Department of Mathematics and Systems Analysis

Advancing incorporation of expert knowledge into Bayesian networks

Pekka Laitila

Aalto University

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Advancing incorporation of expert knowledge into Bayesian networks

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Abstract

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Abstract

Bayesian networks (BNs) are used in many areas to support risk management and decision-making under uncertainty. A BN represents probabilistic relationships of variables and allows to explore their interaction through various types of analyses. In applications, a lack of suitable data often necessitates that a BN is constructed at least partly based on the knowledge of a domain expert. Then, in order to manage limited time and the cognitive workload on the expert, it is vital to have efficient means to support the construction process.

This Dissertation elaborates and improves so-called ranked nodes method (RNM) that is used to quantify expert views on the probabilistic relationships of variables, i.e., nodes, of a BN. RNM is designed for nodes with discrete ordinal scales. With such nodes, the relationship of a descendant node and its direct ancestors is defined in a conditional probability table (CPT) that may consist of dozens or hundreds of conditional probabilities. RNM allows the generation of the CPT based on a small number of parameters elicited from the expert. However, the effective use of RNM can be difficult due to a lack of exact guidelines concerning the parameter elicitation and other user-controlled features. Furthermore, there remains ambiguity regarding the underlying theoretical principle of RNM. In addition, a scarcity of knowledge exists on the general ability of CPTs generated with RNM to portray probabilistic relationships appearing in application areas of BNs.

The Dissertation advances RNM with regard to the above shortcomings. The underlying theoretical principle of RNM is clarified and experimental verification is provided on the general practical applicability of the method. The Dissertation also presents novel approaches for the elicitation of RNM parameters. These include separate designs for nodes whose ordinal scales consist of subjective labeled states and for nodes formed by discretizing continuous scales. Two novel approaches are also presented for the discretization of continuous scales of nodes. The first one produces static discretizations that stay intact when a BN is used. The other one involves discretizations updating dynamically during the use of the BN.

The theoretical and experimental insight that the Dissertation provides on RNM clears the way for its further development and helps to justify its deployment in applications. In turn, the novel elicitation and discretization approaches offer thorough and well-structured means for easier as well as more flexible and versatile utilization of RNM in applications. Consequently, the Dissertation also facilitates and promotes the effective and diverse use of BNs in various domains.

Keywords Bayesian networks, Ranked nodes, Probability elicitation, Conditional probability tables, Continuous node discretization

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Tiivistelmä

Bayes-verkkoja (BV:ja) käytetään monilla sovellusaloilla tukemaan riskienhallintaa ja päätöksentekoa epävarmuuden vallitessa. BV kuvaa muuttujien välisiä tilastollisia suhteita ja mahdollistaa niiden välisen vuorovaikutuksen tutkimisen. Käytännön sovelluksissa sopivan datan puute johtaa usein siihen, että BV rakennetaan ainakin osittain substanssiasiantuntijan tietämykseen perustuen. Tällöin rakennusprosessia on tärkeää tukea tehokkailla menetelmillä ajan säästämiseksi ja asiantuntijan liiallisen kognitiivisen kuormituksen välttämiseksi.

Väitöskirjassa tutkitaan ja parannetaan ranked nodes -menetelmää (RNM), jota käytetään kvantifioimaan asiantuntijanäkemyksiä BV:n muuttujien, eli solmujen, tilastollisista suhteista. RNM on kehitetty diskreetin järjestysasteikon omaaville solmuille. Tällaisilla solmuilla jälkeläissolmun ja sen suorien edeltäjäsolmujen välinen suhde määritellään ehdollisen todennäköisyyden taulukossa (ETT), joka voi koostua kymmenistä tai sadoista ehdollisista todennäköisyyksistä. RNM mahdollistaa ETT:n generoimisen perustuen pieneen määrään parametreja, jotka asiantuntija valitsee. RNM:n tehokas käyttö voi olla kuitenkin haastavaa, koska parametrien määräämiseen ja menetelmän muihin käyttäjän päättämiin ominaisuuksiin liittyen ei ole tarjolla tarkkoja ohjeita. Lisäksi RNM:n teoreettinen periaate on epäselvä. Tiedonpuutetta on myös RNM:llä generoitujen ETT:jen yleisestä kyvystä kuvata BV:jen sovellusalueilla esiintyviä tilastollisia suhteita.

Väitöskirja kehittää RNM:ää yllä mainittujen puutteiden osalta. RNM:n teoreettinen periaate selvennetään, ja menetelmän yleisestä käytännön sovellettavuudesta esitetään kokeellista vahvistusta. Väitöskirja esittelee myös uusia menettelytapoja RNM:n parametrien valitsemiseen. Näihin sisältyy erilliset mallit solmuille, joilla on subjektiivisesti nimetyt tilat, ja solmuille, jotka on muodostettu diskretoimalla jatkuvia mitta-asteikkoja. Lisäksi esitellään kaksi uutta menettelytapaa solmujen jatkuvien mitta-asteikkojen diskretoimiseen. Ensimmäinen tuottaa staattisia diskretointeja, jotka pysyvät muuttumattomina BV:a käytettäessä. Toisessa diskretoinnit päivittyvät dynaamisesti BV:n käytön aikana.

Väitöskirjan tarjoama teoreettinen ja kokeellinen ymmärrys RNM:stä mahdollistaa sen jatkokehityksen ja edistää sen käyttöä sovelluksissa. Uudet menettelytavat parametrien valitsemiseen ja jatkuvien mitta-asteikkojen diskretoimiseen tarjoavat hyvin jäsenneltyjä keinoja RNM:n helpompaan, joustavampaan ja monipuolisempaan hyödyntämiseen sovelluksissa. Kokonaisuudessaan väitöskirja helpottaa ja tukee BV:jen tehokasta ja laajaa käyttöä eri sovellusaloilla.

Avainsanat Bayes-verkot, ranked-solmut, todennäköisyyselisitaatio, ehdollisen todennäköisyyden taulukot, jatkuvien solmujen diskretointi

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Helsinki, May 5, 2022

Pekka Laitila

Contents

List of Publications

This doctoral dissertation consists of a summary and the following papers, which are referred to in the text by their numerals.

1. Laitila, Pekka; Virtanen, Kai. 2020. On Theoretical Principle and Practical Applicability of Ranked Nodes Method for Constructing Conditional Probability Tables of Bayesian Networks. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(5), 1943–1955.

2. Laitila, Pekka; Virtanen, Kai. 2016. Improving Construction of Conditional Probability Tables for Ranked Nodes in Bayesian Networks. *IEEE Transactions on Knowledge and Data Engineering*, 28(7), 1691–1705.

3. Laitila, Pekka; Virtanen, Kai. 2021. Portraying Probabilistic Relationships of Continuous Nodes in Bayesian Networks with Ranked Nodes Method. *Decision Support Systems*, 154, 113709, 1–15.

4. Laitila, Pekka; Virtanen, Kai. 2021. Advancing Construction of Conditional Probability Tables of Bayesian Networks with Ranked Nodes Method. *Submitted manuscript*, 25 pages.

Author's Contribution

Paper 1: On Theoretical Principle and Practical Applicability of Ranked Nodes Method for Constructing Conditional Probability Tables of Bayesian Networks

Laitila is the main author. Virtanen initiated the research topic. Laitila carried out the theoretical analysis on the underlying principle of the ranked nodes method. Laitila also designed and executed the experiments on the practical applicability of the method. Laitila wrote the paper together with Virtanen.

Paper 2: Improving Construction of Conditional Probability Tables for Ranked Nodes in Bayesian Networks

Laitila is the main author. Virtanen initiated the research topic. Laitila carried out the theoretical analysis behind the novel approach for the application of the ranked nodes method to continuous nodes. Laitila also designed the illustration of the approach. Laitila wrote the paper together with Virtanen.

Paper 3: Portraying Probabilistic Relationships of Continuous Nodes in Bayesian Networks with Ranked Nodes Method

Laitila is the main author. Laitila proposed the research topic and carried out the theoretical analysis behind the two novel discretization approaches concerning the application of the ranked nodes method to continuous nodes. Laitila also designed the implementations and the numerical examples of the approaches. Laitila wrote the paper under the guidance of Virtanen.

