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Optimal controller design by nonlinear and game theoretic methods

RAIMO P. HÄMÄLÄINEN
Helsinki University of Technology
Otaniemi, Finland

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Abstract

A review and discussion of the general framework of deterministic dynamic decision-making problems from the control theoretic point of view is given. Recent results obtained by the author in developing new solution methods for nonclassical controller problems and in studying their applicability in practical regulator design is summarized. The summarized papers deal with nonlinear and game theoretic problems. The solution methods for nonlinear optimal control problems were based on the use of the theory of polynomial operators. Game theoretic design was considered in connection with two-controller problems and with a new worst case design method. Solution algorithms for general linear-quadratic two-player difference games with different solution concepts and information structures were developed. The practical examples studied include regulators for nonlinear systems, control constrained regulators, and a worst case design approach. The systems considered were an analog simulation of a microbiological fermentation process and a laboratory pilot process representing the headbox of a paper machine. Direct digital control by a minicomputer and computer simulations were studied.

This thesis consists of the present survey and the following papers:

I HALME, A. and HÄMÄLÄINEN, R.P., On the nonlinear regulator problem. Journal of Optimization Theory and Applications **16**(1975)3/4, p. 255...275. Ref. [44].

II HALME, A., HÄMÄLÄINEN, R.P., HEIKKILÄ, O. and LAAKSONEN, O., On synthetizing a state regulator for analytic nonlinear discrete-time systems. International Journal of Control **20**(1974)3, p. 497...515. Ref. [47].

III HÄMÄLÄINEN, R.P. and HALME, A., A solution of nonlinear TPBVP's occuring in optimal control. Automatica **12**(1976)5, p. 403...415. Ref. [64].

IV HÄMÄLÄINEN, R.P., Nash and Stackelberg solutions to general linear-quadratic two-player difference games. Otaniemi 1976. Helsinki University of Technology, Systems Theory Laboratory, Report B 29. Ref. [60].

V HÄMÄLÄINEN, R.P., Difference games with periodic information structures with application to the worst case design of regulators. Otaniemi 1976. Helsinki University of Technology, Systems Theory Laboratory, Report B 34. Accepted for publication in the International Journal of Systems Science. Ref. [61].

The research work reported in papers I-III was carried out in active collaboratorion between A. Halme and the author. The original ideas of applying the polynomial theory to regulator problems were introduced by A. Halme. The co-authors O. Heikkilä and O. Laaksonen in paper II assisted with the implementation of the regulators designed and with the operation of the pilot process. The results presented in papers IV and V were obtained independently by the author.

Preface

The thesis work summarized in this paper was carried out at the Systems Theory Laboratory, Helsinki University of Technology. Most of the work I performed as a research assistant fellow of the Research Council for Technology of the Academy of Finland.

I wish to thank Professor Hans Blomberg, Head of the Systems Theory Laboratory, for his support throughout the work and for providing me excellent working conditions in his laboratory. I also extend my warmest thanks to my teacher Professor Olli Lokki for his continuous interest and encouragement.

I am greatly indebted to my closest co-worker Associate Professor Aarne Halme, who introduced me into the subject of the present study. His expert guidance and stimulating collaboration has been of invaluable importance to me. My cordial thanks are also due to Dr Jussi Orava, who followed the work with interest, supported me with helpful discussions and went through some of the manuscripts with constructive criticism.

I am grateful to the Research Group of the Systems Theory Laboratory as a whole for creating the friendly and inspiring scientific athmosphere in which I could carry out this work. I also wish to thank the personnel of the Control Engineering Laboratory at Tampere University of Technology, and particularly Mr. Osmo Heikkilä, M.Sc., and Mr. Osmo Laaksonen, M.Sc., for assistance and pleasant co-operation when studying the pilot processes.

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Finally I wish to express my most sincere feelings of gratitude to all my friends for their valuable support and encouragement.

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Helsinki, Finland
December 1976

Raimo P. Hämäläinen

Contents

1 Introduction

The rapid development of modern control theory began some twenty years ago. The classical frequency-plane approach has been followed by the multivariable time-domain state-space representation. Intensive research has been carried out on the stability theory and on the algebraic and optimal control theory. Today increasing attention is being paid to applications. It is often said, however, that the mathematical theory has been separated from engineering practice [39]. This is partly due to the vigorous advancements of the theory itself. The fact that PID controllers, which are the tools prevailing in practice when designing stabilizing controllers, will always remain useful for a wide class of simple problems does not mean that advanced theoretical methods would not be needed. The need for the so-called modern theory originates from complicated problems which typically involve multivariable controls and in which more detailed information such as a dynamic model of the system is employed.

A closer examination of the control problems appearing in real systems reveals that one on the main reasons for the small number of successful applications of modern techniques is in fact due to shortcomings in the theories and mathematical methods. For example in optimal controller design the standard linear-quadratic techniques do not suffice, because a typical feature of practical problems is that the systems are nonlinear and very seldom low-dimensional. Moreover, decentralized decision making is often encountered where more than one independent control agent is involved. These characteristics are accompanied by theoretical problems which are not yet sufficiently well known. Disturbances and system uncertainties are present in all practical systems and it is very difficult to obtain reliable information which could be used efficiently in the design process using known methods. Stochastic optimal control can only seldom be employed in a straightforward manner in practical problems because of the difficulties caused by the a priori distributions of the disturbances. Recently some attempts have been made to develop alternative approaches to such problems. Among these are different worst case design methods.

Difficulties are also encountered in problem formulation and modeling. On the one hand it is impossible for a systems theorist alone to find the relevant models and design criteria in the various fields where systems research is done. On the other hand specialists in application fields seldom possess the general picture of the possible ways of approach. Thus intensive team work is needed between the systems theorists and the specialists of the primary problem under consideration in order to obtain results with practical control studies.

The aim of this paper is firstly to review the general framework for deterministic dynamic optimization problems (see [50, 52, 54]) from the control theoretic point of view and discuss some important theoretical and practical aspects related to these problems. In addition to the traditional optimal control problems with one control agent there are problems which include multiple control agents and criteria and in which the information structures of the agents play an essential role. The appearance and characteristics of these general optimal controller design problems in connection with engineering, biological and economic systems are discussed in brief.

Secondly, the aim is to summarize recent results obtained by the author in developing new solution methods for some nonclassical controller problems and in studying their applicability in practical regulator design. In the studies presented efforts have been made to carry the new mathematical theories all the way to the practical level. The original results which are reviewed here are found in references [44, 47, 60, 61, 64].

The method, which has been developed in [44, 47, 64] for the solution of nonlinear problems, is a functional analytic method where the solutions are based on the use of the theory of generalized polynomial operators. The maximum principle approach is considered and the nonlinear two-point boundaryvalue problems (TPBVP) obtained in the optimal control problems were formulated as functional aperators which can be solved by an inverse theorem concerning polynomial operators.

Problems with multiple controllers are considered in [60, 61] and solution algorithms are derived for general linear-quadratic problems with two control agents having different information structures and solution concepts. These results were also used in the development of a new game theoretic worst case design technique.

The studies reported on practical design experiments were made on a laboratory pilot-plant scale and the optimal regulator designs considered include nonlinear system models, control constraints and a worst case approach to system uncertainties.

2 Optimal controller design problems

The history of controller design dates back to antiquity where heuristically designed level controllers were used. The different stages of later development include e.g. the speed regulator of Watt's steam engine and the frequency – domain design techniques which has been in use for the past forty years [33]. Recently there has been a trend to replace techniques based essentially on trial and error by analytical methods using integral criteria. However, there is also extensive work going on in the field of algebraic control theory (see e.g. [14, 21, 116]) and generalized frequency plane methods [102]. One of the basic ideas for introducing performance indices in the design of stabilizing controllers was to obtain certain desirable properties for the system and to simplifity the design process. Although optimal controllers are being studied intensively and straightforward design algorithms developed, questions remain concerning the relevance and applicability of this approach [39, 56, 103]. The class of linear-quadratic – Gaussian controllers has received most attention and the main problems have already been completely solved (see [67]). The solutions obtained are simple and computationally efficient [10], but unfortunately practical controller design often involves more general formulations. This has become more and more evident with the expansion of the field where systems research is being done. Today the most important questions involve nonlinearities, multiple controllers, and large-scale systems. The difficulties and complexities met with are on an essentially higher level than e.g. in the classical linear-quadratic regulator studies. Recent developments have widened perspectives and it has been recognized that optimal control should be viewed as a special case in the framework of dynamic decision making problems which HO has called generalized control theory [50, 52]. In this chapter the general classification of deterministic control problems and some basic properties of the solutions are briefly discussed. Finally typical characteristics of optimal controller design in engineering, biology and economics are outlined.

2.1 Classification

Deterministic optimization problems are classified into different categories according to the number of criteria and the number of control agents or decision makers which appear in the problem definition. In traditional optimal control theory there is only one controller and one performance measure. However, problems with more than one criterion and with multiple independent decision makers can equally well arise in practical controller design. Different alternatives and the names of the corresponding optimization problems in the dynamic case are listed as follows:

	Controllers	*Criteria*
Optimal control	one	one
Zero-sum differential game	two	two, $J_1 = -J_2$
Vector-valued optimization problem	one	multiple
Dynamic team problem	multiple	one
Nonzero-sum differential game	multiple	multiple

Differential game theory was the first generalization of optimal control theory to many person dynamic optimization. Von Neuman's static games, formulated in the forties, laid the foundation for this type of problems. However, the main developments of dynamic game theory were made through advances in decision and control theory. In zero-sum differential games, where the objectives of the two players are completely antagonistic, pursuit-evasion problems have played a dominating role and the most important applications are found in the military area. Besides optimal control this is the field most extensively studied in generalized control theory. For a survey of and introduction to this subject see e.g. [20, 71, 100].

If there is one common criterion and two or more controllers we have a perfectly cooperative situation and we are in the field of team theory. The name is quite illustrative because a team in real life is an organization in which there is a single goal for all the members. The theory was first developed by MARSCHAK and RADNER [84, 85, 101] in the static form for economic applications. Control theorists established the importance of dynamic teams much later [26, 29, 52, 53, 55]. The interesting questions in dynamic team theory arise from the different information sets of the controllers.

Optimization with respect to multiple criteria leads to entirely new kinds of problems which are not encountered in the single criterion case. The main question with a vector valued performance measure is how to devise methods for trading off one criterion against another (see e.g. [34, 78, 80, 104]). When there is only one controller utility theory has to be used to reduce the multiple objectives to one. From the mathematical point of view this can be considered as a way of reformulating the problem. This kind of decision making situation with no common measure for the different objectives does, however, appear frequently in practice.

In nonzero-sum games where each controller has its own criterion the situation is somewhat easier. Unique definitions for different types of solutions are available. Yet difficulties arise as soon as the possibility of cooperative negotiations is allowed. Research on nonzero-sum differential games has recently been extremely intensive (see [15...18, 76, 89, 98, 100, 109, 110]) although it was initiated less than ten years ago by STARR and HO [112, 113] at a time when the theory for the zero-sum case was already highly developed.

2.2 Solutions

After the above classification it is necessary to comment shortly on the solutions to these problems. There are five things that have to be fixed before the solution to a general control problem can be determined. These are the specification of system dynamics and constraints, the set of agents (controllers, players, decision makers) and their admissible strategies, the information available to each of the agents, the performance measures and finally the solution concept which is to be used. The effects of changes in a certain specification on the solution usually are highly dependent on the other specifications.

In classical deterministic optimal control problems the linear-quadratic case obtains linear solutions. Nonlinearities are caused e.g. by inequality constraints on the state or control variables, by nonquadratic cost functionals and naturally by nonlinear system models. The meaning of an optimal solution is always well defined and no other concepts are needed.

