Convergence of Non-Normalized Iterative Tâtonnement

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Abstract

Global convergence conditions for iterative tâtonnement with the additional requirements that prices stay strictly positive and their changes are bounded are given and convergence is shown when the excess demand function has the gross substitute property and curves appropriately around the equilibrium. Furthermore, this paper introduces a new, second order, form of the weak axiom of revealed preferences; a condition which also implies convergence. It is shown that this condition holds when the excess demand function is strongly monotone.

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1 Introduction

The latest results on discrete time price adjustment processes are mostly negative: discrete time processes may fail to converge and they may exhibit periodic or even chaotic behavior, see Bala and Majumdar (1992), Day and Pianigiani (1991), Goeree et al. (1998), Mukherji (1999), Saari (1995), and Tuinstra (2000). This paper shows that a simple iterative process avoids these phenomena and converges globally under conditions that are only slightly stronger than those required for the continuous time tâtonnement process.

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Tâtonnement processes are usually interpreted as auctions, where a fictitious agent, Walrasian auctioneer, sets the prices until an equilibrium is reached and the trades are made. The main purpose of such processes is to explain how an economy comes to its equilibrium. In addition to this, a discrete time process could provide a practical auctioning method for solving resource allocation problems.

The need for analyzing discrete time price adjustment processes has been long recognized. Samuelson, who formulated the tâtonnement process in continuous time with a set of differential equations, observes the following, see Samuelson (1947, p. 286):

> " The types of functional equations which have been most studied are those defined by differential equations, difference equations, and integral equations, and mixed varieties. The first of these possesses the most highly developed theory and provides valuable examples of various principles. Since economic observations consist essentially of series defined for integral values of time, the second category of difference equations is perhaps of greatest interest to the theoretical economist."

Some discrete time alternatives for the continuous time tâtonnement process have been suggested in the economics literature. Uzawa (1960) has analyzed an iterative process for the normalized excess demand, where the price of one of the commodities, numéraire, is set to a constant and only the rest of the prices are adjusted. There are, however, some negative results on normalized processes. Saari (1985) has shown that for any normalized iterative process there are always economies for which the process fails to converge. Furthermore, according to Goeree et al. (1998) a rather general class of normalized discrete time processes exhibits periodic and chaotic behavior. Tuinstra (2000) demonstrates similar results for a multiplicative process.

Many authors have noticed that in most cases the results on the continuous time process do not hold for the discrete time process. Arrow and Hahn (1971, Section 12.8) argue that a discrete time version of the non-normalized process converges to any given neighborhood of the set of equilibria when the iteration parameter and initial prices are chosen appropriately. The corresponding continuous time process, however, has significantly better convergence properties. Indeed, satisfactory convergence results for non-normalized discrete time tâtonnement are lacking.

This paper studies fixed-point iteration with the additional requirements that prices stay strictly positive and the difference between the old and the new prices is bounded. Since only the value of the excess demand function is used in updating the prices, the process has minimal informational requirements. Moreover, the process has the property that if a commodity has positive excess demand, its price rises, and if the excess demand is negative the price falls. Hence, the process can be interpreted as a discrete time alternative of the non-normalized continuous time tâtonnement process. We emphasize that it is not, however, based on approximating the continuous time process.

It is well known that the continuous time process converges globally under the gross substitution property and the weak axiom of revealed preferences, see Arrow et al. (1959) and Arrow and Hurwicz (1958). Here it is shown that the iterative process converges when in addition to gross substitutability the hypersurface obtained from the excess demand function has positive normal curvature to all tangent directions. Furthermore, a second order form of the weak axiom of revealed preferences is introduced. Together with some common properties of excess demand functions this condition implies the convergence of the iterative process, too. It will be shown that the second order weak axiom holds when the excess demand function is appropriately Lipschitzian and strongly monotone, or the economy has a representative consumer with a strongly concave utility function.

The paper is organized as follows. In Section 2 we present the model for an exchange economy and the iterative adjustment process. The global convergence of the process is analyzed in Section 3. In Section 4 the conditions of Section 3 are applied to show convergence for economies that satisfy the gross substitute property and curve appropriately around the equilibrium. As an example, we demonstrate that Cobb-Douglas economies satisfy these conditions. The convergence of the process is shown under the second order weak axiom of revealed preferences in Section 5. The relationship of this condition to monotone mappings is also analyzed.

2 The Model

2.1 Excess Demand Function

An exchange economy with m consumers and n commodities, $m,\ n \geq 2$, is described by the preference relations $\succeq^i$, $i = 1, \ldots, m$, defined on $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x \geq 0\}$, and the endowment vectors $w^i = (w^i_1, \ldots, w^i_n) \geq 0$, $i = 1, \ldots, m$. In this model, subscript denotes the component of a vector and superscript denotes the consumer. Furthermore, $x \geq 0$ means that $x_j \geq 0$ for all j.

Given a price vector $p > 0$, that is $p \in \mathbb{R}^n_+ = \{p \in \mathbb{R}^n : p_j > 0\ \forall j\}$, the demand function $x^i(p)$ for consumer i is the maximizer of $\succeq^i$ over the budget set, which means that $x^i(p) \succeq^i x$ holds for all $x \in \{x \in \mathbb{R}^n : p \cdot x \leq p \cdot w^i,\ x \geq 0\}$, where

$p \cdot w^i = \sum_j p_j w_j^i$. The equilibrium of the economy is a price vector p^* for which

$$z(p^*) = \sum_{i=1}^{m} [x^i(p^*) - w^i] = 0. \tag{1}$$

The mapping z is called the excess demand function of the economy and it will be assumed to have the following properties:

(P1) z is single valued and continuous for all $p > 0$.
(P2) z satisfies Walras' law: $p \cdot z(p) = 0$ for all $p > 0$.
(P3) z is homogeneous of degree zero: $z(\alpha p) = z(p)$ for all $\alpha > 0$.
(P4) There is a scalar $\nu < 0$ such that $z_j(p) > \nu$ for all j and $p > 0$.
(P5) It holds that
$$\lim_{p^k \to p} [\max_{j \in J_p} z_j(p^k)] = \infty,$$
when $p^k > 0$, $p \neq 0$ and $J_p = \{j : p_j = 0\} \neq \emptyset$.

Homogeneity is an elementary property that an excess demand function has because the consumers' budget sets stay the same when the budget constraints are multiplied with positive constants. Walras' law and continuity result from the consumers' maximization problems when the preferences are strictly convex and locally non-satiated.

The property (P4) means that all the component functions of z are bounded from below on $\mathbb{R}^n_+$. An excess demand function has this property because the consumers' net supply of any commodity cannot exceed the total endowment. According to (P5) all the commodities are desirable in the sense that when some of them become free, the excess demand becomes infinitely large at least for some of those commodities. This is the case, for example, when there is a positive total amount of all the commodities and the consumers have strongly monotone preferences. When z has the properties (P1)–(P5), the economy has at least a ray of equilibrium prices. See, e.g., Mas-Colell et al. (1995, Chapter 17) for more about the properties of excess demand functions.

2.2 Iterative Price Adjustment Processes

To be economically meaningful a tâtonnement process should not require other information than prices and the corresponding excess demand, it should satisfy the *law of demand*, according to which prices should increase for commodities with excess demand and fall in the opposite case, and the process should converge under economically relevant conditions. The simplest continuous time process that satisfies these properties was introduced by Samuelson (1947) and is described by the differential equation

$$\dot{p}(t) = z(p(t)), \tag{2}$$

where $\dot{p}(t)$ is the time derivative of $p(t)$. This process is usually interpreted as an auction run by a fictitious agent, a Walrasian auctioneer, who sets the prices until an equilibrium is reached and the trades are made.

It can be shown that under the following condition (C1), the process (2) is globally stable, i.e., it converges to an equilibrium for any positive initial prices. The stability condition can be stated as follows:

(C1) there is $p^* > 0$ that solves (1) and satisfies $p^* \cdot z(p) > 0$ for all $p > 0$ for which $z(p) \neq 0$.

The importance of this condition to the stability of the process (2) was first noticed in Arrow and Hurwicz (1958) and Arrow et al. (1959), where the set of equilibria was assumed to be unique up to a positive scalar multiple, i.e., a unique ray. It was further shown Arrow and Hurwicz (1960) that (C1) implies the stability even though the set of equilibria is not a unique ray.