Paper 4: Advancing Construction of Conditional Probability Tables of Bayesian Networks with Ranked Nodes Method

Laitila is the main author. Laitila and Virtanen proposed the research topic. Laitila carried out the theoretical analysis behind the novel framework concerning the application of the ranked nodes method to nodes with subjective labeled states. Laitila also designed and programmed the related computer implementations and illustrations. Laitila wrote the paper together with Virtanen.

1. Introduction

Risk management and decision-making under uncertainty are common challenges in business and public administration. Often the framework of a decision-making problem consists of various types of factors and variables whose mutual probabilistic dependencies may be difficult to know or perceive exactly. For instance, there might not be suitable historical data available, or the relevant data may be difficult to identify. These problems are typical in situations where risks are novel or unprecedented. Instances like these include unique projects, the deployment of new production methods or business models, entering new marketplaces, ecological and economical disasters, governmental conflicts, and terrorist attacks.

Even though there might be a lack of suitable historical data, there is often an abundance of expert insight available, along with diverse information or data on indirectly related factors. In these situations, analysis of risks and decision-making under uncertainty can effectively be supported by Bayesian networks (BNs), see, e.g., [1] or [2]. A BN represents a system of linked components both visually and numerically enabling a rigorous quantification of risks and a clear communication of the components' interaction. The visual side of the BN consists of nodes and directed arcs joining them. The nodes correspond to random variables depicting the system components. The arcs indicate direct probabilistic relationships between the nodes. The numerical side of the BN quantifies the probabilistic relationships indicated by the arcs. As a whole, the BN completely encodes the joint probability distribution of the nodes. This enables the conduction of detailed analyses concerning the nodes' probabilistic interaction with each other. BNs allow combining data and expert knowledge, and there are many software available for their construction and use, like [3], [4], [5], and [6]. The applications of BNs are numerous and cover a wide range of domains, such as medical diagnosis and decision support [7], [8], [9], risk analysis of epidemics [10], [11], ecosystems [12], genetics and biology [13], agriculture [14], [15], industry [16], [17], [18], finance [19], policy and military planning [20], [21], cybersecurity [22], as well as commutation and transport [23], [24], [25], [26].

If a comprehensive data collection is available, both the visual and the numerical side of a BN can be constructed by data-fitting approaches, see, e.g., [27], [28], and [29]. However, in practical applications, it is common that the data available is too scarce or unsuitable for the needs of the BN construction [30], [31]. In such cases, the entire BN or some parts of it must be constructed by expert elicitation involving subjective assessments of a domain expert. The most challenging part of this process is typically the quantification of the probabilistic relationships of the nodes [1], [2], [32], [33]. Usually, the nodes have discrete scales, whereby the probabilistic relationship between a descendant (called a child node) and its direct ancestors (called parent nodes) is defined in a conditional probability table (CPT). A single CPT may consist of dozens or even hundreds of conditional probabilities. Therefore, it is often impossible for the domain expert to assess the required probabilities either due to time constraints or mental fatigue [34], [33].

This Dissertation elaborates and improves so-called ranked nodes method (RNM) [31] that is a semi-automated means to define CPTs for nodes with discrete ordinal scales. Compared to assessing the CPT elements individually, RNM provides significant reduction to the elicitation burden of the expert. In RNM, the expert first assesses a small number of parameters that describe the probabilistic interaction between a child node and its parent nodes. The CPT of the child node is then generated based on these parameters for further verification. RNM is implemented in a BN software [4] and it has been utilized in several applications including, e.g., software defect prediction [35], supplier selection in automobile industry [18], and risk management of epidemics [10].

While RNM is actively utilized in applications, the efficient use of the method is hampered by various shortcomings. First, there remains ambiguity regarding the theoretical foundation of RNM. As the exact principle governing the results obtained with RNM is unclear, one may be suspicious of its deployment. In addition, RNM involves a user-controlled parameter whose role is not detailed in the literature. As the parameter affects the generated CPTs, one may be confused on which value to assign for it during the use of RNM. Second, there is a lack of studies on how well in general CPTs constructed with RNM can represent probabilistic relationships occurring in practical BN applications, and which user-made choices affect this ability the most. Third, there does not exist exact guidelines for the elicitation of RNM parameters from the expert. Therefore, the parameters must be determined through trial and error, which can be cumbersome and lacks rigor. Fourth, there are no guidelines concerning the application of RNM to nodes whose ordinal scales are formed by discretizing continuous scales. With such nodes, ignorance on the functioning of RNM may render it incapable of portraying the probabilistic views of the expert. In this setting, lack of technical insight on RNM can also lead to the generation of CPTs that produce misleading results in the analyses carried out with the BN.

The aim of this Dissertation is to overcome the above shortcomings concerning RNM and develop the method further for more versatile use. On one hand, this objective is achieved by theoretical analysis and experimental studying of RNM. On the other hand, it is realized through the development of novel approaches for the practical use of RNM. Overall, the Dissertation enhances the methodology for establishing probabilistic relationships in BNs based on expert elicitation. The contributions of the Dissertation are directly applicable also to influence diagrams that are decision-theory extensions of BNs [36]. Therefore, in a wider perspective, the Dissertation facilitates and promotes the effective and diverse use of both BNs and influence diagrams to support risk management and decision-making under uncertainty in various domains.

The rest of this summary article is structured as follows. Section 2 presents the methodological background. This includes explaining the basic idea of BNs and influence diagrams, a general overview of the construction of BNs and CPTs by expert elicitation, and a more in-depth explanation of the functioning and challenges of RNM. Section 3 presents the key contributions of Papers 1–4 of the Dissertation. Section 4 concludes and suggests topics for future research.

2. Methodological background

2.1 Bayesian networks

2.1.1 Bayesian networks with discrete nodes

Bayesian networks (BNs), see, e.g., [1] or [2], are directed acyclic graphs in which nodes represent random variables and arcs indicate their direct dependencies. Figure 1 presents an example BN that is adapted from [37] and concerns an air bombing run. The BN depicts how the bombing accuracy and the effectiveness of the enemy's ground-to-air defenses determine how well the bombers manage to hit their targets.

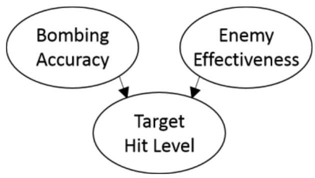

Figure 1. Example BN concerning an air bombing run.

In order to perform probabilistic analyses with the BN, the direct relationships indicated by the arcs must be quantified. Typically, the nodes have discrete scales whereby the quantification happens through conditional probability tables (CPTs). A CPT defines the conditional probability distribution of the child node for all the combinations of states of the parent nodes. If a node does not have any parents, a prior probability distribution must be defined. With the example BN, given that all the nodes have discrete scales of the form {Low, Medium, High}, Tables 1(a) and 1(b) present the prior distributions of *Bombing Accuracy* and *Enemy Effectiveness* whereas Table 1(c) presents the CPT of *Target Hit Level*. Table 1(a) indicates that high levels of bombing accuracy are generally more likely whereas Table 1(b) portrays how all levels of enemy effectiveness are considered equally probable. In turn, Table 1(c) implies that the targets are hit better with increasing levels of bombing accuracy and decreasing levels of enemy effectiveness.

Table 1. Prior distributions of *Bombing Accuracy* (a) and *Enemy Effectiveness* (b) along with the CPT of *Target Hit Level* (c).

