of the efficient market hypothesis; the market is trying to minimize certain information functional. As the efficient market hypothesis [14] argues that the prices of securities contain all relevant available information, the stochastic optimal control problem reflects this idea in terms of a dynamic optimization program, where the market tries to embed any new information in prices as swiftly and efficiently as possible.

3.3 Publication III

Although quantum mechanics remains to this day quite elusive in terms of interpretations, the power of stochastic optimal control can be harvested to explain some properties of QM. The stochastic approach to quantum mechanics was initiated by the Hungarian physicist Imre Fényes in [19]. Richard Feynman used the principle of least action to formulate quantum mechanics already in 1948, [17]. Further work on stochastic quantum mechanics has been done especially by Nelson, [34], Yasue [48] and Papiez [37]. The statistical interpretation of quantum mechanics is given in [3].

In Publication III, the main contributions in terms of literature are two-fold; first of all, the imaginary structure or the Wick rotation of time variable is argued to stem from the metric determinant. Second, the Stueckelberg field equation is derived as a linearized Hamilton-Jacobi-Bellman equation of the stochastic optimal control model. From the Stueckelberg field equation, the Telegrapher's equation and ultimately Schrödinger equations are derived in a concise manner. The Telegrapher's PDE is known to be related directly to the Dirac equation, and thus links to the Dirac equation are provided as well.

3.4 Publication IV

The stochastic optimal control framework of quantum mechanics in Publication III is used to provide an alternative interpretation for the Heisenberg uncertainty principle. In the original formulation of the Heisenberg uncertainty principle, [23], the uncertainty is linked to the possible interference caused by the experiment. The main contribution of Publication IV is to show that the stochastic approach to quantum mechanics allows one to understand the uncertainty principle through thermodynamic equilibrium. The content of the uncertainty principle is, in Publication IV, that the product of the standard deviations has a lower bound in terms of the stationary distribution due to covariance stemming from optimal policy. For to have a small standard deviation for the position, one would need to have a steep value function to constrain and confine the particle within a small location in terms of the stationary distribution. A steep and a localized value function would then imply a large standard deviation for the gradient of the value function. The contribution is to therefore to clarify that if one seeks an equilibrium system which has a very deep potential well and thus good localization and confinement properties, one needs to accept the large variability in the gradient of the value function. This understanding of the uncertainty principle then implies that the lower bound for the product of standard deviations has nothing to do with experiments interfering with the set-up, but it is merely a natural trade-off implied by the linear–quadratic structure of the stochastic optimal control program. The results in Publication IV seem to therefore strongly support the statistical or the ensemble interpretation of quantum mechanics, put forward by, for example by Ballentine [3] and Sir Karl Popper [39].

3.5 Publication V

Since the inception of General Relativity, it has been an open issue whether electromagnetism can be fitted naturally into a one unified framework, from which both gravitation and electromagnetism could be understood, [46]. In Publication V, it is shown how with a certain coupling of the electromagnetic four-potential to the metric tensor, the field equations of EM, that is, Maxwell's equations can be recovered. As is well-known, the field equations of General Relativity can be obtained as a stationarity condition for the Einstein-Hilbert action. This is the principle of least action in General Relativity.

In Publication V, it is shown how the condition of Ricci-flatness leads directly to Maxwell's equations. The key contribution is the novel coupling of the electromagnetic four-potential to the geometry of the spacetime itself. This coupling then shows that the classical Lagrangian in electrodynamics is actually just the scalar curvature of the spacetime. The other two Maxwell's equations are recovered from the Bianchi identity. Furthermore, it is shown how the Einstein field equation is the stationary condition for the Einstein-Hilbert action irrespective of the sign of the metric determinant, which also gives further support to the choice of the sign of the metric determinant in Publication III. Finally, the Weyl curvature of the spacetime is linked to the four-current and thus to electric and magnetic fields. In other words, according to the theory put forward, four-currents induce Weyl curvature on the spacetime locally.

4. Discussion

This Dissertation provides evidence on the broad applicability of the principle of least action and (stochastic) dynamic programming in natural and man-made systems. It is interesting that natural physical laws can be represented with such economic efficiency formulations, in which nature acts as if it was maximizing efficiency and minimizing costs. This same teleological feature can be seen also outside the context of natural science, in particular in finance theory. Rational fiscal policy can also be engineered concisely by utilizing stochastic dynamic programming.

Publication I demonstrates and explores how the volatility of the interest-growth differential affects government debt dynamics with hidden fiscal multipliers and hidden credit risk premiums. The results demonstrate that the risks stemming from the premiums and multipliers and volatility can cause the debt ratio to vary and to diverge quite rapidly, to even unsustainable levels. Therefore, further stochastic modeling of the risks in terms of government debt is needed. The clear limitations of the model in Publication I relate to the focus on theoretical considerations. Unfortunately, it is not foreseeable that complete high-frequency data on variables like growth, debt, or primary balance would be available even in the foreseeable future. The debt dynamics in the present model is modeled in a continuous-time setting and therefore in order to estimate for example the risk index properly, theoretically one should have data sampled at very high frequency. Currently, nominal GDP growth rates are reported on a quarterly basis, for example.