The convergence condition (C1) can be interpreted as the weak axiom of revealed preferences between the equilibrium p^* and any disequilibrium price vector. An excess demand function satisfies this condition in three important cases: (i) when there is no trade at equilibrium, (ii) when the excess demand function satisfies the weak axiom of revealed preferences for any pair of price vectors, or (iii) when it has the gross substitute property. The latter two properties will be discussed in detail in Sections 4 and 5.

The simplest discrete time alternative for the process (2) is the fixed-point iteration

$$p^{k+1} = p^k + z(p^k), \tag{3}$$

where k is the iteration index that corresponds to the time instants at which the prices are adjusted. The main argument for analyzing (3) instead of (2) is that the auction, which a price adjustment process aims to characterize, proceeds in discrete time instants. This paper studies (3) with the additional assumptions that prices stay positive and their changes are bounded. A way to implement such process in practice is given in the following section.

To obtain non-negative prices we could update p_j^k as follows

$$p_j^{k+1} = \max\{0, p_j^k + \mu z_j(p^k)\}, \tag{4}$$

where μ is a positive constant. The convergence of this process has been analyzed by Uzawa (1960) when the prices are normalized so that the price of one commodity is set to a constant and only the prices of other commodities are adjusted. In essence, it has been shown that under gross substitution there is a choice of μ such that the process converges. The corresponding non-normalized process converges to any given neighborhood of the equilibrium ray with some choice of μ and with p^0 chosen such that the prices remain strictly positive during the process, see Arrow and Hahn (1971, Section 12.8). In addition to the limitations on the choice of μ and p^0, the drawback of the process (4) is that due to (P5) the excess demand function is not finite if some prices become zero.

2.3 Fixed-Point Iteration with Positive Prices

It is commonly known that the discrete time process (4) does not converge under the same assumptions as the continuous time process (2). For example, the convergence of the process (4) depends on the choice of parameter μ. Moreover, the normalized discrete time processes tend to exhibit chaotic behavior. The aim of this paper is to show that a modification of fixed-point iteration (3) converges under condition that are remarkably close to the convergence conditions of the continuous time process. Indeed, for numerical or computational considerations the difference of these conditions are negligible.

As mentioned earlier, prices should stay strictly positive. Other requirement we need is that their changes are bounded, i.e., there is $M > 0$ such that $\|p^{k+1} - p^k\| \leq M$. This assumption is needed to show the convergence and it is quite reasonable. Namely, it means that ever increasing price changes do not occur, or prices cannot change arbitrarily fast between two periods. Note that this condition does not mean that prices should be bounded themselves. Moreover, the bound M can be arbitrarily large.

A process that satisfies the two aforementioned requirements can be defined by the following formula

$$p^{k+1} = p^k + \mu_k z(p^k), \tag{5}$$

where the parameter μ_k is updated as follows:

Step 1 a scalar $\gamma_k > 0$ is chosen such that $p^k + \gamma_k z(p^k) > 0$, and $\gamma_k = \gamma_{k-1}$ for $k \geq 1$ if $p^k + \gamma_{k-1} z(p^k) > 0$,
Step 2 $\mu_k = \min\{\gamma_k, M/\|z(p^k)\|\}$, where $M > 0$.

The purpose of the first step is to guarantee that the new prices are positive and the second step guarantees bounded price changes. When $p^k > 0$ there is a positive number γ_k such that $p^k + \gamma_k z(p^k) > 0$. One way to find an appropriate γ_k in numerical considerations is to choose $\gamma_k = (1/2)^l$ where l is the smallest integer for which $p^k + (1/2)^l z(p^k) > 0$. It follows from the first step that when the initial prices are positive, i.e., $p^0 > 0$, then all the prices obtained during the process are positive as well. The second step guarantees that $\mu_k z(p^k)$ is bounded in the Euclidean norm $\|\cdot\|$. As a result the change of the price vector is bounded, namely $\|p^{k+1} - p^k\| = \|\mu_k z(p^k)\| \leq M$, where is an arbitrarily chosen positive number. Note that according to the two steps, μ_k is updated only if it is necessary for obtaining positive prices or for keeping the changes bounded by M. Hence, it may well happen that these steps are never implemented during the actual process.

The process (5) satisfies the law of demand and prices are adjusted in proportion to their excess demands in a similar way as in the process (2). There is, however, an important difference between the prices obtained from the two processes. Namely, it follows from Walras' law that for the process (5) we have $\|p^{k+1}\| > \|p^k\|$ when $z(p^k) \neq 0$, whereas $\|p(t)\| = \|p(0)\|$ for the process (2).

If γ_k went to zero, then the sequence of prices obtained from (5) could become arbitrarily close to the path obtained from (2). If this happened, the process (5) would be an approximation of the process (2) for large k, and we could expect the two processes to converge under the same conditions. In the following section we shall see that μ_k does not converge to zero when the process (5) converges, which means that the process (5) does not approximate (2). The convergence conditions of the two processes are, however, very close to each other.

3 Convergence Analysis

In this section we give general convergence conditions for the process (5). These conditions will be applied in Sections 4 and 5 to show convergence when z has some more specific economic properties. We prove that the process (5) converges when z has the properties (P1)–(P5) and satisfies (C1), see Section 2.2, together with (C2) as stated below. In the condition (C2) vector p^* is the same equilibrium vector for which (C1) holds and $E_\varepsilon = \{p \in \mathbb{R}^n_+ : \|z(p)\| < \varepsilon\}$. The convergence condition (C2) is stated as follows:

(C2) there are positive scalars ε and σ such that $p^* \cdot z(p) \geq \sigma\|z(p)\|^2$ for all $p \in E_\varepsilon$.

Section 5 introduces a slightly strengthened form of the weak axiom of revealed preferences and show that it implies (C2) analogously as the weak axiom implies (C1).

Let us next examine the geometrical interpretation of conditions (C1) and (C2). The condition (C1) means that the hyperplane $\{x \in \mathbb{R}^n : p^* \cdot x = 0\}$ supports the set $\{x \in \mathbb{R}^n : x = z(p),\ p > 0\}$, see Figure 1. The condition (C2) means that this set is at least locally, around the origin, inside a ball which has its center at the ray of solutions $\{p : p = \lambda p^*,\ \lambda > 0\}$. This can be seen writing $p^* \cdot z \geq \sigma\|z\|^2$ equivalently as $\|p^*/(2\sigma) - z\| \leq \|p^*/(2\sigma)\|$. In Section

3.1 we show that for a regular economy (C2) means that the hypersurface obtained from the excess demand function is not too flat around the origin. Indeed, as σ goes to zero, z is allowed to become flatter, i.e., (C1) is obtained as the limit from (C2).

The way in which the parameter μ_k is updated guarantees that the norm of the scaled excess demand $\mu_k z$ is bounded by the constant M. As a result the scaled excess demand is for all $p > 0$ inside a ball centered at the ray of solutions. These geometrical ideas are illustrated in Figure 1, where $\sigma = 1/2$ and $\lambda = 1$.

Let us state the main convergence theorem that will be used in showing the other convergence results of this paper.

Theorem 1. *Let z have the properties (P1)–(P5) and satisfy the conditions (C1)–(C2). Then the process (5) converges to an equilibrium for any $p^0 > 0$. If there is a unique ray of equilibria, then there is $N \geq 0$ such that convergence is monotonical when $k \geq N$.*

The monotonical convergence of the sequence $\{p^k\}_k$ to $\tilde{p}$ means that $\|p^k - \tilde{p}\| \to 0$, when $k \to \infty$, and if $p^k \neq \tilde{p}$, then $\|p^{k+1} - \tilde{p}\| < \|p^k - \tilde{p}\|$.

The following lemmas are used in the proof of Theorem 1. Here we let $B(p^*, \varepsilon)$ denote the closed ball with radius $\varepsilon > 0$ centered at p^*, i.e., $B(p^*, \varepsilon) = \{x \in \mathbb{R}^n : \|x - p^*\| \leq \varepsilon\}$.

