Representing the importance of locations in spatial decision analysis

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Abstract

This thesis explores the usefulness of different types of incomplete spatial preference information in spatial decision analysis, where the consequences of the decision are associated with a spatial region. The objective is to provide decision support and obtain a sufficient understanding of the decision maker's preferences with as little effort as possible.

The preferences of the decision maker are represented through spatial weights, which describe the importance of different locations. Preference information is said to be incomplete if these weights are not known exactly. The decision alternatives are compared, and the concept of dominance is used to find inferior alternatives that are then eliminated. The more detailed the preference information is, the more alternatives can be eliminated.

In this thesis, the utilization of incomplete preference information is based on dividing the spatial region of interest into subregions. Three factors are considered: the division into subregions, the distribution of the spatial weights inside each subregion, and the total spatial weights of the subregions. Two different test problems are solved using different types of incomplete spatial preference information. The usefulness of different representations of the decision maker's preferences is evaluated based on the number of remaining decision alternatives and the values given by two decision rules. A representation is more useful than another if the number of remaining alternatives is smaller and the values of the recommendations given by the decision rules are higher.

Some types of preference information reduce the number of remaining decision alternatives more effectively than others. The results are affected by the way in which the division into subregions is performed, as well as the number of subregions and the information on the total weights of the subregions. Information about the weight distribution inside a subregion seems to be particularly important in providing decision support.

Keywords spatial decision analysis, incomplete preference information, dominance, spatial weights

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Tiivistelmä

Tässä kandidaatintyössä tutkitaan erilaisten epätäydellisten preferenssi-informaatioiden hyödyllisyyttä spatiaalisessa päätösanalyysissa, jossa päätöksen seuraukset liittyvät maantieteelliseen alueeseen. Tavoitteena on tarjota tukea päätöksentekoon siten, että päätöksentekijän preferensseistä saadaan riittävän hyvä kuva mahdollisimman vähällä vaivalla.

Päätöksentekijän preferenssejä esitetään spatiaalipainoilla, jotka kuvaavat sijaintien tärkeyttä. Preferenssi-informaation sanotaan olevan epätäydellistä, kun näitä painokertoimia ei tunneta tarkasti. Päätösvaihtoehtoja vertaillaan, ja dominanssia käytetään huonompien vaihtoehtojen karsimiseen. Mitä tarkempaa epätäydellinen preferenssi-informaatio on, sitä useampi vaihtoehto voidaan sulkea pois jatkotarkastelusta.

Tässä työssä epätäydellisen preferenssi-informaation hyödyntäminen perustuu siihen, että tarkastelualue jaetaan osa-alueisiin. Työssä tarkastellaan kolmea tekijää, jotka ovat jako osa-alueisiin, spatiaalipainojen jakautuminen osa-alueiden sisällä sekä osa-alueiden kokonaispainot. Kaksi eri testiongelmaa ratkaistaan hyödyntämällä erilaisia epätäydellisiä spatiaalisia preferenssi-informaatioita. Preferenssi-informaation eri esitystapojen hyödyllisyyttä arvioidaan jäljelle jäävien vaihtoehtojen määrän sekä kahden päätössäännön antamien suositusten arvojen perusteella. Preferenssi-informaatio on sitä hyödyllisempää, mitä vähemmän vaihtoehtoja jää jäljelle ja mitä korkeampia arvoja päätössääntöjen antamat suositukset saavat.

Jotkin epätäydellisen preferenssi-informaation esitystavat vähentävät jäljelle jäävien vaihtoehtojen lukumäärää tehokkaammin kuin toiset. Tuloksiin vaikuttavat osa-aluejaon toteutustapa, osa-alueiden lukumäärä sekä tapa, jolla tieto osa-alueiden kokonaispainoista on esitetty. Osa-alueen sisäistä painojakaumaa koskeva tieto vaikuttaa olevan erityisen tärkeä tekijä.

Avainsanat spatiaalinen päätösanalyysi, epätäydellinen preferenssi-informaatio, dominanssi, spatiaalipainot

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1 Introduction

Making decisions is an essential part of life. We may have to decide what to eat for breakfast, whom to vote for, or which university to apply to. Making these decisions requires comparing the alternatives available to us, and selecting the best one according to our preferences – whether this analysis is carried out consciously or not. In spatial decision analysis, the decisions and their consequences are associated with a geographical region in some way. A typical example is selecting where to position a fire station in a city. The response time from the station is not constant across the whole region, but varies depending on the location of the destination.

In decision analysis, value functions can be used to compare alternatives. A value function gives each alternative a numerical value that corresponds to the preferences of the decision maker (DM). If the DM's preferences satisfy certain axioms, there exists a value function that corresponds to these preferences. However, constructing this value function is not trivial. A well-known additive multi-attribute value function is presented by Keeney and Raiffa (1976). An additive value function is also used in this thesis, but values are added over different locations rather than attributes.

The preferences of the DM may not be known exactly, or in some cases considering every aspect of the problem thoroughly can be an infeasible task (see, e.g., Ferretti and Montibeller, 2016). This has lead to methods that use incomplete information on the DM's preferences to provide decision support (see, e.g., Kirkwood and Sarin, 1985; Salo and Hämäläinen, 1992; Athanassopoulos and Podinovski, 1997; Salo and Hämäläinen, 2001, 2010). Set choice problems under incomplete information are discussed by, e.g., Podinovski (2010). Furthermore, Punkka and Salo (2013) address preference programming with incomplete ordinal information.

As Keller and Simon (2019) point out, the complexity of spatial decision problems exceeds that of the more traditional ones, since the decision's outcomes vary across a geographical region. Simon et al. (2014) and Harju et al. (2019) present decision models in the spatial context to address this matter. In these models, preference information is represented by weights that are assigned to each spatial location and attribute within the region of interest. Due to the possibly large number of locations, exact preference information on spatial weights is not always easily available. The use of incomplete preference information in spatial context is discussed by Harju et al. (2019).

This thesis follows the spatial models by Simon et al. (2014) and Harju et al. (2019). To address the incomplete information on spatial weights, the region of interest is divided into subregions. The DM's preferences are then represented based on these subregions. The ways in which incomplete spatial preference information can be provided are many. The DM can, e.g., order the subregions based on their importance, set some limits for the total weights of the subregions, or provide information on the weight distribution within each subregion. This thesis explores which are the most useful ways of representing the incomplete spatial preference information. The objective is to obtain a sufficient understanding of the DM's preferences with as little effort from the DM as possible.

This thesis is structured as follows. Section 2 provides an introduction to spatial decision analysis and incomplete spatial preference information. Section 3 discusses how incomplete preference information is utilized to provide decision support and presents the computational aspects of this thesis. In Section 4, the usefulness of different types of incomplete spatial preference information is explored in two test problems. The results of these experiments are discussed in Section 5. Finally, Section 6 concludes the work.

2 Introduction to spatial decision analysis

2.1 Preliminaries

In this thesis, the spatial region of interest is represented by the set S. Each location s belongs to this region, $s \in S$. The discrete model assumes that the set S consists of a finite number of locations $s_1, s_2, \ldots, s_n$. In the non-discrete model, the number of locations in S is infinite. The subsets $S' \subseteq S$ of the region are referred to as subregions. The set of all possible consequences is denoted by C, and all consequences c of a decision are elements of this set, $c \in C$. A decision alternative $z \in Z$ is a function that assigns a consequence to each location within the region of interest. The set of all decision alternatives is thus denoted by $Z = \{z \mid z : S \to C\}$ (Harju et al., 2019).

Consider the problem of deciding where to position a fire station with the objective of providing help across a city as fast as possible. Similar types of examples of choosing positions for fire stations have been presented by Simon et al. (2014) and Honkasaari (2016). The locations s represent the points on the map, and the consequences $z(s)$ describe the response time from the fire station to each location. For a fire station at (x, y) the response time could be formulated as $z(s) = \sqrt{(x - s_x)^2 + (y - s_y)^2}$, assuming proportionality to the distance between the fire station and the location in question. Even though in this example $z(s) \in C = [0, \infty[$, the set of consequences can also be finite.

In decision analysis, the DM's preferences are represented by the preference relation

$$z \succsim z'. \tag{1}$$

The notation indicates weak preference: the decision alternative z is at least as preferable as z' (see, e.g. French, 1986).

In the fire station example above, a short response time is desirable. However, when two alternatives are compared, one fire station position can be better for some locations on the map, and the other position may be better for some other locations. Comparing the alternatives directly is thus not easy, and therefore a more analytical approach is required.

2.2 Spatial value function

A value function $V : Z \to \mathbb{R}$ is said to represent the preference relation (1) if the following equivalence holds

$$V(z) \geq V(z') \Leftrightarrow z \succsim z'. \tag{2}$$

The value function thus describes the relative preferability of the decision alternatives (see, e.g., Keeney and Raiffa, 1976).

In practice, there almost always exists some function that represents the DM's preferences. However, discovering this function, or even the functional form, can be difficult. Simon et al. (2014) and Harju et al. (2019) present conditions for representing the DM's preferences with a spatial value function, but these conditions are not discussed in this thesis. It is assumed that the preferences can be represented with a spatial value function.

This thesis follows the spatial decision model by Simon et al. (2014), extended by Harju et al. (2019). In the discrete model, the number of locations is finite, i.e., consider locations $s_i \in S$, where $i \in I = \{1, 2, \dots, n\}$, and the additive spatial value function is

$$V(z) = \sum_{i=1}^{n} a_i v(z(s_i)). \tag{3}$$

In the model, a_i is the spatial weight describing the importance of the location s_i. The spatial weights are non-negative and they sum to one, i.e., $\sum_{i=1}^{n} a_i = 1$. The function $v : C \to \mathbb{R}$ is the consequence value function that assigns a scalar value $v(c)$ to each consequence $c \in C$. The consequence value function does not depend on the location, but is the same across the whole region. A usual convention, which is followed in this thesis as well, is scaling it such that $v(c) \in [0, 1]$ for all $c \in C$.

2.3 Incomplete preference information

When the exact spatial weights a_i are known, comparing the decision alternatives with the spatial value function $V(z)$ is straightforward. However, as the number of locations s_i increases, defining all spatial weights becomes a laborious, and likely a very challenging task. The DM might not be sure about the exact spatial weights of the locations in question, or there may simply be too many locations to consider. When the exact weights are not known, the preference information is said to be incomplete.

2.3.1 Set of feasible weights

When the preference information is incomplete, the decision alternatives are compared based on a set of feasible weights (see, e.g., Salo and Hämäläinen, 1992). The feasible weights satisfy the available preference information, and the exact weights are an element of this set.

The set of feasible spatial weights is denoted by $A \subseteq A^0$. The base set A^0 contains all the possible spatial weighting vectors a (Harju et al., 2019),

$$A^0 = \{a \in [0,1]^n \mid \sum_{i=1}^{n} a_i = 1\}. \tag{4}$$

The DM provides preference information via preference statements that are then interpreted as mathematical constraints. The set of feasible weights A consists of all the weights that do not contradict these preference statements.

The introduction of new preference statements narrows the set of feasible spatial weights, and the set A is replaced by its subset $A' \subseteq A$. If no preference statements are given, the set of feasible weights is equal to the base set, i.e., $A = A^0$. If the set of feasible weights is singleton, $A = \{a\}$, the spatial preference information is complete.

2.3.2 Dominance

The value of the additive spatial value function $V(z)$ depends on the spatial weights a_i. Thus, when spatial preference information is incomplete, the values of $V(z)$ cannot be uniquely defined. However, the minimum and maximum values with respect to the set of feasible weights can be obtained.

The set of all decision alternatives Z consists of all functions $Z : S \to C$, as mentioned in Section 2.1. However, only a fraction of these functions represent some concrete alternative. In the fire station example, the set of all decision alternatives contains functions that do not correspond to any real position candidate for the fire station, such as $z(x, y) = 1$. Thus, instead of comparing all the possible decision alternatives in Z, the consideration is limited to the set of concrete alternatives, denoted by $\tilde{Z} \subseteq Z$.

