Analysis of the Constraint Proposal Method for Two-Party Negotiations

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Abstract

In the constraint proposal method a mediator locates points at which the two decision makers have joint tangent hyperplanes. We give conditions under which these points are Pareto optimal and we prove that under these conditions the mediator's problem has a solution, when the number of issues in the negotiation is two or any odd number greater than two. In practice, the mediator adjusts a hyperplane going through a reference point until the decision makers' most preferred alternatives on the hyperplane coincide. We give local convergence conditions for fixed-point iteration as an adjustment process. We also discuss the relationship of exchange economies and the constraint proposal method, and the possible ways of using the method.

Key words: Group decisions and negotiations, negotiation support method, Pareto optimality, existence of solution, fixed-point iteration

1 Introduction

In this paper we consider a two-party negotiation over two or more continuous issues. For example, the negotiation could be on allocating resources, such as money and labor force, between two units of a company. The purpose

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of negotiation support methods in such settings is to locate Pareto optimal points among which the decision makers (DMs), e.g., the units of a company, can choose an agreement. Ehtamo et al. (1999a) have recently formalized an interactive method for finding Pareto points by means of joint tangent hyperplanes. The method is called the constraint proposal method. In this paper we focus on three major questions: does the method produce Pareto-optimal points, does it lead to a problem that has a solution, and can the joint tangent hyperplanes be found with fixed-point iteration.

The idea of locating Pareto solutions by finding the joint tangent was first presented for oligopoly games by Ehtamo et al. (1996) and Verkama et al. (1996). Teich et al. (1995), Ehtamo et al. (1999a), and Heiskanen et al. (2001) extend the approach to negotiation settings, where an impartial mediator tries to find joint tangent hyperplanes. The method is based on the geometrical observation that under some concavity assumptions there is a jointly tangential hyperplane for the DMs' indifference contours at a Pareto optimal point.

In practice, the mediator adjusts a hyperplane going through a predetermined reference point until the DMs' most preferred alternatives on the hyperplane coincide. We show that reference points chosen from the line connecting the DMs' global optima produce Pareto optimal points, and the mediator's problem has a solution when the number of issues is either two or any odd number greater than two.

In the theory of oligopolistic markets the joint tangent can be interpreted as a mechanism according to which the members of a cartel can punish each others from deviating the joint optimum, see Osborne (1976). This idea is further generalized to a dynamic resource management problem by Ehtamo and Hämäläinen (1993) and Ehtamo and Hämäläinen (1995), where the parties safeguard themselves with linear strategies against any attempts by the other party to break an agreement. In this paper we use recent results on incentive or contract design framework by Kitti and Ehtamo (2003) to explain the convergence of fixed-point iteration in adjusting the hyperplane constraint.

From the negotiation support point of view, the main benefit of the constraint proposal method is that the DMs' utility functions do not need to be elicited. Second, the method is informationally decentralized in the sense that the DMs do not have to disclose any private information to each other. Other methods with similar properties include, e.g., the heuristic presented by Teich et al. (1996) and the Joint Gains method by Ehtamo et al. (1999b) and Ehtamo et al. (2001). These methods are based on seeking joint improvements from a tentative agreements; an approach, which was first suggested by Raiffa (1982).

The constraint proposal method is implemented in a negotiation support system RAMONA, which has been applied, e.g., to agricultural negotiations between Finnish Government and the Finnish Farmer's Union, see Teich et al. (1995). In RAMONA the hyperplane, on which the DMs are asked their most preferred points, is interpreted as a budget constraint. This interpretation relates the method to exchange economies. We shall briefly discuss the similarities and differences of exchange economies and the constraint proposal method.

The paper is organized as follows. In Section 2 we describe the mediator's problem as a system of equations to be solved and make some observations on the properties of the system. In Section 3 we study the choice of the reference point and Pareto optimality of the solution of the mediator's problem. Conditions for the existence of solution of the mediator's problem are analyzed in Section 4. Adjustment of hyperplane constraint with fixed-point iteration is studied in Section 5. In Section 6 we discuss the relationship of the constraint proposal method and exchange economies. In Section 7 we make some concluding remarks and discuss the possible ways of using the method.

2 Constraint Proposal Method

We assume that there are two DMs, a and b, who negotiate over $n \geq 2$ continuous issues. Let the real numbers $x_1, \dots, x_n$ denote the values of these issues, we also denote $x = (x_1, \dots, x_n)$. The DMs' preferences are characterized with the utility functions $u_a,\ u_b : \mathbb{R}^n \mapsto \mathbb{R}$. These are needed in the mathematical analysis, but the negotiation method itself does not require these functions to be explicitly known. In this paper we use the following assumptions on the DMs' value functions:

(A1) u_a and u_b have unique global optima at $\bar{x}^a$ and $\bar{x}^b$, respectively, and $\bar{x}^a \neq \bar{x}^b$,
(A2) u_a and u_b are continuous,
(A3) u_a and u_b are quasiconcave,
(A4) u_a and u_b are strongly quasiconcave.

The global optima are used for constructing appropriate reference points for the constraint proposal method. We assume that these optimal points are different since otherwise there would be no need to negotiate at all. The continuity of utility functions is crucial when we study the existence of solution for the mediator's problem of finding joint tangent hyperplanes.

Quasiconcavity of function u_i means that the set

$$S_i(y) = \{x \in \mathbb{R}^n : u_i(x) \geq u_i(y)\}$$

is convex for all $y \in \mathbb{R}^n$. Strong quasiconcavity of u_i means that for each x^1, $x^2 \in \mathbb{R}^n$, $x^1 \neq x^2$, we have $u_i(\lambda x^1 + (1-\lambda)x^2) > \min\{u_i(x^1), u_i(x^2)\}$ for all $\lambda \in (0,1)$. Strong quasiconcavity implies quasiconcavity and also assures the uniqueness of the global maximum, see, e.g., Bazaraa et al. (1993, Section 3.5).

The purpose of the constraint proposal method is to locate Pareto optimal solutions that are points where it is not possible to move to any other point without worsening one of the DMs value. Formally, Pareto optimality of point x^* means that there is no x for which

$$u_i(x) \geq u_i(x^*)$$

for $i = a, b$ and the inequality is strict for at least one i.

