Market Conditions under Frictions and without Dynamic Spanning

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MARKET CONDITIONS UNDER FRICTIONS AND WITHOUT DYNAMIC SPANNING*

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ABSTRACT

In this paper we study the conditions for the absence of arbitrage, single agent's optimality, and equilibrium in a market under frictions and with more sources of uncertainty than tradable assets. In other words, we analyze a state-price deflator, i.e., a deflator with the property that the deflated prices of tradable assets are martingales, in the presence of frictions and incompleteness. A unique state-price deflator may exist only on projected markets. It is shown that if the market conditions hold for the projected markets, they hold also for the initial market.

KEYWORDS: Arbitrage, equilibrium, incomplete markets, optimization, transaction costs

1. INTRODUCTION

Markets are incomplete, if there exist more than one state-price deflator, i.e., a deflator with the property that the deflated price of a tradable asset is a martingale. If the markets are incomplete, the price of a contingent claim may depend on the state-price deflator with respect to which it is priced. In this paper we extend the framework of absence of arbitrage, optimal portfolio and consumption choice, and security markets equilibrium in complete markets to cover markets with frictions and without dynamic spanning.

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The general framework for arbitrage free condition in complete markets is derived in Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981), and Cox and Huang (1986). Merton (1969, 1971), Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) have solved optimal portfolio and consumption choice problem in complete markets. Security markets equilibrium in complete markets is studied, e.g., in Breeden (1979), Duffie and Zame (1989), and Huang (1987). Jouini and Kallal (1995) derive the arbitrage free condition under transaction costs. They show that, in the presence of transaction costs, the absence of arbitrage condition is equivalent to the existence of an equivalent probability measure that transforms some process between the bid and ask price processes into a martingale. Hedging and portfolio optimization under transaction costs are analyzed in Cvitanic and Karatzas (1996). Cvitanic and Karatzas (1993) consider hedging in the presence of general closed, convex constraint sets of portfolio processes, and their methodologies can also be applied to the case of different interest rates for borrowing and lending. The single agent's optimality in an incomplete market is studied, e.g., in Cvitanic and Karatzas (1992), Duffie and Sun (1990), and Leland (1985). Grossman and Shiller (1982) and Back (1991) derive market equilibrium without dynamic spanning condition. These models start the analysis from the state-price deflators implicitly given by single agents, and then derive the excess expected rates of return on all securities from the covariance of returns with aggregate consumption increments and the 'market-risk-aversion' constant.

In this paper we consider the same kind of problems that are studied in Jouini and Kallal (1995) and Cvitanic and Karatzas (1992, 1993, 1996), and we utilize their framework. In contrast to Jouni and Kallal, we consider also markets without dynamic spanning, and we derive the market conditions by using the quotient space of prices of tradable assets. The initial stochastic variables are projected into a new space, on which markets are complete and frictionless, and we can employ the framework of complete markets. These projected markets are the same kind of fictitious markets that are used in Cvitanic and Karatzas (1992, 1993, 1996). In addition to Cvitanic and Karatzas, we let the volatility processes of tradable assets differ between various fictitious markets, and we also consider general market frictions, for instance frictions in getting information and time-varying transaction costs. We show that the market conditions hold in the initial economy if they hold in the projected markets.

The rest of the paper is organized as follows: Section 2 defines the framework used in the paper, and Section 3 derives the projected market. Section 4 derives the market equilibrium conditions, and Section 5 concludes.

2. MODEL

We explore an economy where instruments are traded continuously within a time horizon $[0,t]$. There is a finite set of tradable assets, H, where $|H| = k$, and a set of agents, denoted by M. An agent $m \in M$ is defined by a nonzero consumption endowment process e_m and a strictly increasing utility function U_m.

In describing the probabilistic structure of the economy corresponding to an agent $m \in M$, we refer to an underlying probability space $(\Omega, F^{(n_m)}, P_m)$, along with the standard filtration $\{F_t^{(n_m)} : t \in [0,t], n_m < \infty\}$. Here Ω is a set, $F^{(n_m)}$ is a σ-algebra of subsets of Ω generated by an n_m-dimensional Brownian motion $(B_m^1, \ldots, B_m^{n_m})$, and P_m is a probability measure on $F^{(n_m)}$. The probability spaces may differ between various agents, e.g., because there may be frictions in getting market information and because the portfolios of different agents may partly depend on various sources of uncertainty. We denote by $L_s^{n_m}$ the class of functions $f : [0,t] \times \Omega \to \mathbf{R}^s$ in a Hilbert space such that

i) $(t,\omega) \mapsto f(t,\omega)$ is $\boldsymbol{B} \times F^{(n_m)}$ -measurable, where $\boldsymbol{B}$ denotes the Borel σ-algebra on $[0, t]$

ii) $f(t,\omega)$ is $F_t^{(n_m)}$-adapted.

iii) each coordinate f_i of f in $\mathbf{R}^s$ satisfies $E_m\left[\int_0^T f_i(t,\omega)^2\, dt\right] < \infty$ for all $T \in [0,t]$, where E_m denotes the expectation with respect to P_m. Hereafter the index of E is omitted.

The following assumptions characterize more our economy.

ASSUMPTION A1: *The possible tradable asset prices that are discounted by risk-free interest rate for an agent* $m \in M$ *are given by the closed space* $X_m \subset L_k^n$. *The prices are such that the bid prices are always lower or equal to the corresponding ask prices.*

Assumption A1 means that there does not have to be a unique price vector representing the tradable assets. This is due to frictions in the market and/or bid-ask spread for interest rates. In our model the bid price is the selling price after transaction costs and other frictions, and the ask price is the buying price after the market frictions. The price processes depend on agents, because market frictions may differ between various agents. For instance, due to the frictions in the distribution of information various agents may have different process estimates. The price processes are also functions of the portfolio processes, because transaction costs usually depend on the portfolio increments. The prices may depend also on the agent's actions. This happens when the investor is a significant player in the market. The processes in X_m are the price processes of our fictitious markets corresponding to the agent $m \in M$. The following assumption is used in the calculation of optimal consumption and trading strategies.

ASSUMPTION A2: *For all* $m \in M$ *there exists a price process in* X_m *such that given the price process the agent m is not willing to trade in the market.*

Assumption A2 implies that all the agents have an option to trade in the market and they can stop the trading for a time if they want.

