



## Decision Support

## Modeling project preferences in multiattribute portfolio decision analysis

Tommi Tervonen<sup>a,\*</sup>, Juuso Liesiö<sup>b</sup>, Ahti Salo<sup>c</sup><sup>a</sup> Evidera, London, United Kingdom<sup>b</sup> Department of Information and Service Economy, Aalto University School of Business, Finland<sup>c</sup> Systems Analysis Laboratory, Department of Mathematics and Systems Analysis, Aalto University School of Science, Finland

## ARTICLE INFO

## Article history:

Received 17 February 2016

Accepted 25 April 2017

Available online 2 May 2017

## Keywords:

Multiple criteria analysis

Project portfolio selection

Multi-attribute value theory

Preference modeling

## ABSTRACT

When choosing a portfolio of projects with a multi-attribute weighting model, it is necessary to elicit trade-off statements about how important these attributes are relative to each other. Such statements correspond to weight constraints, and thus impact on which project portfolios are potentially optimal or non-dominated in view of the resulting set of feasible attribute weights. In this paper, we extend earlier preference elicitation approaches by allowing the decision maker to make direct statements about the selection and rejection of individual projects. We convert such project preference statements to weight information by determining the weights for which (i) the selected project is included in all potentially optimal or non-dominated portfolios, or (ii) the rejected project is not included in any potentially optimal or non-dominated portfolio. We prove that the two complementary selection rules will exclude exactly the same set of weights. However, analyses that apply the dominance structure often lead to multiple, mutually exclusive feasible weight sets, and therefore the approach based on potential optimality is more relevant for practical decision analysis. We also propose ex ante value of information measures to guide the elicitation of project preference statements, and illustrate our results by analyzing a real case on the selection of infrastructure maintenance projects.

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## 1. Introduction

The decision of allocating resources to a subset of many proposals is important in public administration and private firms which launch new products, invest in infrastructure projects and make commitments to policy actions. For this problem class, Portfolio Decision Analysis (PDA) (Saló, Keisler, & Morton, 2011) offers a collection of theory and methods. The use of PDA methods for project portfolio selection is based on (i) the development of a decision model which captures the salient properties of the available project proposals and the preferences of the Decision Maker (DM) for risk and multiple objectives, and (ii) the solution of a mathematical (integer) optimization problem which helps to determine the most preferred portfolios subject to the relevant constraints. PDA methods are widely employed in practice, and numerous high-impact applications have been reported in contexts

such as R&D project selection (Grushka-Cockayne, de Reyck, & Degraeve, 2008; Phillips & Bana e Costa, 2007; Toppila, Liesiö, & Saló, 2011), healthcare capital budgeting (Kleinmuntz, 2007), military resource allocation (Ewing, Tarantino, & Parnell, 2006), and infrastructure asset management (Mild, Liesiö, & Saló, 2015).

Project portfolio selection usually involves multiple attributes for evaluating the proposals. In order to lower the DM's cognitive load in providing information about the exact attribute trade-offs (weights), much research has been carried out to develop methods in which the DM can provide incomplete preference information (Argyris, Figueira, & Morton, 2011; Fliedner & Liesiö, 2016; Liesiö, Mild, & Saló, 2007, 2008; Lourenço, Morton, & Bana e Costa, 2012). Many of these methods resemble those for choosing the best alternative out of many proposals (Argyris, Morton, & Figueira, 2015; Hazen, 1986; Punkka & Saló, 2013; Saló & Hämäläinen, 1992; Weber, 1987). For instance, instead of requiring the DM to provide exact attribute weights, she can make a holistic assessment of two (real or hypothetical) projects and state that she prefers the first project to the second. Such a statement corresponds to a linear weight constraint that bounds the set of feasible weights.

\* Corresponding author.

E-mail addresses: [tommi.tervonen@evidera.com](mailto:tommi.tervonen@evidera.com), [tommi@sm.aalto.fi](mailto:tommi@sm.aalto.fi) (T. Tervonen), [juuso.liesio@aalto.fi](mailto:juuso.liesio@aalto.fi) (J. Liesiö), [ahsti.salo@aalto.fi](mailto:ahsti.salo@aalto.fi) (A. Saló).

With incomplete preference information, no portfolio is typically optimal for all feasible weights. Hence, plenty of research has focused identifying portfolios that are defensible alternatives in view of incomplete information. Probably the two most widely used concepts are non-dominated and potentially optimal portfolios (see, e.g. Liesiö et al., 2007; Lourenço et al., 2012; Liesiö & Punkka, 2014; Argyris et al., 2015; Fligner & Liesiö, 2016). The concepts are not identical: A feasible portfolio is non-dominated (or efficient) if no other feasible portfolio provides greater or equal value for all feasible weights, whereas a potentially optimal (also convex efficient) maximizes the overall value for some feasible weights. Both of these solution concepts can be used to provide well-founded decision recommendations on the project-level. In particular, there usually exists projects that are included in all of the potentially optimal or non-dominated portfolios. Such projects should be selected, because if the available incomplete information were to be refined so that the feasible weight space would contain a single weight vector, the resulting optimal portfolio for this weight vector would contain all such projects. Conversely, based on the same rationale, all projects which do not belong to any potentially optimal or non-dominated portfolio should be rejected.

Many decision support tools applied in practice allow the DM to iteratively select or reject projects included in some but not all of the non-dominated/potentially optimal portfolios, to construct the final portfolio (Kleinmuntz, 2007; Mild et al., 2015). These tools are heuristic in the sense that they do not model what implications these project preference statements have on the set of feasible weights and, in fact, the current literature offers no formal models to capture such implications. This can be seen as a major shortcoming, because such models could be used to inform DM about implicit judgments on the attributes' importance that are implied by the project preference statements, assuming that the model is consistent. Furthermore, such information would be needed to examine whether the project preference statements are consistent with preferences elicited through standard trade-off questions involving project or portfolio consequences.

The importance of analyzing the implications of project preference statements on the set of feasible attribute weights is further motivated by the apparent cognitive complexity of making such statements. Even in a simple setting with a linear portfolio value function, a DM selecting a project into the portfolio has to, in theory, simultaneously take into account the project's score profile across all attributes, how this profile is in line with attributes' importance, and consider the project resource consumption. In more complicated problems with non-linear portfolio value function and project interactions, the DM may also have to consider how well the project consequences supplement those of other projects included in the portfolio, and whether including the project enables utilizing some synergy effects. Despite these general challenges, it is possible that some decision support processes could benefit from the use of project preference statements if proper methodological support was available. In fact, behavioral research on standard multiattribute single alternative choice problems suggests that holistic preference elicitation can lead to more consistent weights than direct methods (Korhonen, Silvennoinen, Wallenius, & Öörni, 2013).

In this paper, we take the first step to bridge this apparent gap in the PDA toolset by developing approaches for modeling project preference statements as sets of feasible weights. Specifically, we consider two alternative approaches based on analyzing sets of (i) potentially optimal portfolios and (ii) non-dominated portfolios. We identify challenges with the approach based on analyzing sets of non-dominated portfolios, and show that it is as informative as the approach that analyzes potentially optimal portfolios. We also show how commonly used project performance indexes can be extended to build useful *ex ante* measures that support the elicitation

of additional project preference statements, and illustrate how these indexes can be used to guide the preference elicitation process. Finally, we demonstrate our approaches by analyzing a high-impact application on infrastructure asset management.

Our contributions advance the theory and practice of PDA in several ways. First, to our best knowledge, we provide the first theoretical basis for modeling project preference statements by using the concepts of dominance and potential optimality to derive weight information in portfolio problems. Given that modeling preference statements concerning selection and rejection of multi-attribute alternatives in choose-one-out-of-many decision problems has attracted much methodological and applied research (Corrente, Greco, Kadziński, & Słowiński, 2013; Greco, Mousseau, & Słowiński, 2008; Kadziński & Słowiński, 2015; Kadziński & Tervonen, 2013a; 2013b; Kadziński, Tervonen, & Figueira, 2015; Spliet & Tervonen, 2014; Tervonen, Sepehr, & Kadziński, 2015), this contribution has the potential to open a new stream of methodological PDA research. Second, modeling project preference statements as constraints on the feasible attribute weights makes it possible to use these statements in combination with other approaches for eliciting incomplete weight information (see e.g. Liesiö et al., 2007). Finally, our methods can be readily implemented to enhance existing processes and decision support tools for multi-attribute project portfolio selection.

The approach developed here is based on the assumption that project preference statements reveal meaningful information about the attribute weights. By meaningful we mean, that the set of feasible weights implied by the inclusion of a project in, or exclusion of a project from the portfolio, is consistent with the DM's preferences on trade-offs among the attributes. Whether this assumption holds in practice can be debated, but the theory developed here provides techniques for testing this assumption empirically. In particular, the models developed in this paper can be used to translate project preference statements into a set of feasible attribute weights. This set can be then compared to that obtained from the DM's preference statements on attribute trade-offs.

The rest of the paper is structured as follows. Section 2 introduces the additive value model for multi-attribute project portfolio selection and defines the concepts of potential optimality and dominance. Section 3 models project preference statements in terms of constraints on the set of feasible weights, and examines the structure of the resulting feasible weight set. Section 4 develops measures for assessing the 'strength' of these statements in providing additional preference information, and shows how these measures can be used to support portfolio decision processes. Section 5 addresses computational aspects. Section 6 presents an example analysis based on real-life data, and Section 7 concludes by discussing the main results.

## 2. Multiattribute project portfolio selection with incomplete preference information

Let there be  $m$  project proposals  $X = \{x^1, \dots, x^m\}$  which are evaluated on multiple attributes  $i = 1, \dots, n$ , and denote the performance (score) of project  $x^j$  on attribute  $i$  by  $v_i^j$ . A project portfolio  $p \subseteq X$  is a subset of the  $m$  project proposals, and the set of all possible portfolios is the power set  $P = 2^X$ . In what follows, we assume that the overall value of a portfolio can be expressed as

$$V(p, w) = \sum_{i=1}^n w_i V_i \left( \sum_{x^j \in p} v_i^j \right), \quad (1)$$

where the attribute-specific portfolio value functions  $V_1, \dots, V_n$  are assumed to be strictly increasing. The functional (1) form can model non-constant marginal attribute-specific portfolio values, and is thus more general than the widely applied additive-

linear value function  $V(p, w) = \sum_{x^j \in p} \sum_{i=1}^n w_i v_i^j$  (Golabi, Kirkwood, & Sicherman 1981). This is a special case of (1) in which each  $V_i(\cdot)$  is an identity function.

We assume that *feasible portfolios* are those which satisfy  $q$  linear inequalities defined by the coefficient matrix  $A \in \mathbb{R}^{q \times m}$  ( $a_i^j = [A]_{ij}$ ) and the vector  $B = [b_1, \dots, b_q]^T \in \mathbb{R}^q$ . That is,

$$P_F = \{p \in P \mid Az(p) \leq B\}, \quad (2)$$

where  $z(\cdot)$  is a bijection  $z: P \rightarrow \{0, 1\}^m$  such that  $z_j(p) = 1$  if  $x^j \in p$  and  $z_j(p) = 0$  if  $x^j \notin p$ . For instance, with  $m = 5$  projects, portfolio  $p = \{x^1, x^2, x^4\}$  corresponds to  $z(p) = [1, 1, 0, 1, 0]^T$ . For example, if there is a single budget constraint  $R$  and the cost of the  $j$ -th project is  $c_j$ , then we have  $l = 1$ ,  $a_1^j = c_j$  and  $b_1 = R$  in (2). Non-additive synergy and cannibalization effects among projects can be modeled through linear constraints, which involve dummy projects that represent project interactions (see, e.g., Stummer & Heidenberger 2003).

Given precise values for the attribute weights  $w$ , the best feasible portfolio is the one which maximizes the overall value (1). This portfolio is the solution to the Non-linear Zero-One Programming Problem

$$\begin{aligned} \max_{p \in P_F} V(p, w) \\ = \max_{z(p)} \left\{ \sum_{i=1}^n w_i V_i \left( \sum_{j=1}^m z_j(p) v_i^j \right) \mid Az(p) \leq B, z(p) \in \{0, 1\}^m \right\}. \end{aligned} \quad (3)$$

### 2.1. Incomplete preference information

Providing complete weight information can be cognitively demanding for the DM (Weber & Borchering, 1993). Several methods have been proposed to lower the cognitive load by allowing the DM to provide incomplete preference statements (see e.g. Salo & Hämäläinen, 1992; Park & Kim, 1997; Mustajoki, Hämäläinen, & Salo, 2005). These define linear constraints that must be satisfied by the closed set of feasible weights  $W \subseteq W^0$ . For instance, with  $n = 2$  attributes, stating that improving the project's score from the worst to the best level on the first attribute is (weakly) preferred over the similar swing on the second attribute corresponds to the feasible weight set  $W = \{w \in W^0 \mid w_1 \geq w_2\}$ . Often such sets are convex, although incomplete ordinal statements can lead to non-convex sets as well (Punkka & Salo, 2013).