In the constraint proposal method an impartial mediator tries to locate a hyperplane going through a given reference point such that the DMs' most preferred alternatives on that hyperplane coincide. When this happens the hyperplane is tangential to the DMs' indifference curves at the point in question; see Figure 1, where the hyperplane is simply a line. If all the DMs' more preferred points are on the opposite sides of the hyperplane as in Figure 1, then the point is Pareto optimal.

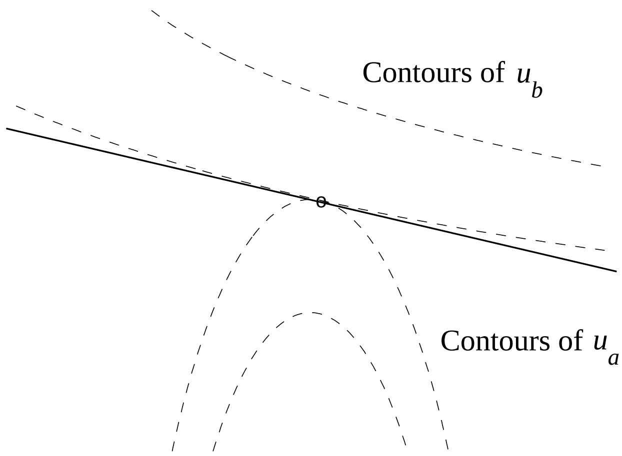

Figure 1. A Pareto optimal point and a joint tangent hyperplane.

Let us now formulate the mediator's problem mathematically. First, the mediator chooses a reference point r and defines a hyperplane

$$H(p,r) = \{x \in \mathbb{R}^n : p \cdot (x - r) = 0\}$$

going through the reference point. The normal of the hyperplane is denoted with p and $p \cdot x$ denotes the usual inner product of vectors p and x. The mediator asks the DMs to give their most preferred points on the hyperplane. These points solve

$$\max_{x \in H(p,r)} u_i(x), \;\; i = a, b. \tag{1}$$

Knowing the optimal answers the mediator then updates the hyperplane. The procedure is repeated until the most preferred points coincide within some predetermined tolerance. We shall turn back to the adjustment of the hyperplane in Section 5.

Let $X_i(p,r)$, $i = a, b$, denote the solutions to (1). The mediator's problem can be formulated as follows: for fixed r find p such that

$$X_a(p,r) \cap X_b(p,r) \neq \emptyset. \tag{2}$$

When (1) has a unique solution, i.e., X_a and X_b consist of single points, then the mediator's problem can be formulated equivalently as the following system of equations to be solved for p:

$$F(p) = x^a(p,r) - x^b(p,r) = 0, \tag{3}$$

where $x^i(p,r)$, $i = a, b$, denotes the unique solution of (1). Recall that under strong quasiconcavity the solution of (1) is unique.

By solving (3) with different reference points different Pareto solutions can be obtained. This can be done in practice, e.g., by sliding the reference point as suggested by Ehtamo et al. (1999a). Under some concavity assumptions for u_a and u_b the resulting Pareto optimal points vary lower semicontinuously as the reference point is changed, see Heiskanen et al. (2001, Theorem 5).

In this paper we assume that when solving for their most preferred points the DMs do not have other constraints than the hyperplane given by the mediator. There could be some other constraints as well, e.g., in a resource allocation problem the amounts of the resources could be limited. Nevertheless, adding the same compact and convex constraint set to the DMs' optimization problems would not affect the mathematical properties of the problem. To ease the notation we neglect all these additional constraints. See Heiskanen (2001) for the use of the constraint proposal method in negotiations with additional constraints.

Let us now make observations on the properties of F. These properties are needed in the analysis in the following sections. First, because any parallel normal vectors define the same hyperplane, F is degree zero homogeneous, i.e.,

(P1) $F(p^1) = F(\alpha p^1)$ for all $\alpha \neq 0$.

In particular, if $F(p^*) = 0$ then $F(\alpha p^*) = 0$ for all $\alpha \neq 0$, which means that the mediator's problem has at least a ray of solutions if it has one solution. This holds for both formulations (2) and (3) of the mediator's problem.

Second, since $x^i(p, r) \in H(p, r)$ for $i = a, b$, it follows that F satisfies a condition which is known as Walras' law in microeconomics literature:

(P2) $p \cdot F(p) = 0$ for all $p \neq 0$.

We shall see that Walras' law plays an important role in the analysis of the constraint proposal method. It is also a property that does not hold for the multi-party generalization of the method considered by Heiskanen et al. (2001) and Heiskanen (2001). Hence, most of the results of this paper cannot be generalized to a multi-party setting with the same techniques as used in this paper. The interpretation of Walras' law is further discussed in Section 6.

2.1 Example: Quadratic Utility Functions

Let us assume that the utility functions are of the form

$$u_i(x) = -\sum_{j=1}^{n} \alpha_j^i (x_j - \bar{x}_j^i),$$

where $\alpha_j^i > 0$ for $j = 1, \ldots, n$ and $i = a, b$. By solving the optimality conditions of (1) we get that the DM i's responses for given hyperplane constraint are given by the formula

$$x_j^i(p, r) = [p \cdot (r - \bar{x}^i)]p_j/[\alpha_j^i \sum_k (p_k^2/\alpha_k^i)] + \bar{x}_j^i, \tag{4}$$

for $j = 1, \ldots, n$ and $i = a, b$.

To illustrate the geometrical ideas behind the constraint proposal method let us now consider the two dimensional case and set $\alpha_1^a = \alpha_2^b = 15$, $\alpha_2^a = \alpha_1^b = 1$, $\bar{x}^a = (0, 0)$, $\bar{x}^b = (2, 2)$, and let us choose the reference point $r = (2, 0)$. The contours of the utility functions are illustrated in the left part of Figure 2; dotted lines represent the contours of u_a and dashed lines represent the contours of u_b. The resulting optimal solution functions $x^a(p, r)$ and $x^b(p, r)$ are illustrated in the figure by solid lines. The optimal solution functions, given by (4), are

$$x^a(p, r) = (2p_1^2, 30p_1p_2)/(p_1^2 + 15p_2^2),$$
$$x^b(p, r) = (30(p_1^2 - p_1p_2) + 2p_2^2, 30p_1^2)\,/(15p_1^2 + p_2^2).$$

The resulting F, defined by (3), is drawn in the right part of Figure 2.