Additional preference statements may reduce the set of feasible weights and can thus narrow the set of possible values for $V(z)$. These minimum and maximum values are then used to rule out inferior alternatives and provide decision recommendations. If the minimum value of some alternative is greater than the maximum value of some other alternative, the first alternative is better. If the value intervals overlap, the preference order of two alternatives can in some cases be obtained using the concept of dominance (see, e.g., Keeney and Raiffa, 1976).

Dominance describes if a decision alternative is inferior to another according to the DM's preferences. An alternative z dominates z', if the following conditions hold:

$$\begin{cases} V(z) \geq V(z'), \text{ for all } a \in A \\ V(z) > V(z'), \text{ for some } a \in A. \end{cases} \tag{5}$$

An alternative thus dominates another, if its value is greater or equal for all feasible weights $a \in A$, and strictly greater for some feasible weight.

The alternatives are divided into dominated and non-dominated. The DM should only consider non-dominated alternatives, since for every dominated alternative there exists at least one better alternative. The set of non-dominated alternatives, denoted by Z_{ND}, is often reduced when additional preference statements are introduced.

2.3.3 Decision rules

The incomplete preference information does not necessarily reduce the number of non-dominated alternatives to one. If no further preference information is sensibly available, decision rules can be used to obtain recommendations as to which of the non-dominated alternatives to choose. However, there is no guarantee that a decision rule will find the best alternative. These rules of thumb are only guidelines that should not be relied on excessively.

Salo and Hämäläinen (2001, 2010) present multiple different decision rules which can be used to obtain decision recommendations. In the test cases of Section 4 of this thesis, the following two decision rules are considered:

(i) *Central values*: Choose the alternative $z \in \tilde{Z}$ for which the sum of the maximum and minimum values is largest, i.e., $[\max_{a \in A} V(z) + \min_{a \in A} V(z)] \geq [\max_{a \in A} V(z') + \min_{a \in A} V(z')]$, for all $z' \in \tilde{Z}$.

(ii) *Minimax regret*: Choose the alternative $z \in \tilde{Z}$ for which the maximum regret, that is, the largest difference between $V(z)$ and other alternatives, is smallest, i.e., $\max_{a \in A}[V(z') - V(z)] \leq \max_{a \in A}[V(z') - V(z'')]$, for all $z', z'' \in \tilde{Z}$.

3 Utilization of incomplete spatial preference information

In this section, ways of representing spatial preference information are discussed and the computational aspects of establishing dominance and computing recommendations by decision rules are presented. The aim is that the methods used are both understandable for the DM and computationally reasonable.

3.1 Representing incomplete spatial preference information

3.1.1 Division into subregions

In order to obtain information about the spatial weights, the region of interest is divided into subregions. Whereas specifying a value for every spatial weight a_i can be a laborious task, the DM may find it easier to consider the relative importance of the subregions. Regarding the division, there are two main points that will be explored in the problems of Section 4: how to perform the division, and how does changing the number of subregions affect the results, i.e., the number of non-dominated alternatives and the values given by the decision rules.

The subregions can in theory be of any shape or consist of multiple separate parts, and the number of subregions is only restricted by the condition that every subregion must contain at least one location. However, for simplicity, only the following two divisions are considered in the test problems of Section 4:

(i) The most straightforward option is to divide the region into rectangles of equal shape and size. However, assessing the subregion weights with this type of division can be cognitively difficult for the DM, since the subregion borders do not likely represent any true borders between, e.g., different city districts or counties.

(ii) Another option is to divide the region of interest into subregions such that the subregions covering more important areas are smaller. Even though this latter division requires some information about the subregion weights, the DM often has some understanding about the relative importance of the subregions. Also in a real life situation, the DM would likely participate in the division process.

3.1.2 Weight distribution inside a subregion

The DM may give some statements about how the importance of locations varies inside a subregion. This information may narrow the possible values of $V(z)$, and can thus be useful in providing decision support. The weight of a subregion can be concentrated in one location, distributed evenly across the subregion, or something between these two extremes.

The default situation is that the DM provides no information about the weight distribution within the subregion. The minimum and maximum values are obtained when all the weight of the subregion is concentrated in one point (Harju et al., 2019). This approach thus provides the lowest minimum and the highest maximum for $V(z)$.

The second alternative is that the weight is distributed evenly across the subregion, i.e., every location within the subregion has the same spatial weight. However, the assumption of the weight being evenly distributed across the subregion may be hard to justify.

If the situation is something between the previous two cases, the DM may describe the weight distribution within the subregion with a smoothness parameter that is given a value between 0 and 1 for each subregion. In this thesis, two definitions for this parameter are considered, according to which the information is intrepreted (M. Harju, personal communication, June 2020). According to both definitions, the parameter value 0 corresponds to the default case, where all weight of the subregion is concentrated in one location. The parameter value 1 in turn implies equal weight distribution across the subregion.

According to the first definition, the smoothness parameter gives a lower bound for the ratio of a weight of a location and the average weight of the locations over the subregion. Let us denote this parameter for subregion S^k with λ_k. When minimizing, a weighted average of minimum and mean is used. For maximization, the weighted average is taken of maximum and mean.

According to the second definition, the smoothness parameter gives a lower bound for the ratio of the weights of two locations within the subregion. Let us denote this parameter for the subregion S^k with ψ_k. For example, if the parameter value is $\psi_k = 1/2$, the largest spatial weight within the subregion cannot be more than twice the smallest one. When minimizing (maximizing), every location of the subregion is given one of two weights whose ratio is ψ_k. Denote the number of locations in the subregion S^k with r_k. Then, μ_k locations with the lowest (highest) consequence value are given the higher weight and the remaining $r_k - \mu_k$ locations are given the lower weight.

The problem with the first definition of the smoothness parameter, λ_k, lies in that the average of the spatial weights within the subregion might not be that understandable as a concept for the DM. On the other hand, this first definition is computationally convenient. The second definition, ψ_k, may be more intuitive for the DM, but has a drawback: the optimal value of μ_k cannot be deduced without exploring all possible values $\mu_k = 1, \ldots, r_k - 1$ and then choosing the one yielding the best result. The optimization of weights inside a subregion in the cases presented in this section is discussed in more detail in Section 3.2.1.

3.1.3 Regional spatial weights

The spatial weight α of a subregion S' describes its relative importance, and is the sum of the weights of the locations within the subregion,

$$\alpha(S') = \sum_{i \in I | s_i \in S'} a_i. \tag{6}$$

After the region of interest has been divided into subregions, as described in Section 3.1.1, the DM may give some preference statements concerning the subregion weights.

The preference information on subregion weights can be ordinal, i.e., the DM may order the subregions by importance. If there are subregions S^1, S^2 and S^3, the DM may for example state that $\alpha(S^2) \geq \alpha(S^1) \geq \alpha(S^3)$. This ordinal information could also be given in a more detailed format. That is, the DM may for example state that subregion S^2 is at least twice as important as subregion S^1. This would result in the inequality $\alpha(S^2) \geq 2\alpha(S^1)$. Giving such detailed statements, however, might not be easy for the DM.

Instead of, or in addition to, ordinal information, the DM may give some intervals for the subregion weights. For example, if the DM knows that the weight of subregion S^1 lies within the interval 10–15 %, the resulting inequalities are $\alpha(S^1) \geq 0.10$ and $\alpha(S^1) \leq 0.15$.

3.2 Computational aspects

Whether or not the decision alternative z dominates z' can be established computationally by solving a set of optimization problems as in non-spatial problems (see, e.g., Salo and Hämäläinen, 1992; Athanassopoulos and Podinovski, 1997). These optimization problems consist of minimizing and maximizing the value difference $V(z) - V(z')$. The conditions (5) imply that alternative z dominates z' if the minimum is non-negative and the maximum is positive. Computing the recommendations by decision rules is also done by solving linear optimization problems, as will be described in Section 3.2.2.

When computing dominance and the recommendations by decision rules, both the weight distributions inside the subregions and the subregion weights have to be considered. It turns out that these two can be treated separately. Assume that the region of interest S is divided into ℓ subregions S^k, $k \in \{1, 2, \ldots, \ell\}$. Let $f_i \in \mathbb{R}, i \in I = \{1, 2, \ldots, n\}$, be real-valued numbers, and let $I^k \subseteq I$ be the indices of the locations of subregion S^k. Then

$$\sum_{i=1}^{n} a_i f_i = \sum_{k=1}^{\ell} \sum_{i \in I^k} a_i f_i = \sum_{k=1}^{\ell} \sum_{i \in I^k} a_i \frac{\sum_{i \in I^k} a_i f_i}{\sum_{i \in I^k} a_i} = \sum_{k=1}^{\ell} \alpha(S^k) \frac{\sum_{i \in I^k} a_i f_i}{\sum_{i \in I^k} a_i}. \quad (7)$$

If the numbers f_i represent the consequence values $v(z(s_i))$, the fraction in the resulting expression can be interpreted as a weighted average of the consequence values over the locations within the subregion S^k. The subregion weight $\alpha(S^k)$ and this fraction are in a sense independent of each other, but when minimizing or maximizing, the optimal value of $\alpha(S^k)$ depends on the values of this fraction for different subregions. The weight distribution inside a subregion must thus be considered first, as it affects the optimization of subregion weights.

3.2.1 Addressing the weight distribution inside a subregion

The expression $\sum_{i \in I^k} a_i f_i$ is minimized differently with respect to the feasible spatial weights $a \in A$, depending on what is known about the weight distribution within each subregion S^k. The four possible cases of the weight distribution inside each subregion are described in Section 3.1.2, and the computational aspects are presented in this section.

If no information about the weight distribution within the subregion is available, or if it is known that all weight is concentrated in one location, the minimization can be expressed in terms of subregion weights $\alpha(S^k)$ as follows

$$\min_{a \in A} \sum_{i \in I^k} a_i f_i = \sum_{i \in I^k} a_i \min_{i \in I^k} f_i = \alpha(S^k) \min_{i \in I^k} f_i. \tag{8}$$

Denote the number of locations within the subregion S^k with r_k. If the weight is distributed evenly across the subregion, the minimization becomes

$$\min_{a \in A} \sum_{i \in I^k} a_i f_i = \sum_{i \in I^k} a_i \sum_{i \in I^k} \frac{f_i}{r_k} = \alpha(S^k) \sum_{i \in I^k} \frac{f_i}{r_k}. \tag{9}$$

If the weight inside the subregion is neither evenly distributed nor concentrated in one location, the smoothness parameter is used. According to the first definition in Section 3.1.2, the parameter λ_k sets a lower bound for the ratio of minimum and mean. The minimization is thus

$$\min_{a \in A} \sum_{i \in I^k} a_i f_i = \alpha(S^k) \left[\lambda_k \sum_{i \in I^k} \frac{f_i}{r_k} + (1 - \lambda_k) \min_{i \in I^k} f_i \right]. \tag{10}$$

As discussed in Section 3.1.2, the second definition of the smoothness parameter, ψ_k, gives a lower bound for the ratio of the weights of two locations within the subregion. This is computationally more challenging than the first definition, but might be more intuitive for the DM. The higher weight is defined as

$$\gamma_k = \frac{1}{\mu_k + \psi_k (r_k - \mu_k)}. \tag{11}$$

Since the ratio of the lower weight and the higher weight is by definition the value of the smoothness parameter ψ_k, the lower weight is $\psi_k \gamma_k$. Reorder $f_i, i \in I^k$, in increasing order, and denote these reordered numbers with $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{r_k}$, such that $\tilde{f}_1 \leq \tilde{f}_2 \leq \cdots \leq \tilde{f}_{r_k}$. The minimization then becomes

$$\min_{a \in A} \sum_{i \in I^k} a_i f_i = \min_{\mu_k \in \{1, \ldots, r_k - 1\}} \alpha(S^k) \left[\gamma_k \sum_{i=1}^{\mu_k} \tilde{f}_i + \psi_k \gamma_k \sum_{i=\mu_k+1}^{r_k} \tilde{f}_i \right]. \tag{12}$$

Maximization of the expression $\sum_{i \in I^k} a_i f_i$ in the four different weight distribution cases is computed similarly to minimization in (8)–(10) and (12), but only maximizing instead of minimizing.