In the multiple controller case there are nonlinear solutions even to linear-quadratic problems [16]. This fact is related to the information structures of the controllers and it is an important feature which has to be taken into account when the solutions are considered. The explanation is quite simple. Consider for example a dynamic two-agent linear-quadratic problem where one of the controllers acts first. His optimal affine control strategy is unique in value but because of the dynamics there may still be different representations available for it. Since the other controller cannot assume that its co-player uses a linear control law the resultant optimization problem faced by this controller need no longer be quadratic, which again leads to nonlinear solutions. The entire question of information becomes central when more than one controller is involved. The most typical information structures are the open-loop and the closed-loop (or perfect memory) cases. In the former case the decisions are time functions which are solely based on information of the initial state. The perfect memory information structure allows the use of more general state dependent strategies. It is only in classical deterministic optimal control that these information structures always yield identical trajectories and costs for the solutions. In the multicontroller case we have to distinguish clearly between the values of the controls as functions of time and the control strategies which are functions of the state variables.

The properties mentioned above are also manifested in the fact that the safe Principle of Optimality does not extend directly to general multiple criteria problems.

When nonzero-sum differential games are considered the central role played by information is also reflected in the need for more specific definitions of what is meant by a solution. If we assume that the players do not cooperate and prefer a safe solution, then the Nash equilibrium strategy is adopted. The Nash strategy secures each player against unilateral attempts of one controller to improve his individual performance further [88, 112, 113]. The Stackelberg solution of a two-player game assumes that the roles, i.e. the information available to the players, are different. There is a leader and there is a follower (see e.g. [109...111]). The setting is illustrated by a situation where the leader knows the follower's criterion but the follower is not aware of the criterion of the leader. Another corresponding situation is one where the follower is forced to announce his strategy first due to different speeds in computing the strategy or by the dominating strength of the leader. The Stackelberg solution is obtained so that the leader optimizes his strategy first by simultaneously taking into account the rational responses of the follower. In fact, the leader is the only decision maker in the game. Yet, even if one can choose the roles, it is not necessarily the best choice to be the leader [15].

The set of noninferior or Pareto optimal solutions is considered when information exchange and cooperation are allowed. The noninferiority property appears so that any deviation from the solution cannot result in simultaneous improvement of the performance of all the controllers.

Solving for the Pareto optimal set is equal to solving a single controller problem with a vector valued criterion [34, 78, 104, 107, 117]. Elements belonging to the Pareto optimal set are found by solving a family of standard optimal control problems where the scalar cost criterion is given as a weighted sum of the individual criteria of the controllers. A selection or negotiation procedure is further needed to obtain the most satisfactory set of weights which determines the preferred solution.

The question of the value of information easily arises in connection with multi-controller systems. The heuristic definition would be the best one can do with the information minus the best one can do without it. However, a deeper insight into this subject reveals many intricate problems [55]. There are also the possibilities of coalition formation and bargaining when many player games are considered. These situations naturally lead to new solutions (see e.g. [30, 80]).

2.3 Optimal controllers in engineering

As was mentioned above, the practice in engineering controller design relies mostly on the classical PID techniques. The theoretical advancements and the development of efficient computational devices such as microprocessors have made it possible to take the first steps in applying modern multivariable optimal control theory to the stabilization problems in engineering. It can be said that the success of this new approach in the design process depends greatly on the ability of the engineer to understand the physics of the problem and to translate the physical requirements and constraints into mathematical language. However, there are a number of questions such as the handling of nonlinearities and configurations with multiple controllers, which need further theoretical studies.

From the practical point of view one of the difficult problems in the application of »optimal» design methods is the selection and interpretation of the performance criteria. Stabilizing controllers are the most common ones used in engineering. The system on which the design is based is usually a perturbation model and the cost a quadratic functional of the perturbations of the state and control vectors from the ideal desired values. Such an objective function is very seldom a direct measure of the real cost caused by the state deviations and the correcting control efforts, the weighting matrices being merely design instruments. Yet to a certain extent there is practical and theoretical justification for using the optimal design procedures because the tuning becomes simple and the resulting regulators are computationally attractive and the stability questions are easily taken care of [10]. As a curiosity one can note that even a PID-type approach can be realized by optimal controllers [9, 24].

Recently the field where dynamic systems analysis and control theory has been studied has rapidly expanded outside the classical areas of electrical and space engineering. Stabilizing and optimizing control is currently a subject of active research e.g. in biotechnical and environmental engineering (see e.g. [41, 45, 46, 62...66] and the references therein). Dynamic optimization has also become an important factor in production control and in large-scale systems [4, 42, 54, 86].

2.4 Optimal controllers in biology

Most dynamic biological phenomena involve control processes in one way or another. The efforts made to explain and understand the performance of biological control systems have sometimes led to the use of optimal control theory. To speak of optimality in connection with biology is somewhat dangereous because it may give rise to teleological interpretations. However, it has been possible to model the operation of some physiological subsystems with optimal controllers (see e.g. [57]). A general review and discussion of these problems appears in a recent article [58]. There is a principal difference in the way optimal control theory is used in engineering and biology. In the former the studies aim at synthesis and in the latter they deal with analysis. Design aspects are met with when optimal control models are utilized when operating artificial assistive devices or when optimizing man machine systems. Optimal control models are being employed to an increasing extent in the analysis of human performance and decision making in general human operator systems.

2.5 Optimal controllers in economic systems

The solid foundation for model construction which natural laws provide in engineering and biology is not usually available in economic analysis. This has somewhat impeded the development of quantitative dynamic optimization research although dynamics and optimization are inherent characteristics of many economic systems. Not until recently have control theorists and mathematical economists started to co-operate and found that they have a common field of interest in dynamic decision problems. This has inspired intensive research in both East and West which is manifested in the increasing number of symposia [68, 69], review articles [6, 12, 36, 37, 73], special issues [2, 3, 83, 114], and textbooks [7, 19] dealing with control and decision making in management and economic systems. The questions studied range from optimal advertisement strategies [108] and competition between enterprises [112] to the planning of optimal economic growth [35, 70] and macroeconomic stabilization [73].

In economic problems the design criteria for the controllers are often expressions reflecting some real expenses and profits which can be measured in practice [35]. As in engineering applications the cost functionals used for stabilizing controllers in economics are, however, primarily design instruments [79]. It is an essential characteristic of economic problems that the system dynamics tend to be nonlinear and the cost criteria nonquadratic. Moreover, multiple controllers with different information structures are frequently involved [76, 98].

There has been a lot of discussion on the role of control and systems theory in economics (see. e.g. [8, 31]). Critical comments have been made e.g. on the relevance of discussing optimal modes of action which have been obtained by using approximative quantitative models and exact performance criteria. It is clear that a direct implementation of an optimal decision policy calculated in this way is very seldom appropriate in the day-to-day management routines. Nonetheless, control theoretic methods have to be used when one wants to evaluate e.g. the best result that could be achiered in an organization evolving in time, the effects of the entrance of additional control agents, the effects of different information structures, the price worth paying for additional information, the structure of the optimal decision policy, and so on. These are examples of questions which require more or less quantitative answers. However, even questions of a qualitative nature concerning planning and decision making often need to be approached through exact quantitative analysis.

3 Nonlinear optimal control problems

Nonlinear problems are frequently met with in practical controller design situations, which has given rise to the development of new mathematical tools for handling nonlinearities. The methods presented so far are insufficient and unsatisfactory in certain cases. Contrary to the linear problems, which have neat, elegant solutions in closed form, the exact solution is obtained only for special classes of nonlinear problems. Generally one has to be content with suboptimal solutions. In this section the application of a new tool, the theory of generalized polynomial operators, to solving nonlinear optimal control problems is described [44, 47, 64]. The theory of polynomial operators and especially the local inverse theorem used here were not developed until a few years ago by HALME and ORAVA [43, 48, 49, 93, 94, 95]. The method presented is a functional analytic approach where the models of dynamic systems are interpreted as operators between time function spaces. Recently a corresponding approach using polynomial operators has also been applied to the optimal state estimation of nonlinear systems [92].

Engineering applications very often require the controller be given in the feedback form. From the computational point of view this is a crucial demand. It means that the only practically realistic approaches are analytical solution techniques. The method may be approximate but the requirement is that the initial state is an explicit parameter in the solution. The feedback form representation is also desired in certain nonlinear economic stabilization and optimal growth problems [27, 28, 70]. For example, the structure of the feedback form solution might be of interest although it would in principle be possible to continuously solve successive open-loop problems as the computation times needed are minimal in relation to the time scale of the economic system itself.

There is an extensive body of literature on the purely numerical solutions to optimal nonlinear controllers or correspondingly to nonlinear TPBVP's, which arise from the use of the maximum principle in these problems (see e.g. [38, 87, 96, 99]). These methods have the disadvantage that the solution procedure has to be repeated for every new initial state because of the open-loop representation of the resulting solution. However, these numerical methods should not be ignored in controller design because there are many special applications even in engineering where an open-loop strategy is sufficient, see e.g. [77].

Among the analytic methods of treating the problem are different linearization techniques and methods that restrict the solutions to other fixed configurations and employ parameter optimization (see e.g. [75, 81, 97, 105, 106]). A wide class of methods is based on power series expansions of the solution. The best known of these are perturbation methods where a small parameter ϵ is associated with the nonlinearity and the solution is represented in a power series with respect to ϵ. The parameter is either inherent in the system or is artificially introduced for computational reasons. In the nonlinear regulator problem this method has been used for the solution of the Hamilton-Jacobi equations (see e.g. [13, 40, 72, 115]) and for the solution of the TPBVP obtained via the maximum principle by expanding the state or the co-state equation in a power series with respect to ϵ [72, 90, 91]. The perturbation approach usually results in the solution of successive differential or partial differential equations for the coefficients in the power series. Additional examples of analytical successive approximation methods are found in references [1, 23, 74].

A new power series approach yielding a feedback form solution to a general class of nonlinear TPBVP's occurring in optimal controller design is described in the following sections. First nonlinear TPBVB's are considered and then examples of different design problems are given. The original articles, in which this technique was studied in more detail, are references [44, 47, 64]. The method is based on transforming first the basic form of the TPBVP in question into an equivalent integral equation in the continuous-time case and correspondingly into a summation equation in the discrete-time case by using Green's functions and then solving this equation. This equivalent form is an operator equation over the solution function space and the solution is obtained by inverting the operator. The inverse is found for problems where the operator is of analytic or polynomial type by applying a local inverse theorem concerning polynomial operators.

3.1 Solution by polynomial operators

This presentation of the method for solving nonlinear control problems deals with the discrete-time case which has so far received minor attention in the literature even though in practice controller design is often based on discrete-time models. Our techniques apply equally well to the corresponding continuous time problems [44].

Let the nonlinear TPBVP related to the dynamic optimization problem in question be given in the following normal form

$$y(k+1) - y(k) = F(y(k+1), y(k)) \tag{1}$$

$$My(k_o) + Ny(k_f) = c \tag{2}$$

where the given time interval is $\{k_o, k_o+1, \ldots, k_f\}$ and the state y is an n-vector, M and N are given matrices, c is a given vector and F is an analytic function having a power series representation about the origin

$$F(y(k+1), y(k)) = Ay(k+1) + By(k) + [\text{higher order terms}]. \tag{3}$$

Now, the original problem (1). . .(2) can be transformed into a summation equation, provided the set $\{A, B, M, N\}$ is boundary compatible, i.e. the corresponding linear problem has a unique solution for all c. Applying Green's functions the equivalent representation for the problem is

$$y(k) - \sum_{j=k_o}^{k_f-1} G(k, j; k_o, k_f)[F(y(j+1), y(j)) - Ay(j+1) - By(j)]$$
$$= H(k; k_o, k_f)c \tag{4}$$

where Green's functions G and H are defined by using the state transition matrix of the linearized problem (see [38, 47, 64]). When F is given by a polynomial operator then the left hand side of Equation (4) also has a polynomial structure and (4) can be expressed in the following operator equation form

$$P(y)(k) = H(k; k_o, k_f)c. \tag{5}$$

This operator P can be represented by a power series in a region around the origin of the time function space $(\mathbf{R}^n)^{\{k_o, \ldots, k_f\}}$ and the inverse theorem of polynomial operators can be applied to solve Equation (5). The inverse theorem gives the solution closest to the origin which is also the only one that goes to zero together with c. This is the desired feasible solution when the boundary value problem (1). . .(2) represents the canonical equations of an optimal regulator problem. The solution of (5) is obtained in the series form

$$y(k) = \sum_{p=1}^{\infty} y^{(p)}(k) \tag{6}$$

where the terms $y^{(p)}$ depend on the homogeneous components H_p of the operator P, which are again easily expressed by the aid of the homogeneous components E_k of F. The terms are then given by the following recursive formulas:

$$y^{(1)}(k) = H(k; k_o, k_f)c \tag{7}$$

$$y^{(2)}(k) = \sum_{j=k_o}^{k_f-1} G(k, j; k_o, k_f)E_2(y^{(1)}(j+1), y^{(1)}(j)) \tag{8}$$

$$y^{(3)}(k) = 2\sum_{j=k}^{k_f-1} G(k, j; k_o, k_f)\mathrm{pf}(E_2)((y^{(1)}(j+1), y^{(1)}(j)), (y^{(2)}(j+1), y^{(2)}(j)))$$
$$+ \sum_{j=k_o}^{k_f-1} G(k, j; k_o, k_f)E_3(y^{(1)}(j+1), y^{(1)}(j)) \tag{9}$$

etc.