There are three solution rays to (3):

$$R_1 = \{(p_1, p_2) : (p_1, p_2) = \lambda(1, 1),\ \lambda \neq 0\},$$
$$R_2 = \{(p_1, p_2) : (p_1, p_2) = \lambda\left((2-\sqrt{3})/(2+\sqrt{3}), 1\right),\ \lambda \neq 0\},$$
$$R_3 = \{(p_1, p_2) : (p_1, p_2) = \lambda\left(1, (2-\sqrt{3})/(2+\sqrt{3})\right),\ \lambda \neq 0\}.$$

These rays are illustrated with dashed lines in the right part of Figure 2. A hyperplane going through the reference point $(2, 0)$ and the normal in R_1, R_2, or R_3 gives the joint tangential points $(1, 15)/8$, $(0.00, 0.14)$, and $(1.86, 2.00)$, respectively. These points are the intersection points of the solid lines in the left part of Figure 2, the points marked with circles. All these points are Pareto optimal. In the right part of Figure 2 we also see that Walras' law, (P2), means that $F(p)$ is perpendicular to its argument p.

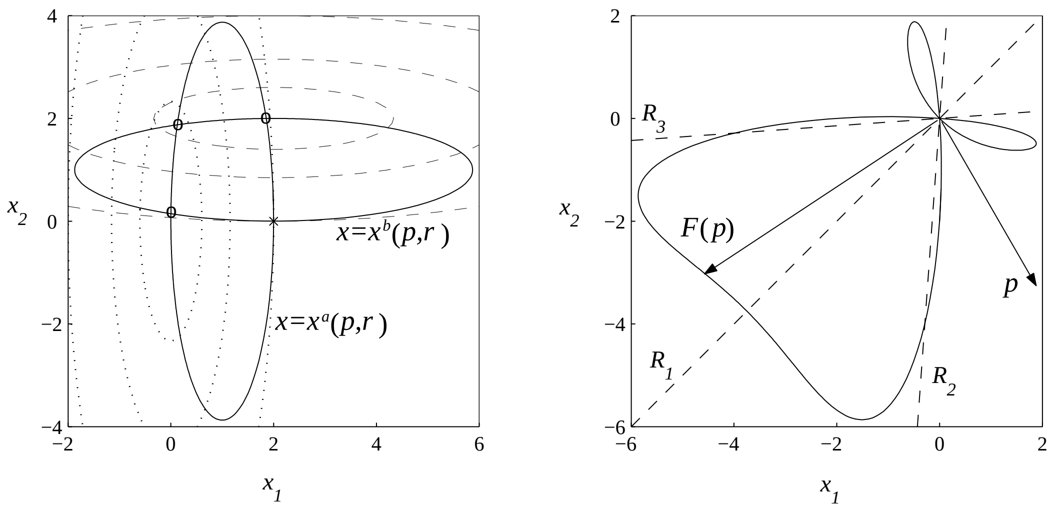

Figure 2. Illustration of $x^a(p, r)$, $x^b(p, r)$, and $F(p)$.

3 Pareto Optimality and the Choice of Reference Points

In this section we first show that under the assumptions (A1)–(A3) for the utility functions all the solutions of the mediator's problem (2) are Pareto optimal when the reference point is chosen from the line connecting the DMs' optima. We also show that all the Pareto points can be obtained by choosing the reference points in this manner.

Let us begin with showing in Lemma 1 that Pareto optimality can be characterized with jointly supporting hyperplanes when the value functions satisfy (A1)–(A3). See, e.g., Yu (1985, Section 3.4) for other Pareto optimality conditions. We use the following notation

$$H^+(p,r) = \{x \in \mathbb{R}^n : p \cdot (x - r) \geq 0\},$$
$$H^-(p,r) = \{x \in \mathbb{R}^n : p \cdot (x - r) \leq 0\}.$$

The proof of Lemma 1 is presented in the appendix.

Lemma 1. *Let the assumptions (A1)–(A3) hold. Then x^* is Pareto optimal if and only if there is $H(p,x^*)$ such that $S_a(x^*) \subset H^+(p,x^*)$ and $S_b(x^*) \subset H^-(p,x^*)$.*

The part (a) of the following proposition tells that the solutions of (2), if there are such, are Pareto optimal when the reference point is chosen from the line connecting the DMs' global optima. The part (b) of the proposition has been proven in Ehtamo et al. (1999a) in the case of differentiable quasiconcave utility functions. The meaning of this results is that all the Pareto points can be obtained by taking reference points from the line connecting the decision makers' global optima. Related results are also given by Heiskanen (2001) for strictly pseudoconcave utility functions. [1]

Proposition 1. *Let the assumptions (A1)–(A3) hold.*

(a) Let $r = \lambda \bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0,1]$. If $x^* \in X_a(p,r) \cap X_b(p,r)$, then x^* is Pareto optimal.
(b) If x^* is Pareto optimal, then there are $r = \lambda \bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0,1]$, and p such that $x^* \in X_a(p,r) \cap X_b(p,r)$.

Proof. Let us begin with the part (a). If $\lambda = 0$ or $\lambda = 1$ the result is obvious. Thus we may suppose that $\lambda \in (0,1)$. By the optimality of x^* we know that $H(p,r)$ is a joint tangent hyperplane of $S_a(x^*)$ and $S_b(x^*)$. It follows that $S_a(x^*)$ and $S_b(x^*)$ belong to opposite halfspaces defined by $H(p,r)$ and hence x^* is Pareto optimal by the quasiconcavity. For example, let us suppose that $S_a(x^*) \subset H^+(p,r)$, i.e., $p \cdot (x - r) \geq 0$ for all $x \in S_a(x^*)$. In particular we have $p \cdot (\bar{x}^a - r) \geq 0$. Observing that

$$\bar{x}^a - r = (1-\lambda)(\bar{x}^a - \bar{x}^b) = (1-\lambda)(r - \bar{x}^b)/\lambda,$$

it follows that $p \cdot (\bar{x}^b - r) \leq 0$. Then $S_a(x^*)$ and $S_b(x^*)$ belong to the opposite halfspaces and Pareto optimality follows from Lemma 1.