3.2.2 Addressing the subregion weights

Recall the definition of the subregion weights (6), and denote them with $\omega_k = \alpha(S^k)$, $\omega \in \Omega$, where Ω is the set of feasible subregion weights. The feasible subregion weights are a subset of the base set Ω^0, which is defined similarly to (4),

$$\Omega^0 = \{\omega \in [0,1]^\ell | \sum_{k=1}^{\ell} \omega_k = 1\}. \tag{13}$$

The DM's preference statements concerning subregion weights, discussed in Section 3.1.3, can be transformed into linear constraints. These constraints are then collected into a $t \times \ell$ matrix Q, where t is the number of constraints. The set of feasible subregion weights Ω satisfies the DM's preference statements and can now be formally defined as

$$\Omega = \{\omega \in \Omega^0 \mid Q\omega \leq \bar{0}\}. \tag{14}$$

The mathematical aspects of the formulation of the constraints and the composition of the matrix Q are presented by Harju et al. (2019) and Anttila (2019).

The optimization problems for establishing dominance are stated using the following definitions.

$$\min_{\omega \in \Omega} V(z) - V(z'), \tag{15}$$

$$\max_{\omega \in \Omega} V(z) - V(z'), \tag{16}$$

$$\text{s.t.} \sum_{k=1}^{\ell} Q_{\phi k}\omega_k \leq 0, \text{ for all } \phi \in \{1, 2, \ldots, t\}, \tag{17}$$

$$\sum_{k=1}^{\ell} \omega_k = 1, \tag{18}$$

$$\omega_k \geq 0. \tag{19}$$

The alternative z thus dominates z' if the minimum (15) is non-negative and the maximum (16) is positive. The same constraints are used both in minimization and maximization. The constraints (17) correspond to the DM's preference statements, and the constraints (18) and (19) in turn follow from the definitions that the subregion weights ω_k sum to one and are non-negative.

The decision rules presented in Section 2.3.3 are applied on the set of non-dominated alternatives. The subregion weights ω_k are constrained similarly as in establishing dominance, and thus the computations are done subject to the constraints (17)–(19). The objective functions are presented below.

(i) *Central values*:

$$\max_{z \in Z_{ND}} \left[\min_{\omega \in \Omega} V(z) + \max_{\omega \in \Omega} V(z)\right]. \tag{20}$$

(ii) *Minimax regret*:

$$\min_{z \in Z_{ND}} \max_{\omega \in \Omega} \max_{z' \in Z_{ND}} V(z') - V(z). \tag{21}$$

Dominance and the recommendations by decision rules can be computed with optimization problems of the same form as the minimization of $\sum_{i \in I^k} a_i f_i$ in Section 3.2.1. If f_i is replaced with the consequence value of the alternative z at location s_i, that is, $v(z(s_i))$, the spatial value function presented in (3) is obtained. Replacing f_i with the difference $v(z(s_i)) - v(z'(s_i))$ in turn results in the value difference between alternatives z and z', namely $V(z) - V(z')$. For example, if the weight is concentrated in one location in each subregion as in (8), the minimization (15) for establishing dominance becomes

$$\min_{\omega \in \Omega} V(z) - V(z') = \min_{\omega \in \Omega} \sum_{k=1}^{\ell} \omega_k \min_{i \in I^k} [v(z(s_i)) - v(z'(s_i))]. \tag{22}$$

Thus, the expressions (8)–(10) and (12) can be used when computing dominance in (15) and (16), and decision rules in (20) and (21).

4 Exploration with test problems

In this section, two different spatial decision problems are examined using various types of incomplete spatial preference information. The usefulness of different factors – ways of representing preference information – is evaluated based on the number of non-dominated decision alternatives. The values given by the recommendations of the decision rules, presented in Section 2.3.3, are also compared to the value of the actual best alternative. All computations are done in MATLAB. Different subregion divisions considered in the two test problems are presented in Appendix A, and the results with different types of incomplete preference information are listed in Appendix B.

In the experiments, the preference information consists of three factors: division into subregions, weight distribution inside a subregion, and regional spatial weights. As described in Section 3.1.1, the division into subregions is performed in two different ways: the first is dividing the region of interest into rectangles of equal shape and size, and the second is dividing the region into subregions such that the subregions are smaller in more important areas. In this thesis, the former is referred to as regular division, and the latter as irregular division.

The effect of the number of subregions on the results is also explored. The different numbers of subregions considered in these test problems are 6, 9, 20, 40 and 100. Increasing the number of subregions even further would not be practical, since then the DM would likely have too many subregions whose importance to consider, and determining constraints for subregion weights would become too laborious a task.

Six different scenarios of weight distribution inside a subregion are considered. These include free distribution, even distribution, and using each of the two definitions of the smoothness parameter in two ways. The first is to take the exact value of the parameter, calculated from the exact weights, and round it down to the nearest tenth. This corresponds to the real-life situation, where the DM can state the parameter values, i.e., lower bounds for the ratios described in Section 3.1.2, somewhat accurately. Another way is to set the smoothness parameter values to 0.5 for each subregion. Taking the second definition, $\psi = 0.5$, this would imply that the maximum spatial weight over the locations inside a subregion can be at most twice the minimum spatial weight.

Five different representations of the subregion weights are compared. These are complete order, 5% weight intervals (e.g., subregion weight lies in the interval 5–10%), complete order together with 5% weight intervals, 1% weight intervals (e.g., subregion weight lies in the interval 3–4%), complete order with 1% weight intervals, and lastly exact weights for reference.

4.1 Fire station positions

4.1.1 Problem formulation and exact solution

Fire coverage is an important factor in the overall safety of a city. Properly located fire stations can help save human lives and be instrumental in preventing severe damage caused by fire. Simon et al. (2014) present a hypothetical problem of selecting positions for three fire stations in a city. This example problem is followed in this section, with the exception of using a discrete model instead a non-discrete model, and having ten predefined position candidates for the fire stations.

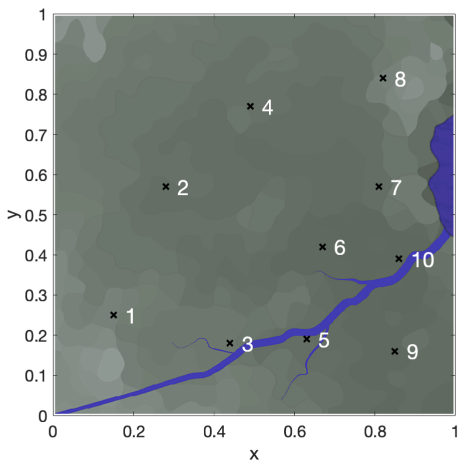

Figure 1: The map and the position candidates for the fire stations.

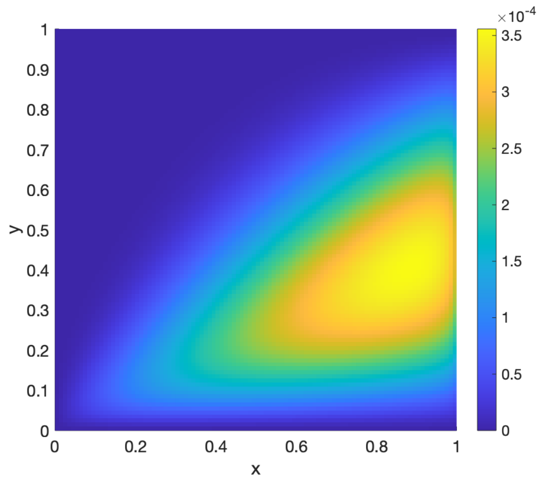

Figure 2: The exact spatial weights a_i of the locations.

The city map is assumed to be square and the dimensions are normalized from 0 to 1 in both x and y. The map resolution is 100×100, i.e., the region is divided into 10,000 locations $s \in S$ in which the consequences are calculated. Ten position candidates for the fire stations are preselected, numbered from 1 to 10. These position candidates are listed in Table 1 and illustrated in Figure 1. There are thus $\binom{10}{3} = 120$ different three-position combinations $z \in \tilde{Z}$, which are the concrete decision alternatives in this problem. A more detailed description of the problem can be found in Simon et al. (2014).

Table 1: The coordinates of the position candidates for the fire stations.

Position candidate	Coordinates (x, y)	Position candidate	Coordinates (x, y)
1	(0.15, 0.25)	6	(0.67, 0.42)
2	(0.28, 0.57)	7	(0.81, 0.57)
3	(0.44, 0.18)	8	(0.82, 0.84)
4	(0.49, 0.77)	9	(0.85, 0.16)
5	(0.63, 0.19)	10	(0.86, 0.39)

Development in the city is concentrated along a river that flows northeast into a bay near the center of the eastern boundary of the city (see Figure 1). The areas near the bay and along the river are the most important, and the western and northwestern parts along with the very boundaries of the city are the least important. The exact spatial weight of a location (x, y) is defined by

$$a(x, y) = \frac{x^{1.1}(1-x)^{0.1}y^{1.5}(1-y)^{(1.425-0.6x)/(0.05+0.4x)}}{B(1.5, (1.425-0.6 \cdot x)/(0.05+0.4 \cdot x))}, \tag{23}$$

where B is the beta function (Simon et al., 2014). The exact spatial weights are illustrated in Figure 2.

The consequences $c \in C = \mathbb{R}_+$ describe the average response time of the fire fighters. Taking into account that for a fraction $\xi = 0.15$ of incidents, the station assigned to respond is not the closest one, the average response time at location $s \in S$ for a combination z becomes

$$z(s) = \sum_{i=1}^{3} \xi^{i-1}(1-\xi) \min\{|s_x - K_x^{(i)}| + |s_y - K_y^{(i)}|, 1\}, \tag{24}$$

where $K^{(i)}$ is the ith closest station from s. When representing the distance between s and $K^{(i)}$, the Manhattan norm is used, since in a metropolitan area one often travels along the gridlines.

The shorter the response time, the smaller the size of the fire that the responder has to fight. Given this connection, the value function for the response time gives large values for the range of response times that will ensure the survival of the buildings.

For times slightly above this range, the value decreases exponentially. Thus, the consequence value function is

$$v(c) = \frac{1 - e^{-3.86(1-c)}}{1 - e^{-3.86}}. \tag{25}$$

The decision alternatives z are assessed with the additive spatial value function $V(z)$ presented in (3). When computing with the exact spatial weights, the best three-station combination consists of the positions 3, 7 and 10, and the value of this alternative is 0.9636. The results obtained with incomplete preference information, i.e., the values of the recommendations of the decision rules, are compared to this value.

4.1.2 Results with incomplete preference information

The region of interest is divided into subregions in two different ways. The regular division into 20 subregions is illustrated in Figure 3. In the irregular division, the city is first divided into three zones: downtown, midtown and uptown. Downtown consists of the most important areas, whereas uptown covers the least important areas. These three zones are then further divided into smaller subregions such that the subregions are smallest in downtown and largest in uptown. Figure 4 presents the irregular division into 20 subregions, with downtown, midtown and uptown denoted with purple, blue and green, respectively. Downtown covers subregions 1–8, midtown subregions 9–17, and uptown subregions 18–20. Other subregion divisions considered in this section can be found in Appendix A.

Tables B2 and B3 in Appendix B present the comparison between different numbers of subregions, and the regular and irregular division. In both tables, the weight distribution inside a subregion is constrained with the smoothness parameter ψ such that the exact lower limit for the ratio of minimum and maximum spatial weights over the subregion is rounded down to the nearest tenth, and the smoothness parameter is given this rounded value. The difference between the two tables concerns the information on subregion weights: in Table B2, the computations are conducted with 1% weight intervals, whereas in Table B3, complete importance order of the subregions is utilized.