ASSUMPTION A3: *The stochastic variables of tradable assets that are discounted by risk-free interest rate corresponding to an agent m Î M follow an Itô stochastic differential equation*

(1) $$dx_m(t) = x_m(t) \times a_{x,m}(t)dt + x_m(t) \times s_{x,m}(t)dB_m(t) \quad \text{for all} \quad x_m(0) \in X_{m,0}, \quad t \in [0,t]$$

where $x_m \in X_m$, $x_m(0)$ *is a random variable which is independent of* $F_\infty^{(n_m)}$ *and* $E\left[|x_m(0)|^2\right] < \infty$, $a_{x,m} \in L_k^{n_m}$, $s_{x,m} \in L_{k \times n_m}^{n_m}$, $B_m \in L_{n_m}^{n_m}$ *is an* n_m*-dimensional Brownian motion on the probability space* $(\Omega, F^{(n_m)}, P_m)$, $n_m \geq k$, *and* $X_{m,0}$ *is the range of processes in* X_m *at time* 0 *corresponding to the agent m. The operator* $\times$ *is defined as follows*

$$x_m(t) \times a_{x,m}(t) = \left[x_m^1(t)a_{x,m}^1(t) \dots x_m^k(t)a_{x,m}^k(t)\right]'$$

$$x_m(t) \times s_{x,m}(t) = \begin{bmatrix} x_m^1(t)s_{x,m}^{1,1}(t) & \dots & x_m^1(t)s_{x,m}^{1,n}(t) \\ \vdots & \ddots & \vdots \\ x_m^k(t)s_{x,m}^{k,1}(t) & \dots & x_m^k(t)s_{x,m}^{k,n}(t) \end{bmatrix}$$

and x' *is the transpose of x.*

We will refer a_x and s_x as the drift and volatility processes of x. Assumption A3 means that there may exist more sources of uncertainty than there are tradable assets. The coordinates of $x(t)$ can be dependent, as they usually are. Combining assumptions A1 and A3 we see that each process in X_m is given by equation (1). For instance, equation (1) holds for the process that first equals the bid price of the tradable asset and then the ask price. This assumption is made in order to simplify our analysis. Comparing A1 and A3 with Cvitanic and Karatzas (1992, 1993) we see that in our model the volatility processes of different fictitious markets may differ and various agents may have distinct process estimates. This is, e.g., due to frictions in getting information, uncertainty in transaction costs, agent specific risks, and uncertainty in illiquid assets.

ASSUMPTION A4: *For each agent* $m \in M$ *there exists a mapping* $y : L_{k \times n_m}^{n_m} \to L_k^{n_m}$ *such that*

(2) $$y(s)(t, x, m, w) = s_{x,m}(t,w)J_{x,m}(t,w) \quad \text{for all} \quad t \in [0,t], \quad x_m \in X_m, \quad a.s.$$

where $J_{x,m} \in L_{n_m}^{n_m}$, $J_{x,m}(t,w) : \mathbf{R}^{k \times n_m} \to \mathbf{R}^k$ *is onto but not necessary one to one,* $J_{x,m}$ *satisfies Novikov's condition*

(3) $$E\left[\exp\left(\frac{1}{2}\int_0^T J_{x,m}(t,w)^2 dt\right)\right] < \infty \quad \text{for all} \quad T \in [0,t]$$

and $y\ (s_{x})(t,x,m,w) = a_{x,m}(t,w)$ *for all* $t \in [0,t]$ *almost surely.*

The situation of Assumption A4 emerges because $s_{x,m} \in L^{n_m}_{k\times n_m}$ may be singular. We will refer to $J_{x,m}$ as the market price of risk. A4 is a central assumption of the paper, because, as we will see, the market conditions hold if it is true. If $s_{x,m}(t)$ is singular for some $t \in [0,t]$, the agent $m \in M$ chooses one market price of risk process for each process in X_m according to his or her tastes about risks among the processes in the market that satisfy A4. If $s_{x,m}(t)$ is invertible for all $t \in [0,t]$, markets are complete and $J_{x,m}$ is unique.

ASSUMPTION A5: *The utility functions of investors are smooth-additive.*

A5 means that the utility function U: $\mathbf{R}_+ \to \mathbf{R}$ of the agent $m \in M$ is defined by

(4) $$U_m(c) = E\left[\int_0^T u_m(c,t)dt\right] \quad \text{for all} \quad T \in [0,t]$$

where c is an adapted non-negative consumption in $L^{n_m}_1$, $u_m : \mathbf{R}_+ \times (0,T) \to \mathbf{R}$ is smooth on $\mathbf{R}_+ \times (0,T)$, and for each $t \in [0,T]$, $u_m(\cdot,t)$: $\mathbf{R}_+ \to \mathbf{R}$ is increasing, strictly concave, with an unbounded partial derivative $\frac{\P u_m(\cdot,t)}{\P c}$ on $\mathbf{R}_+$ satisfying the Inada conditions: $\inf \frac{\P u_m(c,t)}{\P c} = 0$ and $\sup \frac{\P u_m(c,t)}{\P c} = \infty$ for all $t \in [0,T]$.

3. QUOTIENT SPACES

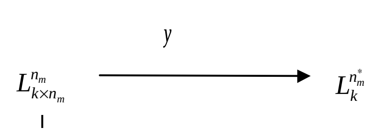
In this section we derive the quotient spaces of $L^{n_m}_{k\times n_m}$ and $L^{n_m}_k$ [for the discussion of quotient spaces see e.g. Kelley (1955)].

Let us define the equivalence relation $R_{x,m}(t)$ by setting $s_1 R_{x,m}(t) s_2$, where $s_1, s_2 \in L^{n_m}_{k\times n_m}$, if $s_1(t,w)J_{x,m}(t,w) = s_2(t,w)J_{x,m}(t,w)$ almost surely, where $J_{x,m}$ is given by Assumption A4, $t \in [0,t]$, $x \in X_m$, and $m \in M$. The range of processes in $L^{n_m}_{k\times n_m}$ can be divided into equivalence classes $r_{x,m}(t,s) = \{s_1 \in L^{n_m}_{k\times n_m}; s_1 R_{x,m}(t) s\}$ for all $t \in [0,t]$, and the set of equivalence classes is denoted by $L^{n_m}_{k\times n_m} / R_{x,m}(\cdot) = \{r_{x,m}(\cdot,s) \mid s \in L^{n_n}_{k\times n_n}\}$ and is called the quotient space of $L^{n_m}_{k\times n_m}$ with respect to $R_{x,m}(\cdot)$. The quotient space is a partition of $L^{n_m}_{k\times n_m}$. That is, every element of $L^{n_m}_{k\times n_m}$ belongs to one and only one class in $L^{n_m}_{k\times n_m} / R_{x,m}(\cdot)$. The continuous mapping $r_{x,m}(t,\cdot) : L^{n_m}_{k\times n_m} \to L^{n_m}_{k\times n_m} / R_{x,m}(t)$ is called projection, and the image of s is the equivalence class into which s belongs. The mapping $r_{x,m}(t,\cdot)$ is a linear surjection which coinduces a s-algebra on $L^{n_m}_{k\times n_m} / R_{x,m}(t)$. Now we can state the following lemma.