Let us now show the part (b). By Lemma 1, Pareto optimality means that there is a hyperplane $H(p,x^*)$ such that $S_a(x^*) \subset H^+(p,x^*)$, and $S_b(x^*) \subset$

[1] Strictly pseudoconcave functions are strongly quasiconcave. Hence, Proposition 1 gives a more general result for two-party negotiations than the results of Heiskanen (2001).

$H^-(p, x^*)$. An appropriate r is now obtained by taking the intersection of the line $\lambda\bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0, 1]$, and $H(p, x^*)$ as the reference point. Hence, we need to show that there is such an intersection point. Let us denote

$$f(\lambda) = p \cdot [\lambda\bar{x}^a + (1-\lambda)\bar{x}^b - r].$$

Because $S_i(x^*)$, $i = a, b$, are convex sets and $\bar{x}^i \in S_i(x^*)$, $i = a, b$, and because $S_a(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$, we know that there are $\delta_1,\ \delta_2 \in [0, 1]$, $\delta_2 < \delta_1$, such that $f(\lambda) > 0$ for all $\lambda \in [\delta_1, 1]$, and $f(\lambda) < 0$ for all $\lambda \in [0, \delta_2]$. Clearly f is a continuous function, so that there is $\lambda^* \in [0, 1]$ such that $f(\lambda^*) = 0$. The result follows by taking $r = \lambda^*\bar{x}^a + (1-\lambda^*)\bar{x}^b$. □

Under assumption (A4) Proposition 1 implies that a Pareto point other than one of the global optima $\bar{x}^a$, $\bar{x}^b$ should have a reference point other than one of these optima. The result does not, however, guarantee that there is always a solution for the mediator's problem even though the reference point is chosen from the line connecting the DMs' optima.

Since any point on a given hyperplane can be taken as a new reference point defining the same hyperplane, the solutions of (2) can be Pareto optimal even though the reference point is not chosen from the line connecting the DMs' optima. It is also easy to find reference points such that at least some of the solutions of (2) fail to be Pareto optimal.

4 Existence of Solution

In this section we show that under the assumptions (A1), (A2), and (A4), the mediator's problem, i.e., equation (3) since (A4) holds, has a solution for any reference point if $n = 2$ or n is odd. In Figure 2 we have an example where the system has three solutions, which are all Pareto optimal; the points market by circles in the left part of Figure 2.

Let us begin with a general existence result for $F(p) = 0$. The proof of the following lemma is given in the appendix and it is based on a fixed-point theorem according to which a continuous mapping from a unit sphere to itself has either a fixed-point or it maps some point to its antipode when n is odd, see (Dugundji, 1966, Corollary 3.3 in Chapter XIV).

Lemma 2. *Let the continuous mapping $F : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{R}^n$ have the properties (P1) and (P2). Then $F(p) = 0$ has at least a ray of solutions when $n = 2$ or $n > 2$ is odd.*

Recall that by a ray of solution we mean that if $F(p^*) = 0$ then $F(\alpha p^*) = 0$ for all $\alpha \neq 0$. Due to homogeneity of $x^a(p, r)$ and $x^b(p, r)$ with respect to their first argument there is at least a ray of solutions for (3) if there is one solution. Similar existence results as that given by Lemma 2 can be found in economics literature, where F is the excess demand function of an exchange economy. In that framework the solution is called competitive equilibrium. There is, however, a significant difference between the results on economic equilibria and the results of this paper. Namely, for exchange economies vector p represents prices and they are assumed to be positive. Moreover, $\|F(p)\|$ becomes infinitely large when some components of p converge to zero. Because of these specific properties, the existence results for exchange economies are based on different deduction than the result of this section, see, e.g., Mas-Colell et al. (1995, Chapter 3). The relationship of exchange economies and the constraint proposal method is discussed in detail in Section 6.

To be able to use Lemma 2 for the mediator's problem we need to show that x^a and x^b are continuous with respect to $p \neq 0$. Lemma 3 gives this result when u_a and u_b are strongly quasiconcave functions with the property that the sets that are preferred to r are compact. These properties hold, e.g., for strictly concave functions that attain their optima. The main characteristics of the problem that guarantee the continuity of F are continuity of the utility functions, single valuedness of the optimal solutions due to strong quasiconcavity, and the continuity of the hyperplane constraint with respect to its normal. The proof of Lemma 3 is given in the appendix.

Lemma 3. *For $i = a, b$ let u_i be a strongly quasiconcave continuous function and let $S_i(r)$ be a compact set. Then $x^i(p, r)$ is continuous with respect to its first argument for all $p \neq 0$.*

When x^a and x^b are continuous with respect to p, then F is continuous, too. The following proposition tells essentially that for the constraint proposal method with two negotiators $F(p) = 0$ has a solution for any reference point. The proposition follows immediately from Lemma 2 and Lemma 3.

Proposition 2. *Let $n = 2$ or $n > 2$ be odd. Let the assumptions (A2) and (A4) hold, and let $S_a(r)$ and $S_b(r)$ be compact sets. Then there is $p^* \neq 0$ such that $x^a(p^*, r) = x^b(p^*, r)$.*

Proposition 2 guarantees that there is a solution for (3) when $n = 2$ or $n > 2$ is odd. Nevertheless, we have not been able to generate a counterexample with n even and concave utility functions, where there is no solution. The technical difficulty with showing the existence of solution for arbitrary n is that x^a and x^b are not continuous at $p = 0$. For exchange economies, which result to similar system of equations, there are stronger results that give the existence for all n. These results are, however, based on properties which do not hold for F in the current problem.

In practice, the constraint proposal method is most suitable when the DMs can easily give their optima on a given hyperplane, which is possible when the number of issues is low. Hence, the method is most suitable when $n = 2$ or $n = 3$. In these cases the mediator's problem has a solution by Proposition 2.

Finally, let us notice that the existence result does not generalize as such to the multi-DM setting because of the structural differences of these problems with the two DM case. For example, in multi-DM setting equation (3) is not defined for linearly dependent parameter vectors, and therefore the resulting F is not continuous.