Irregular division seems to almost always outperform regular division in terms of the number of non-dominated alternatives. The difference is especially large when comparing regular and irregular division into 40 subregions in Table B2: the number of non-dominated alternatives obtained with irregular division is 42, whereas computation with regular division provides 61 non-dominated alternatives. When the information on subregion weights is represented as 1% intervals, the number of non-dominated alternatives first decreases when increasing the number of subregions from 9 to 20 and further to 40. However, the number of non-dominated alternatives obtained with 100 subregions is larger than that obtained with 40 subregions. The

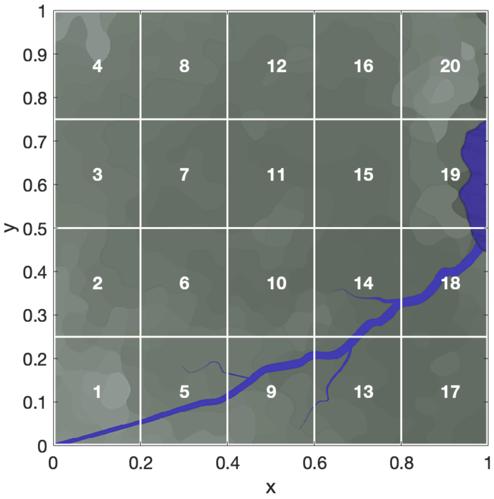

Figure 3: Regular division into 20 subregions.

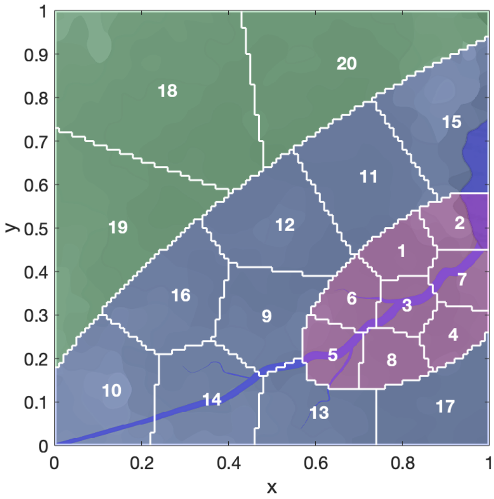

Figure 4: Irregular division into 20 subregions. Downtown, midtown and uptown are denoted with purple, blue and green, respectively.

reason for this is that as the number of subregions increases, the weight of a single subregion decreases. Consequently, most subregion weights are within the interval 0–1% or 1–2%, and thus the 1% weight intervals provide little additional information about the relative importance of the subregions. When the information about subregion weights is represented as a complete order, the number of non-dominated alternatives decreases on every increase of the number of subregions all the way from 9 to 100.

The central values and minimax regret decision rules provide rather good recommendations for both divisions and all numbers of subregions, i.e., the values of the recommendations are quite close to the value of the best alternative. There is not much variation between the two rules. Pointed out by Table B2, the values of the recommendations by the decision rules are slightly higher for smaller number of subregions, but the differences between regular and irregular division are small, so this might only be a coincidence. Table B3 indicates that the central values rule provides slightly better recommendations for larger number of subregions when the region is divided regularly, if the complete importance order of the subregions and 1% intervals for the subregion weights are known.

To explore how the different representations of the weight distribution inside a subregion affect the results, the region of interest is divided irregularly into 20 subregions, as illustrated in Figure 4. The subregion weights for the 20 subregions are given as 1% intervals. These results are presented in Table 2. More extensive results with other representations of the information on subregion weights can be found in Table B1.

Table 2: Fire station positions: comparison of different weight distributions inside a subregion. The computations are carried out with the irregular division into 20 subregions, and the information on subregion weights is represented as 1% intervals.

Weight distribution inside a subregion	Non-dominated alternatives	Central values positions	Central values value	Minimax regret positions	Minimax regret value
Free	103/120	3,8,10	0.9581	3,7,10	0.9636
Exact λ rounded down	88/120	3,8,10	0.9581	3,7,10	0.9636
Exact ψ rounded down	74/120	3,8,10	0.9581	3,7,10	0.9636
$\lambda = 0.5$	61/120	3,7,10	0.9636	3,7,10	0.9636
$\psi = 0.5$	12/120	3,7,10	0.9636	3,7,10	0.9636
Even	3/120	3,7,10	0.9636	3,7,10	0.9636

Free distribution provides the worst results in terms of the number of non-dominated alternatives, as can be seen in Table 2. Larger values of the smoothness parameters λ and ψ reduce the number of non-dominated alternatives more than smaller values, and the number of non-dominated alternatives is smallest when the weight is distributed evenly across the subregion. For example, when the subregion weights are represented as 1% intervals, computation with free distribution provides 88/120 non-dominated alternatives, whereas with the assumption of the weight being distributed evenly across the subregion the number is reduced to only three.

When the exact smoothness parameter values are rounded down to the nearest tenth, the second definition of the parameter, ψ, provides slightly better results, i.e., fewer non-dominated alternatives, than the first one, λ. The improvement between the two definitions is more significant when the parameter values are set to 0.5. This is reasonable, since a lower bound for the ratio of minimum and maximum is always less or equal to the lower bound for the ratio of minimum and mean, and setting $\psi = 0.5$ is thus more restricting than $\lambda = 0.5$.

The values of the recommendations by the central values and minimax regret decision rules improve slightly in a similar order as the number of non-dominated alternatives decreases. When the information on subregion weights is represented as 1% intervals, the central values recommendation is the same as the actual best alternative for even distribution and for the smoothness parameter values $\lambda = 0.5$ and $\psi = 0.5$. With the same information on subregion weights, the minimax regret recommendation is the actual best alternative regardless of how the weight is distributed inside a subregion.

Table 3 presents the comparison of different representations of subregion weights. The computations are conducted with the irregular division into 20 subregions, and the weight distribution inside each subregion is represented with $\psi = 0.5$. More extensive results are listed in Table B1.

The 5% intervals result in the largest number of non-dominated alternatives. When the complete importance order of the subregions is known, the number of non-dominated alternatives is a little smaller. Combining these two results in a significant reduction in the number of non-dominated alternatives. 1% intervals for subregion weights give more information than the wider 5% intervals, and thus provide better results

Table 3: Fire station positions: comparison of different representations of subregion weights. The computations are conducted with the irregular division into 20 subregions, and the weight distribution inside each subregion is represented with $\psi = 0.5$.

Subregion weights	Non-dominated alternatives	Central values positions	Central values value	Minimax regret positions	Minimax regret value
Complete order	56/120	3,4,10	0.9550	3,8,10	0.9581
5% intervals	64/120	3,4,10	0.9550	3,4,10	0.9550
Complete order + 5% intervals	33/120	3,4,10	0.9550	3,4,10	0.9550
1% intervals	12/120	3,7,10	0.9636	3,7,10	0.9636
Complete order + 1% intervals	12/120	3,7,10	0.9636	3,7,10	0.9636
Exact	2/120	3,7,10	0.9636	3,7,10	0.9636

in terms of the number of non-dominated alternatives. These narrower intervals also clearly outperform the combination of complete order and 5% weight intervals. However, the combination of complete order and 1% weight intervals provides hardly any improvement on the results compared to the 1% intervals without ordinal information. The reason for this is that the 1% intervals are narrow enough for deducing the importance order of subregions without it being explicitly stated, if the number of subregions is not too large.

The results provided by the minimax regret rule are similar to those provided by the central values rule. The results improve similarly as the number of non-dominated alternatives decreases: along with the exact subregion weights, 1% weight intervals and the combination of these intervals and complete order provide recommendations with the highest value, whereas the wider 5% intervals provide the worst recommendations, i.e., the values of these recommendations are the lowest. It is worth noticing that the minimax regret recommendation is the actual best alternative when the subregion weights are known exactly, as 1% weight intervals, or as the combination of complete order and 1% intervals, regardless of the weight distribution inside a subregion (see Table B1).

Assuming evenly distributed weight inside each subregion instead of free distribution reduces the number of non-dominated alternatives a lot more than narrowing the weight intervals from 5% to 1% (see Table B1). The number of non-dominated alternatives obtained with 5% weight intervals and free distribution inside each subregion is 117. Assuming even distribution inside each subregion results in 43 non-dominated alternatives, whereas computations with 1% intervals and free distribution result in 103 non-dominated alternatives. On the other hand, 1% weight intervals seem to provide slightly better results with with freely distributed weight than complete order does with the exact values of the smoothness parameter ψ rounded down. The numbers of non-dominated alternatives for these two scenarios are 103 and 106, respectively.

4.2 Radar positioning in air surveillance

4.2.1 Problem formulation and exact solution

In air surveillance, the airspace is scanned periodically in order to detect, locate and track aircraft and missiles. The purpose is to provide protection for the country and its citizens. In this problem, the air force commander of a fictional country is deciding the positions of their air surveillance equipment, with the objective of maximizing surveillance capability across a specific region.

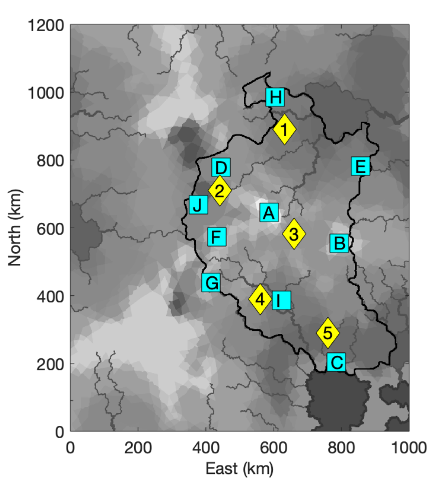

Figure 5: The map and the position candidates for the ground-based radars A–J and the airborne radar 1–5.

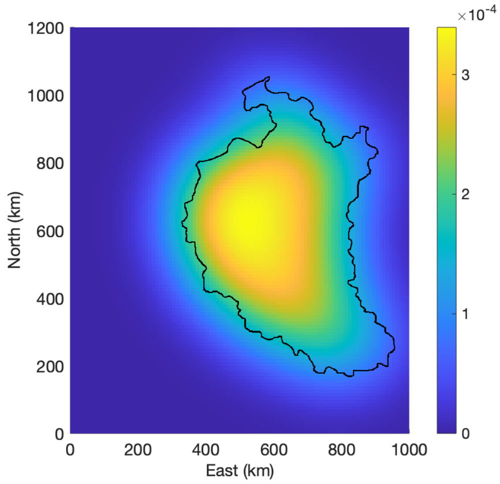

Figure 6: The exact spatial weights a_i of the locations.

There are five ground-based radars and one airborne radar that need to be positioned. The commander has preselected ten position candidates for the ground-based radars, labeled with letters A–J, and five position candidates for the airborne radar, labeled with numbers 1–5. The total number of concrete decision alternatives $z \in \tilde{Z}$, is thus $\binom{10}{5} \cdot \binom{5}{1} = 1260$. The map and the position candidates are presented in Figure 5. The radars have to be located inside the border of the country, which is represented by the black line on the map.

The country has a separate early-warning system that is used for the long-range detection and that makes the first observation of an approaching targets. The five ground-based radars and the one airborne radar are then used for more precise locating of this target. The surveillance capability of the alternate radar groupings is examined over the region S, whose size is 1000 km×1200 km. The map resolution is 100×120, so there are 12,000 locations $s \in S$ in which the consequences of different decision alternatives are calculated. The most important locations are in the inner parts of the country, whereas the locations further away are less important. The exact spatial weights describing the importance of locations are generated to correspond to the exact preferences of the commander, and are illustrated in Figure 6.

The consequence of a decision alternative is the average time between observations. This measure describes the technical performance of different radar groupings, and can get any values greater than zero, $z(s) \in (0, \infty)$ for all $s \in S$. The shorter the time between observations, the better. The consequence value function

$$v(c) = 1 - e^{-10/c}. \tag{26}$$

gives high values for small times between observations, and the values are lower for longer times.

The decision alternatives are again assessed with the additive spatial value function (3). When computing with the exact spatial weights presented in Figure 6, the best grouping consists of positions A, C, F, H and I for the ground-based radars, and the position 3 for the airborne radar. The value of this best alternative is 0.8589. The results obtained with incomplete preference information, i.e, the values of the recommendations of the decision rules, are compared to this value.

4.2.2 Results with incomplete preference information

As in the problem about fire station positions in Section 4.1, the region of interest is first divided into subregions. The regular division into 20 subregions is presented in Figure 7. In the irregular division, the map is first divided into three zones: inner parts of the country, border areas and areas outside the country. These zones are further divided into smaller subregions, such that the subregions are smallest in the inner parts of the country and largest outside the country. Figure 7 illustrates the irregular division into 20 subregions with inner parts of the country, border areas and the areas outside the country denoted with purple, blue and green, respectively. In this division, the inner parts of the country cover subregions 1–6, border areas cover subregions 7–15, and the areas outside the country cover subregions 16–20. Other subregion divisions are presented in Appendix A.