LEMMA 1: *y defined in Assumption A4 induces a continuous linear bijective mapping $y^* : L^{n_m}_{k\times n_m} / R_{x,m}(\cdot) \to L^{n^*_m}_k$, such that*

(5) $$s_{x,m}(t,w)J_{x,m}(t,w) = r_{x,m}\big(t, s_{x,m}(t,w)\big)J^*_{x,m}(t,w) \quad \text{for all} \quad t \in [0,t], \quad x \in X_m, \quad m \in M, \quad a.s.$$

where $J^*_{x,m} \in L^{n^*_m}_{n^*_m}$, $J^*_{x,m}(t,w) : \mathbf{R}^{k\times n^*_m} \to \mathbf{R}^k$ *is a bijection that satisfies the Novikov's condition and* $r_{x,m}(\cdot, s_{x,m}) \in L^{n_m}_{k\times n_m} / R_{x,m}(\cdot)$ *for all* $s_{x,m} \in L^{n_m}_{k\times n_m}$, *and* $n^*_m \le k$. *That is,* $y^*\big(r_{x,m}(t,s_{x,m})\big) = y\ (s_{x,m})$ *and the canonical factorization of* y *is* $y = y^* \circ r_{x,m}$. *Figure* 1 *illustrates the situation.*



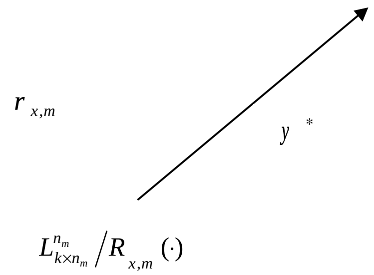


Figure 1. The canonical factorization of *y*

PROOF: First we note that y^* is well defined. That is, $r_{x,m}(t,s_1) = r_{x,m}(t,s_2) \leftrightarrow y(s_1)(t,x,m,w) = y(s_2)(t,x,m,w)$ almost surely, where $s_1, s_2 \in L^{n_m}_{k\times n_m}$, $t \in [0,t]$, $x \in X_m$, and $m \in M$. Because $y : L^{n_m}_{k\times n_m} \to L^{n_m}_k$ is a continuous linear surjection, and $r_{x,m}(t,\cdot)$ is a continuous linear projection, which coinduces the s-algebra of $L^{n_m}_{k\times n_m} / R_{x,m}(t)$, y^* is continuous and linear. It is also a bijection, since y^* is injection by the definition of $R_{x,m}(t)$, and therefore $n^*_m \le k$. This means that there could be no other $J^* \in L^{n^*_m}_{n^*_m}$ that satisfies $a_{x,m}(t,w) = r_{x,m}(t, s_{x,m}(t,w))J^*_{x,m}(t,w)$ almost surely. Because $J_{x,m} \in L^{n_m}_{n_m}$ satisfies the Novikov's condition and $r_{x,m}(t,\cdot)$ is bounded, also $J^*_{x,m}$ satisfies the condition. Now $B^*_m \in L^{n^*_m}_{n^*_m}$ is a process in $\mathbf{R}^{n^*_m}$ on a probability space $(\Omega, F^{(n^*_m)}, P^*_m)$, where $F^{(n^*_m)} \subset F^{(n_m)}$ is the s-algebra generated by B^*_m and $P^*_m : F^{(n^*_m)} \to [0,1]$ is a probability measure on the measure space $(\Omega, F^{(n^*_m)})$. $B^*_m(t)$ is obtained from $B_m(t)$ by eliminating for all $t \in [0,t]$ the same elements that were eliminated from y in order to get y^*. *Q.E.D.*

Lemma 1 means that we shrink the dimension of the Brownian motion in order to get the bijection $J^*_{x,m}$. That is, although we are not able to find a unique market price of risk function in $L^{n_m}_{n_m}$ we can find it in $L^{n^*_m}_{n^*_m}$.

ASSUMPTION A6: $r_{x,m}(\cdot, s)$ *is piecewise constant.*

Assumption A6 ensures that we can integrate with respect to $B^*_m(t)$, since on the piecewise intervals $B^*_m(t)$ is an n^*_m-dimensional Brownian motion.

We denote by $L^{n^*_m}_k / X_m$ the quotient space of $L^{n^*_m}_k$. The classes of $L^{n^*_m}_k / X_m$ are X_m and $\{y\}$, where y belongs to the complement of X_m, i.e., $y \in L^{n^*_m}_k \setminus X_m$. $L^{n^*_m}_k / X_m$ is obtained from $L^{n^*_m}_k$ by projecting X_m into one point, $\tilde{r}_{x,m}(t,\cdot) : L^{n^*_m}_k \to L^{n^*_m}_k / X_m$. That is, $\tilde{r}_{x,m}$ selects one process into which all the processes in X_m are projected. This projection is a surjection, which coinduces the s-algebra of $L^{n^*_m}_k / X_m$.

Now we can define the process of $x^*_m \in L^{n^*_m}_k / X_m$ by

$$dx^*_m(t) = \tilde{r}_{x,m}(t, x_m(t) \times a_{x,m}(t))dt + \tilde{r}_{x,m}(t, x_m(t) \times r_{x,m}(t, s_{x,m}(t)))dB^*_m(t); \tag{6}$$

$$x^*_m(0) \in \mathbf{R}^k \quad \text{for all} \quad t \in [0,t]$$

where $m \in M$, $x^*_m(0)$ satisfies the condition of A3, $a_{x,m} \in L^{n^*_m}_k$, and $r_{x,m}(\cdot, s_{x,m}) \in L^{n^*_m}_{k\times n^*_m}$, and the well-defined state-price deflator $p^*_m \in L^{n^*_m}_1$ by

$$p^*_m(t) = \exp\left\{ -\int_0^t J^*_{x^*,m}(s,w)' dB^*_m(s) - \frac{1}{2}\int_0^t J^*_{x^*,m}(s,w)^2 ds \right\} \quad \text{for all} \quad t \in [0,t] \tag{7}$$

Equation (6) implies that there exists a unique price vector and process for tradable assets in the quotient space $L_k^{n_m^*}/X_m$, and equation (7) means that there exists a unique state-price deflator on $\left(\Omega, F^{(n_m^*)}, P_m^*\right)$ for $x_m^*(t)$. $x_m^*(t)$ is the price process of a fictitious market and p_m^* is the state-price deflator of the market.

