On Arbitrage, Optimal Portfolio and Equilibrium under Frictions and Incomplete Markets

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ON ARBITRAGE, OPTIMAL PORTFOLIO, AND EQUILIBRIUM UNDER INCOMPLETE MARKET AND TRANSACTION COSTS

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ABSTRACT

This thesis considers mathematical finance in an incomplete market with transaction costs. It consists of two practical and two theoretical essays. The practical papers can be seen as an application of optimal portfolio selection and the theoretical papers study the market equilibrium conditions in the presence of incompleteness. The results indicate that the optimal hedging strategy differs significantly from the corresponding strategy in a frictionless market even at one- or two-day trading intervals and that under stochastic real foreign exchange rates international equilibrium is curved. In addition, it is shown that the market conditions hold in an incomplete market under frictions if the conditions hold in the projected markets that exist inside the initial market.

PREFACE

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PUBLICATIONS

The thesis consists of the present summary article and the following papers:

I Keppo, J., “Calling for the True Margin,”
Applied Financial Economics, 1997, 7, 207 – 212.

II Keppo, J. and S. Peura, “Optimal Portfolio Hedging with Nonlinear Derivatives and Transaction Costs”**,** *Computational Economics* (forthcoming).

III Keppo, J., “Optimal Home Currency and the Curved Equilibrium”, Helsinki University of Technology, Systems Analysis Laboratory Research Reports A71, February, 1998.

IV Keppo, J., “Market Conditions under frictions and without Dynamic Spanning”, Helsinki University of Technology, Systems Analysis Laboratory Research Reports A72, February, 1998.

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The results reported in papers I, III, and IV were obtained by the author. The study presented in paper II was carried out in collaboration between S. Peura and the author.

1. INTRODUCTION

The problems of finance vary from corporate policy to the valuation of complex derivative instruments. For instance, in order to carry business a modern company needs to invest in a variety of real assets. The most profitable investment opportunities are found by pricing these real assets, employing the theory of asset pricing. To obtain the necessary funds for the investments firms sell financial assets. These financial securities have a value because they are claims on the firm's real assets, and therefore the firms have to understand how the prices are created in the financial markets.

The three seminal and internally consistent theories upon which mathematical finance is founded are pricing by arbitrage, portfolio and consumption optimization, and equilibrium analysis. Their common theme is how individuals and markets allocate their scarce resources through a price system based on the valuation of risky assets. All these three approaches imply the existence of a linear pricing function, a state-price deflator that defines a relationship between the risk and return of tradable securities. If this state-price deflator is unique, the markets are said to be complete. This uniqueness requires that there be exactly as many tradable assets as sources of uncertainty. Then all tradable assets can be priced by using this pricing function. Correspondingly, the markets are incomplete, if there exist more than one state-price deflator. In an incomplete market the price of a contingent claim may depend on the state-price deflator with respect to which it is priced. Therefore such an elegant theory does not exist in an incomplete market, but it is currently one of the major research agendas.

Stochastic analysis has had a significant impact on the rapid development of financial mathematics. This is because linear stochastic differential equations are found to be efficient in the modeling of asset dynamics. Therefore, applying powerful methods from stochastic analysis and stochastic control to all fields of mathematical finance has become possible. This way new solutions have been derived and earlier models have been formalized in a general framework. For instance, the optimal portfolio selection problem can be solved by employing the stochastic control theory. At the same time, the development of sophisticated analytical and numerical methods has helped to increase the relevance of these developments in the everyday financial practice.

This thesis considers mathematical finance in an incomplete market with transaction costs. The first paper studies margin calculation from the clearing house's perspective and the second essay studies the optimal hedging problem with nonlinear instruments and transaction costs. Both these articles can be seen as an application of optimal portfolio selection. The last two essays consider market conditions in the presence of incompleteness.

This introduction is structured as follows. Section 2 defines the market conditions, Section 3 reviews the related literature, and Section 4 discusses the thesis shortly. Finally Section 5 summarizes the main results.

2. MARKET CONDITIONS

In this section we give the definitions for absence of arbitrage, single agent's optimality, and market equilibrium. The equivalence of all these three market conditions is that they all imply the existence of a state-price deflator. However, each condition has a different path to this existence theorem and therefore they have a different economic interpretation of the deflator.