The notation $\mathrm{pf}(E_2)$ appearing above means the polar form of the homogeneous component E_2. Its definition is given e.g. in [64].

The theory of polynomial operators also provides us with methods for approximating the region of convergence of the solution series (6). However, these aspects will not be treated here (see [44]).

The feedback form solution can easily be obtained from the series (6) since in optimal regulator problems the c vector in the original TPBVP (1) ... (2) is given by the initial condition. Note that the y vector of the general formulation now corresponds to the state-costate vector of the Hamiltonian system. The initial state becomes a parameter of the solution as is desired because c stands in an explicit position in the first order term (7) which again goes recursively to the subsequent terms of the power series. The optimal feedback law is found by replacing the initial time k_o by the current time k, i.e. also the initial state $x(k_o)$ by the current time state $x(k)$, and considering the solution at time points k and $k + 1$. When the discrete maximum principle [25] is applied the optimal control $u(k)$ is given as a function of the costate $p(k+1)$.

The components of the terms of the solution series corresponding to $p(k+1)$ are also expressed as functions of $x(k)$. Inserting the approximation of $p(k+1)$ to the equation of the optimal control in question gives a suboptimal controller for the problem.

The structure of this suboptimal power series solution for a nonlinear regulator is such that although additional higher degree terms are included the lower degree terms of the controller remain unchanged, i.e. the solution is correct up to the degree considered. Further discussion on the effects of truncation are found in [44].

The following sections demonstrate the applicability of this approach in some specific examples of important nonlinear controller design problems. Primary interest is paid to the presentation of the development of the polynomial structure of the related TPBVP's.

3.2 Quadratic design with nonlinear dynamics

Consider a system with nonlinear dynamics of the following form

$$x(k+1) - x(k) = A(x(k))x(k) + B(x(k))u(k) \tag{10}$$

where A and B are matrix functions of the state vector x. (Note that they do not correspond to A and B in Equations (3) and (4)). The assumed nonlinear system is of the general type, where the only restriction is that the control appears linearly. The performance index is the standard quadratic function

$$J(x_o, u) = \frac{1}{2} \sum_{k=k_o}^{k_f} [x(k)^T Q x(k) + u(k)^T R u(k)] \tag{11}$$

with $Q \geqslant 0$, $R > 0$ and free final state $x(k_f)$. Application of the maximum principle yields the following equations for the state and costate vectors

$$\begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} - \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} A(x(k))x(k) + B(x(k))R^{-1}B(x(k))^{\mathrm{T}}p(k+1) \\ Qx(k) + C(x(k), p(k+1))^{\mathrm{T}}p(k+1) \end{bmatrix} \tag{12}$$

where

$$C(x, p) = -\frac{\partial}{\partial x}\,[A(x)x + B(x)u] \text{ with } u = R^{-1}B(x)^{\mathrm{T}}p \tag{13}$$

The boundary conditions are given by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(k_o) \\ p(k_o) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x(k_f) \\ p(k_f) \end{bmatrix} = \begin{bmatrix} x_o \\ 0 \end{bmatrix} \tag{14}$$

and the optimal control is obtained from equation

$$u^{\star}(k) = R^{-1}B(x(k))^{\mathrm{T}}p(k+1). \tag{15}$$

Sometimes the minimum principle is used, resulting in equivalent equations but with different signs in the above representation. The TPBVP (12) ... (14) is of the desired polynomial type if the matrix functions A and B are originally polynomials or if they can be expanded into a power series form. For a continuous-time formulation of this problem in the general time-variant case see [44].

3.3 Bilinear systems

Recently, increasing interest has been shown in the control of the so-called bilinear systems. These form a special subclass of nonlinear problems. A bilinear model is linear separately with respect to state and control; the nonlinearity is due to the cross product terms of these two variables. Besides applications in physics and biology important phenomena are modelved by bilinear equations in economic problems. Variables such as rate of change or percentage of change are more natural for use in economics than the absolute changes in magnitude, an example being the tax rate [5, 22]. The literature contains very few special notes on the design of optimal controllers for bilinear systems. Bilinear dynamics and quadratic criteria lead to nonlinear TPBVP's which are in a convenient polynomial form for solution by the proposed method. Consider, for example, a system with scalar control

$$x(k+1) - x(k) = A\,x(k) + Bu(k) + B_2\,x(k)u(k) \tag{16}$$

and the quadratic criterion (11). Equation (12) now has the following simple form with second and third degree nonlinearities [64]

$$\begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} - \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} 0 & B_1 R^{-1} B_1^{\mathrm{T}} \\ 0 & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} + \begin{bmatrix} A & 0 \\ Q & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \end{bmatrix}$$
$$+ \begin{bmatrix} B_1 R^{-1} x(k)^{\mathrm{T}} B_2^{\mathrm{T}} p(k+1) + B_2 x(k) R^{-1} B_1^{\mathrm{T}} p(k+1) \\ B_2^{\mathrm{T}} p(k+1) R^{-1} B_1^{\mathrm{T}} p(k+1) \end{bmatrix} \tag{17}$$
$$+ \begin{bmatrix} B_2 x(k) R^{-1} x(k)^{\mathrm{T}} B_2^{\mathrm{T}} p(k+1) \\ -B_2^{\mathrm{T}} p(k+1) R^{-1} x(k)^{\mathrm{T}} B_2^{\mathrm{T}} p(k+1) \end{bmatrix}.$$

3.4 Nonquadratic criteria

Nonquadratic cost functions are often encountered when the controller design is based on performance criteria which represent real costs. In practical realizations inequality constraints on the state variables are frequently met with. One way of solving such problems is to consider nonquadratic criteria which include higher order penalty terms for the state constraints.

As an example consider the following controller design problem

$$x(k+1) - x(k) = Ax(k) + Bu(k) \tag{18}$$

$$J(x_0, u) = \frac{1}{2} \sum_{k=k_0}^{k_f} [q(x(k)) + u(k)^{\mathrm{T}} Ru(k)] \tag{19}$$

with final state free, constant matrices A and B, a suitable nonnegative scalar valued function q and $R > 0$. The maximum principle yields the equations

$$\begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} - \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} Ax(k) + BR^{-1} B^{\mathrm{T}} p(k+1) \\ D(x(k))^{\mathrm{T}} - A^{\mathrm{T}} p(k+1) \end{bmatrix} \tag{20}$$

where $D(x) = \frac{1}{2} (\partial q(x)/\partial x)$ with the boundary conditions (14). The TPBVP (20), (14) has a polynomial structure and the proposed solution technique may be employed here if $q(x)$ is a polynomial or if it can be suitably approximated by a power series. The case where $q(x)$ includes only a quadratic and a quartic term is especially easy to deal with. Then the TPBVP has only third degree nonlinear terms and the homogeneous components of the operator become quite simple; moreover, the even degree terms are zero. Previous attempts to solve this problem are few in number and they only deal with the scalar control case (see [64]).

3.5 Bounded controls

In most applications the magnitudes of the control variables have upper bounds e.g. due to undesired side effects or due to the limiting ranges of the operating devices. In spite of this fact stabilizing feedback controllers are designed ignoring the effects of the control constraints. The reason is simply the serious difficulties encountered with analytic solution methods. As far as the author knows, feedback form solutions have not been presented previously. It is only in very special cases where the optimal solution is obtained by saturating the values of the corresponding unconstrained linear regulator, although this solution was once suggested to be valid in general. To illustrate the application of polynomial techniques here let us consider the linear quadratic problem (11), (18) where the components of the control vector u ($\dim u$ = r) are bounded in magnitude by

$$|u_j(k)| \leqslant 1, \qquad j = 1, 2, \ldots, r. \tag{21}$$

The necessary conditions for optimum yield

$$\begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} - \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} A & 0 \\ Q & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} + \begin{bmatrix} B \operatorname{sat}(R^{-1}B^{\mathrm{T}}p(k+1)) \\ -A^{\mathrm{T}}p(k+1) \end{bmatrix} \tag{22}$$

with the boundary conditions (14). The optimal control is now

$$u^{\star}(k) = \operatorname{sat}(R^{-1}B^{\mathrm{T}}p(k+1)). \tag{23}$$

The saturation function appearing in these equations is defined componentwise by

$$\operatorname{sat}\left(\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right)_i = \begin{cases} z_i & \text{, when } |z_i| \leqslant 1 \\ \operatorname{sign} z_i, & \text{when } |z_i| > 1, \quad i = 1, 2, \ldots, n. \end{cases} \tag{24}$$

The solution difficulties are caused by this sat-function. However, the TPBVP (22), (14) can again be transformed into polynomial form by replacing the saturation function componentwise by its polynomial approximation on a given interval. The desired accuracy can be achieved by using polynomials whose degree is high enough. This approximate saturation problem can be assumed to have solutions which are close to the solution of the original problem if the saturation function approximation is done on a sufficiently large interval. Computationally the approximated problem is simple because due to the odd-symmetry of the sat-function the even terms are zero. The feedback law for the controller is obtained by applying the power series solution for the costate p in the controller equation (23) (see [47, 64]).

4 Problems with multiple controllers and game theoretic design

Problems with multiple controllers can arise in different ways, the primary source being decentralized control. Two different types of situation can be found. One of them is the decentralized formulation of a complex system by decomposition so that a set of interacting local subcontrollers is hierarchically coordinated by a higher level supervisory controller [86]. This is either a description of the mathematical procedure used to solve the global problem or the structure of the controller configuration of the decision making system in practice by which one wishes to realize the optimal control of the overall system. It is often hoped that the considerations could be simplified by breaking the overall problem into a number of simple subproblems. In spite of the decentralized representation the aggregate problem usually has to be solved by the co-ordinator who is assumed to possess a complete description of the system [11, 26, 86].

In the other situation the multicontroller structure is given a priori with the controllers having non-identical information and possibly different objectives. This is what is most often meant by a decentralized problem [4, 54]. The appearance of more than one independent control agent is typical of economic systems (see [7, 73, 76, 98]). Market competition between enterprises and macro-economic stabilization by independent agents controlling monetary and fiscal policy can be mentioned as examples. Many kinds of systems with interconnected subprocesses are found in engineering. Consider for example electric networks consisting of several power generator stations which have individual local controllers [54].

The performance criteria of the control agents can generally differ from each other although the overall objective is to stabilize the system. From the local point of view of one agent some of the state components and the agent's own control cost can be considered more important than the others. The problem is thus a nonzero-sum game. For the solution it is reasonable to assume the noncooperative Nash equilibrium solution concept because it gives strategies which are stable against unilateral deviations from the equilibrium. This is often a desirable property in decentralized control systems. The Stackelberg strategy is relevant when one agent has a dominating position. Cooperative solutions can offer better results to the controllers but require centralized decision making which is somewhat conflicting to the original decentralized setting of the problem. Altogether, these concepts are strongly bound to the specific problem in question. If it is only a case of a decentralization of the information available to the controllers with one common performance measure then we are dealing with a dynamic team problem.