(a)

Bombing Accuracy	Low	0.1
	Medium	0.3
	High	0.6

(b)

Enemy Effectiveness	High	0.33
	Medium	0.33
	Low	0.33

(c)

	Bombing Accuracy	Low			Medium			High		
	Enemy Effectiveness	High	Medium	Low	High	Medium	Low	High	Medium	Low
Target Hit Level	Low	0.92	0.72	0.50	0.66	0.15	0.03	0.34	0.02	0.00
	Medium	0.08	0.28	0.50	0.34	0.78	0.69	0.64	0.64	0.15
	High	0.00	0.00	0.01	0.00	0.08	0.28	0.02	0.34	0.85

The graph structure and the CPTs of a BN encode together the joint probability distribution of the nodes. For a BN consisting of nodes $X_1, \dots, X_n$, the joint probability distribution is obtained according to

$$P(X_1, \dots, X_n) = \prod_{i=1}^{n} P(X_i | Pa(X_i)),$$

where $Pa(X_i)$ denotes the parent nodes of X_i. This encoding principle allows one to carry out with BNs various probabilistic analyses about the nodes. A common type of analysis is such that specific states of selected nodes are given 100% probability. This is often called instantiating or observing those nodes, or entering evidence into the BN [2]. Once the selected nodes have been instantiated, the probability distributions of the remaining nodes are updated according to Bayes' theorem, see, e.g., [1], [2]. The updating, known as, e.g., probabilistic inference, propagation of evidence, or belief updating [2], is carried out in BN software through effective algorithms, see, e.g., [29], [38], that utilize the aforementioned encoding principle. These sort of analyses can be used to support both predictive and diagnostic reasoning about the system or phenomenon that is modeled with a BN [39]. In predictive reasoning, evidence about causes is used to update beliefs about effects. The opposite applies for diagnostic reasoning. The ability to support both predictive and diagnostic reasoning makes BNs a useful tool for, e.g., comparing different courses of action or deducing fault sources.

As an illustration, Figure 2(a) displays the example BN when no evidence has been entered. The probability distributions of *Bombing Accuracy* and *Enemy Effectiveness* correspond to those defined in Tables 1(a) and 1(b) whereas *Target Hit Level* has a distribution obtained through the parents' distributions and the CPT in Table 1(c). In Figure 2(b), *Target Hit Level* has been fixed to state *Low* leading into new probability distributions of *Bombing Accuracy* and *Enemy Effectiveness*. The figure indicates that if targets are observed to be hit poorly, one should consider different levels of the bombing accuracy being almost equally probable. In turn, the enemy effectiveness is most likely at high levels. In this regard, Figure 2(b) exemplifies diagnostic reasoning.

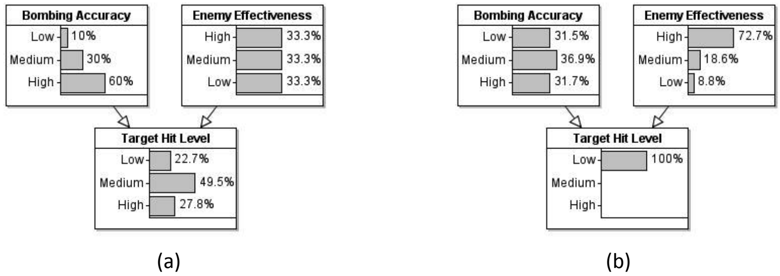

Figure 2. Example BN with (a) no evidence entered and (b) evidence *Target Hit Level = Low*.

2.1.2 Influence diagrams

Influence diagrams [36] are extensions of BNs used in the field of decision analysis. In these models, nodes representing random variables are joined with nodes representing objectives and possible actions of a decision maker. The three different node types are referred to as chance nodes, value nodes, and decision nodes. To elaborate this concept, Figure 3 presents an influence diagram extension of the example BN in Figure 1. The diagram contains two new nodes. The rectangle-shaped decision node *Bombing Altitude*, with states {High, Medium Low}, represents decision alternatives concerning the air bombing run. The diamond-shaped value node *Utility* specifies how much the different outcomes of *Target Hit Level* are valued with regard to each other. In practice, node *Utility* is defined by a utility function that maps the states of *Target Hit Level* into utility numbers representing subjective preferences. Like in the original BN, *Bombing Accuracy*, *Enemy Effectiveness*, and *Target Hit Level* are oval-shaped chance nodes.

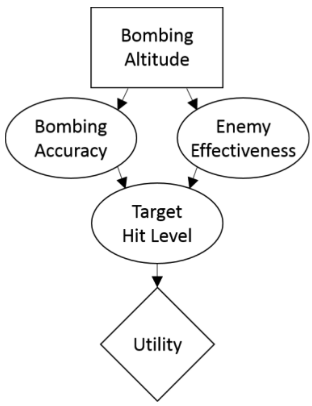

Figure 3. Influence diagram adapted from the example BN in Figure 1.

With influence diagrams, the chance nodes and the value nodes enable the calculation of a probability distribution of utility for each of the decision alternatives. The alternatives can then be ranked from best to worst according to a selected property of the utility distributions. For instance, the ranking can be based on the expected utilities [40] of the decision alternatives. Furthermore, like in BNs, evidence can be entered into influence diagrams by giving specific states of selected chance nodes 100% probability. This ability allows to compare the utility distributions of the decision alternatives with regard to various circumstances represented by the entered evidence.

2.1.3 Representation of continuous variables

In many domains, the system or phenomenon portrayed with a BN includes both discrete and continuous random variables. Yet, most practical BN models include only nodes with discrete scales [30], [41]. In these models, the nodes representing continuous random variables have ordinal scales formed by discretizing continuous scales into consecutive subintervals. The discretizations are static, i.e., they stay intact during the use of the BN. The reason for portraying continuous variables with discrete nodes has a technical origin. The conduction of probabilistic inference in a BN is easier when the dependencies between nodes are defined through CPTs [41]. Therefore, besides discretization, many commonly used BN software provide only limited means for portraying continuous variables, see [42] for a recent overview. Instead of being only a technical limitation, discretization can also be a means to ease the construction of BNs. For instance, a domain expert may find it easier to describe the probabilistic behavior of continuous variables through discretized scales.

The downside of using static discretizations is that it limits the accuracy with which the probabilistic behavior of the underlying continuous variables can be represented. The accuracy can be improved by increasing the granularity of the discretizations. However, it comes at the expense of larger CPTs and an increasing computational burden of probabilistic inference in the BN [41], [43]. Therefore, the challenge with static discretization is to find a suitable trade-off between the desired levels of accuracy and the computational burden regarding the BN.

An effective way to deal with the trade-off challenge of static discretization is the dynamic discretization algorithm [43]. In this approach, probabilistic relationships between child nodes and their parents are established through functional expressions. For example, a child node could be defined to have a continuous probability distribution whose parameters depend on the states of the parent nodes. Based on these relationships, the continuous scales of the nodes are discretized repeatedly by the algorithm during the use of the BN. Whenever evidence is entered into the BN, the algorithm discretizes the nodes iteratively so that the discretizations become denser in those regions of the continuous scales to which the probability mass is concentrated. On each iteration round, the algorithm treats the discretized nodes as standard discrete nodes. This feature of the discretization algorithm enables it to utilize effective standard algorithms of probabilistic inference during the iteration rounds. As the nodes end up having non-uniform discretizations dependent on the entered evidence, it is possible to represent their probability distributions accurately without the need for dense static discretizations.

Besides the static discretization and the dynamic discretization algorithm, several other approaches have been developed to represent continuous variables in BNs, see, e.g., [44] for an overview. The approaches include the conditional linear Gaussian (CLG) model, mixtures of truncated basis functions (MoTBFs), Monte Carlo-based methods, and non-parametric BNs. With the CLG model [45], the conditional probability distribution of each continuous child node is always a Gaussian, i.e., normal, distribution. The mean of the child's Gaussian distribution is a linear combination of the continuous parent nodes. On the other hand, discrete nodes can have only discrete parents in CLG models. MoTBF models [46] generalize the basic static discretization with the idea that instead of a constant-valued function, the probability density function of a continuous node is approximated on each discretization interval with a linear combination of so-called basis-functions. The basis-functions can be, e.g., polynomials or exponential functions. Compared to the static discretization, this approach provides a higher flexibility to approximating the probability density function. In Monte Carlo methods [47], [48], probabilistic inference in the BN is performed through random sampling. With the non-parametric BNs [49], the basic idea is that the joint distribution of the nodes can be defined by the marginal probability distribution of each node and univariate copula functions that describe the probabilistic relationships between directly linked nodes.