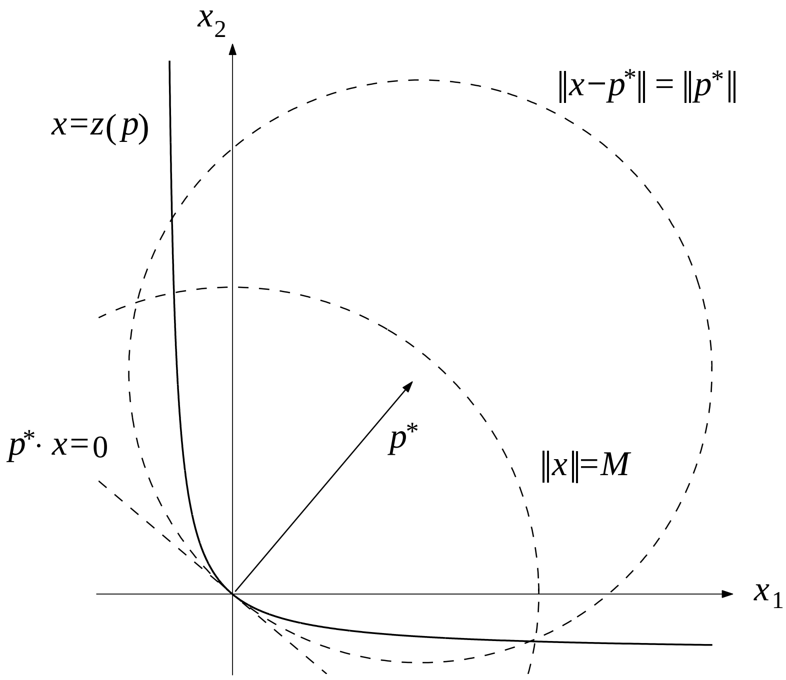

Figure 1. Illustration of the convergence conditions.

Lemma 1. *Let the continuous mapping* $z : B(p^*, r) \mapsto \mathbb{R}^n$ *satisfy Walras' law for all* $p \in B(p^*, r)$*, and let the inequality* $p^* \cdot z(p) \geq \|z(p)\|^2$ *hold for all* $p \in B(p^*, r)$*. If* $p^0 \in B(p^*, r)$ *and* $\mu_k \leq 1$ *for all* k*, then the iteration* $p^{k+1} = p^k + \mu_k z(p^k)$ *converges. When there is* $\bar{\mu}$ *such that* $0 < \bar{\mu} \leq \mu_k$*, the iteration converges to a solution of* $z(p) = 0$*.*

The proof of Lemma 1 is presented in Appendix. The following lemma is for showing that convergence is monotonical when there is a unique ray of equilibria, the proof is in Appendix.

Lemma 2. *Let* z *satisfy the same conditions as in Lemma 1 and let the iteration* $p^{k+1} = p^k + \mu z(p^k)$*,* $\mu > 0$*, converge to a solution* $\tilde{p}$ *for which there is* $\alpha > 0$ *such that*

$$\|z(p)\|^2 \leq 2\alpha z(p) \cdot \tilde{p}$$

for all $p \in B(p^*, r)$*. Then convergence is monotonical.*

The following lemma shows essentially that the convergence condition of Lemma 1 holds for the scaled excess demand that is obtained by adjusting the parameter μ_k as described in steps 1 and 2. The proof is presented in Appendix.

Lemma 3. *If* z *has the properties (P1), (P3)–(P5), and satisfies (C1)–(C2), then there is* $\sigma > 0$ *such that* $p^* \cdot \hat{z}(p) \geq \sigma \|\hat{z}(p)\|^2$ *for all* $p > 0$*, where*

$$\hat{z}(p) = \begin{cases} Mz(p)/\|z(p)\| & \text{if } \|z(p)\| \geq M, \\ z(p) & \text{otherwise.} \end{cases}$$

With the lemmas 1–3 we are ready to prove Theorem 1.

Proof of Theorem 1. Let us first note that the process (5) can be expressed with the formula

$$p^{k+1} = p^k + \lambda_k \hat{z}(p^k),$$

where $\lambda_k = \min\{\gamma_k, 1\}$, and $\hat{z}$ is as defined in Lemma 3. When z has the properties (P1)–(P4) so does $\hat{z}$, and (P5) implies that $\hat{z}$ has the property

(P5') $\lim_{p^k \to p} [\max_{j \in J_p} \hat{z}_j(p^k)] > 0$, when $p \neq 0$, and $J_p = \{j : p_j = 0\} \neq \emptyset$.

Moreover, it is known from Lemma 3 that $p^* \cdot \hat{z}(p) \geq \sigma \|\hat{z}(p)\|^2$ holds for all $p > 0$ when z satisfies (C1)–(C2). Due to homogeneity p^* can be replaced by p^*/σ in (C1) and (C2); hence, without loss of generality we may suppose that $\sigma = 1$. It follows then from Lemma 1 that the iteration converges.

Let us show that due to (P5') the parameter λ_k has positive lower bound that is required in Lemma 1 to obtain convergence to a solution of (1). On the contrary, suppose that $\{\lambda_k\}_k$ has a subsequence that converges to zero. Since the sequence is decreasing this means that the whole sequence converges to zero. It then follows that $p^k \to p$, where some components of p are zero, i.e., $J_p \neq \emptyset$. This can be shown by observing that p^k cannot converge to positive price vector if λ_k is updated infinitely many times, which is the case as $\lambda_k \to 0$. Namely, assume that $\{p^k\}_k$ converges to a point $p > 0$. First, note that there has to be at least one commodity l for which there is a negative subsequence of $\{\hat{z}_l(p^k)\}$. Otherwise λ_k would be updated only finitely many times and it could not converge to zero. Because $p > 0$, for all $\varepsilon \in (0, p_l)$ there is $N_\varepsilon \geq 0$ such that $p_l^k > \varepsilon$ for all $k \geq N_\varepsilon$. By the iteration formula we have $p_l^k + \lambda_k \hat{z}_l(p^k) > \varepsilon$ for all $k \geq N_\varepsilon$ and consequently $\lambda_k > (\varepsilon - p_l^k)/\hat{z}_l(p^k)$ when $\hat{z}_l(p^k) \neq 0$ and $k \geq N_\varepsilon$. For the iteration indices i corresponding to the negative subsequence of $\{\hat{z}_l(p^k)\}_k$ we have $0 < (\varepsilon - p_l^i)/\hat{z}_l(p^i) \to 0$. Then either $p_l^i \to \varepsilon$ or $\hat{z}_l(p^i) \to -\infty$. The first is a contradiction with $p_l^k \to p_l$ and the latter is a contradiction with the convergence of $\{p^k\}_k$ and the continuity of $\hat{z}$. Hence, we have $J_p \neq \emptyset$ and $p_j^k \to 0$ for all $j \in J_p$. Thus, by the continuity of $\hat{z}$ and (P5') there are $l \in J_p$ and $N \geq 0$ such that $p_l^k \to 0$, and $\hat{z}_l(p^k) > 0$ for all $k \geq N$. Now we get from the iteration formula that $p_l^{k+1} > p_l^k$ for all $k \geq N$, which contradicts $p_l^k \to 0$. Hence, λ_k has a positive lower bound and convergence to a solution of (1) follows from Lemma 1.

Let us assume that there is a unique ray of solutions for (1). Then the process (5) converges to a point $\tilde{p} = \beta p^*$, where $\beta > 0$. From Lemma 3 we see that there is $\alpha > 0$ such that for $\alpha\tilde{p}$ we have $2\alpha\tilde{p} \cdot \hat{z}(p) \geq \|\hat{z}(p)\|^2$ for all $p > 0$. We can also note that λ_k is updated only finitely many times, since as shown above p^k cannot converge to a positive price vector if λ_k is updated infinitely many times. Hence, there is N such that $\lambda_k = \lambda_N$ for all $k \geq N$. Lemma 2 then implies monotonical convergence for $k \geq N$. □

Let us make some observations on the proof of Theorem 1. First, it was shown that the parameter γ_k does not converge to zero, which essentially means that the process (5) does not approximate (2) for large k.

Second, suppose the condition

$$p^* \cdot z(p) \geq \sigma \|z(p)\|^2 \tag{6}$$

holds for all $p > 0$ and z has the properties (P1)–(P3) and (P5'), see the proof Theorem 1, then Lemma 3 is not needed in showing the convergence of the process (5). Moreover, in that case we can set $\mu_k = \gamma_k$ in step 2, because z is bounded due to (6), namely $\|z(p)\| \leq \|p^*\|/\sigma$. Boundedness is, however, in contradiction with (P5), according to which the excess demand becomes infinitely large when some of the commodities become free. Therefore, it is reasonable to suppose that (6) holds only locally; that is exactly what the condition (C2) says.