Another source of multiple controller problems is the design and solution techniques where a fictitious auxiliary control agent is introduced, game theoretic worst case design being an important and illustrative example. The aim is to design a controller which would preserve good performance even in the presence of unpredictable disturbances. Worst case design is based on the solution of a game between the controller and an anticontroller representing a fictitious intelligent disturbance [61].

The saddle point condition in optimization theory has also given rise to game theoretic interpretations where the Lagrange multiplier is considered a fictitious player. Moreover, there are a number of solution algorithms for hierarchical coordination and decentralized control problems where the discrepancy or interaction term is given the role of an antagonizing fictitious controller [32, 82, 86].

In the following sections we discuss the formulation of design problems with two controllers from the game theoretic point of view. Special attention is paid to the role of the controllers' information structures. Some new solution algorithms to general linear quadratic difference games are dealt with and a game theoretic worst case design method is described. A more detailed treatment of these questions including references to previous literature is found in the author's articles [59...61].

4.1 Nonzero-sum difference games

In Section 2.2 the solutions of nonzero-sum differential games were already discussed. Here the definition of the necessary concepts in connection with two-player games shall be shortly described. It is illustrative first to consider an example of a static minimization game, shown in Figure 1 in the lines of [109], where the players' costs J_1 and J_2 are convex with respect to their scalar controls u and v corresponding to players 1 and 2 respectively. The broken lines denoted by u° and v° going through the individual minimal points 0_1 and 0_2 represent the locus of optimal controls of one player for fixed values of the control of the other player. The intersection N of these lines, if it exists, determines the Nash equilibrium solution. The Stackelberg strategy S_1 where player 1 is the leader is given by the point of tangency between the constant J_1 contours and the v° line i.e. by the point of v° which gives the smallest cost to player 1. The point S_2 correspondingly represents the Stackelberg solution where player 2 is the leader and player 1 the follower. The cooperative Pareto optimal solution set is the locus of tangency points between the constant cost contours which

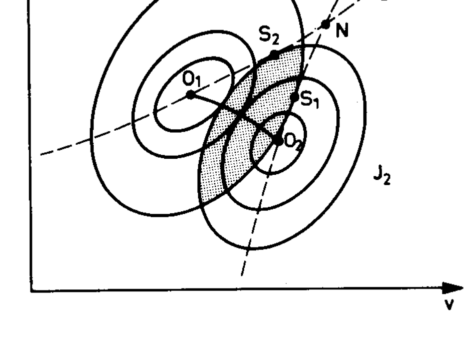

Figure 1. A static two-player game.

connects 0_1 and 0_2. The shaded area represents the intersection of solution sets which give smaller costs to the players than their individual Stackelberg leader's strategies. That part of the Pareto optimal set which lies in the shaded area can be called the set of negotiation solutions. The cooperative solutions are interesting from the point of view of general decision making problems but in controller design mainly noncooperative situations are considered because of their decentralized character and because they are conceptually clear and have straightforward solutions.

In reference [60] discrete-time deterministic problems are treated and we first define the information $z_i^{(k)}$ of the past and present values of the state vector that player i has access to at stage k. The *perfect memory* information structure where all the past states are available is expressed for each $k \epsilon K$ by

$$z_i^{(k)} = \{x(0), x(1), ..., x(k)\} \tag{25}$$

where $K = \{0, 1, ..., N-1\}$ is the set of time points of the interval considered. In the *zero-memory* case only the current time value of the state is known, that is

$$z_i^{(k)} = \{x(k)\}. \tag{26}$$

The *open-loop* information structure means that only the initial state is available:

$$z_i^{(k)} = \{x(0)\} \quad \text{for all } k \epsilon K. \tag{27}$$

The decisions of the players depend on the available information at different stages and the control laws $\gamma_i^{(k)}(z_i^{(k)})$ are picked from the given class of admissible strategies at each stage. To distinguish between controls depending on the perfect memory and the zero-memory information the former are called *closed-loop* and the latter *feedback* strategies. The pair of strategy sequences $((\gamma_1^{(0)\star}, ..., \gamma_1^{(N-1)\star}), (\gamma_2^{(0)\star}, ..., \gamma_2^{(N-1)\star}))$ is a *Nash solution* to the game if the following inequalities hold for all admissible $\gamma_i^{(k)}$, $k \epsilon K$

$$\begin{aligned} & J_1[(\gamma_1^{(0)\star}, ..., \gamma_1^{(N-1)\star}), (\gamma_2^{(0)\star}, ..., \gamma_2^{(N-1)\star})] \\ & \leqslant J_1[(\gamma_1^{(0)}, ..., \gamma_1^{(N-1)}), (\gamma_2^{(0)\star}, ..., \gamma_2^{(N-1)\star})] \end{aligned} \tag{28}$$

$$\begin{aligned} & J_2[\gamma_1^{(0)\star}, ..., \gamma_1^{(N-1)\star}), (\gamma_2^{(0)\star}, ..., \gamma_2^{(N-1)\star})] \\ & \leqslant J_2[(\gamma_1^{(0)\star}, ..., \gamma_1^{(N-1)\star}), (\gamma_2^{(0)}, ..., \gamma_2^{(N-1)})]. \end{aligned} \tag{29}$$

In practice the stagewise definition of the solution is often more important. Instead of the two inequalities (28) and (29) for the strategy sequences the stagewise definition requires that the corresponding inequalities are satisfied at each stage with controls at other stages being fixed to the optimal strategies. The stagewise definition of solutions is convenient from the computational point of view because recursive dynamic programming type techniques are available. For the definition of the stagewise Nash and Stackelberg solutions see [60].

4.2 Solution algorithms for general linear-quadratic problems

The standard linear-quadratic two player difference game has been studied extensively in the literature. Recently, interest has primarily been paid to uniqueness questions and to the structure of the solution strategies while important computational questions have remained untreated (see e.g. [16...18]). Here we shall consider the general quadratic case with all the cross terms included and review the solution algorithms developed for these games with different information structures [59, 60].

Consider the linear system

$$x(k+1) = A(k)x(k) + B_1(k)u(k) + B_2(k)v(k), \quad x(0) = x_o \tag{30}$$

where the state x is an n-vector, the control of player 1, u, is a p-vector and the control of player 2, v, is a q-vector, the time varying matrices A, B_1 and B_2 being of suitable dimensions. The performance measure for player i is given by

$$\begin{aligned} J_i = {} & \tfrac{1}{2}x^T(N)S_i x(N) + \tfrac{1}{2}\sum_{k=0}^{N-1} [x^T(k)Q_i(k)x(k) + 2x^T(k)M_{i1}(k)u(k) \\ & + 2x^T(k)M_{i2}(k)v(k) + u^T(k)R_{i1}(k)u(k) + 2u^T(k)N_i(k)v(k) \\ & + v^T(k)R_{i2}(k)v(k)] \quad , \qquad i = 1, 2 \end{aligned} \tag{31}$$

where S_i, $Q_i(k), R_{12}(k), R_{21}(k) \geqslant 0$ and $R_{11}(k), R_{22}(k) > 0$ for all k. These general quadratic criteria are encountered when for example the performance measures of the players are defined in terms of the output of the controlled system.

4.2.1 Open-loop solutions

When determining the open-loop solutions the cost criteria have to be expressed in terms of the initial state vector. This is accomplished by introducing augmented vectors

$$\bar{x} \triangleq [x^T(1) \mid x^T(2) \mid \dots \mid x^T(N)]^T \tag{32}$$

$$\bar{x}_o \triangleq [x^T(0) \,\vdots\, x^T(0) \,\vdots\, \ldots \,\vdots\, x^T(0)]^T \tag{33}$$

$$\bar{u} \triangleq [u^T(0) \,\vdots\, u^T(1) \,\vdots\, \ldots \,\vdots\, u^T(N-1)]^T \tag{34}$$

$$\bar{v} \triangleq [v^T(0) \,\vdots\, v^T(1) \,\vdots\, \ldots \,\vdots\, v^T(N-1)]^T \tag{35}$$

by which it is possible to transform the problem into a static form depending only on x_o. Both the Nash and the Stackelberg strategies to this game are linear functions of the initial state given in the augmented form by

$$\bar{u}^\star = -\bar{H}_u \bar{x}_o \tag{36}$$

$$\bar{v}^\star = -\bar{H}_v \bar{x}_o \tag{37}$$

where the $\bar{H}_u$ and $\bar{H}_v$ matrices can be obtained from the same type of formulas for both solution concepts [60]. The difference is that in the Stackelberg case certain additional terms enter the definition of the auxiliary matrices which are related to the player being the leader. A corresponding augmented vector representation for the Nash open-loop strategy in the more simple standard linear-quadratic case has already been presented earlier [17]. However, the practical applicability of solution algorithms which involve operations such as inversions of augmented matrices decreases when the number of stages in the interval of play increases. It would therefore be useful to have the solutions in a form where high dimensional matrix equations are transformed into a series of low dimensional equations. The author has shown [60] that such a representation is indeed available for the Nash open-loop solution. No equally simple recursive procedure is, however, found for the Stackelberg open-loop solution. The low dimensional algorithm for the Nash open-loop strategy gives the solutions first stage by stage in a feedback form

$$u^\star(k) = -H_u(k)\, x(k) \tag{38}$$

$$v^\star(k) = -H_v(k)\, x(k) \tag{39}$$

where the feedback gains $H_u(k)$ and $H_v(k)$ are defined in terms of matrices $P_1(k+1)$ and $P_2(k+1)$ which are again obtained by solving recursively a couple of asymmetric Riccati-type matrix difference equations

$$\begin{aligned} P_1(k) &= Q_1(k) - M_{11}(k)\, H_u(k) - M_{12}(k)\, H_v(k) \\ &+ A^T(k) P_1(k+1)[A(k) - B_1(k) H_u(k) - B_2(k) H_v(k)] \end{aligned} \tag{40}$$

$$P_2(k) = Q_2(k) - M_{21}(k)\,H_u(k) - M_{22}(k)\,H_v(k)$$

$$+ A^{\mathrm{T}}(k)\,P_2(k+1)[A(k) - B_1(k)\,H_u\,(k) - B_2(k)\,H_v(k)] \tag{41}$$

with $P_{\mathrm{i}}(\mathrm{N}) = S_{\mathrm{i}}$, i = 1, 2.

The solution algorithm proceeds stagewise backwards so that at a stage k the gains $H_u(k)$ and $H_v(k)$ are first determined using $P_1(k+1)$ and $P_2(k+1)$ which are known from the preceding stage. Then the gains are used to obtain $P_1(k)$ and $P_2(k)$ from the difference equations (40) and (41). When the feedback gains have been computed in this manner for the whole interval the state transition matrix is obtained from a forward recursion and used to yield the open-loop form representation for the strategies.

4.2.2 Feedback solutions

The feedback solutions for a game are probably the most important ones from the practical point of view. For example in the design of decentralized multiple controller systems it is natural to assume that at each stage the optimization of controls is solely based on the current-time values of the state vector. Linear strategies of the form (38) ...(39) are again obtained for both the Nash and the Stackelberg solutions. In this case the related difference equations are symmetric:

$$P_1(k) = Q_1 - M_{11}H_u - H_u^{T}M_{11}^{\mathrm{T}} - M_{12}H_v - H_v^{\mathrm{T}}M_{12}^{\mathrm{T}}$$

$$+ H_u^{\mathrm{T}}R_{11}H_u + H_u^{\mathrm{T}}N_1H_v + H_v^{\mathrm{T}}N_1^{\mathrm{T}}H_u + H_v^{\mathrm{T}}R_{12}H_v \tag{42}$$

$$+ [A - B_1H_u - B_2H_v]^{\mathrm{T}}P_1(k+1)[A - B_1H_u - B_2H_v]$$

and

$$P_2(k) = Q_2 - M_{21}H_u - H_u^{\mathrm{T}}M_{21}^{\mathrm{T}} - M_{22}H_v - H_v^{\mathrm{T}}M_{22}^{\mathrm{T}}$$

$$+ H_u^{\mathrm{T}}R_{21}H_u + H_u^{\mathrm{T}}N_2H_v + H_v^{\mathrm{T}}N_2^{\mathrm{T}}H_u + H_v^{\mathrm{T}}R_{22}H_v \tag{43}$$

$$+ [A - B_1H_u - B_2H_v]^{\mathrm{T}}P_2(k+1)[A - B_1H_u - B_2H_v]$$

with $P_{\mathrm{i}}(\mathrm{N}) = S_{\mathrm{i}}$, i = 1, 2.