The arbitrage-free condition means that there does not exist a zero investment portfolio that yields only positive earnings and strictly positive earnings with strictly positive probability. So, if the markets are arbitrage-free, then there is no possibility of 'free lunch'. By employing the techniques of stochastic analysis the absence of 'free lunch' is equivalent to the existence of a state-price deflator with the property that each price process is a martingale after multiplication by the deflator. This is the foundation of the theory of pricing by arbitrage. The state-price deflator defines an equivalent probability measure and contingent claims can, by definition, be priced by taking their expected value with respect to this equivalent martingale measure.

The optimal consumption and portfolio choice problem is a selection of a consumption process that maximizes the agent's utility and a trading strategy that finances this consumption process. It is assumed that the agent's decisions have no effect on market prices. The probabilities given by the equivalent probability measure can be interpreted as the intertemporal marginal rates of the substitution of an agent maximizing his expected utility. In the optimal consumption and portfolio selection, the state-price deflator is used to transfer the problem into an explicit consumption choice problem. From this problem the solutions of optimal consumption and trading strategy are obtained from two linear parabolic partial differential equations.

A security-spot market equilibrium is a collection of security prices, consumption commodity prices, the agents' optimal consumption processes and trading strategies such that excess supply is zero in security and consumption commodity markets, when the processes of security and consumption commodity prices are given. That is, the prices of the economy are such that supply equals demand and that the allocation of each agent is optimal. The state-price deflator is used to remove the average rates of return of different securities, and it defines a linear relationship between the risk and return of tradable securities. Further, in a real economy the price of a consumption commodity can be seen as the state-price deflator of the economy.

3. Related Literature

In this section we review some literature on the absence of arbitrage, single agent's optimality, and market equilibrium in both complete and incomplete markets.

The theory of asset pricing has roots in Arrow (1953), Arrow and Debreu (1954), Samuelson (1965), Black and Scholes (1973), Merton (1973), and Cox and Ross (1976). Harrison and Kreps (1979), Harrison and Pliska (1981), and Kreps (1981) have formalized the earlier models in a general framework. In all these models the security markets are assumed to be complete. The main result of these papers is the martingale definition of the arbitrage-free condition.

In Jouini and Kallal (1995) the arbitrage-free condition is derived in the presence of transaction costs. They show that the arbitrage-free condition is equivalent to the existence of an equivalent probability measure that transforms some process between the bid and ask price processes of traded securities into a martingale. The pricing in incomplete markets is considered, e.g., in Karatzas and Kou (1996), Föllmer and Sonderman (1986), and Föllmer and Scheiwer (1991).

Portfolio optimization under uncertainty originates from the static models of Markowitz (1952, 1958) and Tobin (1958). The discrete multiperiod model can be found in Samuelson (1969). Merton (1969, 1971) analyzes the optimal consumption and portfolio choice problem and its solution using the continuous-time stochastic control in finite and infinite horizon settings. The martingale method to solve the optimal consumption and portfolio choice have been developed in Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987). In these models the utility maximization problem is considered by martingale methods and without the need of imposing any Markovian assumptions.

The single agent's optimality in the presence of transaction costs is studied, e.g., in Constantinides (1979, 1986), Cvitanic and Karatzas (1996), Duffie and Sun (1990), Leland (1985), and Shreve and Soner (1994). For example, they show that in the presence of transaction costs, the optimal trading strategy involves trading in discrete time intervals and, under particular circumstances, the length of the trading interval may even optimally be chosen as fixed. Cvitanic and Karatzas (1992, 1993) consider, e.g., general closed, convex constraints on portfolio proportions, and different interest rates for borrowing and lending.

Sharpe (1965) and Lintner (1965) derive static equilibrium models for capital asset pricing under uncertainty. The consumption-based capital asset pricing model is derived in Breeden (1979). The basic framework for deriving the security-spot market equilibrium in a continuous-time setting can be found, e.g., in Duffie (1992). Duffie and Zame (1989) have proved the existence of an Arrow-Debreu equilibrium in the case of a smooth-additive utility function. Huang (1987) has derived an equilibrium model with a smooth-additive utility function.

The equilibrium in an incomplete market is studied in Grossman and Shiller (1982) and Back (1991). These models start the analysis from the state-price deflators implicitly given by single agents, and then derive the excess expected rates of return on all securities from the covariance of returns with aggregate consumption increments and the 'market-risk-aversion' constant.