Papers 2 and 3 of this Dissertation address the application of the ranked nodes method (RNM) to nodes whose ordinal scales are formed by discretizing continuous scales. While both papers elaborate the use of static discretizations, Paper 3 also presents a novel approach combining the use of RNM with the dynamic discretization algorithm. This pairing is convenient because the way nodes' probabilistic relationships are defined in RNM readily allows the application of the dynamic discretization algorithm. In addition, both RNM and the algorithm are implemented in an existing BN software (AgenaRisk [4]), which facilitates the deployment of the approach. By contrast, with MoTBF models and non-parametric BNs discussed above, one should develop a separate routine to estimate the parameters necessary in those models from the RNM parameters. The CLG model is incapable altogether to represent the range of probabilistic relationships that can be established with RNM. With Monte Carlo methods, probabilistic inference may become computationally expensive if the nodes' probabilistic relationships are defined using deterministic functions [1], [43]. This can be problematic as RNM specifically involves defining nodes' probabilistic relationships with deterministic functions.

2.2 Construction of Bayesian networks by expert elicitation

In practical applications, it is common that the entire BN or some parts of it must be constructed by expert elicitation due to a lack of suitable data. The construction requires that a domain expert defines the structure of the BN, i.e., the necessary nodes and arcs. In addition, the expert must quantify the probabilistic relationships indicated by the arcs. These two phases of the BN construction are next discussed.

2.2.1 Defining structure

The first task in defining the structure of a BN is identifying the nodes to be included [1], [2], [32], [33]. To select the relevant nodes, it is beneficial to have the expert describe and explain the system or phenomenon that is to be portrayed with the BN [32]. The description can reveal, e.g., causal processes that indicate necessary nodes. Related to this idea, it may be worthwhile to consider the system or phenomenon through different types of variables present [2]. Three main types of variables are defined in [2]. First, there are problem variables. These are the variables whose posterior probability distributions are of interest given observations of other variables in the system. Usually, observations of the problem variables are not available. Next, there are information variables. These are the variables of which observations may be available. The information variables can represent background information concerning the problem variables. On the other hand, they can depict symptom information influenced by the problem variables. The final group are mediating variables. These are unobservable variables through which the other types of variables may interact with each other. When considering the nodes to be included, it is useful that the expert understands the purpose for which the BN is constructed [1], [32]. This helps to define such nodes that the BN represents the underlying system or phenomenon at a suitable level of detail. In general, it is recommendable that the number of nodes in the BN is kept to the minimum [1], [2]. With fewer nodes, there will be less probabilistic relationships to be evaluated and probabilistic inference in the BN will be less complex.

Along with or after identifying the nodes of the BN, also the states of the nodes should be defined [1], [2], [32], [33]. As discussed in Section 2.1.3, the nodes are often given finite sets of discrete states. In this case, the states must be defined so that the probabilistic relationships of the nodes can be represented precisely enough with regard to the purpose for which the BN is constructed [1], [2]. If a continuous scale is deemed necessary for a node, one may apply the approaches briefly described in Section 2.1.3.

Once the nodes of the BN are defined, the ones having a direct probabilistic relationship should be connected to each other with directed arcs. Considering causal relationships between the nodes is generally a good guiding principle for defining the necessary arcs [1], [2], [32]. Then, arcs are drawn from nodes representing causes to nodes representing effects. Two specific approaches for establishing the arcs are discussed in [2]. The first one, referred to as the basic approach in [2], is to utilize the idea of nodes representing problem, information (background and symptom), and mediating variables concerning the underlying system or phenomenon. These variables typically encode causal beliefs that dictate the arc connections as follows. Background variables have problem and symptom variables as children. In addition, problem variables are parents of symptom variables. Mediating variables are often children of background and problem variables, and parents of symptom variables.

The other approach discussed in [2] for defining the arcs of the BN is the use of so-called idioms [50], which are addressed extensively also in [1]. Idioms are BN fragments that represent generic types of probabilistic relationships. Among the most common idioms are cause-consequence, measurement, and definitional/synthesis idioms. The cause-consequence idiom measures the uncertainty of a causal process with observable consequences. The example BN in Figure 1 represents this idiom with *Bombing Accuracy* and *Enemy Effectiveness* being causes and *Target Hit Level* being the observable consequence. The measurement idiom describes the uncertainty about the accuracy of any type of measurement. The definitional/synthesis idiom represents the synthesis or combination of many nodes into a single node for the purpose of organizing the BN. It also models the deterministic or uncertain definitions between variables.

The use of idioms promotes thinking in terms of semantic relations among a small group of variables rather than in terms of nodes and arcs. Structuring the BN is thus moved to a higher level of abstraction, leaving detailed arc selection to happen automatically through the predefined structures of the idioms [2]. Combining idioms provides a way to form BN fragments that may be constructed separately from each other. The fragments can then be combined to form large-scale BNs. In this regard, idioms exemplify the use of modular network fragments in the BN construction, which is the core idea of a framework called object-oriented Bayesian networks, see, e.g., [51], [52], [1], [2], and [50]. In this framework, BN fragments are objects representing specific instances of generic network classes. The objects can be linked to each other through input and output nodes enabling the construction of a main BN model from smaller sub-models.

In practice, defining the structure of a BN is likely to be an iterative process, where nodes and arcs are added, removed and updated as one tries to arrive to a form that portrays beliefs about the underlying system or phenomenon at a desired and manageable level of detail [2], [32], [33]. It is also possible that a need to modify the structure of the BN becomes evident only when the probabilistic relationships of the nodes are being quantified and something new is realized about their interaction mechanics [1], [2].

2.2.2 Quantifying probabilistic relationships

While defining the structure of a BN can be a non-trivial iterative process, quantifying the probabilistic relationships of the nodes is generally regarded to be the more difficult part of the BN construction [1], [2], [32], [33]. As discussed in Section 2.1.3, in BN applications, the nodes typically have discrete states. In this case, the quantification of their probabilistic dependencies requires defining CPTs. The size of the CPT of a child node grows exponentially with the number of the parent nodes. Therefore, the CPT may consist of dozens or even hundreds of conditional probabilities. For instance, if a child node and its three parent nodes have five states each, the CPT consists of 625 elements. Assessing the necessary probabilities even for a single CPT can therefore be impossible for the expert either due to the cognitive strain or the scarcity of time [33], [34].

To deal with the elicitation challenge of CPTs, their construction is often carried out through methods referred to as parametric probability distributions [33], canonical models [53], canonical distributions [38], and filling-up methods [30], [54]. Common to these methods is that they allow constructing a CPT through parameters that are assessed by a domain expert and whose number is significantly smaller than the number of elements in the CPT. Table 2 lists well-known parametric methods along with information on their features. RNM elaborated in this Dissertation is one of them. Next, starting from RNM, an overview of the parametric methods included in Table 2 is presented.

Table 2. Features of parametric methods for construction of CPTs concerning non-binary nodes.

Method	$N\|n, m^*$	$N\|$ $n = 3$, $m = 5$	Parameters assessed: Probabilities	Parameters assessed: Weights of parent nodes	Parameters assessed: Dispersion parameter	Parameters assessed: Other	Optional number of parameters to use
Hassall et al.	n	3		X			
RNM**	$n+1$	4		X	X		X
Inter-Beta**	$2(m-1)+n$	11	X	X		X	X
Røed et al.	$(m-1)n$	12	X		X		
WSA	m^2-m+n	23	X	X			
EBBN	m^2-m+2n	26	X				
Likelihood	$(n+2)m+1$	26	X			X	
Func. interpol.	$2^n(m-1)$	32	X				
Noisy-MAX	$n(m-1)^2$	48	X				
Cain***	$n(m-1)^2$	48	X				
Chin et al.	$n(m^2-m)$	60				X	

*$N|n, m$ is the number of parameters elicited when a child node and its n parents have m states each.

** The numbers of parameters correspond to default forms of use of the methods.

*** The method does not provide a computational routine for the construction of a CPT when the child node has more than three states, i.e., $m > 3$.