Third, constructing an example where the process (5) fails to converge should be rather easy since there are such examples for the process (2), see, e.g., Scarf (1960). More interesting question is whether there are excess demand functions that satisfy (C1) but for which the iterative process does not converge. Conditions (C1) and (C2) guarantee that the sequence of prices obtained from the process (5) is bounded. Hence, we could expect the sequence of prices to be unbounded if only (C1) holds. This would be natural in view of results by Arrow and Hahn (1971), according to which (C1) implies convergence to any given neighborhood of the equilibrium ray but not necessarily to an equilibrium.

3.1 Curvature and Convergence

Theorem 1 shows that the process (5) converges when the set $\{x \in \mathbb{R}^n : x = z(p),\ p > 0\}$ is included in a specific ball at least around the origin. This property holds in Figure 1 because this set is not too flat around the origin. This section characterizes more closely the relationship between the convergence and the geometry of the hypersurface defined by a regular excess demand function.

Let us first define a parameterized hypersurface that can be obtained from an excess demand function z. Because z is homogeneous, one of the commodities, e.g., the last one, can be selected as a numéraire, which means that the price of this commodity is set to a constant and the other prices are considered as relative prices with respect to the price of this commodity. Let $\bar{p} \in \mathbb{R}^{n-1}$ denote the price vector that is obtained by dropping the last price of p. As a result we can define a mapping $\bar{z} : \mathbb{R}^{n-1}_{+} \mapsto \mathbb{R}^n$ by setting $\bar{z}(\bar{p}) = z(\bar{p}, 1)$. This mapping is a parameterized hypersurface in $\mathbb{R}^n$ and $\{x \in \mathbb{R}^n : x = \bar{z}(\bar{p}),\ \bar{p} > 0\}$ is the actual hypersurface obtained from z. Note that $z(p) = \bar{z}(\bar{p})$ when $p = (\bar{p}, 1)$, but due to homogeneity z as such is not an appropriate parameterized hypersurface.

In the rest of this section it will be assumed that $\bar{z}$ is twice continuously differentiable. Let $\nabla_j \bar{z}(\bar{p})$ denote the vector that is obtained by differentiating the component functions of $\bar{z}$ with respect to j'th argument. These vectors are the row vectors of the Jacobian matrix $\nabla \bar{z}(\bar{p})$ and we use them to define the regular points of the parameterized hypersurface $\bar{z}$.

Definition 1. Point $\bar{p}$ is a regular point of $\bar{z}$ if $\nabla_1 \bar{z}(\bar{p}), \ldots, \nabla_{n-1} \bar{z}(\bar{p})$ are linearly independent. A parameterized hypersurface $\bar{z}$ is said to be regular if all points $\bar{p} > 0$ are regular points of $\bar{z}$.

Let $N(\bar{p})$ be the unit normal of the tangent space of $\bar{z}$ at $\bar{p}$, i.e., the normal of the set $\{x : x = \nabla \bar{z}(\bar{p})d,\ d \in \mathbb{R}^{n-1}\}$. At a regular point $\bar{p}$ the tangent space of $\bar{z}$ is $n-1$ dimensional subspace, a hyperplane, spanned by the vectors $\nabla_j \bar{z}(\bar{p})$,

$j = 1, \ldots, n-1$. It follows from Walras' law that $N(\bar{p}^*) = p^*/\|p^*\|$, where $\bar{p}^* = (p_1^*/p_n^*, \ldots, p_{n-1}^*/p_n^*)$. Namely, Walras' law implies that $p \cdot \nabla z(p) = -z(p)$, which gives that $p^* \cdot [\nabla z(p^*)d] = -p^* \cdot z(p^*) = 0$, i.e., p^* is perpendicular to all tangent directions at p^*.

The normal curvature of a parameterized hypersurface can be defined as follows.

Definition 2. Let $\bar{p} > 0$ be a regular point of a parameterized hypersurface $\bar{z}$. The normal curvature of $\bar{z}$ at $\bar{p}$ to a tangent direction $\nabla \bar{z}(\bar{p})d$, $d \neq 0$, is

$$\kappa(d;\bar{p}) = \sum_{k=1}^{n} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left[N_k(\bar{p}) \frac{\partial^2 \bar{z}_k(\bar{p})}{\partial \bar{p}_i \partial \bar{p}_j} d_i d_j \right] / \|d\|^2. \tag{7}$$

Normal curvature measures how the normal direction of the hypersurface changes when moving from $\bar{z}(\bar{p})$ to a tangent direction. The change of the normal direction describes how the hypersurface curves at $\bar{z}(\bar{p})$. See, e.g., Spivak (1979, Sections 7.C–D) on deriving (7) from the basics of differential geometry.[2] In this paper (7) is taken as the definition of normal curvature.

The following lemma shows that the positive normal curvature of $\bar{z}$ to all tangent directions at equilibrium $\bar{p}^*$ is necessary and sufficient condition for z to satisfy (C2), when the regular parameterized hypersurface $\bar{z}$ is twice continuously differentiable. The proof is given in Appendix.

Lemma 4. *Let z be a twice continuously differentiable excess demand function having the properties (P1)–(P3) and an equilibrium at $p^* = (\bar{p}^*, 1)$, and let $\bar{z}$ be a regular parameterized hypersurface. Then z satisfies (C2) if and only if $\bar{z}$ has positive normal curvature at $\bar{p}^*$ to all tangent directions.*

From Theorem 1 and Lemma 4 we can prove the following convergence result according to which (P1)–(P5) together with positive normal curvature of $\bar{z}$ at $\bar{p}^*$ to all tangent directions guarantees the convergence of the process (5). Note that due to regularity and (C1) there is a unique ray of equilibria, which implies monotonical convergence.

Theorem 2. *Let z be twice continuously differentiable regular excess demand function that has the properties (P1)–(P5) and satisfies (C1) for $p^* = (\bar{p}^*, 1)$. Furthermore, let the normal curvature of the regular parameterized hypersurface $\bar{z}$ be positive at $\bar{p}^*$ to all tangent directions. Then the process (5) converges to an equilibrium for any $p^0 > 0$ and there is N such that convergence*

[2] In fact $\kappa(d;\bar{p}) = II(v,v)/I(v)$, where I is the first fundamental form, II is the second fundamental form, and $v = \nabla \bar{z}(\bar{p})d$. The expression (7) for the normal curvature follows from the properties of the second fundamental form.

is monotonical when $k \geq N$.

4 Convergence under the Gross Substitute Property

A differentiable excess demand function z is said to have the gross substitute property if $\partial z_j(p)/\partial p_i > 0$ for $j \neq i$. This property means that when the price of some commodity increases, the demand for other commodities grows. For such an excess demand function, the equilibrium is characterized by the system (1) and if p^* is an equilibrium then $p^* > 0$. Moreover, the set of equilibria is a unique ray, see, e.g., Arrow et al. (1959, Lemma 4).

It can be shown that under the gross substitute property the excess demand function satisfies (C1), see, e.g., Arrow et al. (1959, Lemma 5). It follows that the continuous time process (2) converges when the excess demand function has the gross substitute property. The following proposition shows a related result on the convergence of the process (5) when z satisfies (C2) in addition to having the gross substitute property. The condition (C2) can be replaced by the assumption that the parameterized hypersurface $\bar{z}$ has positive normal curvature at equilibrium to all tangent directions.

Proposition 1. *Let z be a differentiable excess demand function with the properties (P2)–(P4) and the gross substitute property. Let $p^* = (\bar{p}^*, 1)$ be an equilibrium.*

(a) If z satisfies (C2), then the process (5) converges to an equilibrium for any $p^0 > 0$.
(b) If $\bar{z}$ is twice continuously differentiable and has positive normal curvature to all tangent directions at $\bar{p}^*$, then the process (5) converges to an equilibrium for any $p^0 > 0$.

In both cases there is N such that convergence is monotonical when $k \geq N$.

Proposition 1 is based on theorems 1 and 2 and the following lemmas. Lemma 5 shows that in the gross substitute case z has the property (P5). The proof of Lemma 5 is presented in Appendix.

Lemma 5. *Suppose that z is homogeneous, satisfies Walras' law and has the gross substitute property. Then z has the property (P5).*

Lemma 6 shows that in the gross substitute case $\bar{z}$ is regular. The result follows from the well known fact that the rank of the Jacobian $\nabla z(p)$ is $n-1$ for all $p > 0$ when z has the gross substitute property, see, e.g., Hildenbrand and Kirman (1988, Section 6.4).