We illustrate our framework with an example. Let the process of $x_m(t)$ be defined as follows

(8)
$$dx_m(t) = x_m(t)dt + \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix} dB_m^1(t) \\ dB_m^2(t)\end{bmatrix} \quad \text{for all} \quad t \in [0,1]$$
$$dx_m(t) = x_m(t)dt + x_m(t)dB_m^2(t) \quad \text{for all} \quad t \in [1, t]$$

where $B_m^1(t)$ and $B_m^2(t)$ are independent Brownian motion processes, $t > 1$, and $m \in M$. The market price of risk vector is defined as

(9)
$$J_m(t) = \begin{bmatrix}1 \\ 0\end{bmatrix} \quad \text{for all} \quad t \in [0,1]$$
$$J_m(t) = \begin{bmatrix}0 \\ 1\end{bmatrix} \quad \text{for all} \quad t \in [1, t]$$

In this case the process of $x_m^*(t)$ is

(10)
$$dx_m^*(t) = x_m^*(t)dt + x_m^*(t)dB_m^1(t) \quad \text{for all} \quad t \in [0,1]$$
$$dx_m^*(t) = x_m^*(t)dt + x_m^*(t)dB_m^2(t) \quad \text{for all} \quad t \in [1, t]$$

4. ARBITRAGE, OPTIMALITY, AND EQUILIBRIUM

In this section we consider the absence of arbitrage condition, single agent's optimality, and security market's equilibrium. From the previous section we get the following lemma.

LEMMA 2: *There exists* $x_m \in X_m$ *and* $p_m \in L_1^{n_m}$ *such that* $x_m(t, w)$ *is a martingale on* $(\Omega, F^{(n_m)}, Q_m)$, *where* Q_m *is an equivalent martingale measure with Radon-Nikodym derivative*

(11)
$$\frac{dQ_m}{dP_m} = p_m(t) \quad \text{on} \quad F_t^{(n_m)},$$

$m \in M$, *and* $p_m(t) = \exp\left\{-\int_0^t J_{x,m}(s, w)dB_m(s) - \frac{1}{2}\int_0^t J_{x,m}(s, w)^2 ds\right\}$, *if and only if* $x_m^*(t, w)$ *is a martingale on* $(\Omega, F^{(n_m^*)}, Q_m^*)$, *where* Q_m^* *is an equivalent martingale measure with Radon-Nikodym derivative*

(12)
$$\frac{dQ_m^*}{dP_m^*} = p_m^*(t) \quad \text{on} \quad F_t^{(n_m^*)}$$

and $p_m^*(t)$ *is given by equation* (7).

PROOF: If $p_m(t, w)x_m(t, w)$, which process is

(13)
$$d\left(p_m(t)x_m(t)\right) = p_m(t)x_m(t) \times \left[a_{x,m}(t) - s_{x,m}(t)J_{x,m}(t)\right]dt + p_m(t)\left[x_m(t) \times s_{x,m}(t) - x_m(t)J_{x,m}(t)'\right]dB_m(t)$$

is a martingale, then also $p_m^*(t)(w)x_m^*(t, w)$, which process is

(14)
$$d\left(p_m^*(t)x_m^*(t)\right) = p_m^*(t)x_m^*(t) \times \left[\tilde{r}_{x,m}\left(t, a_{x,m}(t)\right) - \tilde{r}_{x,m}\left(t, r_{x,m}\left(t, s_{x,m}(t)\right)\right)J_{x^*,m}^*(t)\right]dt + p_m^*(t)\left[x_m^*(t) \times \tilde{r}_{x,m}\left(t, r_{x,m}\left(t, s_{x,m}(t)\right)\right) - x_m^*(t)J_{x^*,m}^*(t)'\right]dB_m^*(t)$$

is a martingale. This is because

(15) $$\sigma_{x,m}(t,\omega)\pi_{x,m}(t)=\rho_{x,m}\left(t,\sigma_{x,m}(t,\omega)\right)\pi^{*}_{x^{*},m}(t) \quad \text{for all} \quad t\in[0,\tau], \quad a.s.$$

and $\tilde{\rho}_{x,m}$ projects all the processes in X_m into x_m. That is

(16) $$\tilde{\rho}_{x,m}\left(t,\alpha_{x,m}(t)\right)-\tilde{\rho}_{x,m}\left(t,\rho_{x,m}\left(t,\sigma_{x,m}(t)\right)\right)\pi^{*}_{x^{*},m}(t)=\alpha_{x,m}(t)-\sigma_{x,m}(t)\pi_{x,m}(t) \quad \text{for all} \quad t\in[0,\tau], \quad a.s.$$

Conversely, if $\pi^{*}_{m}(t,\omega)x^{*}_{m}(t,\omega)$ is a martingale then there exists $x_m\in X_m$ and $\pi_m\in L^{n}_{1}$ such that (16) holds and according to Assumption A4

(17) $$\alpha_{x,m}(t,\omega)=\sigma_{x,m}(t,\omega)\pi_{x,m}(t,\omega) \quad \text{for all} \quad t\in[0,\tau], \quad a.s.$$

This gives $\pi_m(t,\omega)x_m(t,\omega)$ is also a martingale and $E^{Q_m}[x(t)]=E^{Q^{*}_{m}}[x^{*}(t)]$, where E^{Q_m} means the expectation with respect to Q_m, for all $t\in[0,\tau]$. *Q.E.D.*

Lemma 2 means that if there exists $x_m\in X_m$ and Q_m such that $x_m(t,\omega)$ is a martingale on $(\Omega,F^{(n_m)},Q_m)$ then also $x^{*}_{m}(t,\omega)$ is a martingale on $\left(\Omega,F^{(n^{*}_{m})},Q^{*}_{m}\right)$ and vice versa. This yields the fact that the sufficient condition for the absence of arbitrage on $(\Omega,F^{(n_m)},P_m)$ is that there exists a unique state-price deflator on $\left(\Omega,F^{(n^{*}_{m})},P^{*}_{m}\right)$ that is given by (7) for all $m\in M$. Given Assumption A4 each process of X_m is a martingale with respect to its own martingale measure. Therefore in our analysis Lemma 2 only fixes the martingale measure Q^{*}_{m}. The arbitrage-free condition is proved in Theorem 1.