5 Adjustment of Hyperplane Constraint

The basic idea of the constraint proposal method is that the mediator proposes the negotiators a hyperplane and asks their optimal points on the plane. If the points are significantly different the mediator updates the normal of the hyperplane with using the DMs' current and possibly other previous optimal choices. Ehtamo et al. (1999a) have suggested fixed-point iteration for updating the normal of the hyperplane constraint. The main advantage of this iteration is that the mediator can adjust the hyperplane on the basis of the DMs' optimal answers for given normal. For example, the derivatives of the mapping F need not be approximated. Although fixed-point iteration has been successfully applied by several authors dealing with the constraint proposal method, e.g., Ehtamo et al. (1999a), an explicit convergence proof is lacking. Our aim in this section is to remedy this matter.

In fixed-point iteration the normal p^k is updated in proportion to the value of F as follows:

$$p^{k+1} = p^k + \mu F(p^k), \tag{5}$$

where $\mu > 0$ is a fixed parameter. If the difference of two successive normals is small, then F is close to zero and an approximate solution has been found. Due to the properties (P1) and (P2) fixed-point iteration can also be applied to a normalized system, where one of the components of p is set to a non-zero constant and only the rest of the components are updated. We do not, however, consider the normalized procedure in this paper because it is not clear whether it makes the process more stable or not. The results for the non-normalized process do not hold for the normalized one because Walras' law does not hold if one of the equations is dropped.

The following result on the convergence of fixed-point iteration is shown by Kitti and Ehtamo (2003). Here we denote $B(p^*, r) = \{x \in \mathbb{R}^n : \|x - p^*\| \leq r\}$, $r > 0$.

Lemma 4. *Let the continuous mapping $F : B(p^*, r) \mapsto \mathbb{R}^n$, satisfy (P2) and* *the inequality*

$$p^* \cdot F(p) \geq \|F(p)\|^2 \tag{6}$$

for all $p \in B(p^*, r)$*. If* $p^0 \in B(p^*, r)$ *then (5) converges to a solution of (3). If (5) converges to a solution* $\tilde{p}$ *for which there is* $\alpha > 0$ *such that*

$$\|F(p)\|^2 \leq 2\alpha F(p) \cdot \tilde{p}$$

for all $p \in B(p^*, r)$*, then the convergence is monotonical.*

The above lemma assumes continuity, (P2), and (6). As shown earlier F is continuous when the utility functions are continuous and strongly quasiconcave. Hence, an additional property to obtain convergence is the inequality (6), which geometrically means that the hypersurface $\{x \in \mathbb{R}^n : x = F(p), p \neq 0\}$ curves enough at the origin. More specifically, (6) is equivalent to

$$\|p^*/2 - F(p)\| \leq \|p^*/2\|.$$

Thus, $F(p)$ is inside a ball centered at the ray defined by p^*. This is illustrated in Figure 3, where $F(p)$ is indeed inside a ball, represented with the dashed line, for p chosen from the vicinity of $p^*/2$.

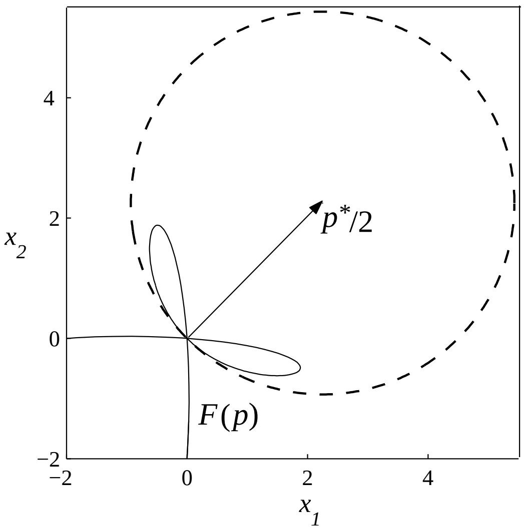

Figure 3. Illustration of convergence condition (6).

Unfortunately, the concavity assumptions do not imply (6) even though (6) seems to be a generic feature. Kitti (2004) shows that when the parameterized hypersurface obtained from F has non-zero normal curvature to all its tangent directions, then (6) is is satisfied. This curvature condition can be formulated for the second derivatives of F, and this is done in Section 5.1. The above Lemma on the convergence of (5) gives a local convergence result. Nevertheless, numerical tests suggest that the iteration converges globally, i.e., for all initial normals. A possible explanation is that $\mathbf{R}^n$ can be divided into regions corresponding to different solution rays and in these regions (6) holds.

In this section we give a more detailed characterization for (6) and derive a simple algebraic test for the convergence of (5). The test is based on examining whether the normal curvature of the hypersurface obtained from F is positive to all tangent directions.

Let us first define some basic concepts of differential geometry. Let us assume that the last component of p is equal to one, i.e., $p_n = 1$. Let us denote $p = (\bar{p}, 1)$, where $\bar{p} \in \mathbb{R}^{n-1}$, and set $\bar{F}(\bar{p}) = F(\bar{p}, 1)$. Mapping $\bar{F} : \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$ is the parameterized hypersurface obtained from F. To define the normal curvature of $\bar{F}$ at $\bar{p}$ we need to assume that it is twice continuously differentiable and regular. Regularity means that the vectors $\nabla_1 \bar{F}(\bar{p}), \ldots, \nabla_{n-1} \bar{F}(\bar{p})$ are linearly independent. Here $\nabla_j \bar{F}(\bar{p})$ denotes the vector that is obtained by differentiating the component functions of $\bar{F}$ with respect to j'th argument. Furthermore, we let $N(\bar{p})$ denote the normal of the hypersurface at $\bar{p}$. It follows from Walras' law that $N(\bar{p}^*) = p^*/\|p^*\|$, when $F(p^*) = 0$ and $p^* = (\bar{p^*}, 1)$, see, e.g., Kitti (2004, Section 3.1).

The normal curvature of $\bar{F}$ at $\bar{p}$ to a tangent direction $\bar{F}(\bar{p})d$, $d \neq 0$, is a function $\kappa(d; \bar{p}) = [(Ld) \cdot d]/\|d\|^2$ that depends on $N(\bar{p})$ and the second derivatives of $\bar{F}$. Here L denotes the matrix $L = AB^{-1}$, where $A = [N(\bar{p}) \cdot \nabla_{ij} \bar{F}(\bar{p})]_{i,j}$, $B = [\nabla_i \bar{F}(\bar{p}) \cdot \nabla_j \bar{F}(\bar{p})]_{i,j}$, and $\nabla_{i,j} \bar{F}$ denotes the vector obtained by differentiating the component functions of $\bar{F}$ with p_i and p_j. The notation $[a_{i,j}]_{i,j}$ for a matrix means that the component of the matrix in i'th row of j'th column is $a_{i,j}$. For the derivation of the formula for the normal curvature see, e.g., Spivak (1979, Section 7.C–D). It is shown in Kitti (2004, Lemma 4) that (6) holds around $p^* = (\bar{p}^*, 1)$ if $\kappa(d; \bar{p}^*) > 0$ for $d \neq 0$. Hence, we can formulate the following proposition.