Lemma 6. *When z has the gross substitute property all points $\bar{p} > 0$ are regular points of $\bar{z}$.*

The result (a) of Proposition 1 follows from Theorem 1 and Lemma 5. The result (b) follows from Theorem 2 and Lemma 6. Furthermore, under gross substitution there is a unique ray of solutions, so that convergence is monotonical in both cases.

4.1 Cobb-Douglas Economy

In this section the convergence of the process (5) is explicitly shown for an economy in which the consumers' preferences are characterized by Cobb-Douglas utility functions that are of the form

$$u_i(x) = \prod_{j=1}^{n} x_j^{a_{i,j}},$$

where $a_{i,j} > 0$ and $\sum_j a_{i,j} = 1$ for all $i = 1, \dots, m$. It follows from each consumer's optimization problem that the j'th component of the consumer i's demand function is $x_j^i(p) = a_{i,j}(p \cdot w^i)/p_j$. Thus, the excess demand for the j'th commodity is $z_j(p) = (p \cdot q^j)/p_j - t_j$, where $q^j = \sum_i a_{i,j} w^i$ and $t_j = \sum_i w_j^i$. Let us suppose that $q_j^i > 0$ for all i, j, for example because $w_i^j > 0$ for all i, j. It can be seen that $\partial z_j(p)/\partial p_i = q_i^j/p_j > 0$ when $i \neq j$, i.e., the excess demand function of the Cobb-Douglas economy z has the the gross substitute property. Moreover, z has the properties (P1)–(P4).

For the convergence of the process (5) to an equilibrium, we need to show that the normal curvature of $\bar{z}$ is positive at $\bar{p}^*$ to all tangent directions. Let us begin with deriving the derivatives of $\bar{z}$ up to second order. The first derivatives of $\bar{z}$ at $\bar{p}$ are

$$\frac{\partial \bar{z}_j(\bar{p})}{\partial \bar{p}_k} = \begin{cases} q_k^n & \text{if } j = n, \\ (q_j^j \bar{p}_j - p \cdot q_j)/(\bar{p}_j)^2 & \text{if } k = j < n, \\ q_k^j/\bar{p}_k & \text{if } k \neq j < n, \end{cases}$$

where $p = (\bar{p}, 1)$, and the second derivatives are

$$\frac{\partial^2 \bar{z}_j(\bar{p})}{\partial \bar{p}_k \partial \bar{p}_l} = \begin{cases} 0 & \text{if } k,\ l \neq j \text{ or } j = n, \\ -2(q_j^j \bar{p}_j - p \cdot q_k)/(\bar{p}_j)^3 & \text{if } k = l = j < n, \\ -q_l^j/(\bar{p}_j)^2 & \text{if } k = j,\ l \neq j < n, \\ -q_k^j/(\bar{p}_j)^2 & \text{if } l = j,\ k \neq j < n. \end{cases}$$

Let us assume for simplicity that the unique ray of equilibria is $\{\lambda(1, \dots, 1) : \lambda > 0\}$. It can be shown that the general case, where (C1) holds for some equilibrium p^*, can be transformed such that (C1) holds for the transformed excess demand function with $\lambda(1,\ldots,1)$ in place of p^*, see Arrow et al. (1959, Section 3.1.1.0). Let us denote $p^* = (1,\ldots,1) = (\bar{p}^*, 1)$. It follows that $t_j = p^* \cdot q^j = \sum_{k=1}^n q_k^j$.

The normal curvature of $\bar{z}$ at $\bar{p}^*$ to a tangent direction defined by $d \in \mathbb{R}^{n-1}$, $d \neq 0$, is

$$\begin{aligned}\kappa(d;\bar{p}^*) &= 2\sum_{j=1}^{n-1}\left[(d_j)^2 t_j - \sum_{k=1}^{n-1}(q_k^j d_j d_k)\right] / \left[(n-1)^{1/2}\|d\|^2\right] \\ &= 2f(d)/\left[(n-1)^{1/2}\|d\|^2\right].\end{aligned}$$

To prove that κ is positive it is enough to show that the function

$$f(d) = \sum_j \left[(d_j)^2 t_j - \sum_k (q_k^j d_j d_k)\right]$$

is positive for all $d \in \mathbb{R}^{n-1}$ for which $\|d\| = \rho$. It turns out that the unique minimizer of f is $d = 0$ and $f(d) > f(0) = 0$ when $d \neq 0$.

The necessary condition for the minimum of f over $\mathbb{R}^{n-1}$ is

$$\frac{\partial f(d)}{\partial d_j} = 2(t_j - q_j^j)d_j - \sum_{\substack{k=1\\k\neq j}}^{n-1}(q_k^j d_k) = 0$$

for all j. Clearly, $d = 0$ satisfies this condition. Let us now show that $f(d)$ is strictly convex function by which it follows that the necessary condition is sufficient and $f(d) > f(0) = 0$ for all $d \neq 0$.

To see that $f(d)$ is strictly convex it is enough to show that its Hessian matrix is positive definite. The entry in the j'th row and k'th column of the Hessian is

$$b_{j,k} = \frac{\partial^2 f(d)}{\partial d_j \partial d_k} = \begin{cases} 2(t_j - q_j^j) & \text{if } j = k, \\ -q_k^j & \text{if } j \neq k. \end{cases} \tag{8}$$

The positive definiteness of the Hessian matrix follows from the observation that the Hessian is strictly positively diagonally dominant, which means that $b_{j,j} > 0$ and $|b_{j,j}| > \sum_{k\neq j}|b_{j,k}|$ for all $j = 1,\ldots,n-1$. From (8) we see that $|b_{j,k}| = q_k^j$ for $j \neq k$, and $b_{j,j} = |b_{j,j}| = 2(t_j - q_j^j)$, so that the Hessian has positive diagonal entries. Furthermore, it can be seen that the Hessian is, indeed, diagonally dominant:

$$|b_{j,j}| - \sum_{k\neq j}|b_{j,k}| = 2\left[(\sum_{k=1}^{n} q_k^j) - q_j^j\right] - \sum_{\substack{k=1\\k\neq j}}^{n-1} q_k^j = 2q_n^j + \sum_{\substack{k=1\\k\neq j}}^{n-1} q_k^j > 0.$$

As a conclusion $\kappa(d;\bar{p}^*) > 0$ holds for all $d \neq 0$. The convergence of the process (5) for a Cobb-Douglas economy follows then from Proposition 1.

5 Second Order Weak Axiom of Revealed Preferences

An excess demand function is said to satisfy the weak axiom of revealed preferences if for any pair of price vectors p^1 and p^2 for which $z(p^1) \neq z(p^2)$ it holds that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) > 0.$$

The interpretation of the WA is that if p^1 is revealed preferred to p^2, which means that the value of $z(p^2)$ with prices p^1 is negative, then p^2 cannot be revealed preferred to p^1. It can be seen that the WA implies (C1); hence, also the stability of the continuous time process (2). As was seen in Section 3 the excess demand function has to satisfy (C2) to obtain convergence for the process (5). Hence, we define a strengthened form of the WA, called the second order weak axiom of revealed preferences, which implies (C2) analogously as the WA implies (C1).

Definition 3. An excess demand function z satisfies the second order weak axiom of revealed preferences (SWA) if for any $p^2 > 0$ there is $\sigma > 0$ such that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) \geq \sigma \|z(p^1) - z(p^2)\|^2$$

The SWA means that if p^1 is revealed preferred to p^2, then p^2 is not revealed preferred to p^1 and the value of $z(p^1)$ with prices p^2 is bounded from below in proportion to the differences of the excess demands $\|z(p^1) - z(p^2)\|^2$. Note that in the definition of the SWA the constant σ depends on p^2.

It can be seen that the SWA implies (6) for all $p > 0$. Thus, we could say that (6) means that the second order weak axiom holds between the equilibrium vector p^* and any other price vector. As explained in Section 3 if z satisfies (6) for all $p > 0$, then it is bounded. Because excess demand functions are not necessarily bounded it is reasonable to assume that the SWA holds only around the equilibria. We say that z satisfies the SWA on E_ε if the SWA holds for all $p^1,\ p^2 \in E_\varepsilon$.

As a corollary of Theorem 1 we obtain the following convergence result for economies that satisfy the SWA.