The presented algorithms have been developed into a form where the Nash and Stackelberg feedback strategies can be determined using the same procedures. Only some additional terms enter the formulas defining the gain matrices $H_u(k)$ and $H_v(k)$ when the Stackelberg solution is considered [60].

4.2.3 Open-closed solutions

The information structures of the control agents are often different, e.g. in economic problems, so that we have to consider open-closed strategies. So far solution algorithms for these problems have received minor attention in the literature, probably due to the complexity of these problems. Suppose now that player 1 has access to open-loop information and player 2 to perfect memory information. The solution procedure is initiated by deriving first the strategy of player 2 when the control of player 1 is fixed to an arbitrary open-loop strategy. The form of the optimization problem faced now by player 2 when choosing his closed-loop strategy will be the same even if he plays a Nash strategy or if he plays Stackelberg follower's strategy. The Stackelberg problem where the leader has perfect memory information is not considered because the solution cannot be found using the stagewise solution techniques. The author has shown [60] that the closed-loop optimal control of player 2 which minimizes J_2 with a fixed open-loop strategy for player 1 is the following affine feedback law

$$v^{\star}(k) = -G_2^{-1}(k)F_2(k)x(k) + t(k) \tag{44}$$

where $t(k)$ depends on the initial state vector through the open-loop strategy of player 1. The vector $t(k)$ and the matrices $G_2(k)$ and $F_2(k)$ are defined in terms of $P_2(k+1)$ which satisfies the symmetric matrix Riccati difference equation

$$\begin{aligned} P_2(k) = {} & Q_2(k)+A^{\mathrm{T}}(k)P_2(k+1)A(k)-[M_{22}^{\mathrm{T}}(k)+B_2^{\mathrm{T}}(k)P_2(k+1)A(k)]^{\mathrm{T}} \\ & \cdot[R_{22}(k)+B_2^{\mathrm{T}}(k)P_2(k+1)\,B_2(k+1)]^{-1}\,[M_{22}^{\mathrm{T}}(k)+B_2^{\mathrm{T}}(k)P_2(k+1)A(k)] \end{aligned} \tag{45}$$

with $P_2(\mathrm{N}) = S_2$.

It may be noted that in this form the affine feedback law obtained resembles the solution of the classical linear-quadratic tracking problem of regulator theory.

When deriving the open-loop strategy of player 1 the current time state has to be eliminated and the problem must again be transformed into the static form by augmentation techniques in such a way that the structure of player 2's strategy is suitably incorporated. A linear augmented form solution of the type (36) is obtained for the Nash and Stackelberg leader's strategy. The solution of the whole problem is thus partly expressed by augmented form equations. As in the open-loop problem it has been possible to develop recurrence equations replacing the augmented matrices of the Nash solution [60]. The open-loop strategy of player 2 and the $t(k)$ term depending on it in (44) is again first given in the feedback form by

$$u^{\star}(k) = -H_u(k)\,x(k) \tag{46}$$

$$t^{\star}(k) = -H_t(k)\,x(k) \tag{47}$$

where the feedback gains $H_u(k)$ and $H_t(k)$ are defined in terms of $P_1(k+1)$ and $P_3(k+1)$ which satisfy the pair of coupled asymmetric difference equations

$$\begin{aligned} P_1(k) &= Q_1^\circ(k) - M_{11}^\circ(k)\,H_u(k) - M_{12}^\circ(k)\,H_t(k) \\ &\quad + A^{\circ\mathrm{T}}(k)P_1(k+1)[A^\circ(k) - B_1(k)H_u(k) - B_2(k)H_t(k)] \end{aligned} \tag{48}$$

$$P_3(k) = K_2(k)H_u(k) + A^{\circ\mathrm{T}}(k)P_3(k+1)[A^\circ(k) - B_1(k)H_u(k) - B_2(k)H_t(k)] \tag{49}$$

with $P_1(\mathrm{N}) = S_1$ and $P_3(\mathrm{N}) = 0$.

This pair of equations is moreover implicitly coupled with the independent difference equation for P_2 (45). The computational algorithm proceeds here in the same manner as in the open-loop case. First the matrices P_1, P_2 and P_3 are determined backwards in time and secondly a forward recursion is solved to obtain the open-loop representation for $u^\star(k)$ and $t^\star(k)$.

The solution to this problem presented here is the only one of practical interest although it is not unique due to the closed-loop information structure of one of the players. Uniqueness of this solution would be guaranteed if a random disturbance term were included in the formulation of the system dynamics. [17].

4.3 Periodic information structures

Up to these days the literature has been lacking almost completely the treatment of multiple controller problems with periodicity in the information structures. Successive economic planning periods give an example of a practical situation where problems with periodic information structures arise in a natural way [73]. Other examples include coordination procedures for large organizations which result in corresponding problems [26].

The two periodic information structures of main importance are called *periodic open-loop* and *periodic perfect memory* [61]. In the former case the decision maker knows at each stage only the value of the state vector at the beginning of the current period. Formally we have for each $k \in \{0, 1, \ldots, \mathrm{N}-1\}$

$$\begin{aligned} &z_i^{(k)} = \{x(\mathrm{m}L)\} \\ &mL \leqslant k < mL + L, \quad m \in \{0, 1, \ldots, \mathrm{M}-1\} \end{aligned} \tag{50}$$

where mL is the last updating stage, L is the lentgh of the period and M is the total number of periods in the game with ML = N. Correspondingly the periodic perfect memory information is defined by

$$z_i^{(k)} = \{x(mL), x(mL+1), ..., x(k)\}$$

$$mL \leqslant k < mL + L, \quad m \in \{0, 1, ..., M-1\}. \tag{51}$$

In this case the decision maker has a perfect memory information structure within the period but the memory is periodically cleared every L:th stage. A correspondingly defined periodic zero-memory information structure would naturally not differ from the standard zero-memory case.

By using the recursive algorithms described in Sections 4.2.1 and 4.2.3 it is possible to obtain the Nash solutions to linear-quadratic games with periodic information structrures. There are two interesting problems. One is a game where both of the players have access to periodic open-loop information. The other is a game where one of the players has access to periodic open-loop and the other to periodic perfect memory information. It is assumed that the length of the period is the same for both players and that the updatings occur simultaneously. Special cases with nonidentical timing and different lengths of the periods have not been considered. The Nash solution procedures are based on the linear feedback form representations of the open-closed strategies, which make it possible to apply dynamic programming type techniques because a quadratic expression of the current time state can be developed for the optimal cost-to-go at each updating stage. It is shown in [61] how the solution of the game (30)...(31) with periodic open-open or periodic open-closed information is obtained by repeatedly solving open-loop games or open-closed games, respectively, whose duration equals the length of the information updating period. The difference between the successive games is that the terminal cost terms in the performance criteria are changed from period to period. Computationally this means that the terminal conditions of the related sets of difference equations (40)...(41) and (45), (48)...(49) above, are repeatedly updated. The updating values are again obtained from a pair of coupled difference equations which are similar to (42)...(43). The same comment on the uniqueness which was made in connection with the open-closed games applies here, too. Corresponding algorithms for Stackelberg strategies cannot be given because the feedback form representation of the open-loop and open-closed solutions are not available.

4.4 A two controller formulation for worst case design

It was established quite early that differential games provide potential tools when approaching worst case design problems. Worst case design of controllers is proposed as an alternative approach to avoid the difficulties related to the a priori distributions of disturbances which are encountered in stochastic optimal control. This method is sometimes also used when the system parameters are subject to unpredictable changes. Suboptimal worst case design methods are well motivated in many engineering and economic problems, although economists have so far paid no attention to these techniques. The basic idea is to replace the nominal optimal design by a suboptimal one which results in a less sensitive performance in cases of bad disturbances and parameter changes or which is computationally simpler to deal with than the original stochastic design problem. Game theoretic worst case design is a deterministic formulation where the disturbance is given the role of an anticontroller which is trying to oppose the actions of the controller. Thus, instead of the nominal optimal controller we consider a suboptimal controller determined by the solution of a two-player game between the control agent and the disturbance agent. If completely antagonist criteria, i.e. zero-sum games, are considered the result is usually too pessimistic. The methods previously presented to restrict the possibilities of the anticontroller and to obtain less pessimistic controllers have not lead to satisfactory practical design procedures (see [61] for a complete list of references).

The author's formulation [61] of the worst case design problem assumes that the information available to the anticontroller is restricted to the periodic open-loop type so that the disturbances entering the system are considered time functions of given length which are only periodically re-evaluated with respect to the values of the system state. The performance criteria of the controller and the anticontroller are conflicting and the design is based on the Nash strategies of the resulting nonzero-sum game with periodic open-closed strategies. In the design procedure the length of the information updating period of the anticontroller is a design parameter. It is used to restrict the possibilities of the disturbance so that the resulting suboptimal worst case controller would not be too pessimistic. In practice it is easy to use the proposed method because the design procedure can be initiated from the nominal deterministic optimal controller solution which is obtained as a limiting case of the worst case solutions.

4.5 Nonlinear nonzero-sum differential games

The computation of approximative solutions to nonzero-sum differential games is considerably more difficult than the computation of solutions to control problems or to two-person zero-sum differential games. The basic difficulty with Nash and Stackelberg solutions is that they depend on the type of strategies (e.g. open-loop or closed-loop) assumed for the players. Thus it is very difficult to know to what strategy direct iterative numerical algorithms will converge. The same difficulty is found with methods based on an iterative solution of the related Hamiltonian system because for a closed-loop problem the partial derivatives of one player's Hamiltonian with respect to the other players' control are not zero. However, an analytic power series method resembling the one proposed here has been shown to be applicable for solving the Nash feedback strategies [89]. By omitting a more general consideration of the potential applicability of the polynomial operator approach to nonlinear game problems it is sufficient to note that it can be directly applied to determine the nonlinear Pareto optimal strategies in nonzero-sum games. Consider for example an n-player game with performance criteria J_i, i = 1, 2, ... n. Members belonging to the Pareto optimal set are found by solving the following problem with a scalar cost

$$J = \sum_{i=1}^{n} \lambda_i J_i\,, \quad \lambda_i > 0, \quad \sum_{i=1}^{n} \lambda_i = 1. \tag{52}$$

for different values of λ_i. This is equivalent to the solution of standard optimal control problems. Thus all nonlinear differential game problems corresponding to the different types of nonlinear control problems discussed in Chapter 3 can be solved by the present method when Pareto optimal strategies are considered.

5 Practical design examples

In the works considered the mathematical development of new algorithms for controller design problems was only one part of the research objective. The other part, which is of equal importance, was to test the applicability of the methods in connection with practical numerical examples. The consideration of trivial scalar examples seldom reveals the real weaknesses and the computational burden of the procedures. In this chapter the solution of different controller problems is discussed using the methods described in the previous sections. The performance of the controllers obtained was studied by computer simulation and by implementing them in the direct digital control of pilot processes.