Proposition 3. *Let F satisfy (P1) and (P2), and let $\bar{F}$ be regular and twice continuously differentiable. Then F satisfies (6) around $p^* = (\bar{p}^*, 1)$, $F(p^*) = 0$, if and only if $\bar{F}$ has positive normal curvature to all tangent directions at $\bar{p}^*$.*

We can use the result of Proposition 3 to derive an algebraic test for the convergence. Namely, if we find the minimal value of the normal curvature over the unit sphere we can see from its sign whether (6) holds. Indeed, the critical points of κ over the unit sphere correspond to so called principal curvatures. These critical points are exactly the eigenvectors of L and the eigenvalues are the principal curvatures. Hence, we can test (6) numerically by computing the eigenvalues of L. If these eigenvalues are positive we know that (6) holds for p^*, and if they are negative then (6) holds for $-p^*$. Notice, however, that this test requires that p^* is known.

For example, in the two-dimensional quadratic case of Section 2.1 we have the following positive principal curvatures corresponding to the three ray of solutions: $\kappa_1 = 16\sqrt{2}/45$, $\kappa_2 = \kappa_3 = \sqrt{14}/45$. Hence, condition (6) holds when p is chosen close enough to any of the the three solution rays illustrated in Figure 2.

6 Constraint Proposal Method and Exchange Economies

In this section we discuss the relationship of the constraint proposal method and exchange economies. In an exchange economy there is a number of consumers with initial allocations of some resources. Given prices for the resources each consumer is willing to buy a bundle that maximizes his utility under his budget, which is the monetary value of his initial bundle. The maximizing bundle is called the consumer's demand function. A vector of prices is an equilibrium if the total demand equals the total supply of the resources which is simply the sum of the initial allocations. Under some economic conditions the equilibrium prices can be found by a simple auctioning process, where an auctioneer adjusts the prices until an equilibrium is reached but no trades are made during the adjustment process. See, e.g., (Mas-Colell et al., 1995, Part IV) for more about the basic properties of exchange economies.

The problem of finding a Pareto solution for the negotiation can be interpreted as a resource allocation problem, where the decision makers are sharing their total dispute $w = \bar{x}^a - \bar{x}^b$. The initial allocation of the total dispute is defined by the reference point $r = \lambda\bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0, 1]$; the proportion of the total dispute for the first DM is λ and $(1-\lambda)$ for the second DM. Moreover, decision maker i gets at the least value $u_i(r)$ as the outcome from the negotiation.

The constraint proposal method can be interpreted as an auctioning process, where the mediator acts as an auctioneer who tries to find a Pareto optimal allocation of the total dispute w. See Kitti (2004) for an iterative process for finding the Walrasian equilibrium. The relationship to resource allocation can be explicitly seen by making the following transform of variables: $y^a = x - \bar{x}^a$, $y^b = \bar{x}^b - x$. The DMs' optimization problems are then of the form

$$\max_{y^i} U_i(y^i) \text{ s.t. } p \cdot (y^i - \lambda_i w) = 0, \; i = a, b, \tag{7}$$

where $U_a(y^a) = u_a(y^a + \bar{x}^a)$, $U_b(y^b) = u_b(\bar{x}^b - y^b)$, and $\lambda_a + \lambda_b = 1$, $\lambda_i \in [0, 1]$ for $i = a, b$.

Let p denote a price vector of n-resources, which correspond to the issues, and $\lambda_i w$ denote the initial endowment that the DM i has. Then we can interpret the linear constraint in (7) as a budget identity. Moreover, the point $y^i(p)$,

$i = a, b$, that solves (7) is the DM's demand function for the resources, and $\sum_i (y^i(p) - \lambda_i w)$ is the excess demand of the resources. Similarly as $F(p)$, the excess demand satisfies Walras' law, which means now that the monetary value of the excess demand is zero. Homogeneity of the excess demand function means that only the relative prices of the resources matter.

As the above discussion demonstrates the mediator's problem in the constraint proposal method is remarkably close to the resource allocation problems of exchange economies. Indeed, the part (a) of Proposition 1 corresponds to the first fundamental welfare theorem in microeconomics and the part (b) corresponds to the second fundamental welfare theorem. According to the first fundamental theorem a price equilibrium is Pareto optimal and according to the latter there is a price equilibrium corresponding to a Pareto solution, see Mas-Colell et al. (1995).

There are some important differences between the constraint proposal method and exchange economies. In an exchange economy the demand functions are not defined if some of the prices are negative. Moreover, the demand for a resource usually grows infinitely large as its prices go to zero, i.e., the utility functions do not have global optima and (A1) does not hold. In the constraint proposal method p can have negative components as well and there is no reason to assume the DMs' responses to satisfy any boundary conditions for zero components of p.

Due to the aforementioned differences, the results for exchange economies are not applicable for the constraint proposal method. For example, exchange economies can be shown to have an equilibrium for any number of resources, but such a result requires aforementioned boundary conditions for zero prices. Pareto optimality results of Section 3 are also based on different assumptions than the welfare theorems for exchange economies.

7 Discussion

7.1 General Remarks

In this paper we have analyzed the choice of the reference point in the constraint proposal method. We have shown that the method produces Pareto optimal points when the mediator chooses the reference point from the line connecting the DMs' optima and all the Pareto points can be produced in this manner. Moreover, we have proven that the mediator's problem has a solution when the number of issues is two or any odd number greater than two. In essence, these results mean that the constraint proposal method is not just a heuristic approach for finding Pareto solutions, but it indeed gives Pareto optimal points.

To find a joint tangent hyperplane the mediator has to solve a system of equations. A suitable method for that purpose is fixed-point iteration, which requires only the DMs' last optima to update the hyperplane. In this paper we have given local convergence conditions for fixed-point iteration and a numerical convergence test.