In line with existing theory, we model incomplete preference information through a non-empty, closed set of feasible weights  $W \subseteq W^0$ , which is derived from all preference statements, including the project preference statements. We do not, however, require the feasible weight set to be convex or even connected, but instead assume that the set has a non-empty interior that spans the whole set. Specifically, the interior of the set  $W$ , denoted by  $\text{int}(W)$ , includes those points  $w \in W$  around which one can form

an  $\epsilon$ -neighborhood  $N_\epsilon(w) = \{w' \in W^0 \mid \|w' - w\|_2 < \epsilon\}$  contained entirely in  $W$ , i.e.,

$$\text{int}(W) = \{w \in W \mid \exists \epsilon > 0 \text{ s.t. } N_\epsilon(w) \subset W\}.$$

In turn, points  $w \in W^0$  whose  $\epsilon$ -neighborhood includes a point from the set  $W$  for any positive value of  $\epsilon$  make up the set's closure  $\text{cl}(W)$ , i.e.,

$$\text{cl}(W) = \{w \in W^0 \mid N_\epsilon(w) \cap W \neq \emptyset \forall \epsilon > 0\}.$$

With these concepts, a feasible weight set is defined to be a non-empty subset of  $W^0$ , which has its interior's closure equal to the set itself.

**Definition 1.** The set of *feasible weight sets* is

$$\mathcal{W} = \{W \subseteq W^0 \mid \text{cl}(\text{int}(W)) = W, W \neq \emptyset\}.$$

Fig. 1 illustrates the structure of  $\mathcal{W}$ . The set  $W^1$  consisting of two points and a line segment does not belong to  $\mathcal{W}$ . This is since its interior is an empty set, which implies that the closure is also empty. The set  $W^2$  consists of two triangles and a separate line segment, but the interior of  $W^2$  consists only of interior points of the two triangles. Hence, the closure does not include the line segment and therefore  $W^2$  does not belong to  $\mathcal{W}$ . Only the set  $W^3$  belongs to  $\mathcal{W}$  since any point in  $W^3$  is arbitrarily close to an interior point (and, equivalently, closure of its interior is the set itself).

The requirement stated in Definition 1 poses no limits for practical application of such sets in modeling preferences. Feasible weight sets proposed in the literature almost without exception satisfy the assumptions of  $\mathcal{W}$ . Any DM preferences that are modeled as regions in the weight space that do not contain the neighboring interior points, such as the line segment in Fig. 1, can be extended to include arbitrarily small neighborhoods without affecting the portfolio decision recommendations. This extended set would then belong to  $\mathcal{W}$ . With incomplete preference information, the portfolio which solves problem (3) may be different for different selections of weight vector  $w$  from the set of feasible weights. In this case, it is warranted to focus on the *non-dominated* portfolios.

**Definition 2.** Portfolio  $p$  *dominates*  $p'$  with respect to weight set  $W \in \mathcal{W}$ , denoted by  $p \succ_W p'$ , if  $V(p, w) \geq V(p', w)$  for all  $w \in W$  and  $V(p, w) > V(p', w)$  for some  $w \in W$ . Furthermore, the set of *non-dominated portfolios* w.r.t. weight set  $W$  is

$$P_N(W) = \{p \in P_F \mid \nexists p' \in P_F \text{ such that } p' \succ_W p\}.$$

The concept of dominance is illustrated in Fig. 2, which shows four feasible portfolios that are all non-dominated except for  $p^4$ . If the DM were to select a dominated portfolio, then one could identify a non-dominated portfolio which yields at least as much value for all weights and more value for some weights.

A portfolio can be non-dominated even if it is not optimal for any weights (cf. portfolio  $p^2$  in Fig. 2). Another widely used opti-

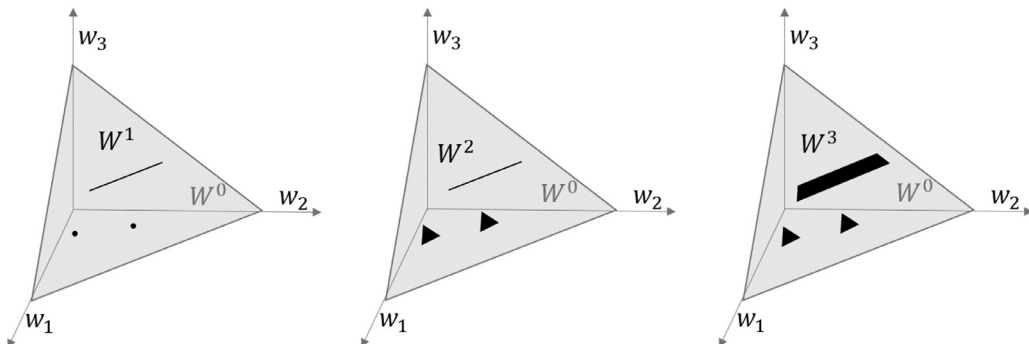
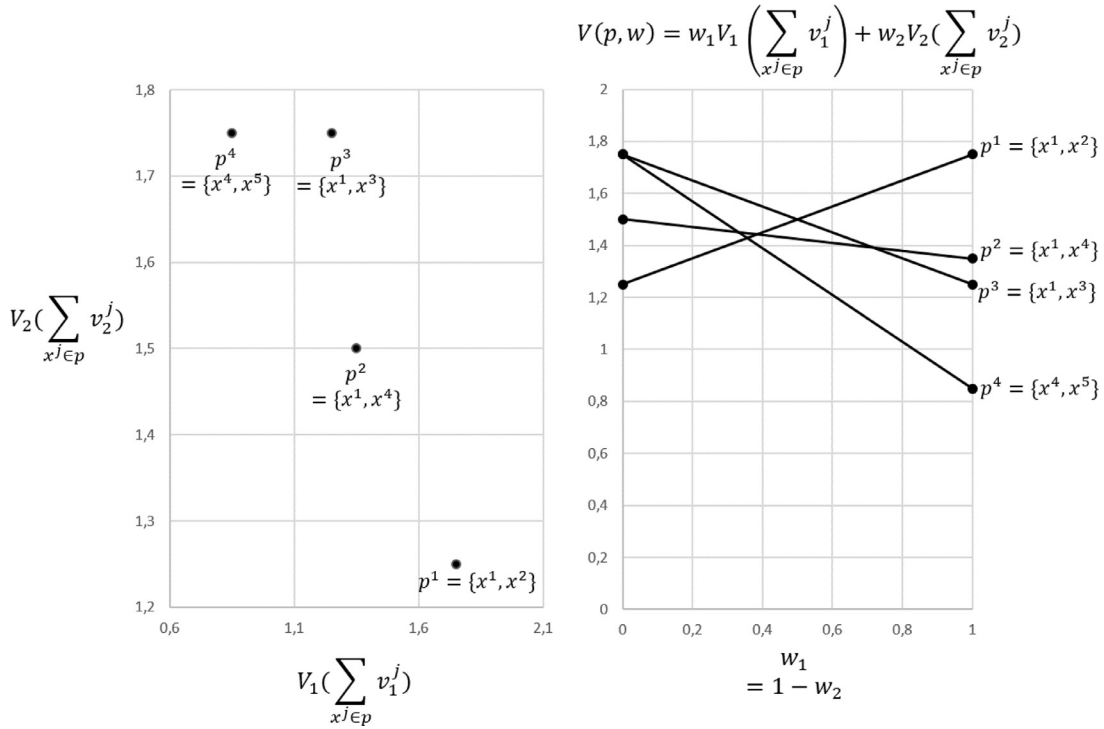


Fig. 1. Three subsets of the 2-simplex  $W^0$  such that  $W^1 \subset W^2 \subset W^3$ , and only  $W^3$  belongs to  $\mathcal{W}$ .



**Fig. 2.** Attribute-specific values of portfolios  $p^1, \dots, p^4$  (left) and their overall values as a function of attribute weights  $w \in W = W^0 = \{(w_1, w_2) \in \mathbb{R}_+^2 | w_1 + w_2 = 1\}$  (right). The set of non-dominated portfolios is  $P_N(W) = \{p^1, p^2, p^3\}$  and the set of potentially optimal portfolios is  $P_O(W) = \{p^1, p^3\}$ .

maximality concept under incomplete information is potential optimality, which focuses on portfolios that have the highest overall value for *some* weights (Hazen, 1986). However, there can be situations in which portfolios have an equal value for some set of weights  $W' \subset W$ , while one dominates the other w.r.t.  $W$  (cf. portfolio  $p^4$  in Fig. 2). Hence, we define a portfolio to be potentially optimal if it has the highest value for a subset of feasible weights with a non-empty interior.

**Definition 3.** A feasible portfolio  $p \in P_F$  is *potentially optimal* w.r.t.  $W \in \mathcal{W}$  if

$$\text{int}(\{w \in W | V(p, w) \geq V(p', w) \forall p' \in P_F\}) \neq \emptyset. \quad (4)$$

The set of *potentially optimal portfolios* w.r.t.  $W$  is denoted by  $P_O(W)$ .

In particular, if a portfolio is potentially optimal with regard to some feasible weight set  $W \in \mathcal{W}$ , then the closure of set (4), i.e.,

$$\hat{W}(p, W) = \text{cl}(\text{int}(\{w \in W | V(p, w) \geq V(p', w) \forall p' \in P_F\})), \quad (5)$$

also belongs to the set of feasible weight sets  $\mathcal{W}$  given by Definition 1. Furthermore, for any feasible weight set in  $\mathcal{W}$  there always exists at least one potentially optimal portfolio, and each potentially optimal portfolio is also non-dominated, as stated by the following lemma.

**Lemma 1.** Let  $W \in \mathcal{W}$ . Then  $\emptyset \neq P_O(W) \subseteq P_N(W)$ .

Portfolios that are optimal for a set of weights with an empty interior are not included in the set  $P_O(W)$ . This is a modeling choice which has the potential drawback of excluding some portfolios that the DM might be interested in. However, arguably these excluded portfolios are not particularly robust as the set of weights for which they maximize value has a dimension less than  $n$ . In particular, the optimality of these portfolios is sensitive to errors in the preference elicitation. Furthermore, for any weights for which these excluded portfolios maximize value, there exists another portfolio in the set  $P_O(W)$  that yields equal value.

Both the set of potentially optimal and the set of non-dominated portfolios can be used to provide recommendations for project selection and rejection. Specifically, projects can be classified into (i) *core projects* that are included in all, (ii) *exterior projects* that are not included in any, and (iii) *borderline projects* that are included in some but not all potentially optimal or non-dominated portfolios (Liesiö et al., 2007).

**Definition 4.** The sets of core, borderline and exterior projects based on potentially optimal portfolios w.r.t.  $W \in \mathcal{W}$  are

$$X_O^C(W) = \{x^k \in X | x^k \in p \forall p \in P_O(W)\}$$

$$X_O^B(W) = \{x^k \in X | \exists p, p' \in P_O(W) \text{ such that } x^k \in p, x^k \notin p'\}$$

$$X_O^E(W) = \{x^k \in X | x^k \notin p \forall p \in P_O(W)\},$$

respectively.

**Definition 5.** The sets of core, borderline and exterior projects based on non-dominated portfolios w.r.t.  $W \in \mathcal{W}$  are

$$X_N^C(W) = \{x^k \in X | x^k \in p \forall p \in P_N(W)\}$$

$$X_N^B(W) = \{x^k \in X | \exists p, p' \in P_N(W) \text{ such that } x^k \in p, x^k \notin p'\}$$

$$X_N^E(W) = \{x^k \in X | x^k \notin p \forall p \in P_N(W)\},$$

respectively.

Table 1 identifies the five projects  $x^1, \dots, x^5$  in Fig. 2 as core, borderline or exterior projects, based on both potentially optimal and non-dominated portfolios. By Lemma 1, the potentially optimal portfolios are also non-dominated, and thus the classification of projects has the following general properties.

**Lemma 2.** Let  $W \in \mathcal{W}$ . Then

$$X_N^C(W) \subseteq X_O^C(W)$$

$$X_N^B(W) \supseteq X_O^B(W)$$

$$X_N^E(W) \subseteq X_O^E(W).$$

As a result of introducing additional preference statements, the revised set of feasible weights  $W'$  may become smaller but not



**Table 1**

Project classification in the example of Fig. 2. For the weight set  $W' = \{(w_1, w_2) \in W^0 | w_1 \geq 0.5\}$  the classification based on non-dominated and potentially optimal portfolios is equivalent since  $P_N(W') = P_O(W') = \{p^1\}$ .