Proposition 2. *Let z be an excess demand function that has the properties (P1)–(P5), and let p^* be an equilibrium. If z satisfies the WA for all $p > 0$ and the SWA on E_ε, then the process (5) converges to an equilibrium for any $p^0 > 0$.*

It is well known that the WA holds when the excess demand function is monotone or has a representative consumer with an appropriate preference relation. This arises the question whether there are similar economic conditions which imply the SWA. This section shows that if the excess demand function is a strongly monotone and Lipschitz continuous mapping, then it satisfies the SWA. Furthermore, if the economy has a representative consumer, whose preferences are characterized by a strongly concave utility function, then the excess demand function satisfies the SWA. The latter result is based on the strong monotonicity of the gradient mapping of a strongly concave function. It follows from Proposition 2 that when z has the properties (P1)–(P5) and satisfies one of the conditions presented in this section, then the process (5) converges globally to an equilibrium.

We first define some monotonicity concepts. Below, I denotes the $n \times n$ identity matrix.

Definition 4. Let S be a convex set in $\mathbb{R}^n$. Mapping $F : S \mapsto \mathbb{R}^n$ is monotone on S if the inequality $(p^1 - p^2) \cdot [F(p^1) - F(p^2)] < 0$ holds for all p^1, $p^2 \in S$ whenever $F(p^1) \neq F(p^2)$. If there is $\sigma > 0$ such that $F + \sigma I$ is monotone on S, then F is said to be strongly monotone on S.

Because excess demand functions are homogeneous, it is reasonable to define monotonicity for them by restricting the monotonicity condition to those price vectors that are somehow comparable to each other. An appropriate monotonicity concept for excess demand functions is obtained by requiring that the monotonicity condition holds for z with a pair of prices p^1 and p^2 if for some vector $y > 0$ we have $p^1 - p^2 \in T_y = \{x \in \mathbb{R}^n : y \cdot x = 0\}$. The condition $p^1 - p^2 \in T_y$ means that the value of commodity bundle y is the same for prices p^1 and p^2. Geometrically monotonicity means that the vector of price changes and the vector of demand changes point to the opposite half spaces. It can be shown that the excess demand function of a large economy, in which there is a continuum of consumers, is monotone when the income distribution of the economy has certain properties, see Hildenbrand (1983).

It is well known that when z is monotone in the sense that the monotonicity condition holds when $p^1 - p^2 \in T_y$, then z satisfies the WA. The SWA is related to the strong monotonicity of the excess demand function analogously, which is shown in Proposition 3, where in addition to monotonicity z is assumed to be Lipschitz continuous in the sense of the following definition. Note that due to homogeneity z cannot satisfy the ordinary Lipschitz condition $\|z(p^1) - z(p^2)\| \leq L\|p^1 - p^2\|$.

Definition 5. An excess demand function z is Lipschitz continuous on the cone $C \subset \mathbb{R}^n_+$ relative to vector $y > 0$ if there is a constant $L > 0$ such that the inequality $\|z(p^1) - z(p^2)\| \leq L\|\alpha_1 p^1 - \alpha_2 p^2\|$ holds for all p^1, $p^2 \in C$ when α_1, $\alpha_2 > 0$ satisfy $\alpha_k p^k - y \in T_y$ for $k = 1, 2$.

The following proposition shows that Lipschitz continuity and strong monotonicity imply the SWA.

Proposition 3. *Let the excess demand function z be Lipschitz continuous on E_ε relative to $y > 0$, and strongly monotone for all p^1, $p^2 \in E_\varepsilon$ that satisfy $p^1 - p^2 \in T_y$. Then z satisfies the SWA on E_ε.*

Proof. Let p^1, $p^2 \in E_\varepsilon$, $p^1 \neq p^2$, and $p^1 \cdot z(p^2) \leq 0$. Moreover, let the positive coefficients α_1 and α_2 be such that $\alpha_k p^k - y \in T_y$ for $k = 1, 2$. In that case we have $\alpha_1 p^1 - \alpha_2 p^2 \in T_y$ and $\alpha_1 p^1$, $\alpha_2 p^2 \in E_\varepsilon$. Note that E_ε is a cone, i.e., $\alpha p \in E_\varepsilon$ for all $\alpha > 0$. By homogeneity it holds that $z(\alpha_k p^k) = z(p^k)$ for $k = 1, 2$. From strong monotonicity and Walras' law we obtain

$$\begin{aligned} -\alpha_1 p^1 \cdot z(p^2) - \alpha_2 p^2 \cdot z(p^1) &= (\alpha_1 p^1 - \alpha_2 p^2) \cdot [z(p^1) - z(p^2)] \\ &\leq -\sigma\|\alpha_1 p^1 - \alpha_2 p^2\|^2. \end{aligned}$$

It follows that

$$\alpha_2 p^2 \cdot z(p^1) - \sigma\|\alpha_1 p^1 - \alpha_2 p^2\|^2 \geq -\alpha_1 p^1 \cdot z(p^2) \geq 0.$$

From the Lipschitz continuity relative to y we get

$$\sigma\|z(p^1) - z(p^2)\|^2/L^2 \leq \sigma\|\alpha_1 p^1 - \alpha_2 p^2\|^2 \leq \alpha_2 p^2 \cdot z(p^1).$$

Hence, the SWA condition holds with the constant $\sigma/(L^2\alpha_2)$. □

Due to the result of Proposition 3 it would be natural to call the SWA as the strong axiom of revealed preferences. Strong axiom, however, usually refers to the following indirect form of the WA: for any $N \geq 2$ the inequalities $p^k \cdot z(p^{k+1}) \leq 0$, $k = 1, \ldots, N-1$, imply that $p^N \cdot z(p^1) > 0$.

There is another relationship between the SWA and strongly monotone mappings in addition to the one described above. Namely, if the economy has a representative consumer whose preferences can be characterized by a locally strongly concave utility function u (see the definition below), then the economy satisfies the SWA around the equilibrium ray. A representative consumer means a preference relation for which $\sum_i x^i(p)$ equals the demand function obtained by maximizing this preference relation under the budget constraint $p \cdot (x - \sum_i w^i) \leq 0$.

Definition 6. A differentiable function u is strongly concave on a convex set

S if ∇u is strongly monotone on S.[3]

See, e.g., Rockafellar and Wets (1998, Section 12.H) for more about strongly monotone mappings and convex functions. In the framework of exchange economies Shannon and Zame (2002) have utilized strong concavity to show determinacy of equilibrium.[4] Note that differentiable strictly concave functions could be defined similarly by requiring the gradient mapping to be monotone.

The relationship of the SWA and the representative consumer with a strongly concave utility function is stated in the following proposition. In addition to strong concavity we need local nonsatiation, which means that in any environment of a commodity bundle there are more desirable bundles. This condition guarantees that Walras' law is satisfied.

Proposition 4. *Let a locally nonsatiated preference relation be characterized by a strictly concave utility function u that is strongly concave on $B(\sum_i w^i, \delta)$. Let $x(p)$ be the demand function that is obtained by maximizing $u(x)$ subject to the budget constraint $p \cdot (x - \sum_i w^i) \leq 0$ and let the prices be positive at equilibrium. Then the excess demand function $z(p) = x(p) - \sum_i w^i$ satisfies the SWA on E_ε for some $\varepsilon > 0$.*

Proof. First, note that the demand function $x(p)$ is continuous because u is a strictly concave function, see, e.g., Hildenbrand and Kirman (1988, Proposition 3.1). Under local nonsatiation and concavity, the necessary and sufficient optimality condition for maximizing u over the budget $p^k \cdot (x - \sum_i w^i) \leq 0$ is $\nabla u(x(p)) = \lambda_k p^k$ for some $\lambda_k > 0$ when $x(p^k) > 0$. Because prices are positive at the equilibrium, i.e., $p^* > 0$, there is $\bar{\varepsilon} > 0$ such that $x(p) > 0$ when $p \in E_{\bar{\varepsilon}}$. Note that we have $x(p^1) - x(p^2) = z(p^1) - z(p^2)$, and local nonsatiation implies Walras' law. These facts and strong concavity yield

$$\begin{aligned} \left[\nabla u\left(x(p^1)\right) - \nabla u\left(x(p^2)\right)\right] \cdot \left[x(p^1) - x(p^2)\right] &= \\ \left(\lambda_1 p^1 - \lambda_2 p^2\right) \cdot \left[z(p^1) - z(p^2)\right] &= \\ -\lambda_1 p^1 \cdot z(p^2) - \lambda_2 p^2 \cdot z(p^1) &\leq -\sigma \|z(p^1) - z(p^2)\|^2. \end{aligned}$$

By rearranging the terms in the bottom line and dividing with λ_2 we obtain

$$p^2 \cdot z(p^1) - (\sigma/\lambda_2)\|z(p^1) - z(p^2)\|^2 \geq -(\lambda_1/\lambda_2) p^1 \cdot z(p^2) \geq 0,$$

where the latter inequality holds when p^1 is revealed preferred to p^2. Thus, z satisfies the SWA when p^1 and p^2 are chosen such that $x(p^1),\ x(p^2) \in$

[3] In a non-differentiable case the definition is the same except that the gradient is replaced with subgradient. Differentiability is assumed here for simplicity.
[4] Shannon and Zame (2002) call strong concavity as quadratic concavity.