The comparison of the regular and irregular division, and different numbers of subregions, is presented in Tables B5 and B6 in Appendix B. In both tables, the weight distribution inside a subregion is represented with $\psi = 0.5$. In Table B5, the computations are conducted with 1% weight intervals, whereas in Table B6, these 1% intervals are accompanied with the complete importance order of the subregions.

In terms of the number of non-dominated alternatives, irregular division almost invariably provides better results than regular division, the only exception being when the region is divided into six subregions. For 9 and 20 subregions, the numbers of non-dominated alternatives obtained with irregular division are approximately half of those obtained with regular division.

When the area is divided regularly, the number of non-dominated alternatives mainly increases as the number of subregions grows. If the information on subregion weights is represented as 1% intervals, these increases are rather large: for example, doubling

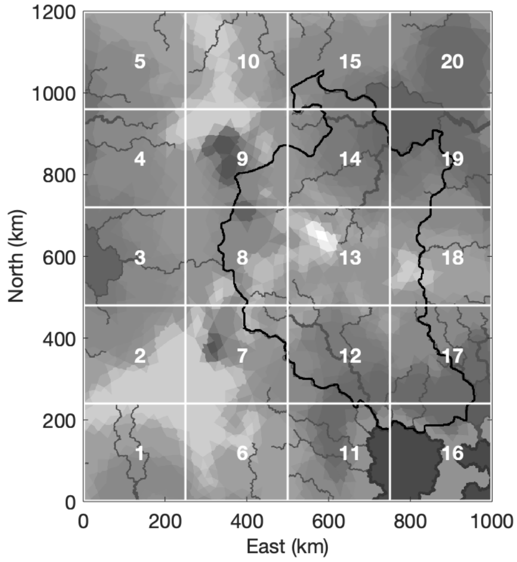

Figure 7: Regular division into 20 subregions.

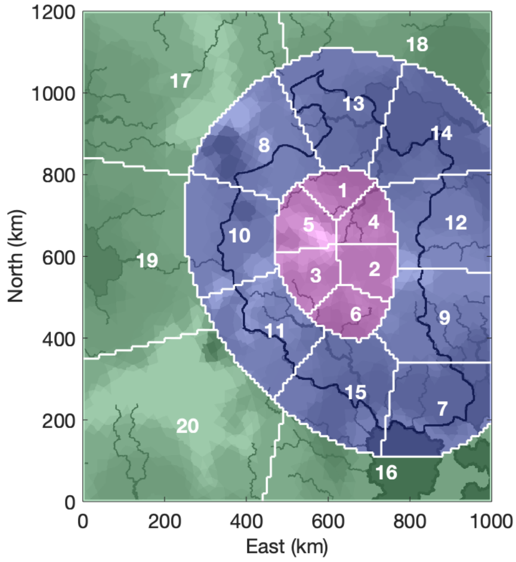

Figure 8: Irregular division into 20 subregions. The inner parts of the country, border areas and the areas outside the country are denoted with purple, blue and green, respectively.

the number of subregions from 20 to 40 increases the number of non-dominated alternatives from 277 to 449. As noted before, the numbers of non-dominated alternatives obtained with irregular division are smaller, but otherwise behave quite similarly. If the complete order of the subregion weights is known in addition to the 1% weight intervals, the increases in the numbers of non-dominated alternatives are not as significant as when only the 1% weight intervals are known.

In terms of the values of the recommendations of the central values and minimax regret decision rules, the values obtained with irregular division are higher than those obtained with regular division, especially when the number of subregions is small. When the region is divided irregularly, the values do not differ much. If the division is regular, the recommendations by the two decision rules seem to generally improve, i.e., the values increase, as the number of subregions increases.

To explore the effect of different representations of the weight distribution inside a subregion on the results, the region of interest is divided irregularly into 20 subregions, as in Figure 8. Table 4 presents the results for the situation when the information on subregion weights is represented as 1% weight intervals. The results with other representations of subregion weights are presented in Table B4 in Appendix B.

Regardless of how the information on subregion weights is represented, free distribution does not reduce the number of non-dominated alternatives at all, that is, all the decision alternatives remain non-dominated – even if the subregion weights are known

Table 4: Radar positioning: comparison of different weight distributions within a subregion. The computations are conducted with the irregular division into 20 subregions, and the information on subregion weights is represented as 1% intervals.

Weight distribution inside a subregion	Non-dominated alternatives	Central values labels	Central values value	Minimax regret labels	Minimax regret value
Free	1260/1260	CEHIJ3	0.8411	BCGHJ3	0.8427
Exact λ rounded down	1245/1260	BCHIJ3	0.8466	BCHIJ3	0.8466
Exact ψ rounded down	1149/1260	AEHIJ3	0.8516	BCHIJ3	0.8466
$\lambda = 0.5$	1065/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
$\psi = 0.5$	125/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Even	3/1260	ACHIJ3	0.8587	ACHIJ3	0.8587

exactly. Similarly to the problem about fire station positions, the computations done with evenly distributed weight inside each subregion result in the smallest number of non-dominated alternatives. When 1% intervals of the subregion weights are known, the smoothness parameter values rounded down to the nearest tenth bring only a little help in reducing the number of non-dominated alternatives.

The only representations of the weight distribution inside a subregion that significantly reduce the number of non-dominated alternatives are when $\psi = 0.5$ or when the weight is evenly distributed inside each subregion. Assuming the former, the number of non-dominated alternatives is 125, i.e., a tenth of the number of decision alternatives obtained with free distribution. Assuming even weight distribution inside each subregion in turn results in only three non-dominated alternative. The second definition of the smoothness parameter, ψ, seems again to be more useful than the first one, λ.

The central values recommendations seem to improve as the smoothness parameters are given higher values. For example, with 1% intervals and λ, the central values recommendation is the actual best alternative, even though the number of non-dominated alternatives is as high as 1065. The values of the recommendations by the minimax regret rule are rather similar to those provided by the central values rule.

Table 5 presents the comparison of different representations of subregion weights. The computations are conducted with the irregular division into 20 subregions, and the weight distribution inside each subregion is represented with $\psi = 0.5$. More extensive results can be found in Table B4 in Appendix B.

In terms of the number of non-dominated alternatives, 5% weight intervals perform worst, resulting in 935 non-dominated alternatives. The results obtained with complete order are slightly better. Narrowing the weight intervals from 5% to 1% intervals reduces the number of non-dominated alternatives to 125, and adding information about the complete importance order to this reduces the number to 112. For comparison, knowing the exact subregion weights would result in 48 non-dominated alternatives.

If the smoothness parameter values are rounded down from the exact ratios, the value of the recommendation by the central values rule is lowest when the information on subregion weights is represented as complete importance order without information about any weight intervals (see Table B4). Otherwise, the values of the recommendations by the central values rule do not differ much between different representations of subregion weights. The values of the recommendations by the minimax regret rule are generally lowest when the information on subregion weights is represented as 5% intervals, and are higher when the weight intervals are narrower or when the complete importance order is known.

Table 5: Radar positioning: comparison of different representations of subregion weights. The computations are conducted with the irregular division into 20 subregions, and the weight distribution inside each subregion is represented with $\psi = 0.5$.

Subregion weights	Non-dominated alternatives	Central values labels	Central values value	Minimax regret labels	Minimax regret value
Complete order	828/1260	ACFHJ3	0.8440	AGHIJ3	0.8516
5% intervals	935/1260	ABCHJ3	0.8506	BCFHJ3	0.8433
Complete order + 5% intervals	445/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
1% intervals	125/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Complete order + 1% intervals	112/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Exact	48/1260	ACHIJ3	0.8587	ACHIJ3	0.8587

According to Table B4, giving even some information about the weight distribution inside a subregion is often more useful than more detailed information about subregion weights, e.g., 1% intervals instead of 5% intervals. On the other hand, if the information on subregion weights is represented as 5% intervals, restricting the weight distribution inside a subregion from $\psi = 0.5$ to even distribution is not as useful in terms of the number of non-dominated alternatives as narrowing the intervals to 1% and keeping $\psi = 0.5$.

5 Discussion

Some general observations can be made based on the results obtained in Section 4. In general, the two test problems considered in this thesis provided very similar results to each other. However, it should be noted that the results might be different for other types of spatial decision problems.

Irregular division, i.e., dividing the more important areas into smaller subregions than the less important areas, seems to result in fewer non-dominated alternatives than the regular division. Naturally, in order to divide the region of interest irregularly, some information about the relative importance of the subregions is required. In real-life applications, the DM is likely to participate in the division process, so this is not necessarily a completely unreasonable requirement. In the irregular division, the subregions can for example represent different parts of town, and thus evaluating their importance may be easier compared to the regular division.

It turns out that the number of subregions does not automatically reduce the number of non-dominated alternatives. The usefulness of dividing the region of interest into more and more subregions seems to depend on what type of information on subregion weights is available. For example, if the information on subregion weights is represented by too wide weight intervals with respect to the number of subregions, many regional spatial weights are within the same interval (e.g., 0–5%), and thus not much new information is actually obtained. If the importance order of the alternatives is known, then increasing the number of subregions seems to be more useful.

Narrower weight intervals for the subregion weights accompanied with the importance order seems to be the most useful representation for the information on subregion weights when providing decision support. In both test problems of this thesis, ordinal information outperforms the wider 5% weight intervals, whereas the narrower 1% intervals seem to be more useful than knowing the complete importance order of the subregions.

If no information about the weight distribution inside a subregion is available, the number of non-dominated alternatives is likely to stay high, even if the information on subregion weights is quite detailed. Assuming even weight distribution inside each subregion results in the lowest number of non-dominated alternatives. Using the second definition for the smoothness parameter, ψ, seems to be slightly more useful than the first one, λ. In both test problems, assuming $\psi = 0.5$ for each subregion instead of free distribution reduces the number of non-dominated alternatives more than narrowing the weight intervals from 5% to 1%. This thesis is the first time that the two definitions for the smoothness parameter are tested in practice. According to the results obtained, the smoothness parameter seems to be quite a powerful tool in reducing the number of non-dominated alternatives.

Even though increasing the smoothness parameter values reduces the number of non-dominated alternatives, it must be taken into account that using too high parameter values is not completely risk-free. The concept of dominance is based on the assumption that the actual best alternative is definitely non-dominated. If the smoothness parameter is given a higher value than the actual lower limit for the corresponding ratio, there is no guarantee that the actual best alternative, computed with the exact spatial weights, remains non-dominated. Thus, one has to consider which is more important: the reduction in the number of non-dominated alternatives, or ensuring that the actual best alternative remains non-dominated. In the test problems of Section 4, the risk of excluding the actual best alternative from the set of non-dominated alternatives is present when the weight distribution inside a subregion is represented by $\lambda = 0.5$, $\psi = 0.5$, or when the weight is evenly distributed inside a subregion.

In terms of the values of the recommendations given by the two decision rules, irregular division seems to be better than the regular one. Generally, those representations of incomplete preference information which result in fewer non-dominated alternatives also provide better recommendations by the decision rules, i.e., the values of the recommendations are closer to the value of the best alternative.

When the number of decision alternatives is large, the computations are time-consuming. Dominance is established by pairwise comparisons, the number of which grows fast as the number of decision alternatives increases. The computation of the recommendation by the minimax regret rule can also be computationally heavy. The computation times in the fire station problem range from a few seconds to minutes, and in the problem about radar positioning the range is from minutes to multiple hours. The shorter computation times are achieved when the preference information is quite detailed, and the number of non-dominated alternatives is small (e.g., 2/120 or 3/1260).

When discussing the usefulness of different representations of the incomplete preference information, the DM's viewpoint must not be dismissed. It is important to consider how easy it might be to give preference statements of a certain form, as the objective is to gain a sufficient understanding of the DM's preferences with as little effort as possible. Comparing the importance of two subregions may be easy, but ordering all the subregions by importance is likely very challenging, especially if the number of subregions is large. If the information of subregion weights is represented as weight intervals, the narrower the intervals, the harder the task of specifying these intervals for all subregions. The tradeoffs between the effort required from the DM and the usefulness of the preference information thus need to be carefully considered.