5.1 Continuous time nonlinear problems

The literature on the design of quadratic controllers for nonlinear systems contains a number of common reference examples. One of them is the continuous time optimization problem with the following second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{53}$$

and the performance index

$$J = \frac{1}{2} \int_0^{t_f} (x^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + u^2) \, \mathrm{d}t. \tag{54}$$

The main difficulty in the application of the power-series solution method to continuous time problems is the determination of the state-transition matrix which is needed in the definition of the Green's matrices. In this specific problem (53) ... (54) the state transition matrix is obtained in an analytic form and the solution algorithm proceeds in a straightforward manner [44]. Figure 2 shows the optimal trajectories for both finite and infinite regulation times. The same time invariant steady-state suboptimal solution which is obtained by the present method is found also by a perturbation method in [90]. A somewhat different result is presented in [40]. Compared with the previous methods the foremost advantages of the present techniques are that a time-variant system, time-variant weighting matrices and a finite regulation time do not cause additional difficulties, provided the necessary state transition matrix can be determined. However, all the methods for solving nonlinear problems require substantially more computational effort than is needed in the linear case.

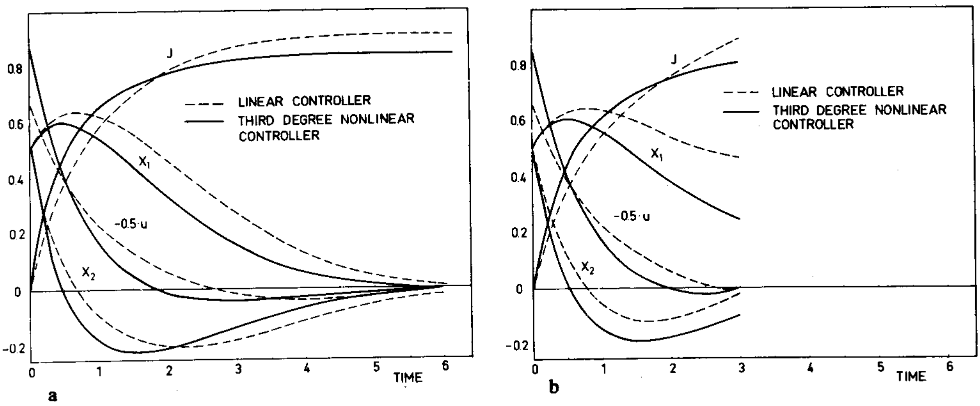

Figure 2. Suboptimal solutions with $x_1(0) = x_2(0) = 0.5$, $\epsilon = 1$ and $t_f = \infty$ (a) and $t_f = 3$ (b) [44].

5.2 Regulators for nonlinear systems

For the following pilot process examples the discrete-time formulation is used because in practice controller design is often based on discrete-time models. The difficulties that may arise with the state transition matrix in continuous-time problems are not met with in the discrete-time case, where recurrence equations can be employed for the determination of the transition matrix. In order to obtain experience with both the relevance of nonlinear design and the applicability of the present solution method the design was initiated by identifying the nonlinear model from measurement data with simulation and regression techniques.

Stabilizing steady-state controllers were studied for an analog computer simulation of a microbiological continuous cultivation process [64] and for a pilot process representing the headbox of a paper machine [47] (see Figure 3). Third degree polynomial models with two state components were used. The headbox systems has a two dimensional control consisting of the water pump and the air pump whereas there is

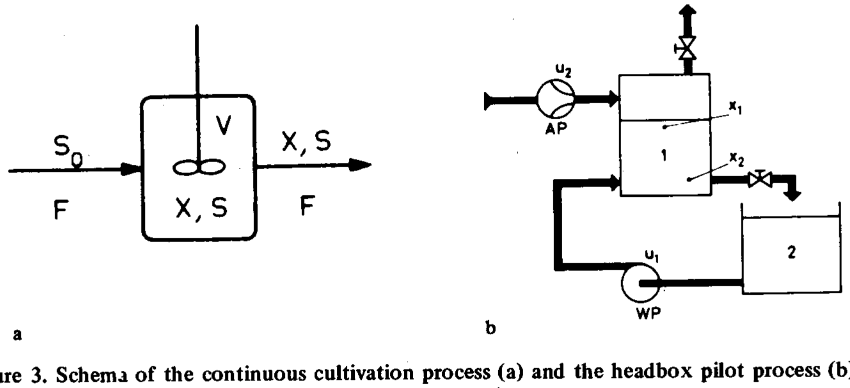

Figure 3. Schema of the continuous cultivation process (a) and the headbox pilot process (b).

only one control variable in the cultivation process, i.e. the dilution rate. The controllers were implemented in a minicomputer which performed the direct digital control of the processes. Third degree truncations of the first three terms of the power series solutions of the related TPBVP's were considered. The resulting regulators then had nonlinear terms up to degree three. Without going into the details of the solution procedure (see [47, 64]) the performance of the nonlinear regulators can be compared with the corresponding linear ones in Figures 4 and 5. The differ-

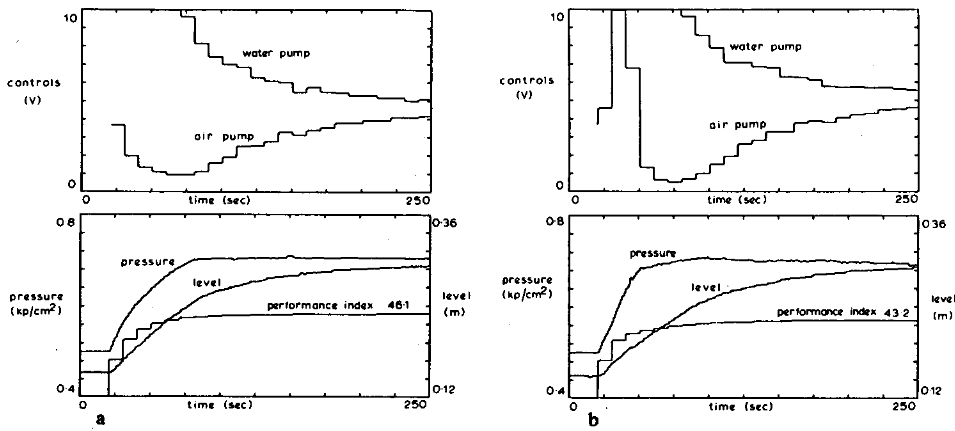

Figure 4. Compensation of a setpoint change in the headbox system with linear regulator (a) and nonlinear regulator (b) [47].

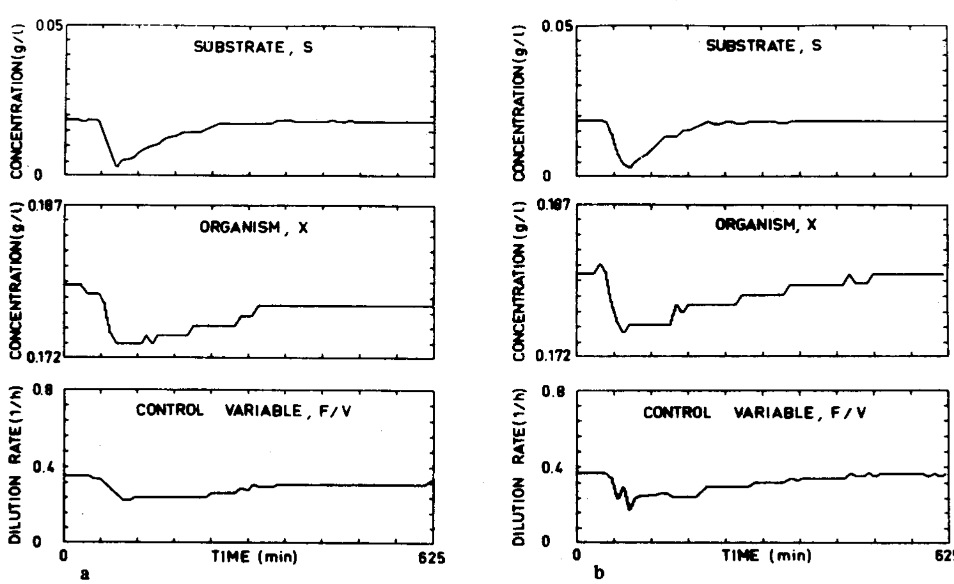

Figure 5. Compensation of a pulse disturbance in the continuous cultivation process with linear regulator (a) and nonlinear regulator (b) [64].

ences are not very striking, but the nonlinear regulators are seen to compensate more rapidly and at a lower cost than the linear regulators. Moreover, the suboptimal nonlinear regulators do not leave steady-state setpoint errors.

5.3 Control constrained regulator

The headbox pilot process was also used when studying control constrained regulators [64]. The control bounds are due to the real upper and lower limits of the driving voltages of the pumps. The design object was to minimize a quadratic criterion with a linearized discrete-time model subject to bounded controls.

In solving this problem a ninth degree polynomial approximation was used for the saturation function in Eq. (22). The solution of the TPBVP includes only odd terms, and the first, third and fifth terms were calculated. Then a fifth degree truncation was used for the costate variable in the feedback law. Compensation of a setpoint change with this suboptimal nonlinear regulator and with the so-called dual mode regulator is shown in Figure 6. The dual mode regulator means the saturated solution of the corresponding linear unconstrained problem. It is clearly seen that the nonlinear regulator employs the saturation value of the air pump only for a short period of time and it is much more rapid than the dual mode regulator. Similar results were also obtained in other compensation experiments.

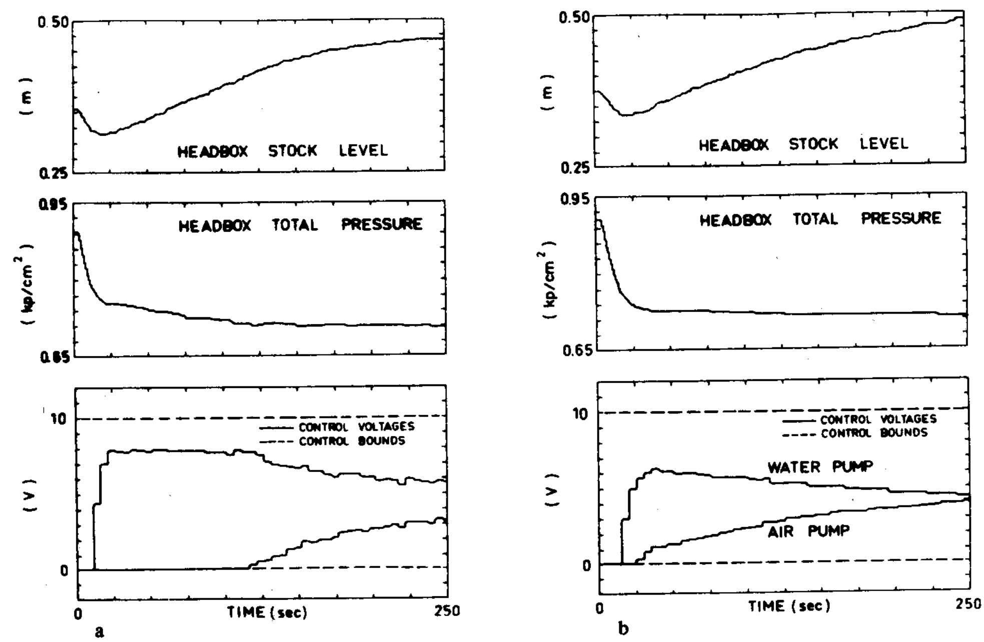

Figure 6. Compensation of a setpoint change in the headbox system with magnitude bounds on the control variables; dual-mode regulator (a), suboptimal nonlinear regulator (b) [64].

Altogether it was shown that the polynomial approximation for the sat-function here gives a good suboptimal solution both in cases where the limiting values have to be used actively and in cases where the control bounds do not become active constraints.

5.4 Worst case design of regulators

In this context practical experimentation is done only with one design procedure which is based on a multiple controller configuration, specifically the game theoretic approach to the worst case design of regulators. Decentralized controllers are not dealt with.

Although game theoretic notions are frequently suggested to be well applicable to worst case design techniques there have not been many studies dealing with the evaluation of the developed methods in connection with practical numerical design problems. Here I shall shortly describe the application of the suggested worst case method (Section 4.4) to the headbox process [61].