$B(\sum_i w^i, \delta)$ and $x(p^1),\ x(p^2) > 0$. It follows from this result and the continuity of x that there is $\varepsilon \leq \bar{\varepsilon}$ such that z satisfies the SWA on $p \in E_\varepsilon$. $\square$

6 Conclusion

An extensive part of literature has concentrated on normalized processes, e.g., processes in which one of the prices is selected as a numéraire and only the rest of them are adjusted. This paper shows that a non-normalized process that is a slight modification of fixed-point iteration $p^{k+1} = p^k + z(p^k)$ converges under conditions that are remarkably close to the continuous time convergence conditions. Indeed, price normalization seems to lead to chaos whereas non-normalized process has better convergence properties.

This paper has also introduced a second order form of the weak axiom of revealed preferences that implies convergence of iterative tâtonnement. This condition has the same economic interpretation as the ordinary weak axiom of revealed preferences but mathematically the condition is more stringent. Actually, the ordinary weak axiom is obtained as a limiting case from the second order version. For practical or numerical considerations the difference between the convergence conditions of the usual continuous time process and the conditions obtained in this paper are quite negligible. This is because the continuous time convergence condition is obtained as the limit from the discrete time convergence condition as the hypersurface defined by the excess demand function becomes flatter.

Appendix: Proofs of the Lemmas

Proof of Lemma 1. Let us first observe that

$$\mu_k p^* \cdot z(p) \geq \|\mu_k z(p)\|^2, \tag{9}$$

when $\mu_k \leq 1$. This can be seen by multiplying both sides of $p^* \cdot z(p) \geq \|z(p)\|^2$ with μ_k^2 and noticing that $\mu_k^2 p^* \cdot z(p) \leq \mu_k p^* \cdot z(p)$ because $\mu_k \leq 1$.

From (9) and Walras' law we have

$$\|p^{k+1} - p^*\|^2 = \|p_k + \mu_k z(p^k) - p^*\|^2 =$$
$$\|\mu_k z(p^k)\|^2 - 2\mu_k z(p^k) \cdot p^* + \|p^k - p^*\|^2 \leq \|p^k - p^*\|^2.$$

Note that p^k belongs to $B(p^*, r)$ for all $k = 0, 1, \ldots$, when $p^0 \in B(p^*, r)$. Therefore, the sequence $\{\|p^k - p^*\|\}_k$ converges and as a result the sequence

$\{\|p^k\|\}_k$ is bounded. From Walras' law it follows that

$$\|p^k\|^2 = \|p^0\|^2 + \sum_{i=0}^{k-1} \mu_i^2 \|z(p^i)\|^2,$$

so that $\{\|p^k\|\}_k$ is a growing and bounded sequence and hence convergent. The iteration formula yields

$$p^k = p^0 + \sum_{i=0}^{k-1} \mu_i z(p^i).$$

Hence, $\|p^0 + \sum_{i=0}^{k-1} \mu_i z(p^i)\|$ converges, too. From the triangular inequality we get

$$\|p^0 + \sum_{i=0}^{k+l} \mu_i z(p^i)\| \geq \left| \|p^0 + \sum_{i=0}^{k} \mu_i z(p^i)\| - \| \sum_{i=k+1}^{k+l} \mu_i z(p^i)\| \right|$$

and we obtain

$$\|p^{k+l} - p^k\| = \| \sum_{i=k+1}^{k+l} \mu_i z(p^i)\| \to 0, \tag{10}$$

when $k \to \infty$ and $l \geq 1$. Thus, $\{p^k\}_k$ is a Cauchy sequence and hence convergent. Let $\tilde{p}$ denote the limit point of this Cauchy sequence.

Let us now show that when $0 < \bar{\mu} \leq \mu_k$ the sequence $\{p^k\}_k$ converges to a solution of $z(p) = 0$. By setting $l = 1$ it follows from (10) that $\mu_k \|z(p^k)\| \to 0$. Because it holds that $\bar{\mu}\|z(p^k)\| \leq \mu_k \|z(p^k)\|$ and z is continuous, we see that $\tilde{p}$ is a solution of $z(p) = 0$. □

Proof of Lemma 2. If $\|z(p)\|^2 \leq 2\alpha z(p) \cdot \tilde{p}$ holds for $\alpha > 0$ then it holds for any $\bar{\alpha} > \alpha$. Specifically, we can choose $\bar{\alpha} > 0$ such that this condition holds for $p^* = \bar{\alpha}\bar{p} - 2\bar{p}$ instead of $\alpha\bar{p}$. Moreover we can take α such that $\|z(p)\|^2 < 2\alpha z(p) \cdot \tilde{p}$ if p is not a solution. Similarly as in Lemma 1 we can deduce that $\|p^{k+1} - p^*\|^2 < \|p^k - p^*\|^2$, and $\|p^{k+1} - \alpha\tilde{p}\|^2 < \|p^k - \alpha\tilde{p}\|^2$ when p^k is not a solution. From parallelogram law we get

$$\|p^k - \alpha\tilde{p}\|^2 + \|p^k - p^*\|^2 = 2\|p^k - \tilde{p}\|^2 + 2(\alpha - 1)\|\tilde{p}\|^2.$$

By rearranging the terms we have

$$\begin{aligned} 2\|p^k - \tilde{p}\|^2 &= 2(\alpha - 1)\|\tilde{p}\|^2 - \|p^k - \alpha\tilde{p}\|^2 - \|p^k - p^*\|^2 \\ &> 2(\alpha - 1)\|\tilde{p}\|^2 - \|p^{k+1} - \alpha\tilde{p}\|^2 - \|p^{k+1} - p^*\|^2 = 2\|p^{k+1} - \tilde{p}\|, \end{aligned}$$

and hence $\{p^k\}_k$ converges monotonically to $\tilde{p}$. □

Proof of Lemma 3. Let z satisfy (C2) on $E_{\bar{\varepsilon}} = \{p \in \mathbb{R}^n_+ : \|z(p)\| < \bar{\varepsilon}\}$ with constant $\bar{\sigma}$. By the homogeneity of excess demand we know that $\hat{z}$ obtains all its values on the unit simplex $\Delta = \{p \in \mathbb{R}^n_+ : \sum_j p_j = 1\}$. Because of (P4) and (P5) it can be seen that $p^* \cdot z(p^k) \to \infty$, when $p^k \to p$ and $J_p \neq \emptyset$. As a result, we have

$$\lim_{p^k \to p} p^* \cdot \hat{z}(p^k) > 0,$$

when $J_p \neq \emptyset$. From this property, continuity, and (C1), it follows that there is $\delta > 0$ such that $p^* \cdot \hat{z}(p) \geq \delta$ for all $p \in \Delta \setminus S$, where $S = \{p \in \Delta : p_j > \varepsilon' \ \forall j = 1, \ldots, n\}$ and $\varepsilon' > 0$ is chosen such that $E_{\bar{\varepsilon}} \cap \Delta \subset S$.