When considering the intuitiveness of the two definitions of the smoothness parameter, the first one, λ, may be a little unclear for the DM: a lower bound for the ratio of minimum and mean might be rather hard to understand, not to mention having to define these values. The second definition, ψ, i.e., a lower bound for the ratio of minimum and maximum spatial weights inside a subregion might be more intuitive. If the number of subregions is large, specifying a smoothness parameter value for each subregion can turn out to be quite a laborious task.

It is not meaningful to state what is a sufficiently small number of non-dominated alternatives, since it naturally depends on the situation. However, for example a hundred alternatives (e.g., radar groupings), is already a large number to consider. Alternative approaches for utilizing the incomplete spatial preference information might turn out to be convenient. One is fitting spatial weights to the incomplete preference information. Simon (2020) also presents a simulation study for evaluating the performance of weight approximation methods in spatial problems. The next steps could be further exploration of these, as well as more extensive testing with the approach considered in this thesis.

6 Conclusion

This thesis explored the usefulness of different representations of incomplete spatial preference information. Three factors were considered: the division into subregions, the distribution of the spatial weights inside a subregion, and the total spatial weights of the subregions. Two test problems were solved using different types of incomplete preference information. The decision alternatives were compared based on the concept of dominance, and two different decision rules were applied to obtain decision recommendations. The usefulness of different representations of incomplete preference information was evaluated based on the number of non-dominated alternatives and the values given by the recommendations by decision rules.

According to the results of the two test problems, irregular division seems to be better than regular. The effect of increasing the number of subregions depends on how the other factors are represented. Information about the weight distribution inside each subregion seems to be important, and the smoothness parameter seems to be quite a powerful tool in providing decision support. If that information is not available, the number of non-dominated alternatives is not likely to decrease very much. When comparing the usefulness of different types of preference statements, it is important to take into account the effort that is required from the DM.

It is worth mentioning that the number of non-dominated alternatives can remain rather high, even if the preference information was quite detailed. Thus, other approaches for utilizing incomplete preference information might turn out to be more convenient in some cases. This thesis only considered two test problems, and therefore the results are not universal and might be different for other types of spatial decision problems. More thorough exploration of the factors considered in this thesis, as well as alternative approaches for utilizing incomplete spatial preference information, could be the next steps in the field.

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A Subregion divisions in the test problems

This appendix presents the subregion divisions used in the test problems of Section 4. Figures A1–A10 concern the problem about fire station positions in Section 4.1, and Figures A11–A20 the radar positioning problem in Section 4.2.

In the irregular divisions concerning the problem about fire station positions, the region of interest is divided into three zones: downtown, midtown and uptown. These zones are then further divided into smaller subregions. In Figures A2, A4, A6, A8 and A10, these three zones are denoted with purple, blue and green, respectively.

In the irregular divisions concerning the problem about radar positioning, the three zones correspond to the inner parts of the country, the border areas and the areas outside the country. In Figures A12, A14, A16, A18 and A20 these zones are denoted with purple, blue and green, respectively.

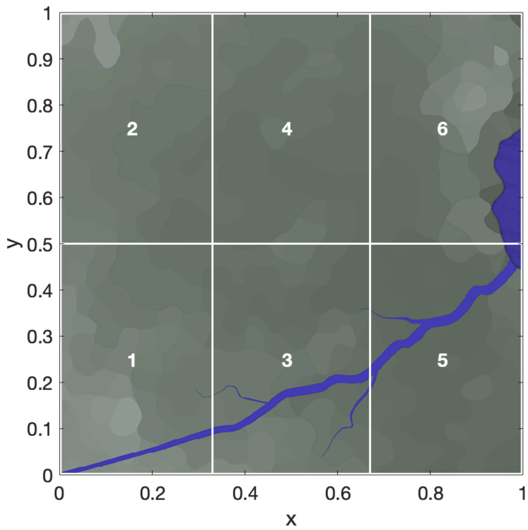

Figure A1: Fire station positions: regular division into 6 subregions.

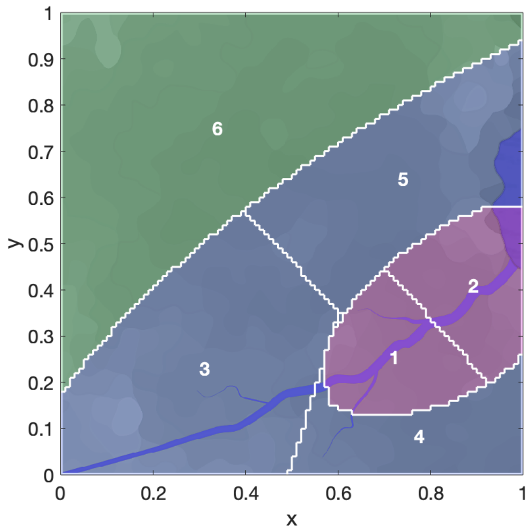

Figure A2: Fire station positions: irregular division into 6 subregions.

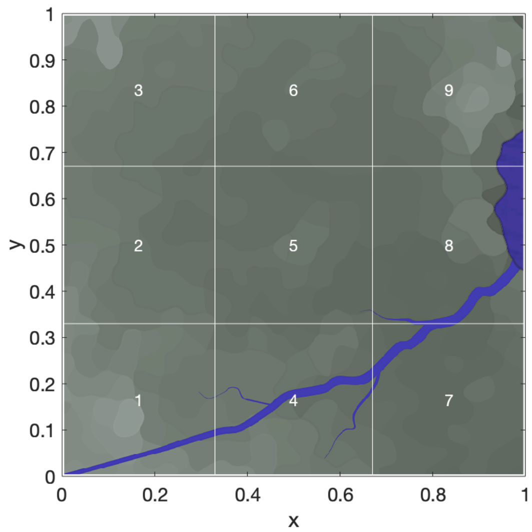

Figure A3: Fire station positions: regular division into 9 subregions.

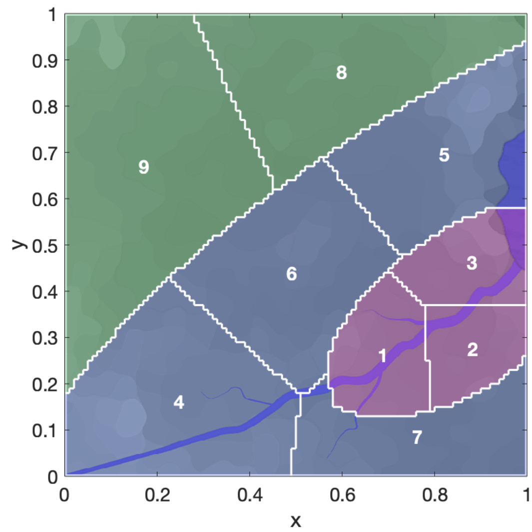

Figure A4: Fire station positions: irregular division into 9 subregions.

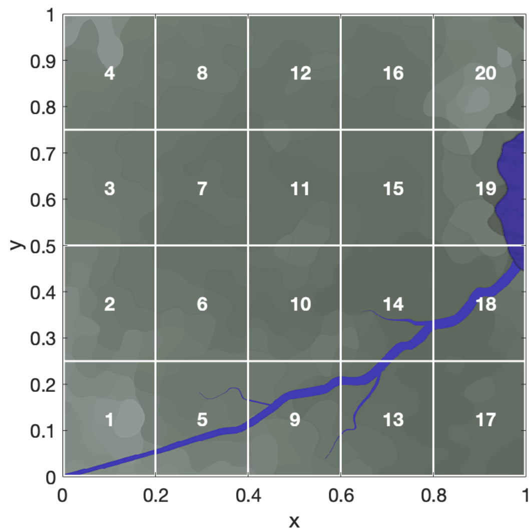

Figure A5: Fire station positions: regular division into 20 subregions.

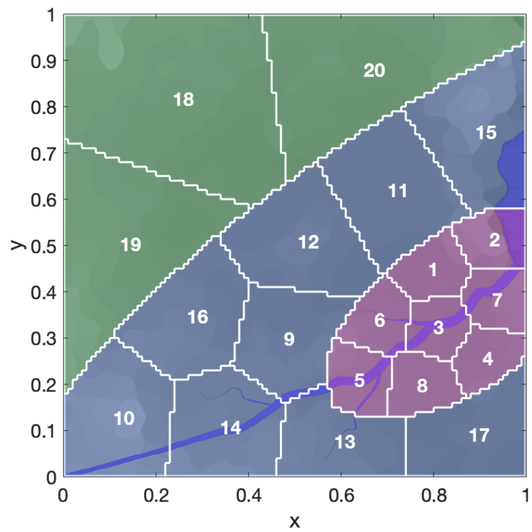

Figure A6: Fire station positions: irregular division into 20 subregions.

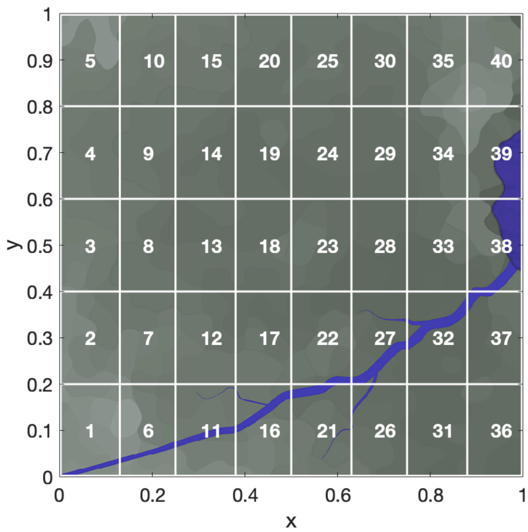

Figure A7: Fire station positions: regular division into 40 subregions.

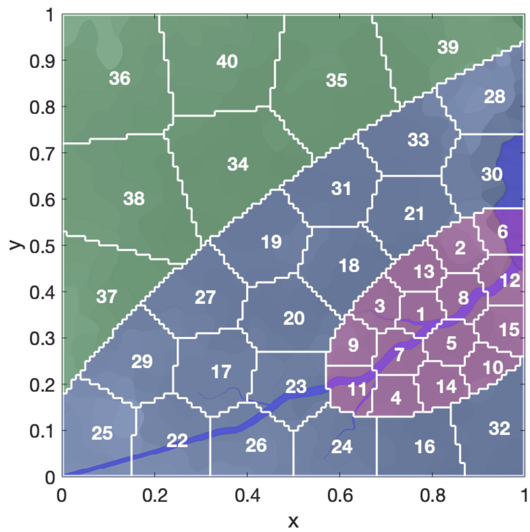

Figure A8: Fire station positions: irregular division into 40 subregions.

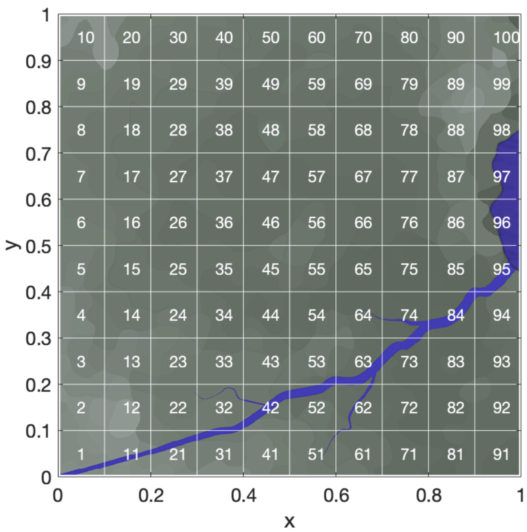

Figure A9: Fire station positions: regular division into 100 subregions.