The results in Publication II indicate that the information minimization control problem can be an operationalization of the Efficient Market Hypothesis. Therefore the transport equation, the backwards Burgers' PDE, could be of use in portfolio management. The backwards Burgers' PDE is the governing law for instantaneous market returns and practical application could be developed for investment companies and portfolio managers. The possible relaxation of the system towards thermal equilibrium and the stationary distribution could also be further studied.

Stochastic mechanics and quantum mechanics can be indeed understood through the prism of stochastic control theory and calculus of variations. In Publication III, it is shown, how one can obtain the relativistically covariant equations of quantum mechanics from a coordinate-invariant stochastic optimal control problem. The general idea is to extend the usual Lagrangian from classical mechanics into a Minkowski spacetime. When this objective is chosen to be minimized on average, over proper time, under a Markov diffusion, the corresponding Hamilton-Jacobi-Bellman equation in a linearized form is exactly the Stueckelberg wave equation from Parameterized Relativistic Dynamics, [15]. In that setting, the logarithm of the wave function is the value function for the stochastic optimal control problem and therefore the wave function is related to the action, as in Feynman's approach to quantum mechanics [17]. The results in Publication III could allow further research to be conducted in order to link canonical relativistic quantum mechanics to the Stueckelberg and Telegrapher's equations, as in the literature there is a link between the Telegrapher's equation and Dirac equation of relativistic quantum mechanics, [21]. The results obtained should stimulate further research on the Stueckelberg PDE [43], which is a four-dimensional extension of the Schrödinger equation. The stochastic optimal control approach to (quantum) mechanics underlines the possibility that at small scales, the spacetime can be seen as if it was fluctuating, see [20]. Nature then obeys a principle of economics or a principle of least action so that the laws of movement can be derived from such an optimal control problem. The Dissertation therefore postulates that fundamental laws of physics can be seen as if nature was an optimal control system. Given that the laws of motion are obtained as a dynamic optimization program, the results indicate that a more general link between laws of physics and computation could be established. Publication IV gives further confirmation of the nature of the uncertainty principle. It reconfirms that the principle is not about the experimenter interfering with measurement, but that it is in an endogenous property of the system instead.

Finally, the results in Publication V indicate that classical field theories can be seen as an instance of the spacetime geometry itself, and this observation helps to understand also why the classical theories of gravitation and electromagnetism share some mathematical features, namely for example that the classical potentials are described by Poisson's equation, an elliptic PDE. If the mathematical model presented in Publication V could be empirically confirmed, there could be an interesting set of engineering applications. The coupling of the Weyl curvature of the spacetime to the four-current and to magnetic and electric fields could allow measuring these effects. The results of Publication V add to the literature on geometrodynamics, [32] and continues the tradition as laid out in the classical articles [40] and [33]. Publication V continues also the work of the Finnish physicist Gunnar Nordström, who was a professor of physics at TKK (presently part of Aalto University) some 100 years ago, [35].

References

[1] A. Agrachev and Y. Sachkov. *Control Theory from the Geometric Viewpoint*. Springer, 2004.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, 1978.

[3] L.E. Ballentine. The statistical interpretation of quantum mechanics. *Rev. Mod. Phys.*, 42:358–381, 1970.

[4] J.L. Basdevan. *Variational Principles in Physics*. Springer, 2007.

[5] R. Bellman. On the application of the theory of dynamic programming to the study of control processes. In *Proc. of the Symposium on Nonlinear Circuit Analysis*, pages 199–213. Polytechnic Institute of Brooklyn, 1956.

[6] D. Bertsekas. *Reinforcement Learning and Optimal Control*. Athena Scientific, 2019.

[7] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Political Econ.*, 81:637–654, 1973.

[8] V. Boltyanskii, R. Gamkrelidze, and L. Pontryagin. On the theory of optimal processes. *Reports of the Academy of Sciences of the USSR*, 110:7–10, 1956.

[9] A. Cadenillas and R. Huamán-Aguilar. Explicit formula for the optimal government debt ceiling. *Ann. Oper. Res.*, 247:415–449, 2016.

[10] A. Cadenillas and R. Huamán-Aguilar. On the failure to reach the optimal government debt ceiling. *Risks*, 6, 2018.

[11] S.M. Carrol. *Spacetime and Geometry*. Addison Wesley, 2004.

[12] M.G. Crandall and P.L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.

[13] E. Dockner, S. Jorgense, N. Van Long, and G. Sorger. *Differential Games in Economics and Management Science*. Cambridge University Press, 2000.

[14] E. Fama. Efficient capital markets: A review of theory and empirical work. *J. Financ.*, 25:383–417, 1970.

[15] J.R Fanchi. Manifestly covariant quantum theory with invariant evolution parameter in relativistic dynamics. *Found. Phys.*, 41:4–32, 2011.

[16] G. Ferrari. On the optimal management of public debt: A singular stochastic control problem. *SIAM J. Control Optim.*, 56:2036–2073, 2018.