	$v_1^j$	$v_2^j$	$X_N^C(W^0)$	$X_N^B(W^0)$	$X_N^E(W^0)$	$X_O^C(W^0)$	$X_O^B(W^0)$	$X_O^E(W^0)$	$X_C^C(W')$	$X_C^B(W')$	$X_C^E(W')$
$x^1$	0.75	0.75	x			x			x		
$x^2$	1	0.5		x			x		x		
$x^3$	0.5	1		x			x				x
$x^4$	0.75	0.6		x				x			x
$x^5$	1	0.25			x			x			x

larger, and thus  $W' \subseteq W$ . If  $W'$  has a non-empty interior (i.e.  $W' \in \mathcal{W}$ ), the sets of non-dominated and potentially optimal portfolios will therefore either become smaller or remain unchanged. Without the assumption of non-empty interior, additional preference statements can enlarge the set of non-dominated portfolios. For instance, if in the example of Fig. 2 the weight set would be reduced to a single point  $W' = \{(0, 1)^T\}$ , then portfolio  $p^4$  would not be dominated by any of the other portfolios, although  $p^3$  has a strictly better value for any infinitely small perturbation of weights from the value  $(0, 1)^T$ . The following lemma formalizes the impact that additional preference statements have on the sets of non-dominated and potentially optimal portfolios, and on the sets of core, borderline and exterior projects.

**Lemma 3.** Let  $W', W \in \mathcal{W}$  such that  $W' \subseteq W$ . Then, for  $*$   $\in \{N, O\}$ ,

$$\begin{aligned} P_*(W') &\subseteq P_*(W) \\ X_*^C(W') &\supseteq X_*^C(W) \\ X_*^B(W') &\subseteq X_*^B(W) \\ X_*^E(W') &\supseteq X_*^E(W). \end{aligned}$$

The DM is advised to select core projects and to reject exterior ones, because core and exterior projects will retain their status even if the set of feasible weights becomes smaller. Furthermore, additional preference statements cannot expand the set of borderline projects. For instance, the last three columns of Table 1 show the classification of projects after adding the preference statement  $w_1 \geq w_2$  to the example in Fig. 2. Specifically, only portfolio  $p^1$  is non-dominated and potentially optimal for the resulting set of feasible weights.

The decision support process is often iterative so that the introduction of additional preference information into the preference model decreases the number of borderline projects. This process will eventually lead to the identification of only few non-dominated or potentially optimal portfolios; it is also possible that only one such portfolio remains. However, the DM may be unable to provide preference statements that reduce the set of feasible portfolios sufficiently. In such situations, she may wish to consider which borderline projects could either be selected into the final portfolio or, alternatively, rejected so that they are excluded from it.

### 3. Modeling project preference statements

This section develops approaches for modeling the DM's statements about which borderline projects should be included in the final portfolio. In particular, we consider two types of statements:

- Statement  $\text{In}(x^k)$ : 'Select project  $x^k$ '.
- Statement  $\text{Out}(x^k)$ : 'Reject project  $x^k$ '.

The developed approach interprets these statements through the feasible weight set, thus enabling the use of project preferences together with standard preference statements. The key challenge with this approach is that unlike standard preference statements, the project preference statements do not contain a comparison of two (hypothetical) portfolios, which could be interpreted

as a constraint for the feasible weights in a straightforward manner. Hence, we seek to identify set of weights for which each non-dominated or potentially optimal portfolio satisfies the statement.

This modeling approach is in line with methods developed for capturing holistic statements in a setting where the objective is to choose one of several decision alternatives (see, e.g., Punkka & Salo, 2013). In particular, each feasible portfolios can be viewed as one decision alternative and the set of possible decision alternatives can be partitioned into two mutually exclusive groups based on any project  $x^k$ : (i) those alternatives that include project  $x^k$ , and (ii) those that do not include project  $x^k$ . Hence, the preference statement  $\text{In}(x^k)$  can be interpreted as the DM stating that only alternatives in group (i) should be considered as possible choices, and therefore the statement should result in such a set of feasible weights that only alternatives belonging to group (i) are non-dominated or potentially optimal. The two approaches based on using the sets of non-dominated or potentially optimal portfolios as the basis of interpreting project preference statements are developed in the following sections.

#### 3.1. Project selection and rejection based on potentially optimal portfolios

Additional project preference statements reduce the set of feasible weights  $W$  to a subset  $W'$ . If  $W'$  is compatible with the statement  $\text{In}(x^k)$ , then it is logical to require that  $x^k$  is contained in all potentially optimal portfolios in  $P_O(W')$ . This requirement ensures that  $x^k$  is a core project, and it imposes no additional assumptions on the preference model. Furthermore,  $W'$  should be the largest such set, i.e., augmenting it by adding other weight vectors would cause  $x^k$  to lose its core classification. Maximality of  $W'$  is required because the resulting weight constraints should exclude only those weights for which  $x^k$  is no longer a core project, even though choosing any subset of  $W'$  would also make  $x^k$  a core project (cf. Lemma 3).

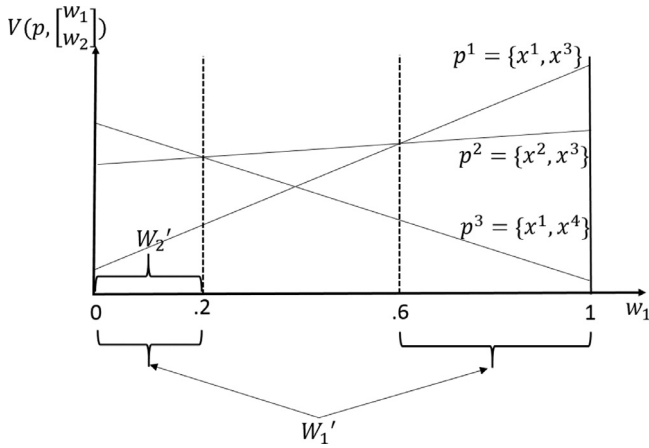
**Definition 6.** Let  $W \in \mathcal{W}$  be the current feasible weight set. The subset of weights  $W' \subseteq W$ ,  $W' \in \mathcal{W}$ , is *PO-compatible* with the preference statement  $\text{In}(x^k)$  if

- (i)  $x^k \in X_O^C(W')$
- (ii)  $x^k \notin X_O^C(W'')$  for any  $W'' \in \mathcal{W}$ , such that  $W' \subset W'' \subseteq W$ .

Clearly, the PO-compatible weight set does not exist if  $x^k$  is an exterior project with regard to the original weight set  $W$ . However, even if  $x^k \in X_O^B(W)$ , but for each feasible weight vector there exists two potentially optimal portfolios such that one of them includes project  $x^k$  and the other one does not, then the PO-compatible weight set does not exist. Otherwise there exists a unique compatible weight set which can be formulated as a finite union of convex subsets of  $W$ . This is formalized by the following theorems, in which  $[p]$  denotes the equivalence class of feasible portfolios whose value is the same for all weights, i.e.,

$$[p] = \{p' \in P_F \mid V(p', w) = V(p, w) \forall w \in W^0\}.$$

**Theorem 1.** Let  $W \in \mathcal{W}$ . There exists a weight set  $W'$  PO-compatible with the preference statement  $\text{In}(x^k)$  if and only if there exists



**Fig. 3.** Values of portfolios  $p^1$ ,  $p^2$  and  $p^3$  as a function of attribute weights  $w \in W = \{(w_1, w_2) \in \mathbb{R}_+^2 \mid w_1 + w_2 = 1\}$ . The set  $W'_1 = \{w \in W \mid w_1 \in [0, 0.2] \cup [0.6, 1]\}$  is PO-compatible with the statement  $\text{In}(x^1)$  and the set  $W'_2 = \{w \in W \mid w_1 \in [0, 0.2]\}$  is PO-compatible with the statement  $\text{Out}(x^3)$ .

$p \in P_O(W)$  such that  $x^k \in p'$  for all  $p' \in [p]$ , and this weight set is given by

$$W' = \bigcup_{\substack{p \in P_O(W) \text{ s.t.} \\ x^k \in p' \forall p' \in [p]}} \hat{W}(p, W). \quad (6)$$

In most applications, not allowing the DM to provide  $\text{In}(x^k)$  statements on exterior projects  $x^k \in X_O^E(W)$  is sufficient to ensure that a PO-compatible weight set exists. However, to be completely safe,  $\text{In}(x^k)$  statements cannot be given about those borderline projects  $x^k \in X_O^B(W)$  that meet the following condition: For each portfolio  $p$  containing  $x^k$ , there exists another portfolio  $p'$  which (i) does not contain  $x^k$ , and (ii) has exactly the same overall value for all weights (i.e.,  $p' \in [p]$ ).

Fig. 3 illustrates Definition 6 and Theorem 1; the set  $W'_1 = \{w \in W \mid w_1 \in [0, 0.2] \cup [0.6, 1]\}$  is PO-compatible with the statement  $\text{In}(x^1)$ . Furthermore, if there existed a portfolio  $p^4 = \{x^2, x^5\} \in [p^1]$ , then for  $\{w \in W \mid w_1 \in [0.6, 1]\}$  both  $p^1$  and  $p^4$  would be potentially optimal, and because  $p^4$  does not contain  $x^1$ , the PO-compatible set would be  $W'_2 = \{w \in W \mid w_1 \in [0, 0.2]\}$ . This example shows that the set of compatible weights may not be connected, because it is formed as a union of the weight sets for which all potentially optimal portfolios include/exclude a specific project. If the set of potentially optimal portfolios  $P_O(W')$  is known, then constructing the set  $W'$  is relatively straightforward, because it suffices to construct the sets  $\hat{W}(p, W)$  defined through linear constraints (cf. Eq. (4)).

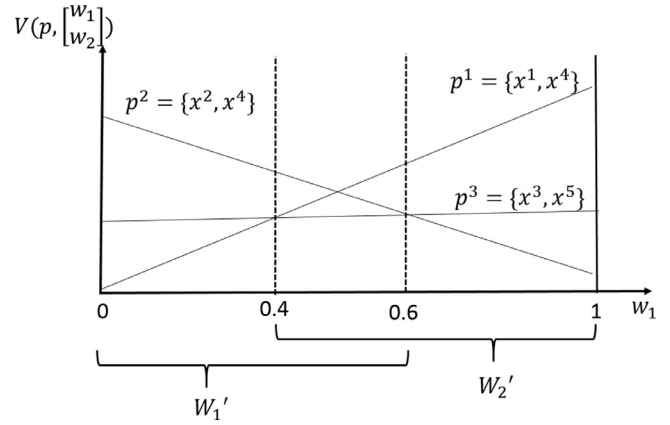
The set of weights compatible with a project rejection statement is defined analogously.

**Definition 7.** Let  $W \in \mathcal{W}$  be the current feasible weight set. The subset of weights  $W' \subseteq W$ ,  $W' \in \mathcal{W}$ , is PO-compatible with the preference statement  $\text{Out}(x^k)$  if

- (i)  $x^k \in X_O^E(W')$
- (ii)  $x^k \notin X_O^E(W'')$  for any  $W'' \in \mathcal{W}$  such that  $W' \subset W'' \subseteq W$ .

**Theorem 2.** Let  $W \in \mathcal{W}$ . There exists weight set  $W'$  PO-compatible with the preference statement  $\text{Out}(x^k)$  if and only if there exists  $p \in P_O(W)$  such that  $x^k \notin p'$  for all  $p' \in [p]$ , and this weight set is given by

$$W' = \bigcup_{\substack{p \in P_O(W) \text{ s.t.} \\ x^k \notin p' \forall p' \in [p]}} \hat{W}(p, W). \quad (7)$$



**Fig. 4.** With the feasible weight set  $W = \{(w_1, w_2) \in \mathbb{R}_+^2 \mid w_1 + w_2 = 1\}$  each of portfolios  $p^1$ ,  $p^2$  and  $p^3$  is non-dominated. Both weight sets  $W'_1$  and  $W'_2$  are ND-compatible with the preference statement  $\text{In}(x^4)$ . These sets are also ND-compatible with the statement  $\text{Out}(x^3)$ . However, the union of these sets is not ND-compatible with either statement since  $W'_1 \cup W'_2 = W$ .

### 3.2. Project selection and rejection based on non-dominated portfolios

Another approach for modeling project preference statements is to interpret them using the set of non-dominated portfolios. Specifically, for a weight set  $W'$  to be compatible with the statement  $\text{In}(x^k)$  ( $\text{Out}(x^k)$ ), project  $x^k$  must be included in (excluded from) each non-dominated portfolio in  $P_N(W')$ .