The basic idea in RNM [31] is that for any combination of states of the parent nodes, the most probable state of the child node is defined by a general rule. The rule is selected by the expert from four alternatives. Within the framework of the selected rule, the parent nodes can have non-equal strengths of influence on the child node. The strengths of the parent nodes are expressed through weights that are also elicited from the expert. In addition, the expert also assigns a parameter describing how dispersed around the mode the probability distribution of the child node is for given states of the parents. The whole CPT may be constructed with a single set of parameters. Alternatively, it can be constructed in parts using part-specific values of the parameters. This feature of RNM is referred to with mark X in the last column of Table 2.

A method presented by Røed et al. [55] is similar to RNM in the sense that the construction of a CPT is based on a functional relationship between the parents and the child node. Moreover, like in RNM, the parents obtain weights reflecting their strengths of influence on the child, and a single parameter defines the dispersion of the probability distributions. However, whereas RNM provides four basic rules to describe the probabilistic relationship of the nodes, the method of Røed et al. uses only one function. This function is similar to a rule alternative of RNM, in which weighted averages are taken of the states of the parent nodes. Also, in a method suggested by Hassall et al. [56], the conditional probability distributions of the child node are calculated utilizing weighted averages of the parent states. However, this method does not involve the expert evaluating the dispersion of the distributions. Furthermore, for a child node with an odd number of states m, the middle state obtains the probability $1/m$ for any combination of the parent states.

Noisy-MAX method [57], [58] is designed for settings in which parent nodes represent individual causes for a common effect, which is represented by the child node. The parameters elicited from the expert are CPT entries indicating the ability of each cause to bring about the effect individually. The rest of the CPT is calculated with the assumption that, in the presence of several causes, each cause affects the child node independently of the others. Noisy-MAX, which handles nodes with multiple ordinal states (i.e., multiple states on an ordinal scale), is a methodological extension of Noisy-OR method [53] designed for binary nodes.

The EBBN method (Elicitation for Bayesian Belief Networks) [59], the weighted sum algorithm (WSA) [60] and the Cain calculator [61] are based on the interpolation of conditional probability distributions. In these methods, the expert first assesses the conditional probability distributions of the child node for so-called anchor combinations of states of the parent nodes. The remaining conditional probability distributions of the CPT are then derived by interpolating between the anchor distributions. Both the anchor state combinations and the interpolation techniques vary between the methods.

The functional interpolation method [62] and the InterBeta method [34] also utilize the principle of interpolation to derive missing probability distributions of a CPT from method-specific anchor distributions assessed by the expert. However, in these methods, the interpolation does not directly involve the probabilities of the anchor distributions. In the functional interpolation method, each anchor distribution is approximated by a normal distribution so that best-fit estimates of the mean and variance parameters are determined. The missing probability distributions of the CPT are formed through normal distributions whose mean and variance parameters are interpolations of the estimates concerning the anchor distributions. The InterBeta method applies a similar principle except that Beta distributions are used instead of normal distributions. The InterBeta method also provides the expert an option to assign weights to parent nodes, their states, or their state combinations. Therefore, the method has mark X in the last column of Table 2. By increasing the weighting detail, the probabilistic relationship of the nodes can be portrayed more accurately.

In the likelihood method [63], the idea is that different state combinations of the parent nodes tend to move the probability distribution of the child node away from a "typical distribution" in a systematic way. The typical distribution, which is assessed by the expert, represents the probability distribution of the child node in the absence of information about the parent nodes. The conditional probability distributions in the CPT are formed by multiplying the typical distribution by likelihood terms. These terms consist of weighting factors that the expert has selected for the states of the child node and the parents. The presentation of the method in [63] lacks any detailed guideline for the elicitation of the weighting factors. Some instruction is provided in [64] along with a remark that the method becomes very complex if the child node has more than three states.

Chin et al. [65] utilize the methodology of the Analytic Hierarchy Process (AHP) [66] for the construction of a CPT. First, the expert performs pairwise comparisons of the probabilities of the states of the child node given the state of an individual parent node. These comparisons are then used to calculate probability distributions of the child node conditioned to single parent nodes. By taking products of these distributions, the final probability distributions of the CPT are obtained.

Table 2 indicates that RNM allows the construction of CPTs with a smaller number of expert-elicited parameters than most other parametric methods. In addition, RNM includes the option to define more parameters to portray the probabilistic relationship of nodes more accurately. Therefore, RNM appears to enable quick quantification of CPTs for verification and their systematic refinement without excessive manual editing of individual CPT elements. The ability to generate CPTs quickly fits well also with the utilization of sensitivity analysis in the construction of a BN. With sensitivity analysis, see, e.g., [67], one can identify the CPT elements to which the BN's behavior shows highest sensitivity. Attention can then be focused on refining these probabilities. Besides the small number of parameters to be elicited, a favorable feature of RNM is that the general rules it involves can help experts to understand and describe the probabilistic relationships between nodes [31]. Furthermore, RNM is implemented in AgenaRisk software [4], which supports its easy deployment. All the aforementioned properties of RNM make it an appealing method for the construction of CPTs in BN applications by expert elicitation. Therefore, it has been selected to be the focus of substantial elaboration in this Dissertation.

2.3 Ranked nodes method

The ranked nodes method (RNM), introduced in a seminal paper [31], is designed for constructing CPTs for a specific class of nodes called ranked nodes. A ranked node is a discrete random variable whose states are expressed with an ordinal scale such that each state can be considered to represent a range of values of a continuous quantity. The ordinal scale may consist of subjective labeled states, like {Low, Medium, High}, or it can be formed by a discretized continuous scale, like {[0 km, 5 km], [5 km, 10 km], [10 km, 20 km]}. All the nodes in the BN in Figure 1 are examples of ranked nodes. This section briefly presents the functioning of RNM and then discusses challenges concerning the method.

2.3.1 Functioning

The construction of a CPT with RNM consists of the following steps:

1. Associate the states of the nodes with subintervals of a unit scale [0, 1].
2. Select a weight expression.
3. Assign weights of the parent nodes.
4. Assign a variance parameter.
5. Compute a CPT using the above settings.
6. Verify the CPT and refine it if needed.

Steps 1–4 and 6 require the involvement of a domain expert, whereas Step 5 is carried out with a computational procedure. The main idea of each step is briefly presented below. A more through description of the steps is found, e.g., from Paper 1 of the Dissertation.

In Step 1, consecutive states of each parent and the child node are associated with consecutive subintervals of the unit scale [0, 1]. In what follows, these subintervals are referred to as state intervals. The state intervals of a single node are of equal width, do not intersect, and cover together the whole unit scale. The order in which the states of the nodes are linked with the state intervals must reflect the direction of influence of the parent nodes on the child node. An underlying assumption in RNM is that the parent nodes affect the child node in a monotonic manner. That is, states of the parent nodes associated with small values on [0, 1] must promote states of the child node also linked with small values on [0, 1]. As an illustration, Figure 4 displays how state intervals could be defined for the states of nodes *Bombing Accuracy*, *Enemy Effectiveness*, and *Target Hit Level* of the example BN in Figure 1. Now, the state intervals indicate that high bombing accuracy and low enemy effectiveness both promote high target hit level.

In Step 2, the expert selects a weight expression that can be seen as a general rule by which the parent nodes affect the child node. Technically, a weight expression is a function with a closed-form mathematical expression. Given a combination of states of the parent nodes, the weight expression maps sample points from the corresponding state intervals to points on the unit scale of the child node. There are four alternative weight expressions: 1) WMEAN, 2) WMIN, 3) WMAX, and 4) MIXMINMAX. With WMEAN, the output is the weighted average of the sample points whereas with MIXMINMAX, it is the weighted average of their minimum and maximum. WMIN and WMAX produce a value that is smaller or larger than the average of the sample points, respectively.

In Step 3, the expert selects weights to the parent nodes. The weights reflect the relative strengths of influence of the parents on the child node. If the weight expression is WMEAN, WMIN or WMAX, all the parent nodes receive their own weight. With MIXMINMAX, only two weights are assigned. For any combination of states of the parent nodes, these two weights become associated with the parents in the most extreme states.