The worst case design approach was in this particular case motivated by the fact that large unpredictable disturbances in the pipe lines occasionally enter the water pump system in practical operating conditions. In the design scheme (see Figure 7) the disturbance was modelled by an additive input to the water pump by which the anticontroller tries to deviate the system from its operating point. The design was based on the linearized discrete-time model of the system and on quadratic criteria. The anticontroller's weight on the state deviation term was opposite to that of the controller with the controls' weights being equal. Comparisons with the nominal optimal regulator showed that in the presence of bad disturbances the performance could be significantly improved by using worst case regulators (see Figure 8). When the disturbance is zero or a zero-mean random signal the control cost for the nominal and worst case regulators are not very different, which shows that the present method does not have the common disadvantage of being too pessimistic.

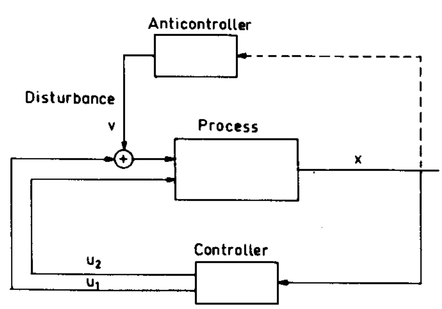

Figure 7. Schema of the worst case design configuration [61].

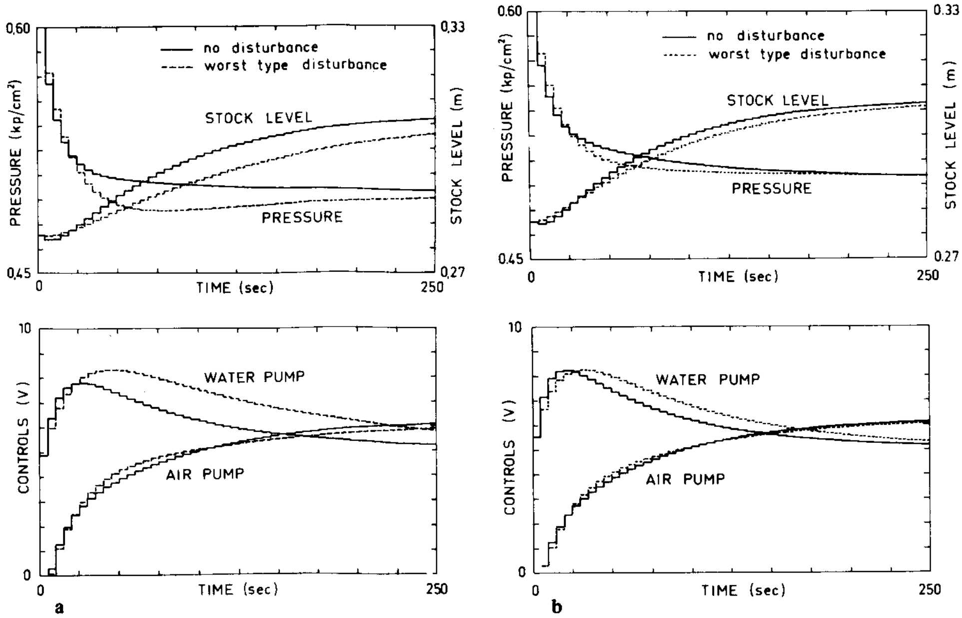

Figure 8. Compensation of a setpoint error using nominal optimal regulator (a) and worst case regulator (b) [61].

6 Concluding remarks

Nonlinearities, multiple control agents, and worst case approaches are among the problems which are of current interest in the field of optimal controller design. The present article reviews recent works of the author dealing with these problems. The main aim of the papers reviewed here, has been to develop solution algorithms. For this reason, formal questions about the existence and uniqueness of the solutions are not considered in greater detail. The presented power series solution method for nonlinear optimal control problems and the recursive algorithms for two-player difference games are computationally straightforward although in both cases the computer programming becomes a rather heavy task. This is a general feature of solution algorithms for this kind of problems. The increasing number of terms and components which have to be handled in the program restricts the use of the presented power-series technique to low-dimensional problems. The examples considered do however demonstrate that the developed nonlinear and game theoretic methods can be successfully employed in practical controller design. Compared with the previous methods the greatest improvements are seen in the results of the examples dealing with the control constrained problem and the worst case design problem. The selection of weights in the performance criteria remains an important but difficult problem related to the design of stabilizing optimal controllers in practice, which has not been considered here.

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References

1. AL'BREKHT, E. G., On the optimal stabilization of nonlinear systems. J. Appl. Math. and Mech. (PMM) **25**(1961)5, p. 1254...1266.
2. Annals of Economic and Social Measurement. Special issues on control theory applications: macroeconomics ... microeconomics. **5**(1976)2, 3.
3. Annals of Economic and Social Measurement. Special issue on stochastic control applications. **4**(1975)2.
4. AOKI, M., On decentralized linear stochastic control problems with quadratic cost. IEEE Trans. Autom. Control **AC-18**(1973)3, p. 243...250.
5. AOKI, M., Some examples of dynamic bilinear models in economics. In: Variable structure systems with application to economics and biology, ed. RUBERTI, A. and MOHLER, R.R. Berlin, Springer Verlag, 1975, p. 163...169.
6. AOKI, M., Stochastic control in economic theory and economic systems. IEEE Trans. Autom. Control **AC-21**(1976)2, p. 213...220.
7. AOKI, M., Optimal control and system theory in dynamic economic analysis. New York, North-Holland Publ. Co., 1976.
8. AOKI, M. and LEIJONHUFVUD, A., Cybernetics and macroeconomics: a comment. Econ. Inq. **14**(1976), p. 251...258.
9. ATHANS, M., On the design of p–i–d controllers using optimal linear regulator theory. Automatica **7**(1971)5, p. 643...647.
10. ATHANS, M., The role and use of the stochastic linear-quadratic-Gaussian problem in control system design. IEEE Trans. Autom. Control **AC-16**(1971)6, p. 529...552.
11. ATHANS, M., Survey of decentralized control methods. Ann. Econ. and Soc. Meas. **4**(1975)2, p. 345...355.
12. ATHANS, M. and KENDRICK, D., Control theory and economics: a survey, forecast, and speculations. IEEE Trans. Autom. Control **AC-19**(1974)5, p. 518...524.
13. BALDWIN, I.F. and SIMS WILLIAMS, I.H., The use of method of perturbations in the synthesis of closed-loop optimal laws for non-linear systems. Automatica **5**(1969)3, p. 357...367.
14. BARNETT, S., Some topics in algebraic systems theory: a survey. Int. J. Control **19**(1974)4, p. 669...688.
15. BASAR, T., On the relative leadership property of Stackelberg strategies. J. Optimiz. Theory and Appl. **11**(1973)6, p. 655...661.

16. BASAR, T., A counterexample in linear-quadratic games: existence of nonlinear Nash solutions. J. Optimiz. Theory and Appl. **14**(1974)4, p. 425...430.

17. BASAR, T., Nash strategies for m-person differential games with mixed information structures. Automatica **11**(1975)5, p. 547...551.

18. BASAR, T. and SELBUZ, H., Properties of Nash solutions of a two-stage nonzero-sum game. IEEE Trans. Autom. Control **AC-21** (1976)1, p. 48...54.

19. BENSOUSSAN, A., HURST, E.G. and NÄSLUND, B., Management applications of modern control theory. New York, North-Holland, Publ.Co, 1974.

20. BERKOVITZ, L.D., A survey of differential games. In: Mathematical theory of control, ed. BALAKRISHNAN, A.V. and NEUSTADT, L.W., New York, Academic Press, 1967.

21. BLOMBERG, H. and YLINEN, R., An operator algebra for analysis and synthesis of feedback and other systems. Otaniemi 1976. Helsinki University of Technology, Systems Theory Laboratory, Report B 31.

22. BROCKETT, R.W., Volterra series and geometric control theory. Automatica **12**(1976)2, p. 167...176.

23. BUKOV, V.N., Approximate solution of the nonlinear problem of analytical control design for the criterion of generalized work. Eng. Cybern. **13**(1975)2, p. 146...150.

24. CALOVIC, M.S., Proportional-integral-derivative realization of optimal-quadratic regulators. Proc. IEE **121**(1974)11, p. 1441...1443.

25. CANON, M.D., CULLUM, C.D. Jr. and POLAK, E., Theory of optimal control and mathematical programming. New York, Mc Graw-Hill, 1970.

26. CHONG, C.Y. and ATHANS, M., On the periodic coordination of linear stochastic systems. Automatica **12**(1976)4, p. 321...335.

27. CHOW, G.C., On the control of nonlinear econometric systems with unknown parameters. Econometrica **44**(1976)4, p. 685...695.

28. CHOW, G., An approach to the feedback control of nonlinear econometric systems. Ann. Econ. and Soc. Meas. **5**(1976)3, p. 297...309.

29. CHU, K.C., Team decision theory and information structures in optimal control problems – Part II. IEEE Trans. Autom. Control **AC-17**(1972)1, p. 22...29.

30. CLEMHOUT, S., LEITMAN, G. and WAN, H.Y. Jr., Equilibrium patterns for bargaining under strike, a differential game model. Proc. 6th IFAC World Congress, Boston, USA, August 1975, paper 16.5.

31. COCHRANE, J.L. and GRAHAM, J.A., Cybernetics and macroeconomics. Econ. Inq. **14**(1976), p. 241...250.

32. COHEN, G., A game theoretic approach to decentralized optimal control. Centre d'Automatique de l'Ecole des Mines de Paris, 1974. (unpublished manuscript).

33. CSAKI, F., A survey of previous and recent trends in control engineering. Per. Polytech., Electr.Eng. **17**(1973)2, p. 99...119.

34. DA CUNHA, N.O. and POLAK, E., Constrained minimization under vector-valued criteria in finite dimensional spaces. J. Math. Anal. and Appl. **19**(1967)1, p. 103...124.

35. DUBOVSKII, S.V. and UZDEMIR, A.P., Optimality criteria and variational approaches in dynamic models of an economy. Autom. and Remote Control **35**(1974)6, p. 951...958.

36. DYUKALOV, A.N., IVANOV, Yu. N. and TOKAREV, V.V., Control theory for economic systems: I Problems of description. Autom. and Remote Control **35**(1974)5, p. 797...818.

37. DYUKALOV, A.N., IVANOV, Yu. N. and TOKAREV, V.V., Control theory for economic systems: II Qualitative features, control schemes. Autom. and Remote Control **35**(1974)6, p. 932...950.

38. FALB, P.L. and de JONG, J.L., Some successive approximation methods in control and oscillation theory. New York, Academic Press, 1969.

39. FOSS, A.S., Critique of chemical process control theory. IEEE Trans. Autom. Control **AC-18**(1973)6, p. 646...652.

40. GARRARD, W.L., Additional results on sub-optimal feedback control of non-linear systems. Int. J. Control **10**(1969)6, p. 657...663.

41. GYLLENBERG, A., HÄMÄLÄINEN, R.P. and HALME, A., Modelling of micro-biological systems for process optimization and control. Otaniemi 1975. Helsinki University of Technology, Systems Theory Laboratory, Report B 26.

42. HAKKALA, L. and BLOMBERG, H., On-line co-ordination under uncertainty of weakly interacting dynamical systems. Automatica **12**(1976)2, p. 185...193.

43. HALME, A., Polynomial operators for nonlinear systems analysis. Acta Polytech. Scand. **Ma 24**, 1972.

44. HALME, A. and HÄMÄLÄINEN, R.P., On the nonlinear regulator problem. J. Optim. Theory and Appl. **16**(1975)3/4, p. 255...275.

45. HALME, A., HÄMÄLÄINEN, R.P., Research related to process analysis and control of the Pekilo-protein process. Proc. Second National Meeting on Biophysics and Biotechnology in Finland, Espoo Finland, February 1976.