**Definition 8.** Let  $W \in \mathcal{W}$  be the feasible weight set. The subset of weights  $W' \subseteq W$ ,  $W' \in \mathcal{W}$ , is ND-compatible with the preference statement  $\text{In}(x^k)$  if

- (i)  $x^k \in X_N^C(W')$
- (ii)  $x^k \notin X_N^C(W'')$  for any  $W'' \in \mathcal{W}$  such that  $W' \subset W'' \subseteq W$ .

**Definition 9.** Let  $W \in \mathcal{W}$  be the feasible weight set. The subset of weights  $W' \subseteq W$ ,  $W' \in \mathcal{W}$ , is ND-compatible with the preference statement  $\text{Out}(x^k)$  if

- (i)  $x^k \in X_N^E(W')$
- (ii)  $x^k \notin X_N^E(W'')$  for any  $W'' \in \mathcal{W}$  such that  $W' \subset W'' \subseteq W$ .

Thus, if  $W'$  is compatible, then  $x^k$  is included in all (none) non-dominated portfolios  $P_N(W')$  and no additional weight vectors can be included in  $W'$  without  $x^k$  losing its core (exterior) classification. Note that an ND-compatible  $W'$  is not necessarily unique and hence there can exist several sets in  $\mathcal{W}$  that satisfy the requirements of Definition 8 or 9. For instance, in Fig. 4, the two sets

$$W'_1 = \{(w_1, w_2)^T \in W^0 \mid w_1 \in [0, 0.6]\}$$

$$W'_2 = \{(w_1, w_2)^T \in W^0 \mid w_1 \in [0.4, 1]\}$$

are ND-compatible with the statement  $\text{Out}(x^3)$ . Both sets are ND-compatible with the statement  $\text{In}(x^4)$ , which shows that when working with non-dominated portfolios, the project preference statements can result in a situation where incomplete preference information corresponds to multiple sets of feasible weights.

A borderline project  $x^k$  can become a core one (included in all non-dominated portfolios) only if every non-dominated portfolio in which  $x^k$  does not belong to becomes dominated by at least one feasible portfolio in which  $x^k$  belongs to. This gives an upper bound on the number of sets that are ND-compatible with the preference statement  $\text{In}(x^k)$ .

**Lemma 4.** Let  $W \in \mathcal{W}$  be the current feasible weight set. The upper bound for the number of subsets of weights that are ND-compatible with the statement  $\text{In}(x^k)$  is  $|P_k^+| |P_k^-|$ , where  $P_k^+ = \{p \in P_N(W) \mid x^k \in p\}$

and  $P_k^- = \{p \in P_N(W) \mid x^k \notin p\}$  are the sets of portfolios in which the borderline project  $x^k$  does and does not belong to, respectively.

The worst-case exponential number of ND-compatible weight sets makes it difficult to explain the results of the analysis to the DM: if the DM questions results of the analysis, there can be a large number of possible sets of mutually exclusive constraints that all explain the results. Furthermore, the weights that can be eliminated from the feasible weight set are equal when project selection statements are interpreted based on potentially optimal portfolios (Definitions 6 and 7) and non-dominated portfolios (Definitions 8 and 9). This is stated in Theorem 3.

**Theorem 3.** Assume that the set  $W' \in \mathcal{W}$  is PO-compatible with the preference statement  $\text{In}(x^k)$ , and that  $W^j \in \mathcal{W}$ ,  $j = 1, 2, \dots, J$ , are all the sets ND-compatible with the preference statement  $\text{In}(x^k)$ . Then  $W' = \bigcup_{j=1}^J W^j$ .

Theorem 3 implies that the information gained from project selection statements is essentially the same in analyses based on non-dominated and potentially optimal portfolios (see e.g. sets that are ND- and PO-compatible with  $\text{In}(x^4)$  in Fig. 4). Recall that holistic preference statements can be used in PDA to narrow down the feasible weight set until a sufficiently informed portfolio selection can be made. Thus, as there are multiple, worst-case exponential number of ND-compatible weight sets but only a single PO-compatible set, from the viewpoint of practical decision analysis with a focus on interacting with the DM, it makes little sense to interpret the project selection statements based on ND-compatible portfolios. For brevity, we state without proof a similar result for project rejection statements: If the set  $W' \in \mathcal{W}$  is PO-compatible with the preference statement  $\text{Out}(x^k)$ , then  $W' = \bigcup_{j=1}^J W^j$ , where  $W^j \in \mathcal{W}$ ,  $j = 1, 2, \dots, J$  are all the sets ND-compatible with the preference statement  $\text{Out}(x^k)$ .

#### 4. Targeting project preference statements

In many project portfolio selection problems there are dozens or hundreds of project candidates (see e.g. Ewing et al., 2006; Mild et al., 2015), and even after some preference information about the attribute weights have been elicited, the number of borderline projects may still be large. It is therefore useful to identify all those borderline projects about which project preference statements are likely to reduce the number of potentially optimal portfolios as much as possible. The DM can be asked to consider if she is willing to select or reject some of the corresponding borderline projects. Furthermore, the decision process should result in selecting a robust portfolio which is optimal (has the largest overall value with the portfolio value model in Eq. (1)) for a large share of feasible attribute weights. Hence, the size of the compatible weight set resulting from the project preference statements is also a relevant criterion for prioritizing borderline projects to be evaluated by the DM.

To support the prioritization of borderline projects in preference elicitation, we adapt and extend two measures that have been suggested in the literature. The first is the *Core Index*, which measures the share of non-dominated portfolios that include a particular project (Liesiö et al., 2007). Because analyses with non-dominated portfolios are not very useful when eliciting project preference statements (see previous Section), we compute core indexes over the set of potentially optimal portfolios. Specifically, the Core Index (CI) of project  $x^k$  with respect to a set of feasible weights  $W \in \mathcal{W}$  is

$$\text{CI}(x^k, W) = \frac{|\{p \in P_0(W) \mid x^k \in p\}|}{|P_0(W)|}. \quad (8)$$

**Table 2**

Classification of projects for selection in ( $\text{In}(x^k)$ ) and rejection from ( $\text{Out}(x^k)$ ) the final portfolio. Bold entries indicate the best candidates for eliciting project preference statements.

$\text{In}(x^k)$	Low CI	High CI
Low AI	Non-robust, large reduction of $P_0(W)$	Non-robust, small reduction of $P_0(W)$
High AI	<b>Robust, large reduction of <math>P_0(W)</math></b>	Robust, small reduction of $P_0(W)$
$\text{Out}(x^k)$	Low CI	High CI
Low AI	Robust, small reduction of $P_0(W)$	<b>Robust, large reduction of <math>P_0(W)</math></b>
High AI	Non-robust, small reduction of $P_0(W)$	Non-robust, large reduction of $P_0(W)$

The CI of any core (exterior) project is equal to one (zero), while the CI of borderline projects is in the open interval (0, 1).

The second measure is based on the *Acceptability Index* introduced by Lahdelma, Hokkanen, and Salminen (1998). The original acceptability index describes the share of weights for which a particular alternative is optimal in single alternative choice problems. For use in PDA, we define the project's Acceptability Index (AI) as the relative size of the subset of weights for which the project is a core one, and thus included in all potentially optimal portfolios (cf. Definition 6). Specifically, given a set of feasible weights  $W \in \mathcal{W}$ , the AI of project  $x^k$  is

$$\text{AI}(x^k, W) = \frac{\text{vol}(W')}{\text{vol}(W)}, \quad (9)$$

where  $W'$  is PO-compatible with  $\text{In}(x^k)$  and  $\text{vol}(\cdot)$  denotes the  $n - 1$ -dimensional volume of the given set.

Core and acceptability indexes can be used for identifying those borderline projects that are good candidates for the elicitation of project preference statements. Specifically, the lower (higher) the CI of project  $x^k$  is, the smaller is the set of potentially optimal portfolios w.r.t. the set of weights compatible with the preference statement  $\text{In}(x^k)$  (statement  $\text{Out}(x^k)$ ). In turn, if a project with a high (low) AI is selected (rejected), then a large share of weights will remain feasible. These results are stated formally in the following lemmas.

**Lemma 5.** Let  $W, W' \in \mathcal{W}$ . If  $W' \subseteq W$  is PO-compatible with the preference statement  $\text{In}(x^k)$ , then

$$\begin{aligned} |P_0(W')| &\leq \text{CI}(x^k, W) |P_0(W)| \\ \text{vol}(W') &= \text{AI}(x^k, W) \text{vol}(W) \end{aligned}$$

**Lemma 6.** Let  $W, W' \in \mathcal{W}$ . If  $W' \subseteq W$  is PO-compatible with the preference statement  $\text{Out}(x^k)$ , then

$$\begin{aligned} |P_0(W')| &\leq (1 - \text{CI}(x^k, W)) |P_0(W)| \\ \text{vol}(W') &\leq (1 - \text{AI}(x^k, W)) \text{vol}(W) \end{aligned}$$

From another point of view, AIs describe the robustness of project preference statements. If a borderline project has a very high acceptability, then alterations of previously elicited preference statements are unlikely to make it an exterior one. If the project has a very low AI, then even a small alteration in one of the previously elicited preference statements could lead it to becoming an exterior one. Therefore, the DM may wish to avoid selecting projects with an extremely low AI, because feasibility of the final portfolio would otherwise be highly dependent on validity of all preference statements. Table 2 classifies project preference statements based on the corresponding core and acceptability indexes. Note, however, that if the DM is sure about including a project with low AI and low CI (or excluding one with high AI and high CI), such a statement should not be discouraged, as it would reduce the set of feasible weights considerably.

Although core and project acceptability indexes can be computed for any  $W \in \mathcal{W}$ , they are not very useful metrics for decision support at the beginning of the analysis when  $W = W^0$ , because the “real” DM preferences may lie on a small region of  $W^0$  which could be formed by eliciting linear inequality constraints. We recommend to start the process by first eliciting incomplete preference information in form of linear weight constraints, and only afterwards to elicit the project preference statements. The core and project acceptability indexes are useful only in this second phase.

Selecting projects on which to provide preference statements can be viewed as a portfolio problem itself. However, it is important to highlight that the core and acceptability indexes of a particular project reflect the marginal effect a statement about this project would have on the weight set, or the number of potentially optimal portfolios, but these indexes cannot as such be used evaluate the effect of combinations of such statements. This is since CI and AI are contingent of the set of feasible weights  $W$ , which will change as the result of each statement. Hence, rather than encouraging DM to provide a sequence of, for instance,  $\ln(\cdot)$  statements on projects with low  $\text{CI}(\cdot, W)$  values, it is better to ask for one statement at a time and update the AI and CI values based on the set of feasible weights  $W'$ , resulting from this single statement.

Continuous update of CI values also provides a way of giving feedback to the DM about the implications of each project preference statement. In particular, after providing a project preference statement, the DM can immediately be provided with the updated CI values, to show which other projects the statement causes to be excluded from, or included in all potentially optimal portfolios. The DM can then reflect on this information and has the option of removing the preference statement if the project implications are not consistent with her intentions. In addition, it is important also to illustrate effects the statements have on the set of feasible weights (e.g., average and range of each weight). These types of feedback are particularly important in real-life applications in which the DM is likely to base the statements at least partly on considerations not explicitly included in the portfolio value model.

## 5. Computational considerations

Applying the developed models for capturing project preferences requires identification of the set of potentially optimal portfolios  $P_O(W)$  to identity sets of core, borderline and exterior projects. The feasible weight set  $W^1 \in \mathcal{W}$  is assumed to capture preference information in form of linear weight constraints. In case no such preference information exists, then  $W^1 = W^0$ . However, the current literature offers no exact algorithms to compute the set of potentially optimal portfolios  $P_O(W^1)$  directly. Hence, we base our approach on first computing the set of non-dominated portfolios  $P_N(W^0)$ , since Lemmas 1 and 3 together imply that  $P_O(W^1) \subseteq P_N(W^1) \subseteq P_N(W^0)$ . Even if the attribute-specific portfolio value functions  $V_i$  in (1) are non-linear, the set  $P_N(W^0)$  can be identified by solving a Multi-Objective Zero-One Linear Programming (MOZOLP) problem as stated by the following lemma.