In addition to the aforementioned results, we have discussed the relationship of the constraint proposal method and exchange economies. We have seen that the mediator's problem in the constraint proposal method can be transformed to a resource allocation problem where the total resource to be shared is the difference of the DMs' optima. We have also pointed out some differences between the economic resource allocation model and the mediator's problem. Due to these differences the results for the constraint proposal method are based on different assumptions and techniques as those for exchange economies.

7.2 Ways of Using the Constraint Proposal Method

The constraint proposal method can be applied in a variety of ways. One way, as suggested by Ehtamo et al. (1999a), is to use the method for finding an approximation for the whole Pareto frontier. The negotiation then becomes distributive along the frontier. The method can also be used in a kind of "post-settlement settlement" fashion; this method was suggested by Raiffa (1982). First the parties negotiate unaided and reach a tentative solution point, not necessarily Pareto optimal, after which they search for a jointly beneficial Pareto optimal solution using one of the available methods, e.g., the constraint proposal method.

Yet, there is at least a third possible way of using the constraint proposal method. Namely, that of first bargaining on a suitable reference point for the method and then using it. We describe such a process briefly. In particular, the bargaining could be restricted to the reference points on the line connecting the DMs' optima. This problem is one dimensional since it is over the choice of parameter λ that defines a point $r(\lambda) = \lambda \bar{x}^a + (1 - \lambda)\bar{x}^b$. Note, however, that the resulting utility points $v_i(\lambda) = u_i(r(\lambda))$, $i = 1, 2$, do not form a line in u_1, u_2-plane; rather they form a rough approximation of the Pareto frontier.

If the negotiation over the "approximate" Pareto frontier results in $r(\bar{\lambda})$, then the DM i is guaranteed to have at least the value $v_i(\bar{\lambda})$ after applying the constraint proposal method with this reference point. Indeed, the constraint proposal method applied with reference point $r(\lambda)$ gives a point that both DMs prefer to it.

The negotiation over the reference points can be considered as a bargaining problem. For example, one may use the axiomatic approach to bargaining initiated by Nash (1950). Nash bargaining solution is obtained by maximizing the product

$$[v_a(\lambda) - d_a] \cdot [v_b(\lambda) - d_b], \tag{8}$$

where d_i is the threat point, which gives the value for the DM i if the bargaining fails. For example, we may take the threat point according to the worst case scenario, where the d_i is chosen to be the value at the other party's optimum, i.e., $d_a = v_a(0)$, $d_b = v_b(1)$. Even though it is not necessarily possible to give both DMs their worst case outcomes, these values can be taken as the threat points.

In practice, the bargaining solution $\bar{\lambda} \in [0, 1]$ can be found approximately by first eliciting the utility functions $v_a(\lambda)$ and $v_b(\lambda)$ within some accuracy. See von Winterfeldt and Edwards (1986, Section 7.3.) for methods of estimating utility functions, such as v_a and v_b, that depend on a single parameter. There is a plethora of efficient methods to perform this task. After having found the approximations of utility functions the bargaining solution can be computed numerically by maximizing (8).

Let us sum up the process of finding a single Pareto optimal point for the negotiation problem:

1. The reference point $r(\bar{\lambda})$ is chosen according to Nash bargaining solution, e.g., by a sequential bargaining process.
2. The mediator finds one solution for (2) with the reference point $r(\bar{\lambda})$ and suggest this point to the DMs.

Proposition 1 guarantees that the above procedure gives a Pareto optimal point if the mediator finds a solution for (2).

As an example, let us consider the same utility functions as in the two dimensional example of Section 2.1. We now obtain $v_a(\lambda) = -64(1 - \lambda)^2$, $v_b(\lambda) = -64\lambda^2$, $d_a = v_a(0) = 0$, and $d_b = v_b(1) = 0$. The optimum of (8) is obtained at $\bar{\lambda} = 1/2$ and $r(\bar{\lambda}) = (1, 1)$, i.e., the reference point is chosen exactly from the middle of the DMs' optima. With this reference point the mediator's problem has three solutions giving the Pareto optimal points $(1, 15)/8$, $(1.74, 1.99)$, and $(0.22, 1.16)$, the latter two being approximate values.

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Appendix: Proofs of the lemmas

Proof of Lemma 1:

Let us first assume that $x^* = \bar{x}^a$, which is a Pareto optimal point, and show that there is a joint tangent hyperplane at this point. Note that the deduction is similar for $x^* = \bar{x}^b$. We have $S_a(x^*) = \{x^*\}$ and by the convexity of $S_b(x^*)$ there is a hyperplane $H(p, x^*)$ such that $S_b(x^*) \subset H^-(p, x^*)$ and because $x^* \in H(p, x^*)$ we see that $S_a(x^*) \subset H(p, x^*) \subset H^+(p, x^*)$.

Let us now show that there is a joint tangent hyperplane for a Pareto optimal point $x^* \neq \bar{x}^i$, $i = a, b$. By the definition of Pareto optimality

$$\mathrm{int}S_a(x^*) \cap \mathrm{int}S_b(x^*) = \emptyset. \tag{9}$$

Note that we have $\mathrm{int}S_i(x^*) \neq \emptyset$ because $\bar{x}^i \in \mathrm{int}S_i(x^*)$, and by the continuity of value functions $\mathrm{int}S_i(x^*) = \{x \in \mathbb{R}^n : u_i(x) > u_i(x^*)\}$. It follows that there is a hyperplane $H(p, x^*)$ such that $\mathrm{int}S_a(x^*) \subset H^+(p, x^*)$ and $\mathrm{int}S_b(x^*) \subset H^-(p, x^*)$, see, e.g., Bazaraa et al. (1993, Theorem 2.4.8). Since the halfspaces are closed sets we have $S_a(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$.