THEOREM 1: *There is no arbitrage in the initial market if and only if there is a state-price deflator on* $\left(\Omega,F^{(n^{*}_{m})},P^{*}_{m}\right)$ *for all* $m\in M$.

PROOF: There is no arbitrage on $\left(\Omega,F^{(n^{*}_{m})},P^{*}_{m}\right)$ if and only if there exists an equivalent martingale measure with Radon-Nikodym derivative π^{*}_{m} [see Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981), and Clark (1993)]. We can write the deflated process of a dynamic portfolio $\theta(t)x_m(t)$, where $\theta'\in L^{n_m}_{k}$ is a trading strategy process on $\left(\Omega,F^{(n_m)},P_m\right)$, as follows

(18) $$\int_0^t \theta(s)d\left[\pi_m(s)x_m(s)\right]=\int_0^t \theta(s)d\left[\pi_m(s)\left(x^{*}_{m}(s)+x^{\tilde{\rho}}_{m}(s)+x^{\rho}_{m}(s)\right)\right] \quad \text{for all} \quad t\in[0,\tau]$$

where $m\in M$, $x^{\tilde{\rho}}_{m}(t)\in\tilde{\rho}^{-1}_{x,m}(t,x^{*}_{m}(t))-x^{*}_{m}(t)$, $x^{\rho}_{m}(t)\in\rho^{-1}_{x,m}(t,x^{*}_{m}(t))-x^{*}_{m}(t)$, and $\pi_m(t)=\exp\left\{-\int_0^t \pi_{x^{1},m}(s,\omega)dB_m(s)-\frac{1}{2}\int_0^t \pi_{x^{1},m}(s,\omega)^2ds\right\}$, $\tilde{\rho}_{x,m}\left(t,x_m(t)\right)=x^{1}_{m}(t)$ for all $x_m\in X_m$, and $x^{1}_{m}\in X_m$.

Because $\pi_m(t)$ is the state price deflator of x^{1}_{m}, we get from Assumption A1 the following condition

(19) $$E\left[\int_0^t \theta(s)d\left[\pi_m(s)x^{\tilde{\rho}}_{m}(s)\right]\right]\le 0 \quad \text{for all} \quad t\in[0,\tau], \quad x_m\in X_m$$

That is, if $x^{\tilde{\rho}}_{m}(t)\le 0$ then $\theta(t)\ge 0$, and if $x^{\tilde{\rho}}_{m}(t)\ge 0$ then $\theta(t)\le 0$. This is due to transaction costs. Combining (18) and (19) we see that $\theta(t)\pi_m(t)x_m(t)$ is a supermartingale. If this holds for all $m\in M$, then there is no arbitrage in the initial market.

Conversely, if we assume that there is no arbitrage in the initial market then there exist $x_m\in X_m$ such that $\pi_m(t)x_m(t)$ is a martingale for all $m\in M$. Equations (18) and (19) ensure that this implies the absence of arbitrage condition. Projecting all the processes in X_m into x_m, we see that there is no arbitrage on $\left(\Omega,F^{(n^{*}_{m})},P^{*}_{m}\right)$, because

$p^*_m(t)x^*_m(t)$ is a martingale for all $m \in M$.

Q.E.D.

Theorem 1 implies the same that is also proved in Jouini and Kallal (1995). That is, the absence of arbitrage condition is equivalent to the existence of an equivalent probability measure that transforms some process between the bid and ask price processes into a martingale. In our analysis all the processes of the initial economy are martingales with respect to their own martingale measures. However, in the proof of Theorem 1 we only need to assume that for all $m \in M$ there exists one process in X_m such that it is a martingale under the equivalent martingale measure.

The problem in finding the optimal hedging, consumption, and portfolio processes under transaction costs and other frictions is the fact that trading can be seen as zero-utility consumption. As mentioned earlier, the process $x_m \in X_m$ is a function of the trading strategy, and therefore also the martingale measure of x_m is a function of the trading strategy. We write explicitly $x_m(t) = x_{m,q}(t)$ and $p_m(t) = p_{m,q}(t)$.

We define the T-maturity upper- and lower- hedging prices of a contingent claim C in our initial economy as

(20) $$C^{up}(t) = \inf_{m\in M}\left\{\inf_{q\in L^n_k}\left\{q(t)x_{m,q}(t) \mid q(T)x_{m,q}(T) \geq C(T,x_1)\ a.s.\right\}\right\}$$

and

(21) $$C^{low}(t) = \sup_{m\in M}\left\{\sup_{q\in L^n_k}\left\{-q(t)x_{m,q}(t) \mid q(T)x_{m,q}(T) \leq C(T,x_1)\ a.s.\right\}\right\}$$

where the portfolios are self-financing, x_1 is the underlying price process, and $0 \leq C^{low}(t) \leq C^{up}(t) \leq \infty$. Because it is possible that the market is incomplete, there may be uncontrollable risks in the agents' portfolio processes. The upper hedging price is finite if it is independent of the uncontrollable risks and the asset prices are independent of the agents' actions. Using the framework of Cvitanic and Karatzas (1993) we get the following proposition.

PROPOSITION 1: *If the upper hedging price is finite, we get*

(22) $$C^{up}(t) = \inf_{m\in M}\left\{\sup\left\{E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]\right\}\right\}$$

and

(23) $$C^{low}(t) = \sup_{m\in M}\left\{\inf\left\{E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]\right\}\right\}$$

where $t \in [0,T]$ *and* $T \in [0,t]$.

PROOF: If Q is the martingale measure of x_1, then $C(t,x_1) = E^Q\left[C(T,x_1) \mid F_t\right]$ is the price in a complete frictionless market. The price on $\left(\Omega, F^{(n^*_m)}, P^*_m\right)$ is now given by

(24) $$C^*_m(t) = E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]$$

Taking supremum over all martingale measures we get

(25) $$C^{up}_m(t) \leq \sup\left\{E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]\right\}$$

where C^{up}_m is the upper price corresponding to agent m. In the same way we get for the lower price $C^{lower}_m(t) \geq \inf\left\{E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]\right\}$.