**Lemma 7.**  $p \in P_N(W^0)$  if and only if  $z(p)$  is a Pareto optimal solution to the  $n$ -objective MOZOLP problem

$$\begin{aligned} & \nu\text{-max}_{z(p)} \left[ \sum_{j=1}^m z_j(p)v_1^j, \sum_{j=1}^m z_j(p)v_2^j, \dots, \sum_{j=1}^m z_j(p)v_n^j \right] \\ & Az(p) \leq B \\ & z(p) \in \{0, 1\}^m \end{aligned} \quad (10)$$

There are several exact algorithms for solving medium-sized MOZOLP problems (e.g.,  $m = 60$ ) in reasonable time (see, e.g., Kiziltan & Yucaoglu 1983; Liesiö, Mild, & Salo 2008; Villarreal & Karwan 1981). Approximate algorithms can be used to solve larger

problems with hundreds of projects (see, e.g., Mild et al. 2015). However, it is important to highlight that the number of projects  $m$  also includes possible dummy projects needed to model interactions among the projects. In case the portfolio value function is additive-linear, i.e.,  $V(p, w) = \sum_{x \in p} \sum_{i=1}^n w_i v_i^x$ , the set of non-dominated portfolios  $P_N(W^1)$  can be identified directly though MOZOLP, which can be faster than solving the problem (10) (see Liesiö et al., 2008, for details).

The set of potentially optimal portfolios  $P_O(W^1)$  can be computed with linear programming by checking, for each portfolio in  $p \in P_N(W^0)$ , whether there exists  $w \in W$  in which the portfolio has the highest overall value among all portfolios in  $P_N(W^0) \setminus [p]$ , where  $[p]$  is the equivalence containing portfolios that yield the same overall value as  $p$  for all weights. This check corresponds to a linear programming problem (LP) as stated by the following lemma.

**Lemma 8.** Let  $p \in P_N(W^0)$  and  $W^1 \subseteq W^0$ . Then  $p \in P_O(W^1)$  if and only if

$$\begin{aligned} & \max_{\substack{d \in \mathbb{R} \\ w \in W^1}} \left\{ d \mid \sum_{i=1}^n w_i \left( V_i \left( \sum_{x \in p} v_i^x \right) - V_i \left( \sum_{x \in p'} v_i^x \right) \right) \right. \\ & \left. \geq d \quad \forall p' \in P_N(W^0) \setminus [p] \right\} > 0. \end{aligned} \quad (11)$$

LP problem (11) has  $n + 1$  continuous decision variables and the number of constraints is linear in the number of non-dominated portfolios  $|P_N(W^0)|$ .

Acceptability index computation with deterministic scores is equal to polytope volume computation, which is known to be #P-hard (Dyer & Freeze, 1991; Lawrence, 1991). Therefore, in higher dimensionality problems their exact computation is intractable and the volumes within (9) have to be estimated numerically (Lahdelma & Salminen, 2001). Tervonen, van Valkenhoef, Baştürk, and Postmus (2013) successfully applied the Markov Chain Monte Carlo hit-and-run technique for sampling weight vectors, allowing to estimate acceptability indexes efficiently when the feasible weight space is convex. For details on the procedure, see Tervonen et al. (2013), van Valkenhoef, Tervonen, and Postmus (2014) and Tervonen and Lahdelma (2007).<sup>1</sup>

The set of weights compatible with given project preference statements is not necessarily convex (Definitions 6 and 7). Therefore, the potentially optimal portfolios cannot be computed with linear programming when such statements are included in the analysis. We propose to estimate the set of potentially optimal portfolios numerically as follows.

1. Construct convex weight space  $W^1$  by restricting  $W^0$  with linear constraints arising from any possible non-project preference statements (such as  $w_1 > w_2$ ).
2. Generate portfolios that are non-dominated for  $W^1$ .
3. Construct the possibly non-convex feasible weight space  $W^2$  by restricting  $W^1$  with project preference statements.
4. Draw a sufficient sample (e.g. 10,000, see Tervonen & Lahdelma, 2007) of weight vectors from  $W^2$  with simple rejection sampling where hit-and-run is used to generate candidate draws from the convex weight space  $W^1$ .
5. Use the final weight samples for estimating the set of potentially optimal portfolios, i.e. those non-dominated ones that have the highest value for at least one of the weight vector draws, and core and acceptability indexes.

<sup>1</sup> An open source R package implementing the hit-and-run procedure is available from CRAN (<https://cran.r-project.org/web/packages/hitandrun/>).



The rejection rate in the above sampling procedure depends on the acceptability indexes of the selected/rejected projects (Lemma 5). The project acceptability indexes provide information about the robustness of the final portfolio choice recommendation, and the DM should avoid selecting projects with extremely low acceptability indexes, or rejecting projects with extremely high acceptability indexes, because such project preference statements are not robust (see Table 2). Therefore, the rejection sampling method is tractable in most practical cases.

## 6. Application to infrastructure maintenance project selection

We illustrate the use of project preference statements with data from a real-life case on infrastructure asset management reported by Mild et al. (2015) in which a dedicated PDA model was used annually to support the selection of a portfolio of bridges for maintenance. This model included 6 attributes to measure the repair urgency of each bridge: Damage Point Sum (DPS), Traffic significance, Carrying deficiency, Width deficiency, exposure to Salt, Visual appearance; and three constraints: Annual maintenance budget, minimum level for the portfolio DPS reduction, and a management capacity constraint limiting the number of bridges in the portfolio. The model results were delivered to DMs with a spreadsheet that included the bridge Core Index values, their attribute-specific scores and other technical information. The spreadsheet allowed the DMs to manually construct the final portfolio by selecting individual bridges and showed the resulting overall portfolio performance in terms of attributes and resource consumption. However, selections made by the DMs were not fed back into the PDA model to update the sets of non-dominated or potentially optimal portfolios and corresponding core index values. Here we illustrate how such selections could be modeled as project preferences statements in the PDA model using the developed approaches.

Although in the original case the model was applied for data sets containing hundreds of bridge projects, for brevity we analyze here a subset of 46 projects from one of these data sets with the constraints scaled accordingly (see Appendix B). This analysis was carried out on a standard laptop (2.93 gigahertz processor, 8 gigabytes memory). RPM-Decisions software, which implements the dynamic programming algorithm of Liesiö, Mild, and Salo (2008), was used to compute the set of non-dominated portfolios  $P_N(W^0)$ . This took approximately 20 seconds. Identification of potentially optimal portfolios  $P_O(W)$  for different sets of feasible weights  $W$  and the computation of acceptability and core indexes were carried out with a custom implementation, which is available online free of charge (Tervonen & Liesiö, 2016). Each set of potentially optimal portfolios  $P_O(W)$  was obtained in less than 4 minutes, and the estimation of the acceptability and core indexes took less than 35 seconds. These results are presented in Table 3. Specifically, the set of potentially optimal portfolios  $P_O(W^0)$  contains 294 portfolios, each of which is a subset of the 24 borderline projects  $X_O^B(W^0)$  as there are no core projects ( $X_O^C(W^0) = \emptyset$ ).

There are some projects (e.g.,  $x^{14}$  and  $x^{30}$ ) with zero AI and non-zero CI. This is due to imprecision in the AI estimation procedure: Some of the potentially optimal portfolios may have very small optimal weight regions, and hence none of the sampled 10,000 weight vectors are in these regions. The differences between exact and Monte Carlo estimated indexes are larger than what has been observed in ranking and classification multiattribute problems (cf. Kadziński & Tervonen, 2013a; 2013b). This is likely due to the larger number of decision alternatives (potentially optimal portfolios) in the current study.

In the second iteration of the analysis, we apply the preference information from the original application, which includes incomplete ordinal statements about the importance of different at-

tributes as well as a weight lower bound of 0.02 to enforce some importance for every attribute. These statements yield the set of feasible weights

$$W^1 = \left\{ w \in \mathbb{R}^6 \mid \begin{aligned} &w_{DPS} \geq w_{traffic} \geq w_i, \text{ for } i \in \{\text{salt, visual}\} \\ &w_{DPS} \geq w_{carry} + w_{width} \geq w_i, \text{ for } i \in \{\text{salt, visual}\} \\ &w_i \geq 0.02 \text{ for } i \in \{\text{DPS, traffic, carry, width, salt, visual}\} \\ &\sum_i w_i = 1 \end{aligned} \right\}. \quad (12)$$

The number of potentially optimal portfolios is  $|P_O(W^1)| = 74$ , which is considerably less than the 294 potentially optimal portfolios of the previous iteration. However, since the number of borderline projects is 21, and there are no core projects ( $X_O^C(W^1) = \emptyset$ ), the DM may be inclined to provide project preferences to obtain more conclusive results. Moreover, core and acceptability indexes can be used to assist in identifying candidates for such statements. For example, project  $x^5$  has a CI of 0.34, but a very low AI of 0.03. Although the DM could select this project into the optimal portfolio, the portfolio recommendation would not be particularly robust with regard to  $\ln(x^5)$  because  $x^5$  is included in the optimal portfolio for only 3% of the weights in  $W^1$ . On the other hand,  $x^{45}$  has CI of 0.22 and AI of 0.62, and is therefore a good candidate for selecting into the final portfolio.

Assume that such considerations result in the DM providing a single project preference statement  $\ln(x^{45})$ , and let  $W^2$  denote the corresponding PO-compatible weight set (Definition 6). Because the AI of project  $x^{45}$  is 0.62, this reduces the size of the feasible weight set by 38%. The CI of project  $x^{45}$  is 0.22, so 78% of the portfolios included in  $P_O(W^1)$  are not included in  $P_O(W^2)$ . Fig. 5 highlights the fact that many projects obtain a unit AI and CI as a result of adding this single project preference statement. The number of borderline projects reduces to  $|X_O^B(W^2)| = 11$ .

Furthermore, the AI of projects  $x^1, x^3, x^6, x^8, x^{18}$  and  $x^{20}$  is close to zero for the weight set  $W^2$ . Assume that after careful examination of the attribute performance scores of these projects, the DM decides that they can be rejected from the optimal portfolio together with project  $x^{17}$ . The AI of project  $x^{17}$  is 0.56, which implies that rejecting it results in 44% weight space reduction. In fact, with only linear constraints for the weights ( $W^1$ ), the  $\text{AI}(x^{17}, W^1) = 0.47$ , which means that this decision is quite robust with regard to variation in preferences expressed by the linear constraints.

Introducing the project preference statements  $\text{Out}(x^k)$ ,  $k \in \{1, 3, 6, 8, 17, 18, 30\}$  results in the PO compatible weight set  $W^3$  with only  $|P_O(W^3)| = 3$  potentially optimal portfolios. The DM could now choose one out of these three portfolios based on their attribute scores. Note that  $p^1$  and  $p^3$  are considerably less robust to changes in weights due to containing  $x^{12}$  with AI of 0.05, and therefore the DM might want to select  $p^2$ .

Minimum, maximum and average of the sampled weights in 4 iterations of the analysis are presented in Tables 4, 5 and 6, respectively. Such descriptive statistics could be used for providing DM information on implications of the preference statements. For some criteria (Width, Salt, Visual), the project preference statements have little bearing on minimum and maximum of the weights, whereas for others (DPS, Traffic, Carry) the project preference statements affect the feasible ranges and the averages. Note that these descriptive statistics are computed based on the sampled weights, that is, they are estimates instead of exact values. For instance, with  $W^0$ , the theoretical maximum of each weight is 1, but none of the 10,000 sampled weight vectors contained values close to 1.

**Table 3**

Project attribute measurements, acceptability (AI) and core indexes (CI) in 4 iterations of the analysis. Last three columns indicate the projects belonging to the three potentially optimal portfolios in the analysis with the feasible weight space  $W^3$ . The projects are sorted according to  $AI(x_j, W^0)$ . AI and CI are only shown for entries where one of these is strictly positive.