[17] R.P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.*, 20(367), 1948.

[18] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, 2006.

[19] I Fényes. Eine wahrscheinlichkeitstheoretische Begründung und Interpretation der Quantenmechanik. *Z. Phys*, 132(1):81–106, 1952.

[20] C. Frederick. Stochastic space-time and quantum theory. *Phys. Rev. D.*, 13:3183–3191, 1976.

[21] B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman. Relativistic extension of the analogy between quantum mechanics and Brownian motion. *Phys. Rev. Lett.*, 53:419–422, 1984.

[22] S. Gerovitch. *From Newspeak to Cyberspeak - A history of Soviet Cybernetics*. The MIT Press, 2002.

[23] W. Heisenberg. Über den anschaulichen inhalt der quantentheoretischen Kinematik und Mechanik. *Z. Phys*, 43:172–198, 1927.

[24] S. Hodges and A. Carverhill. Quasi mean reversion in an efficient stock market: The characterisation of economic equilibria which support Black–Scholes option pricing. *Econ. J.*, 103:395–405, 1993.

[25] R. Huamán-Aguilar and A. Cadenillas. Government debt control: Optimal currency portfolio and payments. *Oper. Res.*, 63:1044–1057, 2015.

[26] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, 1988.

[27] C. Lanczos. *The Variational Principles of Mechanics*. Dover, 1986.

[28] D. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons Inc., 1969.

[29] R.C. Merton. Optimal consumption and portfolio rules in continuous time. *J. Econ. Theory*, 3:373–413, 1971.

[30] P. Mirowski. *More Heat than Light - Economics as Social Physics, Physics as Nature's Economics*. Cambridge University Press, 1989.

[31] P. Mirowski. *Machine Dreams - Economics becomes a Cyborg Science*. Cambridge University Press, 2002.

[32] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. Princeton University Press, 2017.

[33] C.W. Misner and J.A. Wheeler. Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. *Ann. of Phys.*, 2:525–603, 1957.

[34] E. Nelson. Derivation of the Schrödinger equation from Newtonian mechanics. *Phys. Rev.*, 150:1079–1085, 1966.

[35] G. Nordström. Über die Möglichkeit, das Elektromagnetische Feld und das Gravitationsfeld zu vereiningen. *Phys. Zs.*, 15:504–506, 1914.

[36] L. Padulo and M.A. Arbib. *System Theory*. W.B. Saunders Company, 1974.

[37] L. Papiez. Stochastic optimal control and quantum mechanics. *J. Math. Phys.*, 23, 1982.

[38] H. Pesch and M. Plail. The maximum principle of optimal control: A history of ingenious ideas and missed opportunities. *Control. Cybern.*, 38:973–995, 2009.

[39] K.R. Popper. *Quantum Theory and the Schism in Physics*. Psychology Press, 1992.

[40] G.Y. Rainich. Electrodynamics in general relativity. *Trans. Amer. Math. Soc.*, 27, 1925.

[41] A. Rojo and A. Bloch. *The Principle of Least Action - History and Physics*. Cambridge University Press, 2018.

[42] H. Rund. *The Hamilton-Jacobi Theory in the Calculus of Variations*. D. Van Nostrand Company LTD, 1966.

[43] E. C. G. Stueckelberg. *Helv. Phys. Acta.*, 14, 1941.

[44] J.L. Synge and A. Schild. *Tensor Calculus*. Dover, 1978.

[45] V. Terekhovich. Metaphysics of the principle of least action. *Stud. Hist. Philos. M. P.*, 62:189–201, 2018.

[46] V.P. Vizgin. *Unified Field Theories in the First Third of the 20th Century*. Birkhäuser, 1994.

[47] N. Wiener. *Cybernetics: Or Control and Communication in the Animal and the Machine*. The MIT Press, 1948.

[48] K. Yasue. Quantum mechanics and stochastic control theory. *J. Math. Phys.*, 22, 1981.

[49] J. Yong and X.Y. Zhou. *Stochastic Controls - Hamiltonian Systems and HJB Equations*. Springer-Verlag, 1999.

Aalto-DD 136/2021

"The fundamental principle of human action, the law, that is to political economy what the law of gravitation is to physics is that men seek to gratify their desires with the least exertion." - Henry George (1839-1897)

The strive for efficiency is abundant in nature as well as in society. To make best use of scarce resources is the essence of economic considerations. What is rather surprising, is that this simple principle of efficiency is so omnipresent also in our natural world.

This dissertation presents a collection of applications of the principle of least action and stochastic optimal control theory in public economics, mathematical finance and theoretical physics.

9789526405377

ISBN 978-952-64-0537-7 (printed)
ISBN 978-952-64-0538-4 (pdf)
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)

Aalto University
School of Science
Department of Mathematics and Systems Analysis
www.aalto.fi

BUSINESS + ECONOMY

ART + DESIGN + ARCHITECTURE

SCIENCE + TECHNOLOGY

CROSSOVER

DOCTORAL DISSERTATIONS