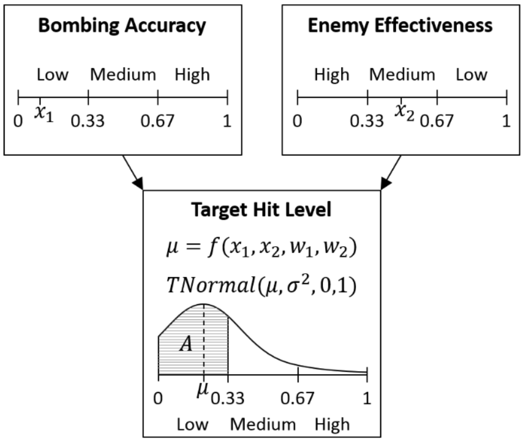

Figure 4. Generation of probability $\boldsymbol{P}$(*Target Hit Level* = *Low* |*Bombing Accuracy* = *Low*, *Enemy Effectiveness* = *Medium*) with RNM. Sample points $\boldsymbol{x_1}$and $\boldsymbol{x_2}$ from the unit scales of the parent nodes are mapped by weight expression $\boldsymbol{f}$ to mean parameter $\boldsymbol{\mu}$ on the unit scale of *Target Hit Level*. Doubly truncated normal distribution $\boldsymbol{TNormal(\mu, \sigma^2, 0, 1)}$ is then integrated over the state interval [0, 0.33], and the result is represented by area $\boldsymbol{A}$. The final probability value is obtained by taking the average of areas similar to $\boldsymbol{A}$ that result from the use of different values of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$.

In Step 4, the expert selects a value for a variance parameter. The larger this parameter is, the flatter each conditional probability distribution in the CPT will be.

In Step 5, the CPT is generated by calculating the conditional probability distributions of the child node for all the combinations of states of the parent nodes with a specific routine. With each combination of the parent states, the routine takes first sample points from the corresponding state intervals. These sample points are then mapped to points on the unit scale of the child node by the weight expression involving the parents' weights. Finally, the conditional probabilities of the child node over its ordinal states are formed by integrating normal distributions truncated to [0, 1] over the state intervals of the child node. The mean parameters of these distributions are the output values of the weight expression. The variance parameter of the distributions is the value selected by the expert in Step 4.

Figure 4 illustrates the generation of the probability P(*Target Hit Level* = *Low* |*Bombing Accuracy* = *Low*, *Enemy Effectiveness* = *Medium*) regarding the example BN in Figure 1. In Figure 4, sample points x_1 and x_2, taken from the appropriate state intervals of the parent nodes, are mapped to mean parameter μ on the unit scale of *Target Hit Level*. The mapping function is the weight expression f involving weights w_1 and w_2 assigned to *Bombing Accuracy* and *Enemy Effectiveness*, respectively. Then, $TNormal(\mu, \sigma^2, 0,1)$ distribution, i.e., normal distribution $N(\mu, \sigma^2)$ truncated to [0, 1], is integrated over the state interval [0, 0.33] of *Target Hit Level*. Here, σ^2 is the variance parameter selected by the expert. The value of the calculated integral is represented by the shaded area A in Figure 4. The final probability value is obtained by taking the average of areas similar to A that result from the use of different sample points taken from the state intervals [0, 0.33] and [0.33, 0.67] of *Bombing Accuracy* and *Enemy Effectiveness*, respectively.

In Step 6, the CPT is verified by checking that some representative conditional distributions of the child node reflect the probabilistic views of the expert. If the correspondence is deemed inadequate, the CPT has to be refined by adjusting the parameters used in RNM or possibly changing probabilities in the CPT by hand. It is also possible to divide the CPT in parts based on the states of selected parent nodes. Each part of the CPT can then be generated with a part-specific weight expression and part-specific values of the weights and the variance parameter.

2.3.2 Shortcomings

The theoretical principle of RNM lacks a detailed disclosure in the existing literature. While the basic idea of RNM is explained in [31] by referring to linear regression, the generation of CPTs is not fully explained. For instance, the generation of a CPT in Step 5 of RNM involves a user-controlled sample size parameter that affects both the values of the resulting probabilities and the time taken by the generation. However, the exact meaning or the value to be assigned to this parameter have not been discussed in the literature predating Papers 1–4 of this Dissertation.

In addition to ambiguity about its underlying principle, there is a shortage of studies on how well RNM allows to portray probabilistic relationships in real-life BN applications. One study [54] explores the ability of CPTs constructed with RNM to portray probabilistic relationships typical in human reliability analysis. However, the study does not consider the option in RNM to construct a CPT in parts. Besides this type of partitioned use of RNM, the ability of RNM-generated CPTs to represent probabilistic relationships of nodes may depend on other user-made choices too. For instance, the default range of weights in AgenaRisk [4] implementation of RNM is [1, 5], which, if adapted, may limit the representation ability of the CPTs.

Another shortcoming is that there are no exact guidelines for the elicitation of the weight expression (Step 2), the weights (Step 3), and the variance parameter (Step 4) from the expert. It is described in [31] how the selection of the weight expression can be supported by asking the expert to assess the mode of the child node for given extreme combinations of the states of the parent nodes. However, this description does not present any precise rules for inferring the suitable weight expression from the elicited information [68]. Therefore, technical insight on RNM is required to conclude whether some weight expression is suitable. Concerning the elicitation of the weights and the variance parameter, there exists no guidance, whereby they must be decided by trial and error. This means that the CPT is generated repeatedly with different values of the weights and the variance parameter until the result is considered satisfactory. For instance, the acceptance of the CPT may be based on the expert inspecting relevant conditional probability distributions. The trial and error procedure is laborious, especially because the effect of the different parameters on the CPT may not be unambiguous to observe and comprehend. There are also no instructions for deciding whether a CPT should be generated in parts and how a partition should be established.

A challenge is also a lack of guidelines concerning the application of RNM to ranked nodes formed by discretizing continuous scales. If the discretizations are conducted in ignorance of the functioning of RNM, it may become impossible to find suitable RNM parameters to portray the probabilistic relationship of the nodes. Moreover, this consequence may not be realized before time is spent in vain in the search of suitable parameter values. Furthermore, if the discretizations are changed during the use of the BN, one may end up with CPTs that produce misleading results in the analyses carried out with the BN.

Like the above discussion indicates, RNM suffers from shortcomings that can hamper its efficient use. This Dissertation is motivated by those shortcomings. The contributions of the Dissertation relate to overcoming them and developing RNM further for easier construction of CPTs, and thereby also BNs, based on expert elicitation.

3. Contributions

This Dissertation studies RNM through theoretical analysis (Papers 1–4) and experiments involving repeated trials and data fitting (Paper 1). Based on the results obtained, Paper 1 clarifies the underlying theoretical principle of RNM and provides new insight on its practical applicability. Paper 2 presents a novel approach concerning the application of RNM to nodes with continuous scales. The approach contains guidelines for the discretization of the continuous scales and the elicitation of RNM parameters. Paper 3 extends the work in Paper 2 by presenting two new discretization approaches that enhance the application of RNM to continuous nodes. Paper 4 presents a framework for the elicitation of RNM parameters concerning ranked nodes defined through ordinal scales with labeled states. Figure 5 displays an overview of the grouping of Papers 1–4 with regard to applied research methods, node types considered, and main contributions. The themes and contributions of Papers 1–4 are summarized in Table 3. The contributions of each paper are discussed in more detail below.