	DPS	Traffic	Carry	Width	Salt	Visual	$W^0$		$W^1$		$W^2$		$W^3$		$P_0(W^3)$		
							AI	CI	AI	CI	AI	CI	AI	CI	$p^1$	$p^2$	$p^3$
43	2.688	4.0	0	0	3.00	0.67	0.76	0.91	1.00	0.78	1.00	1.00	1.00	1.00	1	1	1
10	2.864	2.5	0	0	3.00	0.67	0.70	0.59	0.99	0.77	1.00	1.00	1.00	1.00	1	1	1
11	1.360	4.0	0	0	3.00	0.67	0.53	0.45	0.76	0.39	1.00	1.00	1.00	1.00	1	1	1
34	0.448	4.0	0	3	3.00	0.00	0.47	0.37	0.45	0.34	0.47	0.31	0.95	0.67	1	1	0
41	4.000	2.0	0	0	0.00	0.67	0.47	0.68	0.96	0.88	0.93	0.46	1.00	0.67	0	1	1
45	1.256	4.0	0	0	3.00	0.67	0.46	0.21	0.62	0.22	1.00	1.00	1.00	1.00	1	1	1
12	1.544	0.0	3	4	0.00	0.00	0.43	0.45	0.36	0.69	0.02	0.08	0.05	0.67	1	0	1
5	0.120	0.0	4	4	0.00	0.67	0.42	0.33	0.03	0.34					0	0	0
28	0.856	0.0	4	3	0.00	0.00	0.36	0.35	0.12	0.41	0.00	0.08			0	0	0
3	0.472	0.0	4	2	0.00	2.00	0.33	0.49	0.05	0.22	0.01	0.08			0	0	0
37	0.352	0.0	3	3	0.00	1.33	0.26	0.25							0	0	0
17	1.688	3.0	0	0	3.00	0.67	0.25	0.22	0.47	0.20	0.56	0.38			0	0	0
8	0.368	4.0	0	0	3.00	0.67	0.24	0.04	0.00	0.01	0.00	0.08			0	0	0
24	0.280	0.0	4	3	0.00	0.67	0.08	0.15	0.00	0.11					0	0	0
23	0.224	0.0	2	4	0.00	0.00	0.06	0.06	0.00	0.11					0	0	0
22	1.912	1.0	1	2	0.00	0.67	0.05	0.20	0.19	0.34					0	0	0
35	0.328	4.0	0	0	3.00	0.67	0.04	0.01							0	0	0
1	0.944	3.0	0	1	3.00	0.00	0.03	0.05	0.00	0.04	0.00	0.08			0	0	0
6	1.184	3.0	0	0	3.00	0.67	0.03	0.00	0.00	0.01	0.00	0.08			0	0	0
20	0.824	4.0	0	0	3.00	0.00	0.02	0.04	0.01	0.05	0.01	0.31			0	0	0
18	0.776	4.0	0	0	3.00	0.00	0.01	0.01	0.00	0.01	0.00	0.08			0	0	0
36	1.496	0.0	2	2	0.00	0.67	0.01	0.07	0.01	0.07					0	0	0
30	0.952	3.0	1	0	0.00	0.67	0.00	0.07	0.00	0.01					0	0	0
14	1.968	4.0	0	0	3.00	0.67	0.00	0.01							0	0	0
2	0.648	2.5	0	0	3.00	0.67									0	0	0
4	0.656	1.0	0	0	0.00	0.00									0	0	0
7	4.000	4.0	0	0	3.00	1.33									0	0	0
9	1.176	0.0	1	2	0.00	0.00									0	0	0
13	0.344	3.0	0	0	3.00	0.00									0	0	0
15	1.336	0.0	2	1	0.00	0.67									0	0	0
16	0.512	0.0	0	2	0.00	0.67									0	0	0
19	0.784	0.0	2	2	0.00	0.67									0	0	0
21	0.376	2.0	0	1	0.00	0.67									0	0	0
25	0.184	0.0	0	0	0.00	0.00									0	0	0
26	0.200	0.0	1	2	0.00	0.00									0	0	0
27	0.256	0.0	0	1	0.00	0.00									0	0	0
29	0.360	3.5	0	0	3.00	0.00									0	0	0
31	0.328	4.0	0	0	3.00	0.00									0	0	0
32	0.584	2.5	0	0	0.00	0.67									0	0	0
33	0.464	2.0	0	0	3.00	0.00									0	0	0
38	0.488	3.0	0	0	3.00	0.67									0	0	0
39	0.616	0.0	0	1	0.00	0.67									0	0	0
40	0.200	4.0	0	0	3.00	0.00									0	0	0
42	0.280	3.0	0	0	3.00	0.67									0	0	0
44	1.000	0.0	0	0	0.00	0.67									0	0	0
46	0.448	0.0	2	1	0.00	0.67									0	0	0

## 7. Discussion

In this paper we developed two approaches for modeling the DM's preference statements about which projects should be selected in, or rejected from the final portfolio, as sets of feasible attribute weights compatible with these statements. The first approach is based on identifying a set of feasible weights for which every potentially optimal portfolio contains all the projects the DM has selected, and none of the projects the DM has rejected. The second approach follows a similar logic, but identifies a set of feasible weights such that each non-dominated portfolio contains all the projects the DM has selected, and none of the projects the DM has rejected. Both approaches assume that the portfolio preference statements reveal information about attribute trade-offs. Whether or not this assumption holds in practice is debatable and should be tested for empirically.

Based on the comparison of these two approaches, we have argued that the first approach based on the use of potential optimality may be better suited for decision support, because it provides a unique set of feasible weights compatible with the project pref-

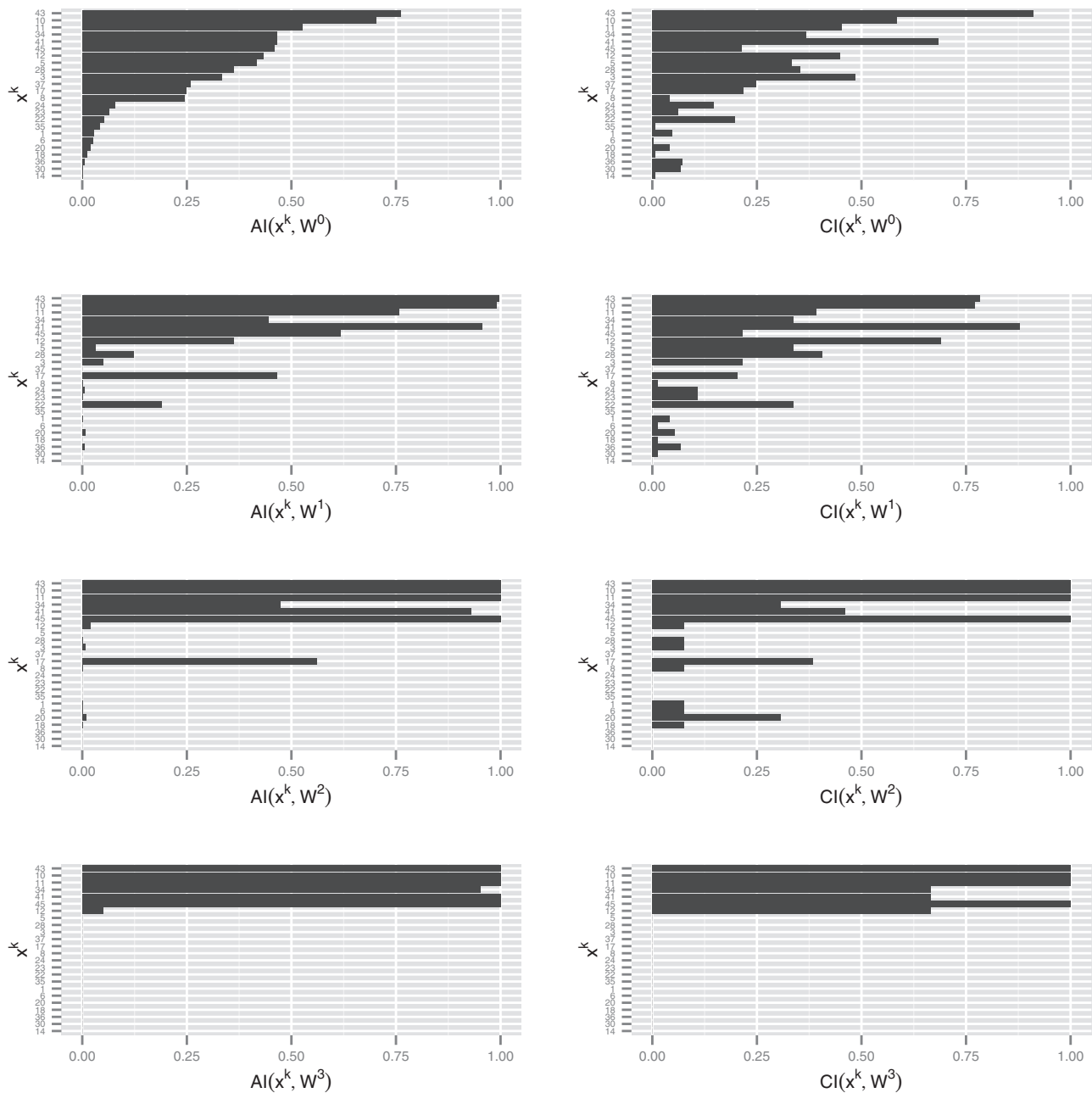
**Table 4**

Minimums of weight samples in 4 iterations of the analysis.

	DPS	Traffic	Carry	Width	Salt	Visual
$W^0$	0.00	0.00	0.00	0.00	0.00	0.00
$W^1$	0.22	0.02	0.02	0.02	0.02	0.02
$W^2$	0.21	0.10	0.02	0.02	0.02	0.02
$W^3$	0.24	0.15	0.02	0.02	0.02	0.02

erence statements. The second approach based on non-dominated portfolios can produce several sets of compatible weights. Although one could argue that in such a setting the union of these sets would be an appropriate way of modeling the preference statements, it turns out that this union is always equal to the feasible weight set obtained from the first model. Finally, in the analysis based on the set of potentially optimal portfolios, project core and acceptability indexes provide a priori information on how much different project preference statements reduce the set of feasible weights and potentially optimal portfolios.

Our results suggest at least two avenues for future research. First, empirical research is needed to analyse to what extent the



**Fig. 5.** Acceptability and core indexes for the four iterations with increasing amount of preference information. The analyses are without preferences ( $W^0$ ), with weight constraints ( $W^1$ ), and with project preference statements  $\text{In}(x^{45})$  ( $W^2$ ) and  $\text{Out}(x^{17}, x^1, x^3, x^6, x^8, x^{18}, x^{20}, x^{28})$  ( $W^3$ ). The figure includes only projects for which  $0 < CI(x^k, W^0) < 1$ .

**Table 5**

Maximums of weight samples in 4 iterations of the analysis.

	DPS	Traffic	Carry	Width	Salt	Visual
$W^0$	0.82	0.84	0.82	0.92	0.86	0.86
$W^1$	0.78	0.41	0.39	0.41	0.24	0.23
$W^2$	0.68	0.44	0.30	0.29	0.23	0.23
$W^3$	0.51	0.41	0.28	0.28	0.20	0.23

DMs' project preference statements actually capture preferences for the attribute-specific project values and how much they are affected by factors that are external to the value model (e.g. geographical location of the bridge in our application). On the one hand, such biases might occur more often with the project preference statements, which focus on actual projects, than with standard techniques, which focus on hypothetical projects with their outcomes set at the most or least preferred levels of the at-

**Table 6**

Averages of weight samples in 4 iterations of the analysis.

	DPS	Traffic	Carry	Width	Salt	Visual
$W^0$	0.17	0.17	0.16	0.17	0.16	0.17
$W^1$	0.41	0.19	0.12	0.12	0.08	0.08
$W^2$	0.38	0.23	0.11	0.10	0.09	0.09
$W^3$	0.35	0.25	0.09	0.14	0.09	0.09

tribute measurement scales. On the other hand, eliciting preferences using real projects may result in higher DM involvement that could in turn lead to less biases. In standard single alternative choice problems, the use of hypothetical reference alternatives has been shown to lead to more consistent preference statements (Vetschera, Weitzel, & Wolfsteiner, 2014), but it is unclear whether this result is generalizable to the PDA context. Second, our results suggest that potential optimality may provide a more intuitive and

readily extendable solution concept than dominance for multiattribute portfolio decision analysis under incomplete information. Yet, the literature on both exact and heuristic algorithms for solving potentially optimal portfolios is almost non-existent compared to wide literature on algorithms for solving non-dominated portfolios (see e.g. Villarreal & Karwan, 1981; Stummer & Heidenberger, 2003; Liesiö et al., 2008).

## Acknowledgments

Juuso Liesiö received funding for this research from the Academy of Finland (Grant number 253583). This research has been partly supported by the project Platform Value Now, funded through the grant 293446 of the Strategic Research Council of the Academy of Finland. The authors acknowledge useful comments from three anonymous reviewers and the editor.

## Appendix A. Proofs

We first establish the following auxiliary lemma that will be used in other proofs.

**Lemma 9.** Let  $W \in \mathcal{W}$ . If  $V(p, w) = V(p', w)$  for all  $w \in W$  then  $V(p, w) = V(p', w)$  for all  $w \in W^0$ , i.e.,  $p \in [p']$ .