3. THESIS

This thesis can be divided into two different parts. The first part consists of two practical papers, “Calling for the True Margin” and “Optimal Portfolio Hedging with Nonlinear Derivatives and Transaction Costs”, that derive numerical models for practitioners in financial markets. These articles can be seen as applications of optimal portfolio selection. The second part consists of two theoretical essays “Optimal Home Currency and the Curved International Equilibrium” and “Market Conditions under Frictions and without Dynamic Spanning”. These papers derive market conditions, i.e., the arbitrage-free condition, single agent’s optimality, and equilibrium, in the presence of some incompleteness.

3.1 MARGIN CALCULATION

The first essay of the thesis considers margin calculation from the clearing house’s perspective. The role of a clearing house is to provide a guarantee for the investors. While the investors are customers of the derivative market place, the actual trades are executed through the members of the exchange. Clearing houses collect margins from the derivative exchange’s customers in order to ensure that all the participants get their earned profits, because the other customer’s trading profit is the other customer’s trading loss in the derivative business.

A clearing house bears two risks, namely market risk and credit risk. The market risk is the risk of a large price change, large enough to cause some of the customers to lose more than their initial margin requirements. The market risk, if realized, causes a liability from these customers to the clearing house. On the other hand, the credit risk is the risk related to this liability. Realization of the market risk leads to an exposure to the credit risk, which in turn may lead to actual losses. For a clearing house to make a loss on a derivative security, the value of the final customer’s portfolio must decrease more than the net margin, and both the final customer and its member must default because members guarantee the customers’ deals as if they were their own.

This work presents a framework for understanding the risks and exposures and suggests a way to use the framework in determining the margin requirements by using a fixed risk-level model. First, the probability distribution of the value of the customer’s portfolio is characterized. Secondly, the loss distribution of the clearing house is derived from this probability distribution and the default probability of the member. Finally, the margin requirement of the investor is set based on the clearing house’s loss distribution. This calculation is an application of the optimal portfolio selection problem with portfolio constraints. In this respect it is related, e.g., to Cvitanic and Karatzas (1992). The optimal margin requirement is equal to the minimum amount of money such that the lowest outcome of the clearing house’s loss distribution with the given fixed risk-level is nonnegative.

The fixed risk-level method is used in order to avoid explicit estimation of the clearing house’s utility function. Using the principle of expected utility maximization a much more general model could be derived. For example, this kind of a model could solve single customer’s margin requirement by taking into account the other customers’ margins. However, such models are difficult and costly to solve and are usually too complex for practical margin calculations.

This essay shows that traditional models, such as using a fixed percentage from the underlying asset, can lead to margin requirements that can be much larger than required in order to maintain a selected risk-level. Thus, the rules of thumb currently used often result in tying up unnecessary amounts of the final customer’s capital.

3.2 OPTIMAL HEDGING

The second essay of the thesis considers optimal portfolio hedging in the presence of transaction costs and nonlinear instruments. Hedging means reducing the risk of a portfolio by employing a trading strategy. The optimality criterion of our model is to maximize the expected utility of terminal wealth. The hedging strategies are conditioned on full information about the joint distribution of asset returns over the entire hedging period. Specifically, we assume that the length of the trading interval has been selected based on some prior grounds pertaining to the tastes and the nature of the investor’s business activities. The analysis also concentrates on the problem of determining the optimal portfolio allocation strategy within the given discretization of the hedging period. The solution is obtained from a two-stage numerical procedure. First, the problem is transformed into a nonlinear programming problem, which utilizes simulated coefficient matrices. Then, the nonlinear programming problem is solved numerically using standard constrained optimization techniques.