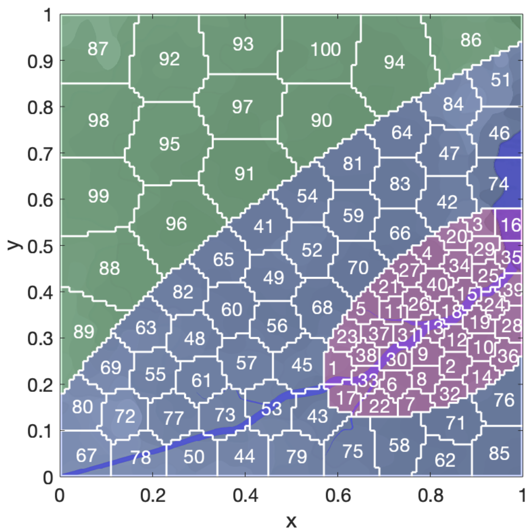

Figure A10: Fire station positions: irregular division into 100 subregions.

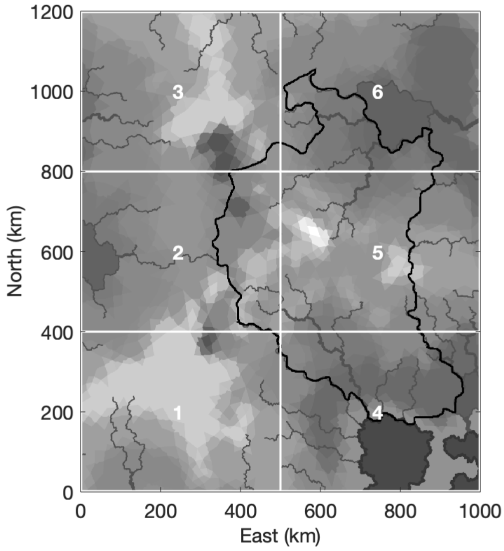

Figure A11: Radar positioning: regular division into 6 subregions.

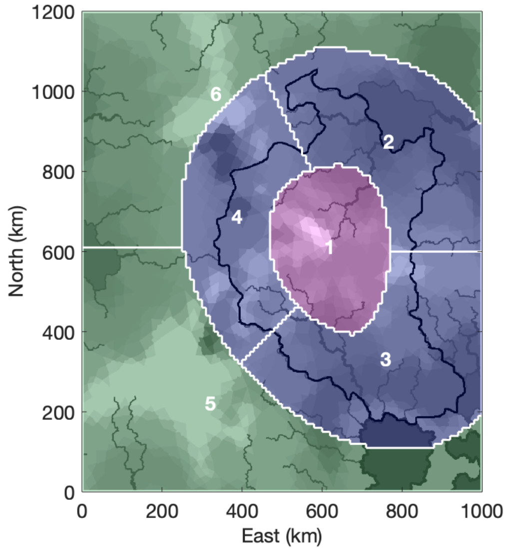

Figure A12: Radar positioning: irregular division into 6 subregions.

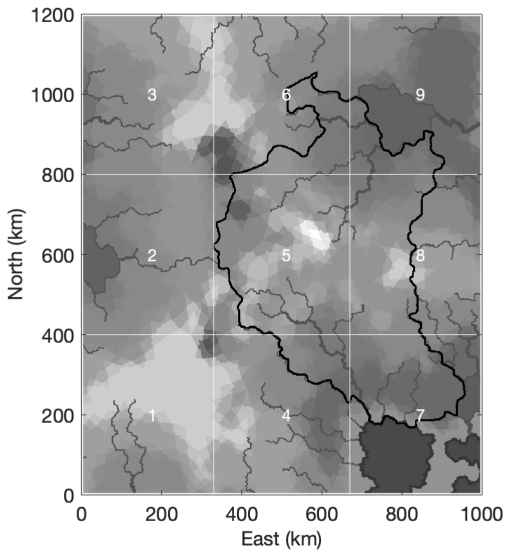

Figure A13: Radar positioning: regular division into 9 subregions.

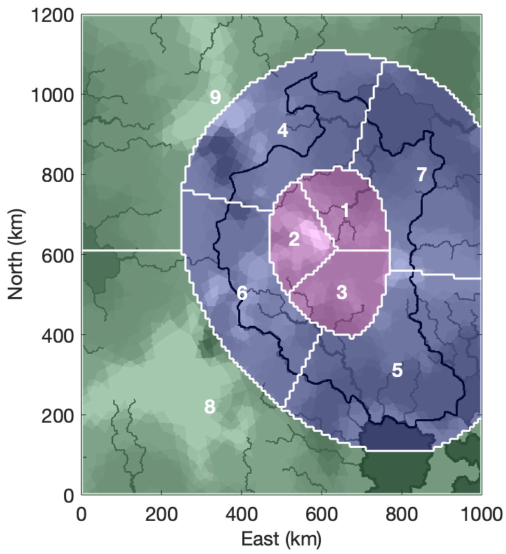

Figure A14: Radar positioning: irregular division into 9 subregions.

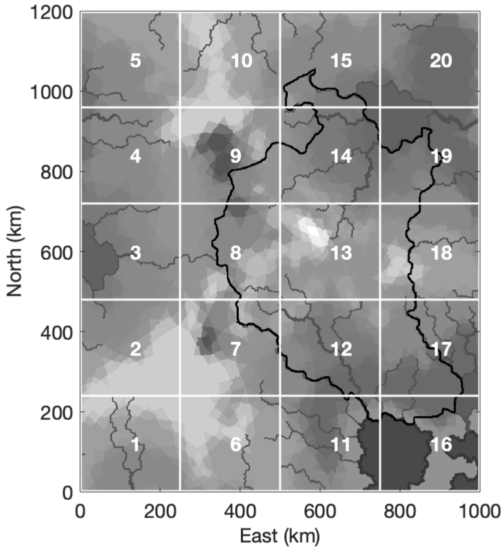

Figure A15: Radar positioning: regular division into 20 subregions.

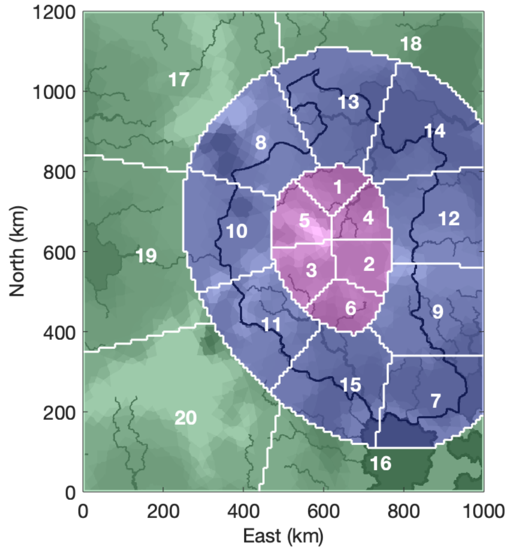

Figure A16: Radar positioning: irregular division into 20 subregions.

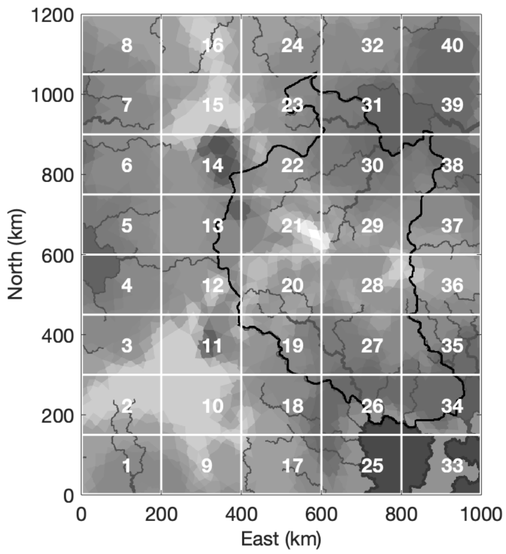

Figure A17: Radar positioning: regular division into 40 subregions.

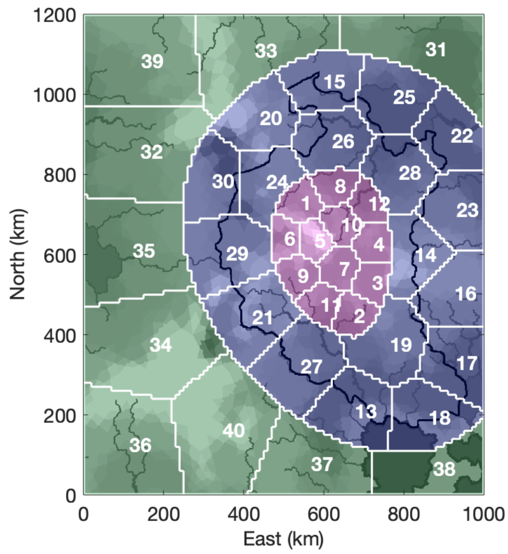

Figure A18: Radar positioning: irregular division into 40 subregions.

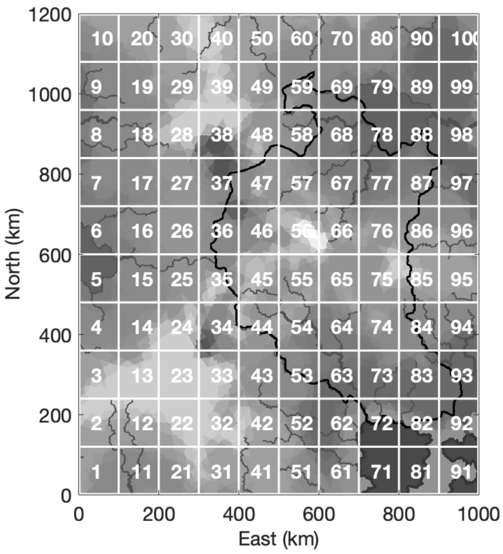

Figure A19: Radar positioning: regular division into 100 subregions.

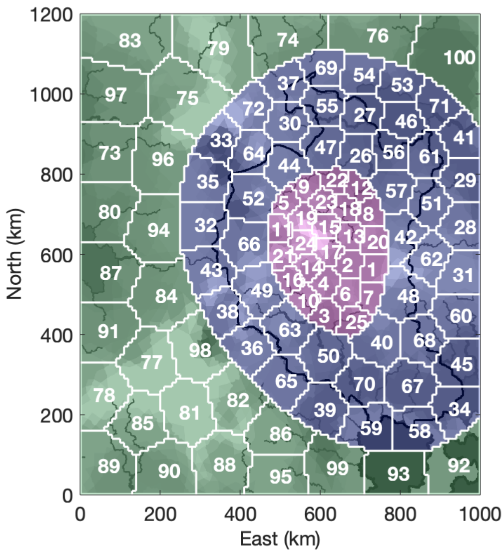

Figure A20: Radar positioning: irregular division into 100 subregions.

B Results of the test problems

This appendix presents the results for the two test problems of this thesis. Tables B1–B3 present the results for the problem of the fire station positions in Section 4.1, and Tables B4–B6 present the results for the problem about radar positioning in Section 4.2.

Table B1 presents the comparison of different representations of subregion weights, as well as the weight distribution inside a subregion in the problem about fire station positions. The computations are carried out with irregular division into 20 subregions. Tables B2 and B3 present the comparison of different numbers of subregions and the regular and irregular division. In both tables, the weight distribution inside a subregion is represented with the smoothness parameter ψ such that the exact lower limit of the ratio of the minimum and maximum spatial weight over the subregion is rounded down to the nearest tenth, and the smoothness parameter is given this value. In Table B2, the information on subregion weights is represented as 1% weight intervals, whereas in Table B3, the information is represented as complete importance order of the 20 subregions.

Table B4 presents the comparison of different representations of subregion weights and the weight distribution inside a subregion in the problem about radar positioning. The computations are carried out with irregular division into 20 subregions. Tables B5 and B6 present the comparison of different numbers of subregions and the regular and irregular divisino. In both tables, the weight distribution inside a subregion is represented with the smoothness parameter $\psi = 0.5$. In Table B5, the information on subregion weights is represented as 1% weight intervals, and in Table B6 the information is represented as the combination of 1% weight intervals and the complete importance order of the subregions.

Table B1: Fire station positions: comparison of different representations of subregion weights and the weigh distribution inside a subregion. The results are obtained with irregular division into 20 subregion.