**Proof.** Take any  $w^0 \in W^0$  and  $w \in \text{int}(W)$ . Then there exists  $\epsilon > 0$  such that  $w' = w + \epsilon(w^0 - w) \in W$ . Since,  $V$  is linear in weights

$$\begin{aligned} 0 &= V(p, w') - V(p', w') = V(p, w) + \epsilon V(p, w^0) - \epsilon V(p, w) \\ &\quad - V(p', w) - \epsilon V(p', w^0) + \epsilon V(p', w) \\ &= (1 - \epsilon) [V(p, w) - V(p', w)] + \epsilon [V(p, w^0) - V(p', w^0)], \end{aligned}$$

$= 0, \text{ since } w \in W$

which implies  $V(p, w^0) = V(p', w^0)$ .  $\square$

### Lemma 1

**Proof.**  $P_0(W) \neq \emptyset$ : We prove  $P_0(W) = \emptyset \Rightarrow \text{int}(W) = \emptyset$ . Assume  $P_0(W) = \emptyset$ , which implies that  $\text{int}(\hat{W}(p, W)) = \emptyset$  for all  $p \in P_F$ . Since each  $\hat{W}(p, W)$  is closed and has an empty interior it is nowhere dense. The Baire category theorem states that a union of nowhere dense sets is nowhere dense, which gives  $\text{int}(\bigcup_{p \in P_F} \hat{W}(p, W)) = \emptyset$ . On the other hand, take any  $w \in W$  and  $p^* \in \arg \max_{p \in P_F} V(p, w)$ , then  $w \in \hat{W}(p^*, W) \subseteq W$ , which implies  $W \subseteq \bigcup_{p \in P_F} \hat{W}(p, W)$ . Thus,  $\text{int}(W) \subseteq \text{int}(\bigcup_{p \in P_F} \hat{W}(p, W)) = \emptyset$ , which implies  $\text{int}(W) = \emptyset$ .

$P_0(W) \subseteq P_N(W)$ : Take any  $p \in P_0(W)$ , which implies  $W' = \text{int}(\hat{W}(p, W)) \neq \emptyset$ . For any  $p' \in P_F$  either (i)  $V(p, w) > V(p', w)$  for some  $w \in W' \subseteq W$ , in which case  $p' \not\succ_W p$ , or (ii)  $V(p, w) = V(p', w)$  for all  $w \in W'$ , in which case Lemma 9 implies  $V(p, w) = V(p', w)$  for all  $w \in W^0 \supseteq W$  and hence  $p' \not\succ_W p$ .  $\square$

### Lemma 2

**Proof.**  $X_N^C(W) \subseteq X_O^C(W)$ : Assume  $x^j \in p \forall p \in P_N(W)$ . Take arbitrary  $p \in P_0(W)$ , then Lemma 1 implies  $p \in P_N(W)$  and hence  $x^j \in p$ , which implies  $x^j \in X_O^C(W)$ .

$X_N^E(W) \subseteq X_O^E(W)$ : Assume  $x^j \notin p \forall p \in P_N(W)$ . Take arbitrary  $p \in P_0(W)$ , then Lemma 1 implies  $p \in P_N(W)$  and hence  $x^j \notin p$ , which implies  $x^j \in X_O^E(W)$ .

$X_N^B(W) \supseteq X_O^B(W)$ :  $x^j \in X_O^B(W) \Rightarrow x^j \notin X_O^E(W) \cup X_O^C(W) \Rightarrow x^j \notin X_N^E(W) \cup X_N^C(W) \Rightarrow x^j \in X_N^B(W)$ .  $\square$

### Lemma 3

**Proof.**  $P_N(W') \subseteq P_N(W)$ : Assume  $p \in P_F$  and  $p \notin P_N(W)$ . Then there exists  $p' \in P_F$  such that  $p' \succ_W p$ , i.e.,  $V(p', w) \geq V(p, w)$  for all  $w \in W' \subseteq W$  and  $V(p', w^*) > V(p, w^*)$  for some  $w^* \in W$ . Take any

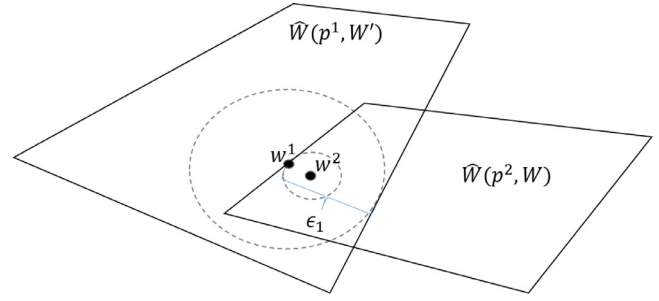


Fig. A.6. Illustration of proof of Theorem 1 part (i).

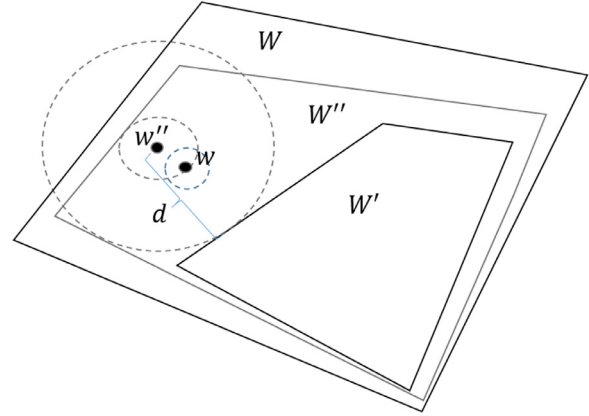


Fig. A.7. Illustration of proof of Theorem 1 part (ii).

$w \in \text{int}(W')$ . Then there exists  $\epsilon \in (0, 1]$  such that  $w' = w + \epsilon(w^* - w) \in W'$ , and the value difference of portfolios  $p'$  and  $p$  then evaluated at  $w' \in W'$  is

$$\begin{aligned} V(p', w') - V(p, w') &= V(p', w) + \epsilon V(p', w^*) - \epsilon V(p, w) \\ &\quad - V(p, w) - \epsilon V(p, w^*) + \epsilon V(p, w) \\ &= (1 - \epsilon) [V(p', w) - V(p, w)] \\ &\quad + \epsilon [V(p', w^*) - V(p, w^*)] > 0, \end{aligned}$$

$\geq 0, \text{ since } w \in W'$

since  $V$  is linear in weights. Hence,  $p' \succ_{W'} p$  and thus  $p' \notin P_N(W')$ .

$P_0(W') \subseteq P_0(W)$ : Assume  $p \notin P_0(W)$  which implies  $\text{int}(\hat{W}(p, W)) = \emptyset$ . Furthermore, since  $W' \subseteq W$ ,  $\hat{W}(p, W') \subseteq \hat{W}(p, W)$ , which implies  $\text{int}(\hat{W}(p, W')) = \emptyset$ , i.e.,  $p \notin P_0(W')$ .

$X_*^C(W') \supseteq X_*^C(W)$ :  $x^j \in X_*^C(W) \Rightarrow x^j \in p \forall p \in P_*(W) \Rightarrow x^j \in p \forall p \in P_*(W') \subseteq P_*(W) \Rightarrow x^j \in X_*^C(W')$ .

$X_*^E(W') \supseteq X_*^E(W)$ :  $x^j \in X_*^E(W) \Rightarrow x^j \notin p \forall p \in P_*(W) \Rightarrow x^j \notin p \forall p \in P_*(W') \subseteq P_*(W) \Rightarrow x^j \in X_*^E(W')$ .

$X_*^B(W') \subseteq X_*^B(W)$ :  $x^j \in X_*^B(W') \Rightarrow x^j \notin X_*^E(W') \cup X_*^C(W') \Rightarrow x^j \notin X_*^E(W) \cup X_*^C(W) \Rightarrow x^j \in X_*^B(W)$ .  $\square$

## Theorem 1

**Proof.** Let

$$W' = \bigcup_{\substack{p \in P_0(W) \\ \text{s.t. } x^k \in p' \forall p' \in [p]}} \hat{W}(p, W). \quad (\text{A.1})$$

By Definition 3  $\hat{W}(p, W) \neq \emptyset$ ,  $p \in P_0(W)$ . Hence the (A.1) can be an empty set if and only if there does not exist  $p \in P_0(W)$  such that  $x^k \in p'$  for all  $p' \in [p]$ . As  $\emptyset \notin \mathcal{W}$  an empty set is not PO-compatible. To prove a non-empty (A.1) is PO-compatible we need to show that satisfies properties (i) and (ii) of Definition 6 (Figs. A.6 and A.7).

(i) Take any  $p^1 \in P_0(W')$  and  $w^1 \in \text{int}(\hat{W}(p^1, W'))$ , which is non-empty by Definition 3. Let  $\epsilon_1 > 0$  be such that  $N_{\epsilon_1}(w^1) \subseteq$



$\hat{W}(p^1, W')$ . Because

$$w^1 \in \hat{W}(p^1, W') \subseteq W' = \bigcup_{\substack{p \in P_0(W) \\ s.t. x^k \in p' \forall p' \in [p]}} \hat{W}(p, W),$$

there must exist  $p^2 \in \{p \in P_0(W) | x^k \in p' \forall p' \in [p]\}$  such that  $w^1 \in \hat{W}(p^2, W)$ . Since  $\hat{W}(p^2, W) \in \mathcal{W}$  there exists  $w^2 \in N_{\epsilon_1/4}(w^1) \cap \text{int}(\hat{W}(p^2, W))$ . Since  $w^2 \in \text{int}(\hat{W}(p^2, W))$  exists  $\epsilon_2 \in (0, \epsilon_1/4]$  such that  $N_{\epsilon_1/4}(w^2) \subseteq \hat{W}(p^2, W)$ . Also  $N_{\epsilon_1/4}(w^2) \subseteq N_{\epsilon_1}(w^1) \subseteq \hat{W}(p^1, W')$ . Hence,  $V(p^1, w) = V(p^2, w)$  for all  $w \in N_{\epsilon_1/4}(w^2)$  and Lemma then 9 implies  $p^1 \in [p^2]$ , which yields  $x^k \in p^1$ . Since the choice of  $p^1 \in P_0(W')$  was arbitrary,  $x^k \in X_0^C(W')$ .

(ii) Take any  $W'' \in \mathcal{W}$  such that  $W' \subset W'' \subset W$ . Take any point in  $w'' \in W'' \setminus W'$  and let  $d = \min_{w' \in W'} \|w'' - w'\|$ . Since  $W'' \in \mathcal{W}$  there exists  $w \in N_{d/4}(w'') \cap \text{int}(W'')$ . Furthermore, there exists  $\epsilon \in (0, d/4)$  such that  $N_\epsilon(w) \subseteq W''$  (since  $w \in \text{int}(W'')$ ) and  $N_\epsilon(w) \cap W' = \emptyset$  (since  $N_\epsilon(w) \subset N_{d/2}(w'')$  and  $N_{d/2}(w'') \cap W' = \emptyset$ ). Lemma 1 implies  $P_0(N_\epsilon(w)) \neq \emptyset$ . Take any  $p \in P_0(N_\epsilon(w))$ . By Lemma 3  $p \in P_0(W'') \subseteq P_0(W)$ . But since  $\hat{W}(p, W) \supseteq N_\epsilon(w)$  is not contained in  $W'$  there must exist  $p' \in [p]$  such that  $x^k \notin p'$ . Hence,  $x^k \notin X_C(W'')$ .  $\square$

### Theorem 2

**Proof.** The proof is equivalent to that of Theorem 1 in which statements ' $x^k \in \cdot$ ' and ' $x^k \notin \cdot$ ' have been replaced by ' $x^k \notin \cdot$ ' and ' $x^k \in \cdot$ ', respectively.  $\square$

### Lemma 4

**Proof.** According to Definition 8, any ND-compatible set  $W' \in \mathcal{W}$  is such, that  $x^k \in X_N^C(W')$ . By Definition 5,  $x^k \in p \forall p \in P_N(W')$ . Therefore, by Definition 2, for each  $p \in P_F$  such that  $x^k \notin p$ ,  $\exists p' \in P_F$  such that  $x^k \in p'$  and  $p' \succ_W p$ . Because  $p, p' \in P_F$ , it follows that  $p, p' \in P_N(W)$ . Thus, for each  $p \in P_k^+$ , there must exist  $p' \in P_k^+$  such that  $p' \succ_W p$ , and there are  $|P_k^+|^{P_k^-}$  such combinations. Some of these combinations might not be valid as there might exist  $W^1, W^2 \in \mathcal{W}$  such that  $W^1 \subset W^2$  and  $x^k \in p \forall p \in P_N(W^1)$ ,  $x^k \in p \forall p \in P_N(W^2)$ , and from these two only  $W^2$  would satisfy the second condition of Definition 8. Therefore  $|P_k^+|^{P_k^-}$  is an upper bound.  $\square$

### Theorem 3

**Proof.**  $W' \subseteq \bigcup_j W^j$ : Take arbitrary  $w^* \in W'$ . Theorem 1 then implies that there exists  $p^* \in P_0(W')$  (i.e.  $\text{int}(\hat{W}(p^*, W')) \neq \emptyset$ ), such that  $w^* \in \hat{W}(p^*, W')$ , and  $x^k \in p$  for all  $p \in [p^*]$ . Now consider the set  $P_N(\hat{W}(p^*, W'))$  which is non-empty by Lemma 1 since  $\text{int}(\hat{W}(p^*, W')) \neq \emptyset$ , and take arbitrary  $p' \in P_N(\hat{W}(p^*, W'))$ . By construction  $V(p^*, w) \geq V(p', w)$  for all  $w \in \hat{W}(p^*, w)$ , but since  $p'$  is non-dominated there cannot exist  $w \in \hat{W}(p^*, W')$  such that  $V(p^*, w) > V(p', w)$ . Hence,  $V(p^*, w) = V(p', w)$  for all  $w \in \hat{W}(p^*, W')$ , and thus Lemma 9 implies  $p' \in [p^*]$ . Hence,  $x^k \in p'$  and since selection of  $p'$  was arbitrary we have  $x^k \in X_N^C(\hat{W}(p^*, W'))$ . By Definition 8  $\hat{W}(p^*, W') \subseteq W^j$  for some  $j$ , which implies  $\bigcup_j W^j \supseteq \hat{W}(p^*, W') \ni w^*$ .