Now we take any self-financing portfolio process that satisfies $q(T)x_{m,q}(T) \geq C(T,x_1)$. Because $E^{Q^*_m}\left[q(T)x_{m,q}(T) \mid F_t^{(n^*_m)}\right] \geq E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]$ for all the projected martingale measures Q^*_m, we get from (19)

(26) $$C^{up}_m(t) \geq \sup\left\{E^{Q^*_m}\left[C(T,x_1) \mid F_t^{(n^*_m)}\right]\right\}$$

Similarly, with the lower hedging price we get $C_m^{lower}(t) \le \inf\left\{E^{Q_m^*}\left[C(T,x_1) \mid F_t^{(n_m^*)}\right]\right\}$ by constructing any self-financing portfolio that satisfies $q(T)x_{m,q}(T) \le C(T,x_1)$.

Taking the infimum over all the upper prices we get the market upper price, and taking the supremum over all the lower prices we get the market lower price.

Q.E.D.

Proposition 1 implies that if the frictions are zero, i.e., the bid and ask prices are equal, for all the agents in M, then the upper- and lower hedging prices are equal if the upper prices are finite.

Now we start to consider the optimal consumption and portfolio processes. Given the adapted consumption endowment process $e_m^* \in L_1^{n_m^*}$ on $\left(\Omega, F^{(n_m^*)}, P_m^*\right)$ there exists a dynamic portfolio $q^*(t)x_{m,q^*}^*(t)$, where $q^*' \in L_k^{n_m^*}$ is a trading strategy process, which finances an adapted consumption process $c_m^* \in L_1^{n_m^*}$ on $\left(\Omega, F^{(n_m^*)}, P_m^*\right)$ if

(27) $$p_m^*(t)q^*(t)x_{m,q^*}^*(t) = \int_0^t p_{m,q^*}^*(s)\left[e_m^*(s) - c_m^*(s)\right]ds + \int_0^t q^*(s)d\left[p_{m,q^*}^*(s)x_{m,q^*}^*(s)\right] \quad \text{for all} \quad t \in [0,T]$$

and $p_{m,q^*}^*(T)q^*(T)x_{m,q^*}^*(T) = 0$, i.e., terminal consumption is zero, where $m \in M$ and $T \in [0,t]$. $e_m, c_m \in L_1^{n_m}$ are the corresponding processes of e_m^* and c_m^* on $(\Omega, F^{(n_m)}, P_m)$.

Equation (27) gives the following lemma.

LEMMA 3: *Given the endowment process $e_m \in L_1^{n_m}$ and any adapted $c_m \in L_1^{n_m}$ there exists a process $q' \in L_k^{n_m}$ financing c if and only if*

(28) $$E\left\{\int_0^T p_{m,q^*}^*(t)\left[c_m^*(t) - e_m^*(t)\right]dt\right\} = 0$$

and

(29) $$\int_0^T p_{m,q}^r(s)c_m^r(s)ds = \int_0^T p_{m,q}^r(s)e_m^r(s)ds + \int_0^T q^*(s)d\left[p_{m,q}^r(s)x_{m,q}^r(s)\right]$$

where $x_{m,q}^r(t) \in r_{x,m}^{-1}(t, x_{m,q^*}^*(t)) - x_{m,q^*}^*(t)$, $c_m^r(t) \in r_{x,m}^{-1}(t, c_m^*(t)) - c_m^*(t)$, $e_m^r(t) \in r_{x,m}^{-1}(t, e_m^*(t)) - e_m^*(t)$, $p_{m,q}^r(t) = p_{m,q}(t) - p_{m,q^*}^*(t)$, $T \in [0,t]$, *and $q = q^*$ almost surely.*

PROOF: See e.g. Cox and Huang (1989) for the proof that Lemma 3 holds for e_m^* and c_m^* on $\left(\Omega, F^{(n_m^*)}, P_m^*\right)$. Now $\tilde{r}_{x,m}$ projects all the processes in X_m into $x_{m,q}$ and we get by using (18) and (27) the following financing condition

(30) $$q(t)\left[p_{m,q^*}^*(t)x_{m,q^*}^*(t) + p_{m,q}^r(t)x_{m,q}^r(t)\right] = \int_0^t p_{m,q^*}^*(s)\left[e_m^*(s) - c_m^*(s)\right]ds + \int_0^t p_{m,q}^r(s)\left[e_m^r(s) - c_m^r(s)\right]ds + \int_0^t q(s)d\left[p_{m,q^*}^*(s)x_{m,q^*}^*(s) + p_{m,q}^r(s)x_{m,q}^r(s)\right] \quad \text{for all} \quad t \in [0,T]$$

and $q(T)\left[p_{m,q^*}^*(T)x_{m,q^*}^*(T) + p_{m,q}^r(T)x_{m,q}^r(T)\right] = 0$, where $q = q^*$. Because q^* finances c_m^* we get from (30) equation (28) and

(31) $$q^*(t)\left[p_{m,q}^r(t)x_{m,q}^r(t)\right] = \int_0^t p_{m,q}^r(s)\left[e_m^r(s) - c_m^r(s)\right]ds + \int_0^t q^*(s)d\left[p_{m,q}^r(s)x_{m,q}^r(s)\right] \quad \text{for all} \quad t \in [0,T]$$

Taking into account the terminal consumption condition we get (29).

Conversely, if q finances c, then (30) holds. Because $x^{r}_{m,q}$ and x^{*}_{m,q^*} are independent processes, we get equations (27) and (31) from (30). Using again the terminal consumption condition and the framework of Cox and Huang (1989) we get (28) and (29).

Q.E.D.

Lemma 3 implies that in an incomplete market there can be uncontrollable risks in the agents' endowment and portfolio processes.

Given Lemma 3 a single agent $m \in M$ faces the following problem on $\left(\Omega, F^{(n^*_m)}, P^*_m\right)$

$$\text{(32)} \qquad \sup_{c^*_m \in L_1^{n^*_m}} U_m\left(c^*\right)$$

subject to

$$\text{(33)} \qquad E\left\{\int_0^T p^{\;*}_{\;m,q^*}(t)\left[c^*_m(t) - e^*_m(t)\right]dt\right\} = 0$$

Now we can state the following theorem.