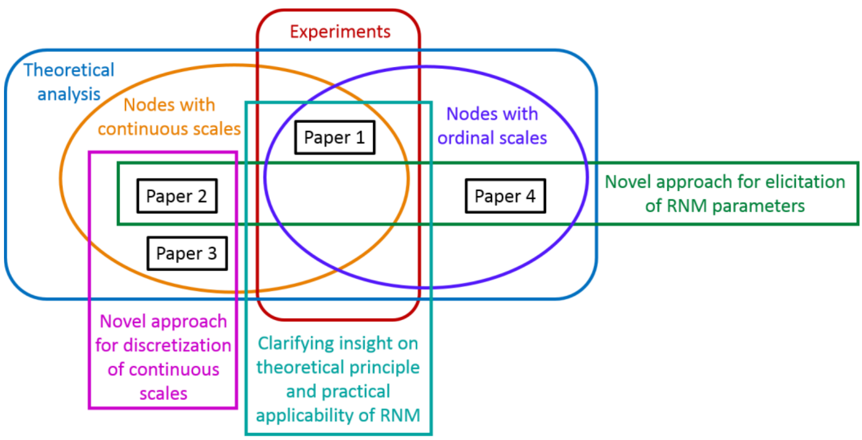

Figure 5. Grouping of Papers 1–4 with regard to applied research methods (rounded rectangles), node types considered (ovals), and main contributions (rectangles).

Table 3. Summary of the themes and contributions of Papers 1–4.

Paper	Theme	Theoretical Contribution	Pragmatic Contribution
1	Theoretical basis of RNM; Usability of RNM in real-life BN applications	Clarification of the underlying theoretical principle of RNM; Experiment on the correspondence between RNM-generated CPTs and the underlying principle; Experiment on the ability of RNM-generated CPTs to represent probabilistic relationships in practical BN applications	Basis for the methodological development of RNM; Justification for the deployment of RNM in practical applications; Advice concerning user-controlled features of RNM
2	Application of RNM to nodes with continuous scales	Exact interpretations of weights of RNM; Feasibility conditions for the weight expressions of RNM	New approach for the discretization of continuous scales of nodes and the elicitation of RNM parameters
3	Application of RNM to nodes with continuous scales	Novel principle of the probabilistic relationship of ranked nodes being encoded by initial RNM-compatible discretizations and RNM parameters	Two new approaches for the discretization of continuous scales of nodes
4	Application of RNM to nodes with discrete ordinal scales consisting of labeled states	Connection between weight intervals of parent nodes and the probability distributions of a child node; Feasibility conditions for the weight expressions of RNM	New framework for the elicitation of RNM parameters; Two MATLAB implementations to facilitate the execution of the framework

3.1 Paper 1

Paper 1 first clarifies the underlying theoretical principle of RNM by showing how the generation of probabilities with RNM is underpinned by a regression model of continuous random variables. Especially, it is shown that as the value of a user-controlled sample size parameter increases, the conditional probabilities generated with RNM converge to probabilities provided by the regression model. Increasing the value of the sample size parameter also increases the computational burden of CPT generation with RNM. Related to this feature, the paper presents an experiment in which CPTs are generated for a child node so that the number of the parent nodes, the number of states of the nodes, and the value of the sample size parameter are varied. The results of the experiment indicate that in typical applications of RNM, one can generate in a matter of seconds CPTs whose elements reflect well probabilities given by the underlying regression model. From the practical point of view, the connection between the regression model and the generated CPTs provides to the use of RNM transparency, which encourages its deployment. Furthermore, the connection means that the regression model can be used as a platform for further development of RNM. This opportunity is utilized in Papers 2–4 of the Dissertation.

Paper 1 also includes an experiment in which CPTs generated with RNM are fitted to 151 CPTs from 19 real-life BN applications. The impact of four factors on the fitting ability is studied. These factors relate to choices that the user of RNM must make while applying the method. The factors are: 1) generating a whole CPT with single values of RNM parameters in comparison to generating it in parts with alternative part-specific parameter values; 2) the ranges allowed for the weights and the variance parameter; 3) the value of the sample size parameter; and 4) a property called elementary RNM-compatibility, which is introduced in Paper 1. A child node and its parents are elementary RNM-compatible if they have an equal number of states and their probabilistic relationship fulfils a feature common to all CPTs generated with RNM for such nodes. Overall, the CPTs generated with RNM are found to provide a good average fit to a large portion of the studied CPTs. The use of part-specific RNM parameters and the elementary RNM-compatibility are discovered to be the factors affecting the accuracy of the fit the most. As a whole, the results of the fitting experiment encourage and help to justify the deployment of RNM in practical applications. Based on the results, the paper also provides advice concerning the use of RNM. Especially, the paper discusses how to recognize and evaluate the necessity to generate a CPT in parts using part-specific RNM parameters. In addition, the paper presents tentative instructions on how to define the states of ranked nodes compatibly with the functioning of RNM.

3.2 Paper 2

Paper 2 presents a novel approach for applying RNM to ranked nodes whose discrete ordinal scales are formed by discretizing continuous scales. The approach consists of three phases. First, there is a guideline for discretizing the continuous scales into ordinal scales that are compatible with the functioning of RNM. In the guideline, the domain expert evaluates the mode of the child node on its continuous scale for given values of the parent nodes. The resulting discretizations involve all the nodes getting the same number of ordinal states. These states provide a basis for the discovery of suitable RNM parameters and the construction of sensible CPTs. Second, there is a guideline for the elicitation of a weight expression and weights of the parent nodes. The guideline allows to determine a suitable weight expression and suitable values of the weights indirectly. The domain expert only has to assess the mode of the child node in specific scenarios defined by the values of the parent nodes. The expert can state the mode either as a point value or as an interval. The guideline is premised on interpretations and feasibility conditions of the weights derived in the paper. Third, there are suggestions of ways to refine a generated CPT after its verification to portray the probabilistic views of the expert more accurately.

Several features of the new approach ease the use of RNM. One such feature are the mode enquiries in the discretization and weight elicitation guidelines. For a domain expert, considering the mode of the child node in various scenarios may be a familiar form of reasoning in the daily work. Therefore, the mode enquiries can decrease the cognitive strain posed on the expert during the use of RNM. Furthermore, they provide for the expert elicitation a clear structure, which facilitates the effective execution of RNM.

In the weight elicitation guideline, a single mode assessment always yields a single weight for each weight expression. In addition, the feasibility conditions of the weights derived in the paper allow automated determination of a suitable weight expression based on the weight values determined. These features have two practical benefits. First, the suitable weight expression and the suitable weights can be determined without deep understanding of the functioning of RNM. This property promotes the use of RNM directly by people who have domain expertise and/or basic knowledge over BNs, but lack insight on methods for CPT construction. Second, as a single weight follows from a single mode assessment, the origins of the weight expression and the weights are straightforward to track. This transparency is helpful if there is a need to explain or justify the weight expression or the weights that are being used, e.g., to different stakeholders related to the BN application in which RNM is utilized. Flexibility in the weight elicitation guideline is provided by the freedom of the expert to specify the mode of the child node as an interval of any width instead of a point value. Interval-valued mode assessments allow to take into account the uncertainty of the expert about the probabilistic relationship of the nodes. The use of mode intervals enables well also group elicitation. The approach is applicable for constructing a CPT either as a whole with a single set of RNM parameters or in parts with part-specific parameter values.

3.3 Paper 3

Paper 3 presents two novel discretization approaches for applying RNM to nodes with continuous scales. These approaches can be seen as extensions of the one presented in Paper 2. In comparison to the latter, the new approaches allow more flexible and diverse application of RNM to nodes with continuous scales. Both of the approaches are based on the original idea that besides the RNM parameters, the probabilistic relationship of a child node and its parents is defined by so-called initial RNM-compatible discretizations. These discretizations are elicited from a domain expert with the discretization guideline presented in Paper 2. Accordingly, they involve all the nodes getting the same number of ordinal states. However, once the initial RNM-compatible discretizations are defined, the new approaches provide options for rediscretizing the nodes in later phases of the CPT construction and even during the use of the BN.

The first new approach is called the static discretization approach as it results in discretizations that stay intact during the use of the BN. After the elicitation of the initial RNM-compatible discretizations, this approach provides the option to rediscretize the nodes both before and after the elicitation of the RNM parameters. At both instants, the nodes can be rediscretized into any non-equal numbers of ordinal states. The rediscretizations before the elicitation of RNM parameters can be selected based on the way the domain expert naturally considers the probabilistic relationship of the nodes. Here, the weight elicitation guideline presented in Paper 2 is usable independent of the chosen rediscretizations. The rediscretizations after the elicitation of RNM parameters can be decided on the basis of the analyses that are to be carried out with the BN. This ability to change the discretizations without the need to re-elicit the RNM parameters is a novel feature that is lacking from the existing practices of applying RNM to nodes with continuous scales.