$W' \supseteq \bigcup_j W^j$ : Take any  $W^j$ . Then by Definition 8  $x^k \in X_N^C(W^j)$  and hence Lemma 2 implies  $x^k \in X_0^C(W^j)$ . By Definition 6,  $W^j \subseteq W'$ . Since the choice of  $W^j$  was arbitrary,  $\bigcup_j W^j \subseteq W'$ .  $\square$

### Lemma 5

**Proof.** By Definition 6,  $x^k \in X_0^C(W')$  which implies  $x^k \in p$  for all  $p \in P_0(W')$ . Thus,

$$|P_0(W')| = |\{p \in P_0(W') | x^k \in p\}| \\ \leq |\{p \in P_0(W) | x^k \in p\}| = |\{p \in P_0(W) | x^k \in p\}| \frac{|P_0(W)|}{|P_0(W)|}$$

$$= \text{CI}(x^k, W) |P_0(W)|,$$

where the inequality holds since  $W' \subseteq W$  and hence  $P_0(W') \subseteq P_0(W)$  by Lemma 3. Note that the inequality is strict if there are two portfolios  $p, p' \in P_0(W)$  such that  $p \in [p]$ , and  $x^k \in p$  but  $x^k \notin p'$ . In this case  $p \notin P_0(W')$  and hence  $|\{p \in P_0(W') | x^k \in p\}| < |\{p \in P_0(W) | x^k \in p\}|$ . The second result follows trivially from (9).  $\square$

### Lemma 6

**Proof.** By Definition 7,  $x^k \in X_0^E(W')$  which implies  $x^k \notin p$  for all  $p \in P_0(W')$ . Thus,

$$|P_0(W')| = |\{p \in P_0(W') | x^k \notin p\}| \leq |\{p \in P_0(W) | x^k \notin p\}| \\ = \left( |P_0(W)| - |\{p \in P_0(W) | x^k \in p\}| \right) \frac{|P_0(W)|}{|P_0(W)|} \\ = (1 - \text{CI}(x^k, W)) |P_0(W)|,$$

where the inequality holds since  $W' \subseteq W$  and hence  $P_0(W') \subseteq P_0(W)$  by Lemma 3.

To prove the second inequality denote  $P^+ = \{p \in P_0(W) | x^k \in p' \forall p' \in [p]\}$  and  $P^- = \{p \in P_0(W) | x^k \notin p' \forall p' \in [p]\}$ . Then by Theorems 1 and 2 PO-compatible weight sets with preference statements  $\text{In}(x^k)$  and  $\text{Out}(x^k)$  are  $W^+ = \bigcup_{p \in P^+} \hat{W}(p, W)$  and  $W^- = \bigcup_{p \in P^-} \hat{W}(p, W)$  respectively.  $W^+ \cap W^-$  is a union of sets  $\hat{W}(p^+, W) \cap \hat{W}(p^-, W)$ ,  $p^- \in P^-$ ,  $p^+ \in P^+$ . For any  $p^+ \in P^+$  and  $p^- \in P^-$  clearly  $p^+ \notin [p^-]$  and hence Lemma 9 implies that  $\text{int}(\hat{W}(p^+, W) \cap \hat{W}(p^-, W)) = \emptyset$ . By Baire category theorem, the union of these also has an empty interior, i.e.,  $\text{int}(W^+ \cap W^-) = \emptyset$ . Thus  $\text{vol}(W^+ \cap W^-) = 0$  which can be used to obtain

$$\text{vol}(W^-) = \text{vol}((W^- \cap W^+) \cup (W^- \cap (W \setminus W^+))) \\ = \text{vol}(W^- \cap (W \setminus W^+)) \leq \text{vol}(W \setminus W^+) \\ = \text{vol}(W) - \text{vol}(W^+) = \text{vol}(W) - \text{vol}(W^+) \frac{\text{vol}(W)}{\text{vol}(W)} \\ = (1 - \text{AI}(x^k, W)) \text{vol}(W).$$

$\square$

### Lemma 7

**Proof.** Denote  $\delta(w) = V(p, w) - V(p', w) = \sum_{i=1}^n w_i (V_i(\sum_{x^j \in p} v_i^j) - V_i(\sum_{x^j \in p'} v_i^j))$  which is clearly linear in  $w$ . Hence, by Definition 2

$$p \succ_W p' \Leftrightarrow \begin{cases} \delta(w) \geq 0 \text{ for all } w \in W \\ \delta(w) > 0 \text{ for some } w \in W \end{cases} \\ \Leftrightarrow \begin{cases} \delta(w) \geq 0 \text{ for all } w \in \text{ext}(W) \\ \delta(w) > 0 \text{ for some } w \in \text{ext}(W), \end{cases} \quad (\text{A.2})$$

since  $\delta(w)$  is linear in  $w$ . Now, since  $W = W^0$  the extreme points are  $w^1, \dots, w^n$  such that  $w_k^k = 1$  and  $w_i^k = 0$  for all  $i \in \{1, \dots, n\} \setminus \{k\}$ . Hence

$$p \succ_{W^0} p' \Leftrightarrow \begin{cases} V_i(\sum_{x^j \in p} v_i^j) \geq V_i(\sum_{x^j \in p'} v_i^j) \text{ for all } i \in \{1, \dots, n\} \\ V_i(\sum_{x^j \in p} v_i^j) \geq V_i(\sum_{x^j \in p'} v_i^j) \text{ for some } i \in \{1, \dots, n\} \end{cases} \quad (\text{A.3})$$

$$\Leftrightarrow \begin{cases} \sum_{x^j \in p} v_i^j \geq \sum_{x^j \in p'} v_i^j \text{ for all } i \in \{1, \dots, n\} \\ \sum_{x^j \in p} v_i^j \geq \sum_{x^j \in p'} v_i^j \text{ for some } i \in \{1, \dots, n\} \end{cases} \quad (\text{A.4})$$

where the last equivalence is due to the fact that each  $V_i$  is a strictly increasing function. Now consider  $p \in P_N(W^0)$  then there does not exist another feasible portfolio that dominates it. Based on (A.3)–(A.4), this holds if and only if  $z(p)$  is a solution to the MOZOLP problem (10), and there does not exist another solution which has a better value in each objective function and a strictly better in at least one. This, by definition, means that  $z(p)$  is a Pareto optimal solution to problem (10).  $\square$

### Lemma 8

**Proof.** ‘ $\Rightarrow$ ’: Assume  $p \in P_0(W^1)$ . Then any  $w \in \text{int}(\hat{W}(p, W^1))$  is a feasible solution for some strictly positive value for  $d$ . ‘ $\Leftarrow$ ’: Assume  $d > 0$ . Then there exists  $w \in W^1$  such that  $V(p, w) > V(p', w)$  for all  $p' \in P_N(W^0) \setminus [p]$ . Since  $V(\cdot, w)$  is linear in  $w$  these strict inequalities have to hold inside some open ball  $N_\epsilon(w^*) \subseteq W^1$ , i.e.,

$$\begin{aligned} N_\epsilon(w^*) &\subseteq \{w \in W^1 \mid V(p, w) > V(p', w) \forall p' \in P_N(W^0) \setminus [p]\} \\ &= \{w \in W^1 \mid V(p, w) \geq V(p', w) \forall p' \in P_N(W^0)\} \\ &= \{w \in W^1 \mid V(p, w) \geq V(p', w) \forall p' \in P_F\}. \end{aligned}$$

Since  $\text{int}(N_\epsilon(w^*)) = N_\epsilon(w^*) \neq \emptyset$ , this result implies that  $\text{int}(\{w \in W^1 \mid V(p, w) \geq V(p', w) \forall p' \in P_F\}) \neq \emptyset$ , which by Definition 3 implies that  $p \in P_0(W^1)$ .  $\square$

### Appendix B. Project performances and costs for the application

See Table B.7 below.

**Table B.7**

Bridge data used in the analysis. The model has three portfolio feasibility constraints: The total cost cannot exceed the budget of  $b_1 = 1,030,260$  euros, the total DPS reduction must be at least  $b_2 = 873$  points, and the maximum number of bridges included in the portfolio is  $b_3 = 6$ .

$j$	DPS ( $v_1^j$ )	Traffic ( $v_2^j$ )	Carry ( $v_3^j$ )	Width ( $v_4^j$ )	Salt ( $v_5^j$ )	Visual ( $v_6^j$ )	Cost ( $a_1^j$ )	DPS Red ( $a_2^j$ )
1	0.944	3.0	0	1	3.00	0.00	9000	71
2	0.648	2.5	0	0	3.00	0.67	59,000	49
3	0.472	0.0	4	2	0.00	2.00	22,000	35
4	0.656	1.0	0	0	0.00	0.00	108,000	49
5	0.120	0.0	4	4	0.00	0.67	19,000	9
6	1.184	3.0	0	0	3.00	0.67	9000	89
7	4.000	4.0	0	0	3.00	1.33	1,117,000	971
8	0.368	4.0	0	0	3.00	0.67	237,000	28
9	1.176	0.0	1	2	0.00	0.00	14,000	88
10	2.864	2.5	0	0	3.00	0.67	36,000	322
11	1.360	4.0	0	0	3.00	0.67	29,000	102
12	1.544	0.0	3	4	0.00	0.00	21,000	116
13	0.344	3.0	0	0	3.00	0.00	31,000	26
14	1.968	4.0	0	0	3.00	0.67	711,000	246
15	1.336	0.0	2	1	0.00	0.67	15,000	100
16	0.512	0.0	0	2	0.00	0.67	16,000	38
17	1.688	3.0	0	0	3.00	0.67	350,000	127
18	0.776	4.0	0	0	3.00	0.00	108,000	58
19	0.784	0.0	2	2	0.00	0.67	18,000	59
20	0.824	4.0	0	0	3.00	0.00	16,000	62
21	0.376	2.0	0	1	0.00	0.67	34,000	28
22	1.912	1.0	1	2	0.00	0.67	11,000	143
23	0.224	0.0	2	4	0.00	0.00	19,000	17
24	0.280	0.0	4	3	0.00	0.67	5000	21
25	0.184	0.0	0	0	0.00	0.00	26,000	14
26	0.200	0.0	1	2	0.00	0.00	17,000	15
27	0.256	0.0	0	1	0.00	0.00	13,000	19
28	0.856	0.0	4	3	0.00	0.00	99,000	96
29	0.360	3.5	0	0	3.00	0.00	30,000	27
30	0.952	3.0	1	0	0.00	0.67	101,000	71
31	0.328	4.0	0	0	3.00	0.00	238,000	25
32	0.584	2.5	0	0	0.00	0.67	17,000	44
33	0.464	2.0	0	0	3.00	0.00	280,000	35
34	0.448	4.0	0	3	3.00	0.00	18,000	34
35	0.328	4.0	0	0	3.00	0.67	20,000	25
36	1.496	0.0	2	2	0.00	0.67	23,000	112
37	0.352	0.0	3	3	0.00	1.33	76000	26
38	0.488	3.0	0	0	3.00	0.67	10,000	37
39	0.616	0.0	0	1	0.00	0.67	81,000	46
40	0.200	4.0	0	0	3.00	0.00	58,000	15
41	4.000	2.0	0	0	0.00	0.67	176,000	627
42	0.280	3.0	0	0	3.00	0.67	26,000	21
43	2.688	4.0	0	0	3.00	0.67	272,000	302
44	1.000	0.0	0	0	0.00	0.67	68,000	75
45	1.256	4.0	0	0	3.00	0.67	8000	94
46	0.448	0.0	2	1	0.00	0.67	12,000	34

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