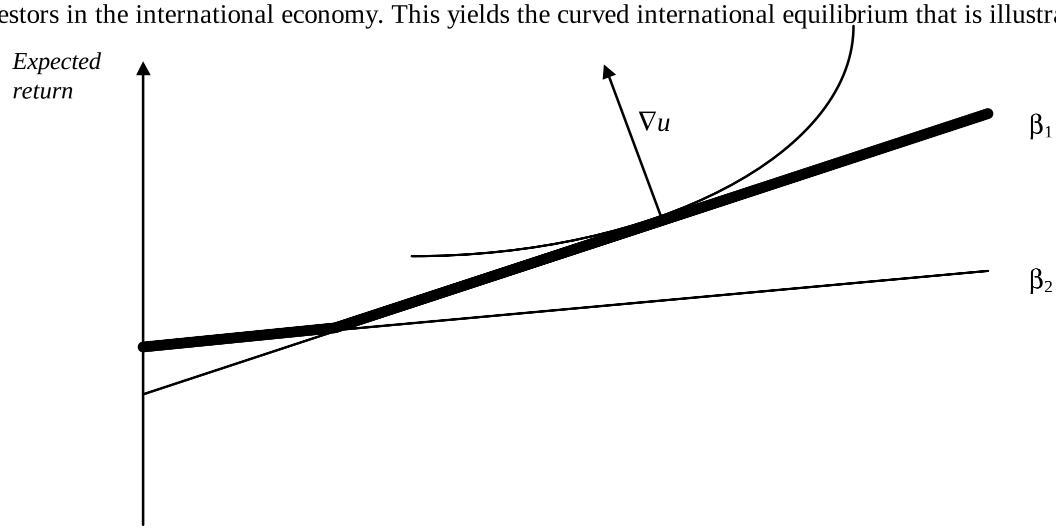
In the solving of dynamic trading strategies, we try to find an approximate solution to a path-dependent dynamic programming problem. The approximation is based on finding the optimal trading strategy among the strategies in which the increments depend on the expected path of the state vector. These strategies are conditioned on a particular subset of the state vector. The approximation reduces the complexity of the problem into computationally manageable levels.

Our dynamic portfolio hedging model is related both to the literature on optimal dynamic portfolio choice, where Merton (1971) has done the seminal work, and to the literature on static portfolio choice (Markowitz, 1958). Our approach is one of hedging behavior models since it assumes non-tradability of the initial portfolio. However, the optimality criteria in the dynamic optimal portfolio selection problem and the dynamic hedging problem are equivalent, and the latter is a general variant of the former, since the hedging problem reduces to the problem of unconstrained optimal portfolio choice in the absence of a fixed initial portfolio. Methodologically, our approach reduces the complicated problem into a non-linear programming problem. This method is similar to the static one-period portfolio selection problem. The model is also related to Cvitanic and Karatzas (1992, 1996), since we also consider the constraints on portfolio proportions.

This essay shows that the optimal hedging behavior differs significantly from delta hedging. This happens even at one- or two-day trading intervals. It also shows how the degree of risk aversion and transaction costs affects the individuals' hedging behavior. These results are difficult to obtain from an entirely analytic model, at least in the current generality of the problem.

3.3 Curved Equilibrium

The third essay considers the international markets. We extend the choice of the optimal portfolio and consumption process to include the selection of an optimal home currency for single agents in a segmented real international economy. We assume that there exist frictions in the trading of consumption commodities between different currencies and that future frictions are uncertain. The uncertainty can be due to an uncertainty about the availability of transporting capacity, the development of technology, and/or the wastage of consumption commodities during the transportation. This yields to the situation where there are stochastic real foreign exchange rates and where the market prices of risks differ between currencies. We derive the explicit relationship between the market price of risk vectors in different currencies. Using these dependencies, we solve the optimal consumption problem for a single agent who is able to freely choose his home currency, i.e., we allow investors to take advantage of the diverging valuation of risks and real interest rates between currencies. We also derive an equilibrium that emerges, when a representative agent of each currency behaves according to the optimization model. In this situation, the market prices of risk do not reflect the risk attitudes of the investors in the particular currency. The prices are simply determined by the interplay of all investors in the international economy. This yields the curved international equilibrium that is illustrated in Figure 1.



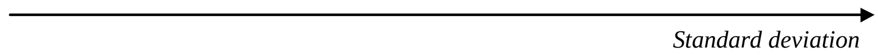


Figure 1: Curved international equilibrium (∇u is the utility gradient of the agent, the thick contour is the international efficient set, b_i is the efficient line in currency i, and i = 1, 2)

Figure 1 shows that for all currencies the national efficient line must equal the international efficient contour at least at one point. Because the international efficient contour is convex our international equilibrium is curved. From Figure 1 we see that the optimal home currency selection is equivalent to finding of the most appropriate national efficient line. The curved equilibrium can be seen as an extension to Breeden (1979) in which the equilibrium yields a linear relationship between risk and return.