The second new approach introduced in Paper 3 is called the dynamic discretization approach. It is complementary to the static approach and usable once the initial RNM-compatible discretizations and the RNM parameters have been defined. The dynamic approach combines the use of RNM with an existing dynamic discretization algorithm [43]. Therefore, the discretizations of the nodes do not stay intact but update repeatedly during the use of the BN. For any evidence entered into the BN, the density of the discretizations increases in those areas of the nodes' continuous scales in which the probability mass is more concentrated. The dynamic non-uniform discretizations enable the portrayal of the probability distributions accurately without a demand for dense uniform discretizations. Thus, if the granularity of the discretization required in the static approach causes computational memory problems, the dynamic approach allows the probabilistic relationship of the nodes to be explored with the desired level of detail. The dynamic approach is useful especially when one would like to enter point-valued evidence into the BN or explore the statistics of the probability distributions, such as quantiles used in risk analysis [69].

Besides presenting the technical ideas of the new discretization approaches, Paper 3 explains and demonstrates how they are implemented and applied with a suitable standard BN software, using AgenaRisk [4] as an illustrative example. The ability to use the new approaches with a well-known BN software facilitates their deployment in practice.

3.4 Paper 4

Paper 4 presents a novel framework for the elicitation of RNM parameters when the ordinal scales of ranked nodes correspond to subjective labeled states.

The framework is designed for a setting in which a child node and its parents all have the same number of ordinal states. Such a setting is common in RNM applications. The framework begins with an initial elicitation procedure, in which the expert has to assess the two most probable states of the child node for specific scenarios defined by the states of the parent nodes. This is followed by a computational procedure that determines a feasible weight expression and a feasible set of weights of the parent nodes which produce CPTs compatible with the assessments of the expert. The computational procedure utilizes weight interval formulas and feasibility conditions of the weights derived in the paper. Finally, in a supplementary elicitation procedure, the expert selects point values for the weights from the feasible set along with the variance parameter. This procedure requires the expert to evaluate the probability distributions of the child node in the same scenarios that are considered in the initial elicitation procedure. The framework covers the construction of a CPT as one with a single set of RNM parameters and also in parts with part-specific parameter values. While the former is the default way, the need to shift to the latter is indicated in the framework clearly by the outcomes of different phases.

To support the application of the framework, the paper provides two MATLAB [70] implementations [71]. The first one is an implementation of the computational procedure. Given the expert assessments acquired in the initial elicitation procedure, it determines the feasible weight expression and the feasible set of weights. If a weight expression is infeasible, the implementation advices how the expert assessments should change for that weight expression to become feasible. The second implementation is designed to support the execution of the supplementary elicitation procedure. Using RNM parameters as its input, it generates and visualizes the probability distributions of the child node for the relevant scenarios.

The practical benefits of the framework are similar to those of the weight elicitation guideline presented in Paper 2 regarding nodes with continuous scales. The initial elicitation procedure and the automated computational procedure enable the determination of a feasible weight expression and a set of feasible weights in a structured way with a low elicitation effort from the domain expert. In the supplementary elicitation procedure, the value of a single weight is specified based on a single scenario. This feature makes the origins of the weights easy to track and explain. Covering the non-partitioned and partitioned ways of using RNM, and indicating the need to change from the former to the latter, are properties of the framework that further ease the application of RNM. These properties are also unique in the sense that, besides the advice presented in Paper 1, the existing literature does not present any guidelines for deciding in which of the two ways RNM should be used. In addition, like the weight elicitation guideline of Paper 2, the framework of Paper 4 also suits well for group elicitation.

4. Discussion

This Dissertation elaborates and develops the ranked nodes method (RNM) in multiple ways that enhance its use in the construction of conditional probability tables (CPTs) for Bayesian networks (BNs) and influence diagrams. With regard to the number of parameters to be elicited from a domain expert, RNM is among the least laborious methods for semi-automated construction of CPTs which is often required in real-life applications. Despite this favorable feature, the use of RNM can be cumbersome or compromised due to a lack of exact guidelines concerning the parameter elicitation and other user-controlled features. Furthermore, there remains ambiguity regarding the underlying theoretical principle of RNM. In addition, a scarcity of knowledge exists on the general ability of CPTs generated with RNM to portray probabilistic relationships appearing in practical BN applications.

The new insights and approaches presented in this Dissertation advance RNM with regard to all of the shortcomings mentioned above. Especially, Paper 1 clarifies the underlying theoretical principle of RNM and provides experimental verification on its general practical applicability. These contributions clear the way for the further development of RNM and help to justify its deployment in applications. In turn, Papers 2–4 present novel approaches for the elicitation of RNM parameters (Papers 2 and 4) and the discretization of continuous nodes into ranked nodes (Papers 2 and 3). The approaches provide thorough and well-structured means for easier as well as more flexible and versatile utilization of RNM in applications. Thus, overall, the Dissertation directly enhances the methodology for the construction of CPTs by expert elicitation. Consequently, the Dissertation also facilitates and promotes the effective and diverse use of BNs and influence diagrams to support risk management and decision-making under uncertainty in various domains.

Paper 3 demonstrates how the new discretization approaches presented therein can be implemented with AgenaRisk software [4]. In addition, Paper 4 provides two MATLAB implementations to support the application of the new framework it presents for the elicitation of RNM parameters. While all these implementations are useful, the deployment of the new approaches presented in Papers 2–4 would benefit from more advanced software realizations. For instance, one could design to AgenaRisk an add-in component that would provide a graphical user interface for the elicitation of RNM parameters and the discretization of continuous nodes according to the new approaches. Moreover, implementations in open-source languages, such as R and Python, would make RNM and the new approaches accessible to a wider audience.

This Dissertation opens up several research avenues concerning further development of RNM. The first theme concerns the validation of the new approaches of using RNM presented in Papers 2–4. The usefulness of the approaches is demonstrated in the papers with illustrative examples. Yet, experiments with humans testing the approaches in various setups are vital to understand better their strengths and weaknesses. In such experiments, knowledge could be acquired on, e.g., how fast, easily and accurately CPTs are constructed by using the approaches in comparison to construction without them.

The second theme is the determination of RNM parameters by data fitting. In the second experiment conducted in Paper 1, these parameters are determined by fitting CPTs generated with RNM to CPTs found from practical BN applications. By further elaborating the means used in the fitting, it might be possible to develop an approach in which the parameters are estimated by combining expert knowledge and any data available. With regard to data concerning continuous variables, the approach would benefit from an automated discretization procedure to form the discrete ordinal scales of ranked nodes. One aspect of this data fitting theme would be to compare how CPTs generated with RNM parameters estimated from data compare to CPTs estimated from the same data through other means.

The third future research theme is further development and exploration of specific features of the approaches presented in Papers 2–4. The first feature concerns the elicitation framework of Paper 4. The framework is designed only for a setting in which the parent nodes and the child node all have the same number of states. While the setting is common in RNM applications, extending the framework to cover nodes with unequal numbers of states would bring more relief to the use of RNM.

The second feature to explore concerns the elicitation of the variance parameter of RNM. In the guidelines of Papers 2 and 4, the variance parameter is elicited from the domain expert by trial and error, which is the present norm. That is, the expert experiments with different values of the parameter until the CPT being constructed appears satisfactory. To complement this practice, one could try to develop an alternative way to elicit the variance parameter from the expert. For instance, before the generation of the CPT, the expert could be asked to quantify specific conditional probability distributions of the child node. The variance parameter would then be determined from these assessments. In this regard, the elicitation of the variance parameter connects back to the theme of determining RNM parameters by data fitting.

The third feature worth more exploration concerns the new discretization approaches presented in Paper 3. They both utilize initial discretizations that are defined by the domain expert and involve a child node and its parents obtaining an equal number of ordinal states. Regarding this feature, it could be explored how the number of initial states of the nodes affects CPTs constructed with the new discretization approaches. This kind of study could help establish recommendations about the suitable number of initial states.

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