Let us now assume that there is a hyperplane $H(p, x^*)$ such that $S_a(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$. By this we have $\mathrm{int}S_a(x^*) \subset \mathrm{int}H^+(p, x^*)$ and $\mathrm{int}S_b(x^*) \subset \mathrm{int}H^-(p, x^*)$, where

$$\mathrm{int}H^+(p, x^*) = \{x \in \mathbb{R}^n : p \cdot (x - x^*) > 0\},$$
$$\mathrm{int}H^-(p, x^*) = \{x \in \mathbb{R}^n : p \cdot (x - x^*) < 0\}.$$

Because $\mathrm{int}H^-(p, x^*) \cap \mathrm{int}H^+(p, x^*) = \emptyset$, we get (9), which means that x^* is Pareto optimal. $\square$

Proof of Lemma 2:

Let $B(y, \rho)$ denote the closed ball having radius $\rho > 0$ centered at y, i.e., $B(y, \rho) = \{x \in \mathbb{R}^n : \|x - y\| \leq \rho\}$, where $\|\cdot\|$ is the Euclidean norm. Moreover, we let $\partial B(y, \rho)$ denote the boundary of this ball.

Let us define a mapping $G : \partial B(0, 1) \mapsto \partial B(0, 1)$ by setting

$$G(p) = \frac{p + F(p)}{(1 + \|f(p)\|^2)^{1/2}}.$$

It follows from Walras' law that $\|G(p)\| = 1$ so that the image of $\partial B(0, 1)$ under G belongs to $\partial B(0, 1)$ itself.

For any $p \in \partial B(0,1)$ the mapping is continuous since F is continuous. Thus, either G has a fixed point or it sends some point to its antipode when n is odd, which follows from a corollary of Poincaré-Brouwer theorem, see, e.g., Dugundji (1966, Corollary 3.3 in Chapter XIV). Hence, there is $p^* \in \partial B(0,1)$ such that $p^* = G(p^*)$ or $p^* = -G(p^*)$. By taking inner product of both sides of these equations with respect to $(1 + \|F(p^*)\|^2)^{1/2}\, F(p^*)$ and applying Walras' law we get $\|F(p^*)\|^2 = 0$ or $-\|F(p^*)\|^2 = 0$, which implies that $F(p^*) = 0$.

Let us assume that $n = 2$ and set $p_1(\phi) = \cos\phi$, $p_2(\phi) = \sin\phi$, $f_i(\phi) = F_i\,(p_1(\phi), p_2(\phi))$, $i = 1, 2$. By the homogeneity $F(p_1(\phi), p_2(\phi))$ obtains all its values when $\phi \in [0, \pi]$ and

$$f_i(\phi_0 + \pi) = f_i(\phi_0),\ \ i = 1, 2. \tag{10}$$

Walras' law can be now written as

$$f_1(\phi)\cos\phi + f_2(\phi)\sin\phi = 0 \tag{11}$$

To see that $F(p) = 0$ has a solution, we need to show that $f_i(\phi) = 0$, $i = 1, 2$, for some $\phi \in [0, \pi]$.

Let us first observe that (11) implies that $f_1(0) = 0$ and $f_2(\pi/2) = 0$. Hence, $F(p)$ has a solution if $f_2(0) = 0$ or $f_1(\pi/2) = 0$, and we may assume that $f_2(0) \neq 0$ and $f_1(\pi/2) \neq 0$. Let us consider the case $f_2(0) > 0$ and $f_1(\pi/2) > 0$. By the continuity of f_i, the positivity of $\sin\phi$ and $\cos\phi$ on $(0, \pi/2)$, and (11) there is $\varepsilon > 0$ such that $f_2(\phi) < 0$ for all $\phi \in (\pi/2 - \varepsilon, \pi/2)$. Then f_2 changes its sign on the interval $(0, \pi/2)$, i.e., there is $\phi^* \in (0, \pi/2)$ such that $f_2(\phi^*) = 0$ and by (11) we have $f_1(\phi^*) = 0$.

If $f_1(\pi/2) < 0$ there is $\varepsilon \in (0, \pi/2)$ such that $f_1(\phi_0) < 0$, where $\phi_0 = \pi/2 + \varepsilon$. Since $\sin\phi_0 > 0$ and $\cos\phi_0 < 0$, (11) implies that $f_2(\phi_0) < 0$. Because $f_2(\pi) = f_2(0) > 0$ and f_2 is continuous, we know that f_2 changes its sign on (ϕ_0, π). Hence, there is ϕ^* such that $f_i(\phi^*) = 0$, $i = 1, 2$. The similar deduction can be made when $f_2(0) < 0$. □

Proof of Lemma 3:

Let us first notice that

$$X_i(p, r) = \arg\max_{x \in \phi_i(p,r)} u_i(x),\ \ i = a, b,$$

where $\phi_i(p, r) = S_i(r) \cap H(p, r)$, i.e., the constraint $x \in H(p, r)$ can be replaced with $x \in \phi_i(p, r)$. This is because the maximization problem has a unique solution that belongs to $S_i(r)$, which is the set of points that are at least as good as r. Note also that ϕ_i is non-empty and because S_i is compact valued so is ϕ_i.

Let us first show that ϕ_i is lower hemicontinuous with respect to $p \neq 0$, i.e., $p^k \to \bar{p}$, $x^k \in \phi_i(p^k, r)$, imply that there is a subsequence $\{x^{k_j}\}_j$ such that $x^{k_j} \to \bar{x} \in \phi_i(\bar{p}, r)$. This is because due to compactness of $S_i(r)$ the sequence $\{x^k\}_k$ has a convergent subsequence and because $p \cdot (x - r)$ is a continuous function, the limit of the subsequence belongs to $\phi_i(\bar{p}, r)$. Second, ϕ_i is upper hemicontinuous with respect to $p \neq 0$, i.e., $p^k \to \bar{p}$, $\bar{x} \in \phi_i(\bar{p}, r)$, imply that there is a sequence $\{x^k\}_k$ with $x^k \in \phi_i(p^k, r)$ for all k such that $x^k \to \bar{x}$. Indeed, such a sequence can be constructed by setting $x^k = \arg\min_{x \in \phi_i(p^k, r)} \|x - \bar{x}\|$.

Because ϕ_i is both upper and lower hemicontinuous, it is continuous. By the Berge's theorem X_i is a closed and upper hemicontinuous set-valued mapping for $p \neq 0$, because it is the set of points that maximize a continuous function u_i over a compact-valued continuous mapping ϕ_i, see, e.g., Border (1985, Theorem 12.1). Strong quasiconcavity implies that X_i is a singleton, and as a single valued upper hemicontinuous mapping X_i is continuous, see Border (1985, Proposition 11.9 (d)). Hence, $x^i(p, r)$ is continuous with respect to its first argument when $p \neq 0$. $\square$

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