3.4 Market Conditions in Incomplete Markets

The fourth essay considers market conditions in an incomplete market and under frictions. Jouini and Kallal (1995) have studied market conditions with transaction costs. In contrast to Jouni and Kallal, we consider also incomplete markets and derive the market conditions by using the quotient space of prices of tradable assets. Specifically, we derive the conditions for the absence of arbitrage, single agent's optimality, and equilibrium in a market with separate bid and ask prices and where there are more sources of uncertainty than there are tradable assets. In this case there exist more than one linear pricing function, i.e., state-price deflator. First we construct a new probability space having less information than the initial probability space. On the new probability space, the markets are complete, and we can employ the framework of complete markets. All the processes of tradable assets are projected into this new space, and it is shown that the market conditions hold in the initial economy if they hold for the projected markets. These projected markets are the same kind of fictitious markets that are used in Cvitanic and Karatzas (1992, 1993, 1996). In addition to Cvitanic and Karatzas, we let the volatility processes of tradable assets differ between various fictitious markets, and we also consider general market frictions. Figure 2 illustrates our model.

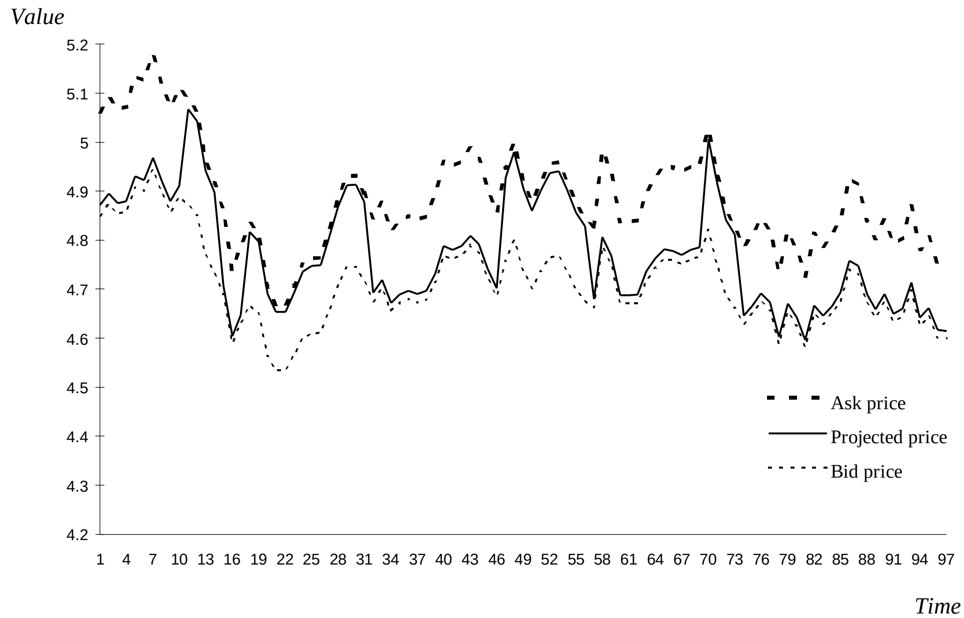


Figure 2: Trajectories of the prices of a tradable asset

In Figure 2 the dashed lines are the trajectories of bid and ask prices of an asset. The solid line is the path into which the bid and ask trajectories are projected. Our framework implies that if the market conditions hold for the projected processes, they hold also for the bid and ask price processes.

4. SUMMARY

Modern mathematical finance considers market conditions by using the methods of stochastic analysis. In complete markets there exists a unique linear pricing function. In an incomplete market there could be many pricing functions and therefore such an elegant theory does not directly exist.

This thesis consists of two practical essays and two theoretical papers. The practical papers can be seen as an application of optimal portfolio selection and the theoretical papers study the market conditions in the presence of incompleteness.

Although it is difficult to predict where the theory will go next, it seems that in the short run great emphasis is devoted to the modeling of incomplete markets under frictions. Most likely, the continuous-time models that are usually based on the Brownian motion will be generalized into the case of abstract information filtration. The derivative industry has also a great interest in these research areas. For instance, investment banks have to consider point processes in order to price default contingent claims. One disadvantage of these approaches is that in many cases the computational problems of the models turn out to be quite hard. Therefore it seems that the research of computational finance will increase in the near future.

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