THEOREM 2: *There exists an optimal consumption and trading strategy on* $\left(\Omega, F^{(n_m)}, P_m\right)$ *for agent* $m \in M$ *if and only if there exist an optimal consumption and trading strategy on* $\left(\Omega, F^{(n^*_m)}, P^*_m\right)$*. The optimal consumption choice* $\hat{c}^*_m \in L_1^{n^*_m}$ *on* $\left(\Omega, F^{(n^*_m)}, P^*_m\right)$ *for agent* $m \in M$ *on time period* $[0,T]$ *is*

$$\text{(34)} \qquad \hat{c}^*_{m,\hat{q}^*}(t) = I_m\left[g^*_{m,\hat{q}^*} p^{\;*}_{\;m,\hat{q}^*}(t), t\right] \quad \text{for all} \quad t \in [0,T]$$

where $I_m[\cdot,t]$ *inverts* $\frac{¶u_m(\cdot,t)}{¶c}$, *meaning that* $I_m\left[\frac{¶u_m(x,t)}{¶c}, t\right] = x$ *for all x and t, and* $g^*_{m,\hat{q}^*} > 0$ *is a Lagrange multiplier satisfying* $E\left(\int_0^T p^{\;*}_{\;m,\hat{q}^*}(t)\left\{I_m\left[g^*_{m,\hat{q}^*} p^{\;*}_{\;m,\hat{q}^*}(t), t\right] - e^*_m(t)\right\}dt\right) = 0$.

The optimal portfolio $\hat{q}^*$ *solves*

$$\text{(35)} \qquad \hat{q}^*(t) = \frac{1}{p^{\;*}_{\;m,\hat{q}^*}(t)} j^{\;*}_{\;m,\hat{q}^*}(t)'\left[x^*_{m,\hat{q}^*}(t) \times \tilde{r}_{x,m}\left(t, r_{x,m}\left(t, s_{x_{\hat{q}},m}(t)\right)\right) - x^*_{m,\hat{q}^*}(t) J^{\;*}_{\;x^*,m}(t)'\right]^{-1}$$

$$\text{for all} \quad t \in [0,T], \quad a.s.$$

where $E\left(\int_0^T p^{\;*}_{\;m,\hat{q}^*}(s)\left\{I\left[g^*_{m,\hat{q}^*} p^{\;*}_{\;m,\hat{q}^*}(s), s\right] - e^*_m(s)\right\}ds \;\middle|\; F^{(n^*)}_{t,m}\right) = \int_0^t j^{\;*}_{\;m,\hat{q}^*}(s)' dB^*_m(s)$. *Further, the optimal consumption and portfolio processes satisfy*

$$\text{(36)} \qquad E\left\{\int_0^T u_m\left(I_m\left[g^*_{m,\hat{q}^*} p^{\;*}_{\;m,\hat{q}^*}(t), t\right], t\right)dt\right\} \geq E\left\{\int_0^T u_m\left(I_m\left[g^*_{m,q^*} p^{\;*}_{\;m,q^*}(t), t\right], t\right)dt\right\}$$

for all q^* *and* $c^*_{m,q^*}(t) = I_m\left[g^*_{m,q^*} p^{\;*}_{\;m,q^*}(t), t\right]$ *that satisfy equations* (34) *and* (35). *We also must have*

$$\text{(37)} \qquad U_m\left(\hat{c}^*_{m,\hat{q}^*}\right) < \infty$$

PROOF: By the saddle point theorem [see e.g. Luenberger (1969)] and strict monotonicity of U_m, the optimal consumption process $c^*_m(t)$ solves the unconstrained problem

$$\text{(38)} \qquad \sup_{c^*_m \in L_1^{n^*}} U_m\left(c^*\right) - g^*_{m,q^*} E\left\{\int_0^T p^{\;*}_{\;m,q^*}(t)\left[c^*_m(t) - e^*_m(t)\right]dt\right\}$$

From Assumption A5 and (38) we get

(39) $$c^{*}_{m,q^{*}}(t) = I_m\left[g^{*}_{m,q^{*}} p^{*}_{m,q^{*}}(t), t\right] \quad \text{for all} \quad t \in [0,T]$$

There exist Lagrange multipliers such that $E\left(\int_0^T p^{*}_{m,q^{*}}(s)\left\{I\left[g^{*}_{m,q^{*}} p^{*}_{m,q^{*}}(s), s\right] - e^{*}_m(s)\right\}ds\right) = 0$ holds since $I_m[\cdot,t]$ is continuous and strictly decreasing, and maps $(0,\infty)$ into itself with $I_m[0+,t] = \infty$ and $I_m[\infty,t] = 0$.

From (14), (27), Lemma 3, and $E\left(\int_0^T p^{*}_{m,q^{*}}(s)\left\{I\left[g^{*}_{m,q^{*}} p^{*}_{m,q^{*}}(s), s\right] - e^{*}_m(s)\right\}ds \;\middle|\; F^{(n^{*})}_{t,m}\right) = \int_0^t j^{*}_{m,q^{*}}(s)' dB^{*}_s$ we get

(40) $$q^{*}(t) = \frac{1}{p^{*}_{m,q^{*}}(t)} j^{*}_{m,q^{*}}(t)'\left[x^{*}_{m,q^{*}}(t) \times r_x\left(t, r\left(t, s_{x_{q^{*}},m}(t)\right)\right) - x^{*}_{m,q^{*}}(t) J^{*}_{x^{*}_{q^{*}},m}(t)'\right]^{-1}$$

The optimal processes have to also satisfy equations (36) and (37), and the solution does not have to be unique.

The optimal consumption and portfolio processes on $\left(\Omega, F^{(n_m)}, P_m\right)$ are $\hat{c}^{*}_{m,\hat{q}^{*}} + c^{r}_{m,\hat{q}^{*}}$ and $\hat{q}^{*}$, where $c^{r}_{m,\hat{q}^{*}}$ solves equation (29). These are optimal processes, because the drift process of $c^{r}_{m,\hat{q}^{*}}$ is equal to zero and the volatility processes of $\hat{c}^{*}_{m,\hat{q}^{*}} + c^{r}_{m,\hat{q}^{*}}$ and $\hat{c}^{*}_{m,\hat{q}^{*}}$ belongs to the same equivalence class with respect to the agent's market price of risk. That is, the agent $m \in M$ is risk neutral with respect to $c^{r}_{m,\hat{q}^{*}}$.

Correspondingly, if there exist optimal strategies on $\left(\Omega, F^{(n_m)}, P_m\right)$, then the optimal consumption has the hollowing representation $c^{*}_{m,q^{*}} + c^{r}_{m,q^{*}}$, where $c^{*}_{m,q^{*}}$ is a consumption strategy on $\left(\Omega, F^{(n^{*})}_m, P^{*}_m\right)$ and $c^{r}_{m,q^{*}}$ solves equation (29). If $c^{*}_{m,q^{*}} \neq \hat{c}^{*}_{m,\hat{q}^{*}}$ for all $\hat{c}^{*}_{m,\hat{q}^{*}}$ that solve (34) - (37), then there exists a consumption strategy that gives more utility or there does not exist an optimal consumption strategy and a trading strategy that finances the consumption process. This is a contradiction, since $c^{*}_{m,q^{*}} + c^{r}_{m,q^{*}}$ is an optimal consumption process, and we get that there exist optimal consumption and portfolio processes on $\left(\Omega, F^{(n^{*}_m)}, P^{*}_m\right)$.