46. HALME, A., HÄMÄLÄINEN, R.P. and GYLLENBERG, A., Modelling, optimization and control of continuous fermentation processes. Tampere 1975. Tampere University of Technology, Control Engineering Laboratory, Research Report 1–1975.

47. HALME, A., HÄMÄLÄINEN, R.P., HEIKKILÄ, O. and LAAKSONEN, O., On synthetizing a state regulator for analytic nonlinear discrete-time systems. Int. J. Control **20**(1974)3, p. 497...515.

48. HALME, A. and ORAVA, J., Generalized polynomial operators for non-linear systems analysis. IEEE Trans. Autom. Control **AC-17**(1972)2, p. 226...228.

49. HALME, A., ORAVA, J. and BLOMBERG, H., Polynomial operators in non-linear systems theory. Int. J. Syst. Sci. **2**(1971)1, p. 25...47.

50. HO, Y.C., Differential games, dynamic optimization and generalized control theory. J. Optimic. Theory and Appl. **6**(1970)3, p. 179...209.

51. HO, Y.C., On the minimax principle and zero-sum stochastic differential games. J. Optimiz. Theory and Appl. **13**(1974)3, p. 343...361.

52. HO, Y.C., Decisions, control, and extensive games. Proc. 6th IFAC World Congress, Boston, USA., August 1975, paper 16.1.

53. HO, Y.C. and CHU, K.C., Team decision theory and information structures in optimal control problems – Part I. IEEE Trans. Autom. Control. **AC-17** (1972)1, p. 15...22.

54. HO, Y.C. and MITTER, S.K., ed., Directions in large-scale systems, many person optimization and decentralized control. New York, Plenum Press, 1976.

55. HO, Y.C. and SUN, F.K., Value of information in two team zero sum problems. J. Optimiz. Theory and Appl. **14**(1974)5, p. 557...571.

56. HOROWITZ, I.M. and SHAKED, U., Superiority of transfer function over state-variable methods in linear time-invariant feedback system design. IEEE Trans. Autom. Control **AC-20**(1975)1, p. 84...97.

57. HÄMÄLÄINEN, R.P., Adaptive control of respiratory mechanics. Trans. ASME, J. Dyn. Syst. Meas. Control. September 1973, p. 327...331.

58. HÄMÄLÄINEN, R.P., Optimization concepts in models of physiological systems. Proc. Third European Meeting on Cybernetics and Systems Research, Wien, Austria, April 1976.

59. HÄMÄLÄINEN, R.P., Recursive algorithms for Nash strategies in two-player difference games. Int. J. Control, to the published.

60. HÄMÄLÄINEN, R.P., Nash and Stackelberg solutions to general linear-quadratic two-player difference games. Otaniemi 1976. Helsinki University of Technology, Systems Theory Laboratory, Report B 29.

61. HÄMÄLÄINEN, R.P., Difference games with periodic information structures with application to the worst case design of regulators. Int. J. Syst. Sci., to be published.

62. HÄMÄLÄINEN, R.P., GYLLENBERG, A. and HALME, A., Systems engineering with microbiological processes. Otaniemi 1975. Helsinki University of Technology, Systems Theory Laboratory, Report C 20 (in Finnish).

63. HÄMÄLÄINEN, R.P. and HALME, A., On the dynamic models and control systems of wastewater treatment processes. Otaniemi 1974. Helsinki University of Technology, Systems Theory Laboratory, Report C 19 (in Finnish).

64. HÄMÄLÄINEN, R.P. and HALME, A., A solution of nonlinear TPBVP's occuring in optimal control. Automatica **12**(1976)5, p. 403...415.

65. HÄMÄLÄINEN, R.P., HALME, A. and GYLLENBERG, A., On the dynamics and control of a continuous fermentation process. Proc. Automation Days –75 of the Finnish Society of Automatic Control, Espoo, Finland, 1975 (in Finnish).

66. HÄMÄLÄINEN, R.P., HALME, A. and GYLLENBERG, A., A control model for activated sludge wastewater treatment process. Proc. 6th IFAC World Congress, Boston, USA, August 1975, paper 61.6.

67. IEEE Transactions Automatic Control. Special issue on linear-quadratic-Gaussian problem. **AC-16**(1971)6.

68. IFAC/IFORS International Conference on Dynamic Modelling and Control of National Economies. IEE Conference publication **101**, 1973.

69. IFAC/IFORS Second International Conderence on Dynamic Modelling and Control in National Economics, Wien, Austria, January 1977.

70. INFANTE, E.F. and STEIN, J.L., Optimal growth with robust feedback control. Rev. Econ. Stud. **XL**(1973)1, p. 47...60.

71. ISAACS, R., Differential games. New York, John Wiley & Sons, 1965.

72. JAMSHIDI, M., A feedback near-optimum control for nonlinear systems. Inf. and Control **32**, September 1976, p. 75...84.

73. KENDRICK, D., Application of control theory to macroeconomics. Ann. Econ. and Soc. Meas. **5**(1976)2, p. 171...190.

74. KRASOVSKY, A.A., A new solution to the problem of a control system analytical design. Automatica **7**(1971)1, p. 45...50.

75. KRIECHBAUM, G.K.L. and NOGES, E., Sub-optimal control of nonlinear dynamical systems via linear approximation. Int. J. Control **13**(1971)6, p. 1183...1195.

76. KYDLAND, F., Decentralized stabilization policies: optimization and the assignment problem. Ann. Econ. and Soc. Meas. **5**(1976)2, p. 249...261.

77. LAUTALA, P.A.J. and ORAVA, P.J., On short term optimal scheduling of hydroelectric power plant chains. Otaniemi 1976. Helsinki University of Technology, Control Engineering Laboratory, Report 2.

78. LIN, J.G., Maximal vectors and multi-objective optimization. J. Optimiz. Theory and Appl. **18**(1976)1, p. 41...63.

79. LIVESEY, D.A., Can macro-economic planning problems ever be treated as a quadratic regulator problem. Proc. IFAC/IFORS International Conference on Dynamic Modelling and Control of National Economics. IEE Conference publication **101**, 1973, p. 1...8.

80. LOGAN, H.L. Jr., Preferred coalitions in cooperative differential games. J. Optimiz. Theory and Appl. **13**(1974)2, p. 186...202.

81. LONGMUIR, A.G. and BOHN, E.V., The synthesis of suboptimal feedback control laws. IEEE Trans. Autom. Control **AC-12**(1967)6, p. 755...758.

82. MAGEIROU, E. and HO, Y.C., Decentralized stabilization via game theoretic methods. Proc. IFAC Symposium on Large Scale Systems Theory and Applications, Udine, Italy, June 1976, p. 467...477.

83. Management Science. Special issue on applied optimal control (1977), to be published.

84. MARSCHAK, J., Elements for a theory of teams. Managem. Sci. **1**(1955), p. 127...137.

85. MARSCHAK, J. and RADNER, R., The economic theory of teams. New Haven, Yale University Press, 1971.

86. MESAROVIC, M.D., MACKO, D. and TAKAHARA, Y., Theory of hierarchical, multilevel systems. New York, Academic Press, 1970.

87. MIELE, A., Recent advances in gradient algorithms for optimal control problems. J. Optimiz. Theory and Appl. **17**(1975)5/6, p. 361...430.

88. NASH, J.F., Non-cooperative games. Ann. Math. **54**(1951)2, p. 286...295.

89. NISHIKAWA, Y., ITAKURA, H. and KITAGAWA, H., Nonzero-sum differential games with nonlinear dynamics. Tokyo 1972. Institute of Electronics and Communication Engineers of Japan, Reports of Professional Group of Nonlinear Problems, NLP 71-18.

90. NISHIKAWA, Y., SANNOMIYA, N. and ITAKURA, H., A method for suboptimal design of nonlinear feedback systems. Automatica **7**(1971)6, p. 703...712.

91. NISHIKAWA, Y., SANNOMIYA, N. and ITAKURA, H., Power-series approximations for the optimal feedback control of noisy nonlinear systems. J. Optimiz. Theory and Appl. **19**(1976)4, p. 633...644.

92. ORAVA, J., State estimation of polynomial type non-linear differential systems. Acta Polytech. Scand. **Ma 26**, 1975.

93. ORAVA, J. and HALME, A., Derivatives of general polynomial operators. Otaniemi 1971. Helsinki University of Technology, Systems Theory Laboratory, Report B 11.

94. ORAVA, J. and HALME, A., Fixed point of polynomial type nonlinear operators. J. Optimiz. Theory and Appl. **13**(1974)2, p. 139...151.

95. ORAVA, J. and HALME, A., Inversion of generalized power series representation. J. Math. Anal. and Appl. **45**(1974)1, p. 136...141.

96. ORAVA, P.J. and LAUTALA, P.A.J., Back-and-forth shooting method for solving two-point boundary value problems. J. Optimiz. Theory and Appl. **18**(1976)4, p. 485...497.

97. PEARSON, I.D., Approximation methods in optimal control. J. Electron. and Control **13**(1962)5, p. 453...469.

98. PINDYCK, R.S., The cost of conflicting objectives in policy formulation. Ann. Econ. and Soc. Meas. **5**(1976)2, p. 239...248.

99. POLAK, E., Computational methods in optimization. New York, Academic Press, 1971.

100. Proceedings of the First International Conference on the Theory and Application of Differential Games. Amherst, Massachusetts, USA, 1969.

101. RADNER, R., Team decision problems. Ann. Math. Stat. **33**(1962)3, p. 857...881.

102. ROSENBROCK, H., State-space and multivariable theory. London, Nelson, 1970.

103. ROSENBROCK, H.H. and Mc MORRAN, P.D., Good, bad or optimal? IEEE Trans. Autom. Control **AC-16**(1971)6, p. 552...554.

104. SALAMA, A.I.A. and GAURISHANKAR, V., Optimal control of systems with a single control and several cost functionals. Int. J. Control **14**(1971)4, p. 705... 725.

105. SALMON, D.M. and KOKOTOVIC, P.V., Design of linear feedback controllers for nonlinear plants. IEEE Trans. Autom. Control **AC-14**(1969), p. 289...292.

106. SANNOMIYA, N., A method for construction of suboptimal non-linear regulators by minimax algorithm. Int. J. Control **18**(1973)2, p. 417...427.

107. SCHMITENDORF, W.E., Cooperative games and vector-valued criteria problems. IEEE Trans. Autom. Control **AC-18**(1973)2, p. 139...143.

108. SETHI, S.P., Optimal control problems in advertising. In: Optimal control theory and its applications II, ed. KIRBY, B.J. Berlin, Springer Verlag, 1974, p. 301... 384.

109. SIMAAN, M. and CRUZ, J.B. Jr., On the Stackelberg strategy in nonzero-sum games. J. Optimiz. Theory and Appl. **11**(1973)5, p. 533...555.

110. SIMAAN, M. and CRUZ, J.B.Jr., Additional aspects of the Stackelberg strategy in nonzero-sum games. J. Optimiz. Theory and Appl. **11**(1973)5, p. 613...627.

111. STACKELBERG, H. von, The theory of market economy. Oxford, Oxford University Press, 1952.

112. STARR, A.W. and HO, Y.C., Nonzero-sum differential games. J. Optimiz. Theory and Appl. **3**(1969)3, p. 184...206.

113. STARR, A.W. and HO, Y.C., Further properties of nonzero-sum differential games. J. Optimiz. Theory and Appl. **3**(1969)4, p. 207...219.

114. Transactions of ASME, Journal of Dynamic Systems, Measurement, and Control. Special issue on ecological and socio-economic systems, March 1976.

115. WERNLI, A. and COOK, G., Suboptimal control for the nonlinear quadratic regulator problem. Automatica **11**(1975)1, p. 75...84.

116. WONHAM, W.M., Linear multivariable control – a geometric approach. Berlin, Springer Verlag, 1974.

117. ZADEH, L.A., Optimality and non-scalar-values performance criteria. IEEE Trans. Autom. Control **AC-8**(1963)1, p. 59...60.

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