Subregion weights	Weight distribution inside a subregion	Non-dominated alternatives	Central values positions	Central values value	Minimax regret positions	Minimax regret value
Complete order	Free	120/120	3,4,10	0.9550	3,4,7	0.9477
Complete order	Exact λ rounded down	114/120	2,3,7	0.9480	3,4,7	0.9477
Complete order	Exact ψ rounded down	106/120	2,3,7	0.9480	3,4,7	0.9477
Complete order	λ = 0.5	102/120	3,4,10	0.9550	3,4,7	0.9477
Complete order	ψ = 0.5	56/120	3,4,10	0.9550	3,8,10	0.9581
Complete order	Even	34/120	3,4,10	0.9550	3,8,10	0.9581
5% intervals	Free	117/120	2,3,7	0.9480	3,4,10	0.9550
5% intervals	Exact λ rounded down	106/120	2,3,7	0.9480	2,3,7	0.9480
5% intervals	Exact ψ rounded down	103/120	2,3,7	0.9480	2,3,7	0.9480
5% intervals	λ = 0.5	97/120	3,4,10	0.9550	3,4,10	0.9550
5% intervals	ψ = 0.5	64/120	3,4,10	0.9550	3,4,10	0.9550
5% intervals	Even	43/120	3,4,10	0.9550	3,4,10	0.9550
Complete order + 5% intervals	Free	108/120	2,3,10	0.9528	3,4,10	0.9550
Complete order + 5% intervals	Exact λ rounded down	98/120	2,3,10	0.9528	3,4,10	0.9550
Complete order + 5% intervals	Exact ψ rounded down	92/120	2,3,10	0.9528	2,3,7	0.9480
Complete order + 5% intervals	λ = 0.5	82/120	3,4,10	0.9550	3,4,10	0.9550
Complete order + 5% intervals	ψ = 0.5	33/120	3,4,10	0.9550	3,4,10	0.9550
Complete order + 5% intervals	Even	12/120	3,4,10	0.9550	3,4,10	0.9550
1% intervals	Free	103/120	3,8,10	0.9581	3,7,10	0.9636
1% intervals	Exact λ rounded down	88/120	3,8,10	0.9581	3,7,10	0.9636
1% intervals	Exact ψ rounded down	74/120	3,8,10	0.9581	3,7,10	0.9636
1% intervals	λ = 0.5	61/120	3,7,10	0.9636	3,7,10	0.9636
1% intervals	ψ = 0.5	12/120	3,7,10	0.9636	3,7,10	0.9636
1% intervals	Even	3/120	3,7,10	0.9636	3,7,10	0.9636
Complete order + 1% intervals	Free	103/120	3,8,10	0.9581	3,7,10	0.9636
Complete order + 1% intervals	Exact λ rounded down	87/120	3,8,10	0.9581	3,7,10	0.9636
Complete order + 1% intervals	Exact ψ rounded down	74/120	3,8,10	0.9581	3,7,10	0.9636
Complete order + 1% intervals	λ = 0.5	59/120	3,7,10	0.9636	3,7,10	0.9636
Complete order + 1% intervals	ψ = 0.5	12/120	3,7,10	0.9636	3,7,10	0.9636
Complete order + 1% intervals	Even	2/120	3,7,10	0.9636	3,7,10	0.9636
Exact	Free	94/120	3,7,10	0.9636	3,7,10	0.9636
Exact	Exact λ rounded down	69/120	3,7,10	0.9636	3,7,10	0.9636
Exact	Exact ψ rounded down	63/120	3,7,10	0.9636	3,7,10	0.9636
Exact	λ = 0.5	35/120	3,7,10	0.9636	3,7,10	0.9636
Exact	ψ = 0.5	2/120	3,7,10	0.9636	3,7,10	0.9636
Exact	Even	1/120	3,7,10	0.9636	3,7,10	0.9636

Table B2: Fire station positions: comparison of different numbers of subregions and the regular and irregular division. The information on subregion weights is represented as 1% weight intervals, and the information on the weight distribution inside a subregion is represented with the exact values of the smoothness parameter ψ rounded down to the nearest tenth.

Number of subregions	Division	Non-dominated alternatives	Central values positions	Central values value	Minimax regret positions	Minimax regret value
6 (3x2)	Regular	116/120	3,8,10	0.9581	3,8,10	0.9581
9 (3x3)	Regular	107/120	3,7,9	0.9617	3,4,10	0.9550
20 (5x4)	Regular	82/120	3,4,10	0.9550	3,4,10	0.9550
40 (8x5)	Regular	61/120	3,4,10	0.9550	3,4,10	0.9550
100 (10x10)	Regular	63/120	2,5,7	0.9537	2,5,7	0.9537
6	Irregular	105/120	1,5,7	0.9570	3,7,10	0.9636
9	Irregular	105/120	3,8,10	0.9581	3,8,10	0.9581
20	Irregular	74/120	3,8,10	0.9581	3,7,10	0.9636
40	Irregular	42/120	3,4,10	0.9550	3,4,10	0.9550
100	Irregular	66/120	3,4,10	0.9550	3,4,10	0.9550

Table B3: Fire station positions: comparison of different numbers of subregions and the regular and irregular division. The information on subregion weights is represented as the complete importance order of the 20 subregions. The information on the weight distribution inside a subregion is represented with the exact values of the smoothness parameter ψ rounded down to the nearest tenth.

Number of subregions	Division	Non-dominated alternatives	Central values positions	Central values value	Minimax regret positions	Minimax regret value
6 (3x2)	Regular	120/120	1,4,9	0.9258	2,5,7	0.9537
9 (3x3)	Regular	120/120	1,4,9	0.9258	2,5,7	0.9537
20 (5x4)	Regular	109/120	1,4,10	0.9402	2,3,7	0.9480
40 (8x5)	Regular	99/120	1,4,10	0.9402	3,4,10	0.9550
100 (10x10)	Regular	69/120	1,4,10	0.9402	3,4,10	0.9550
6	Irregular	120/120	1,7,9	0.9518	3,4,7	0.9477
9	Irregular	120/120	3,4,7	0.9477	2,5,7	0.9537
20	Irregular	106/120	2,3,7	0.9480	3,4,7	0.9477
40	Irregular	81/120	2,3,7	0.9480	2,5,7	0.9537
100	Irregular	52/120	3,4,6	0.9448	2,5,7	0.9537

Table B4: Radar positioning: comparison of different representations of subregion weights and the weigh distribution inside a subregion. The results are obtained with irregular division into 20 subregion.

Subregion weights	Weight distribution inside a subregion	Non-dominated alternatives	Central values labels	Central values value	Minimax regret labels	Minimax regret value
Complete order	Free	1260/1260	ACEHJ3	0.8417	BCGHJ3	0.8427
Complete order	Exact λ rounded down	1260/1260	ABCGJ1	0.8227	ACGHJ3	0.8483
Complete order	Exact ψ rounded down	1260/1260	BCDFJ1	0.8022	BCGHJ3	0.8427
Complete order	$\lambda = 0.5$	1257/1260	ACFHJ3	0.8440	CFHIJ3	0.8407
Complete order	$\psi = 0.5$	828/1260	ACFHJ3	0.8440	AGHIJ3	0.8516
Complete order	Even	310/1260	ACEFJ2	0.8169	AGHIJ3	0.8516
5% intervals	Free	1260/1260	CEHIJ3	0.8411	CEHIJ4	0.8271
5% intervals	Exact λ rounded down	1260/1260	CEHIJ3	0.8411	BCGHJ3	0.8427
5% intervals	Exact ψ rounded down	1255/1260	ACEHJ3	0.8417	BCGHJ3	0.8427
5% intervals	$\lambda = 0.5$	1252/1260	ABCHJ3	0.8506	BCGHJ3	0.8427
5% intervals	$\psi = 0.5$	935/1260	ABCHJ3	0.8506	BCFHJ3	0.8433
5% intervals	Even	426/1260	ABCHJ3	0.8506	BCDFH3	0.8498
Complete order + 5% intervals	Free	1260/1260	ABCHJ3	0.8506	CEHIJ4	0.8271
Complete order + 5% intervals	Exact λ rounded down	1260/1260	ABCHJ3	0.8506	BCGHJ3	0.8427
Complete order + 5% intervals	Exact ψ rounded down	1233/1260	ACHIJ3	0.8587	BCGHJ3	0.8427
Complete order + 5% intervals	$\lambda = 0.5$	1220/1260	ACHIJ3	0.8587	BCGHJ3	0.8427
Complete order + 5% intervals	$\psi = 0.5$	445/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Complete order + 5% intervals	Even	34/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
1% intervals	Free	1260/1260	CEHIJ3	0.8411	BCGHJ3	0.8427
1% intervals	Exact λ rounded down	1245/1260	BCHIJ3	0.8466	BCHIJ3	0.8466
1% intervals	Exact ψ rounded down	1149/1260	AEHIJ3	0.8516	BCHIJ3	0.8466
1% intervals	$\lambda = 0.5$	1065/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
1% intervals	$\psi = 0.5$	125/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
1% intervals	Even	3/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Complete order + 1% intervals	Free	1260/1260	BCHIJ3	0.8466	BCGHJ3	0.8427
Complete order + 1% intervals	Exact λ rounded down	1245/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
Complete order + 1% intervals	Exact ψ rounded down	1143/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
Complete order + 1% intervals	$\lambda = 0.5$	1050/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
Complete order + 1% intervals	$\psi = 0.5$	112/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Complete order + 1% intervals	Even	2/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Exact	Free	1260/1260	CEHIJ3	0.8411	BCHIJ3	0.8466
Exact	Exact λ rounded down	1202/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
Exact	Exact ψ rounded down	1038/1260	AEHIJ3	0.8516	BCHIJ3	0.8466
Exact	$\lambda = 0.5$	903/1260	ACHIJ3	0.8587	BCHIJ3	0.8466
Exact	$\psi = 0.5$	48/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
Exact	Even	1/1260	ACHIJ3	0.8587	ACHIJ3	0.8587

Table B5: Radar positioning: comparison of different numbers of subregions and the regular and irregular division. The information on subregion weights is represented as 1% weight intervals, and the information on the weight distribution inside a subregion is represented with the smoothness parameter $\psi = 0.5$.

Number of subregions	Division	Non-dominated alternatives	Central values labels	Central values value	Minimax regret labels	Minimax regret value
6 (2x3)	Regular	139/1260	BCEHJ4	0.8161	BCEHJ4	0.8161
9 (3x3)	Regular	268/1260	ACEHJ4	0.8397	ABCFH3	0.8537
20 (4x5)	Regular	277/1260	ABHIJ3	0.8577	ACHIJ3	0.8587
40 (5x8)	Regular	449/1260	ABHIJ3	0.8577	BDFHI3	0.8566
100 (10x10)	Regular	836/1260	ADGHI3	0.8543	ACGHJ3	0.8483
6	Irregular	142/120	ABCHJ3	0.8506	ACHIJ3	0.8587
9	Irregular	121/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
20	Irregular	125/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
40	Irregular	263/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
100	Irregular	700/1260	ACHIJ3	0.8587	ACHIJ3	0.8587

Table B6: Radar positioning: comparison of different numbers of subregions and the regular and irregular division. The information on subregion weights is represented as the combination of 1% weight intervals and the complete importance order of the 20 subregions. The information on the weight distribution inside a subregion is represented with the smoothness parameter $\psi = 0.5$.

Number of subregions	Division	Non-dominated alternatives	Central values labels	Central values value	Minimax regret labels	Minimax regret value
6 (2x3)	Regular	139/1260	BCEHJ4	0.8161	BCEHJ4	0.8161
9 (3x3)	Regular	266/1260	ACEHJ4	0.8397	ABCFH3	0.8537
20 (4x5)	Regular	208/1260	ABHIJ3	0.8577	ABHIJ3	0.8577
40 (5x8)	Regular	283/1260	BDFHI3	0.8566	ABHIJ3	0.8577
100 (10x10)	Regular	285/1260	ACFHI3	0.8589	ACHIJ3	0.8587
6	Irregular	142/1260	ABCHJ3	0.8506	ACHIJ3	0.8587
9	Irregular	121/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
20	Irregular	112/1260	ACHIJ3	0.8587	ACHIJ3	0.8587
40	Irregular	176/1260	ACHIJ3	0.8587	ACFHI3	0.8589
100	Irregular	199/1260	ACHIJ3	0.8587	ABHIJ3	0.8577