Q.E.D.

Theorem 2 implies that the form of the optimal consumption trading strategy is the same as the corresponding strategies in complete frictionless market. This complete economy is defined by the state-price deflator implicitly given by equations (34) and (35). Different agents may have distinct state-price deflators, because their frictions, utility functions, and endowment processes may differ. The difficulty in solving equations (34) – (36) is that we require that the corresponding state-price deflator is used with an optimal strategy. That is, we can not just fix the state-price deflator and solve the optimal solution, because usually this leads to strategies that try to take advantage from frictions. If an agent tries to take advantage from the frictions then Assumption A2 is applied and we set the asset price equal to a price between the bid and ask prices such that the optimal portfolio increment is zero. Using Assumption A2, Theorem 2, and the framework of Cvitanic and Karatzas (1996) we get the following proposition.

PROPOSITION 2: If *the optimal consumption and trading strategies of an agent* $m \in M$ *exist then they satisfy*

(42) $$U_m\left(\hat{c}^{*}_{m,\hat{q}^{*}}\right) = \inf\left\{E\left[\int_0^T u_m\left(I_m\left[g^{*}_m p^{*}_m(t), t\right], t\right)dt\right]\right\}$$

where the infimum is taken over all state-price deflators, $T \in [0,t]$, $\hat{q}^{*}$ *is the optimal trading strategy,* $\hat{c}^{*}_{m,\hat{q}^{*}} + c^{r}_{m,\hat{q}^{*}}$ *is the optimal consumption process,* $c^{r}_{m,\hat{q}^{*}}$ *solves equation* (29), *and* g^{*}_m *solves* $E\left(\int_0^T p^{*}_m(s)\left\{I\left[g^{*}_m p^{*}_m(s), s\right] - e^{*}_m(s)\right\}ds\right) = 0$.

PROOF: From Theorem 2 we get $U_m\left(\hat{c}^*_{m,\hat{q}^*}\right) \geq \inf\left\{E\left[\int_0^T u_m\left(I_m\left[g^*_m p^*_m(t),t\right],t\right)dt\right]\right\}$. Now we relax the condition that the corresponding state-price deflator and price process are used with a trading strategy. Let us fix the process x^*_m and derive the optimal consumption and trading strategies. Then the deflated wealth process is given by

$$(43) \qquad \tilde{W}_{x^*_m}(t) = \int_0^t q^*(s)dx^*_m(s) \quad \text{for all} \quad t \in [0,t]$$

where q^* is the trading strategy which finance the optimal consumption in the fictitious market. The wealth process that takes into account the frictions is given by

$$(44) \qquad W_{x^*_m}(t) = \int_0^t q^*(s)d\left[x^*_m(s) + x^{\tilde{r}}_m(s)\right] \quad \text{for all} \quad t \in [0,t]$$

From (19) and (23) we get $W_{x^*_m}(T) \leq \tilde{W}_{x^*_m}(T) = 0$. This gives $E^{Q^*_m}\left[W_{p^*_m}(T) \mid F_t^{(n^*_m)}\right] \leq \inf\left\{E^{Q^*_m}\left[\tilde{W}_{p^*_m}(T) \mid F_t^{(n^*_m)}\right]\right\}$, where the infimum is taken over all martingale measures. Taking into account Theorem 2 we get

$$(45) \qquad U_m\left(\hat{c}^*_{m,\hat{q}^*}\right) \leq \inf\left\{E\left[\int_0^T u_m\left(I_m\left[g^*_m p^*_m(t),t\right],t\right)dt\right]\right\}$$

Q.E.D.

Proposition 2 implies that if there exists a consumption process that gives more utility than the optimal consumption process then there does not exist a portfolio process that finance the consumption. Assuming Assumption A2 we get that the trading is zero if the agent is trying to take advantage from the transaction costs, i.e., if he or she is trying to sell at the ask price or buying at bid price.

Security spot-market equilibrium is a collection

$$(46) \qquad \left\{(x_m, p_m, c_m, q_m) \mid x_m \in X_m, m \in M\right\},$$

such that, given the security-price processes in X_m and the state-price processes, for each agent *m*, (c_m, q_m) solves (32) and (33), and markets clear $\sum_{m\in M} q_m(t) = 0$ and $\sum_{m\in M} c_m(t) - e_m(t) = 0$ for all $t \in [0,T]$. Using Theorem 2 we get the following theorem.

THEOREM 3: *The equilibrium conditions lead to the following a.s. equality*

$$(47) \qquad \sum_{m\in M} e_m(t) = \sum_{m\in M} I_m\left[g_m p_{m,q_m}(t),t\right] \quad \text{for all} \quad t \in [0,T]$$

where $I_m(\cdot,t)$ *inverts* $\frac{\partial u_m(\cdot,t)}{\partial c}$, p_{m,q_m} *is the state-price deflator of agent m* $\in$ *M, and* q_m *is the optimal portfolio process of m.*

PROOF: From (47) we see directly that commodity market clear and each agent has optimal consumption and trading strategy. Theorem 2 gives $\sum_{m\in M} q_m(t) = 0$, i.e., asset markets clear, because $\sum_{m\in M} j_{m,q_m}(t) = 0$. *Q.E.D.*

From Theorem 2 and Theorem 3 we see that the sufficient condition for the existence of an equilibrium on the incomplete market with frictions is that there exist optimal strategies for each agent on his or her complete frictionless market that is inside the initial market. If the initial market is complete and frictionless then $r_{x,m}$ and $\tilde{r}_{x,m}$ are identity mappings. That is, the results of this paper can be seen as an extension to the corresponding theorems in complete markets.

5. SUMMARY

This paper studies the state-price deflator in the presence of frictions and without dynamic spanning. A unique state-price deflator may only exist on projected markets. The existence of the deflator is equivalent to the existence of a surjection that maps the volatility functions onto the drift functions. There is no arbitrage, if there exists a unique state-price deflator on a projected market. If there exists an optimal consumption process and a portfolio that finance the consumption on a projected market then there exists an optimal strategies also on the initial market. Given the optimal consumption process and trading strategy for single agents, there exists an equilibrium for tradable assets.

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