Clearly, the infimum of $p^* \cdot \hat{z}(p)$ over $S \setminus E_{\bar{\varepsilon}}$ is positive, since otherwise $\hat{z}$ would violate (C1). Let $\alpha > 0$ denote this infimum. We have $p^* \cdot \hat{z}(p) \geq \min\{\delta, \alpha\}$ for all $p \in \Delta \setminus E_{\bar{\varepsilon}}$. Because $\|\hat{z}(p)\| \leq M$ we get $p^* \cdot \hat{z}(p) \geq \hat{\sigma}\|\hat{z}(p)\|^2$ for all $p \in \Delta \setminus E_{\bar{\varepsilon}}$ by choosing $\hat{\sigma} < \min\{\delta, \alpha\}/M^2$. The result follows by setting $\sigma = \min\{\bar{\sigma}, \hat{\sigma}\}$. □

Proof of Lemma 4. From Taylor's formula we get

$$\bar{z}_k(\bar{p}^* + d) = \bar{z}_k(\bar{p}^*) + \nabla \bar{z}_k(\bar{p}^*) \cdot d + \frac{1}{2} d \cdot \nabla^2 \bar{z}_k(\bar{p}^*) \cdot d + o(\|d\|^2), \tag{11}$$

where $d \in \mathbb{R}^{n-1}$ is such that $\bar{p}^* + d > 0$ and $o(\|d\|^2)/\|d\|^2 \to 0$ as $\|d\| \to 0$. Here $\nabla \bar{z}_k$ denotes the gradient of k'th component function of $\bar{z}$. Furthermore, vectors are considered as column vectors and x' denotes the transpose of vector x.

Recall that Walras' law gives $N(\bar{p}^*) = p^*/\|p^*\|$. Furthermore, we have

$$\nabla \bar{z}(\bar{p}) = \begin{bmatrix} \nabla \bar{z}_1(\bar{p})' \\ \vdots \\ \nabla \bar{z}_n(\bar{p})' \end{bmatrix} = \begin{bmatrix} \nabla_1 \bar{z}(\bar{p}) & \cdots & \nabla_{n-1} \bar{z}(\bar{p}) \end{bmatrix},$$

and because $\nabla \bar{z}(\bar{p}) d$ is a tangent direction of the parameterized hypersurface $\bar{z}$ at $\bar{z}(\bar{p})$, we get $p^* \cdot [\nabla \bar{z}(\bar{p}^*) d] = 0$. From this together with (11) and $\bar{z}(\bar{p}^*) = 0$ we obtain

$$2p^* \cdot \bar{z}(\bar{p}^* + d) = \sum_{i,j,k} p_k^* \frac{\partial \bar{z}_k(\bar{p}^*)}{\partial \bar{p}_i \partial \bar{p}_j} d_i d_j + o(\|d\|^2),$$

where in the summation i and j run from 1 to $n-1$ and k runs from 1 to n. By the definition of normal curvature this can be written as

$$2p^* \cdot \bar{z}(\bar{p}^* + d) = \|p^*\| \kappa(d; \bar{p}^*) \|d\|^2 + o(\|d\|^2). \tag{12}$$

From Taylor's formula (11) we also get

$$\|\bar{z}(\bar{p}^* + d)\|^2 = \bar{z}(\bar{p}^* + d) \cdot \bar{z}(\bar{p}^* + d) = d \cdot [\nabla \bar{z}(\bar{p}^*)' \nabla \bar{z}(\bar{p}^*) d] + o(\|d\|^2), \tag{13}$$

where $\nabla\bar{z}(\bar{p}^*)'$ is the transpose of the Jacobian matrix.

Because $\bar{p}^*$ is a regular point of $\bar{z}$, the Jacobian is full rank matrix. Therefore, the matrix $A = \nabla\bar{z}(\bar{p}^*)'\nabla\bar{z}(\bar{p}^*)$ is positive definite. It is known from linear algebra that for a symmetric matrix A it holds that

$$\beta_L\|d\|^2 \leq d \cdot (Ad) \leq \beta_U\|d\|^2, \tag{14}$$

where β_L and β_U are the minimal and maximal eigenvalues of A, respectively.

When the curvature is positive to all directions, we have

$$\kappa^* = \min_{\|d\|=\rho} \kappa(d;\bar{p}^*) > 0,$$

because as a continuous function κ attains its minimum over $\partial B(0,\rho) = \{d \in \mathbb{R}^{n-1} : \|d\| = \rho\}$, where $\rho > 0$ is chosen such that $\bar{p}^*+d > 0$ for all $d \in \partial B(0,\rho)$. An appropriate ρ can be found because $p^* > 0$. By choosing $\alpha > \beta_U/(\|p^*\|\kappa^*)$ we obtain the following inequality from (12) and (13):

$$2\alpha p^* \cdot \bar{z}(\bar{p}^* + d) \geq \|\bar{z}(\bar{p}^* + d)\|^2,$$

when $\|d\| \leq \rho$. Hence, z satisfies (C2) around the equilibrium ray with the constant $\sigma = 1/(2\alpha)$.

To conclude the proof it needs to be shown that (C2) implies that $\bar{z}$ has positive normal curvature at $\bar{p}^*$ to all tangent directions. Without loss of generality we may assume that $\sigma = 1$ in (C2). It follows that there is $\rho > 0$ such that $2p^* \cdot \bar{z}(\bar{p}) \geq \|\bar{z}(\bar{p})\|^2$ for all $\bar{p} \in B(\bar{p}^*,\rho)$. From (12), (13), and (14) we get

$$\|p^*\|\kappa(d;\bar{p}^*)\|d\|^2 + o(\|d\|^2) \geq \beta_L\|d\|^2,$$

for all $d \in B(0,\rho)$, and consequently $\kappa(d;\bar{p}^*) \geq \beta_L/\|p^*\| > 0$, i.e., the normal curvature of $\bar{z}$ is positive to all tangent directions at $\bar{p}^*$. □

Proof of Lemma 5. Let us suppose that $p^k \to p$ as $k \to \infty$. Without loss of generality we may suppose that the first l prices of p are zero, i.e., $J_p = \{1,\ldots,l\}$, $l < n$. By homogeneity we can choose an equilibrium vector p^* such that $p_j > p_j^*$ for all $j \notin J_p$. Moreover, there is N such that when $k \geq N$, we have $p_j^k < p_j^*$ for all $j \in J_p$, and $p_j^k > p^*$ for all $j \notin J_p$.

According to Walras' law

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*,\ldots,p_l^*,p_{l+1}^k,\ldots,p_n^k) = -\sum_{j=l+1}^{n} p_j^k z_j(p_1^*,\ldots,p_l^*,p_{l+1}^k,\ldots,p_n^k). \tag{15}$$

The gross substitute property implies that $z_j(p^k) < z_j(p_1^*,\ldots,p_l^*,p_{l+1}^k,\ldots,p_n^k)$

for $j \notin J_p$ and $k \geq N$, because $p_j^k < p_j^*$, $j \in J_p$. Thus, from (15) we obtain

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*, \ldots, p_l^*, p_{l+1}^k, \ldots, p_n^k) < -\sum_{j=l+1}^{n} p_j^k z_j(p^k) = \sum_{j=1}^{l} p_j^k z_j(p^k), \tag{16}$$

where the last equality is from Walras' law.

Let us make a counter assumption that $z_1(p^k), \ldots, z_l(p^k)$ are bounded above. Taking limits from both sides of (16) as $k \to \infty$ yields

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*, \ldots, p_l^*, p_{l+1}, \ldots, p_n) \leq 0, \tag{17}$$

because $\lim_{k\to\infty} \sum_{j=1}^{l} p_j^k z_j(p^k) \leq 0$ by the boundedness. From the gross substitute property, on the other hand, it follows that when $j \in J_p$ we have

$$z_j(p_1^*, \ldots, p_l^*, p_{l+1}, \ldots, p_n) > z_j(p^*) = 0. \tag{18}$$

Recall that p^* was chosen such that $p_j > p_j^*$ for all $j \in J_p$. Clearly, (17) leads to contradiction with (18). Thus, at least one of $z_1(p^k), \ldots, z_l(p^k)$ becomes infinitely large as $p_j^k \to 0$ for all $j \in J_p$. □

References

Arrow, K. J., Block, H. D., Hurwicz, L., 1959. On the stability of the competitive equilibrium, II. Econometrica 27, 82–109.

Arrow, K. J., Hahn, F. H., 1971. General Competitive Analysis. North-Holland, Amsterdam.

Arrow, K. J., Hurwicz, L., 1958. On the stability of the competitive equilibrium, I. Econometrica 26, 522–552.

Arrow, K. J., Hurwicz, L., 1960. Competitive stability under weak gross substitutability: The "Euclidean distance" approach. International Economic Review 1 (1), 38–49.

Bala, V., Majumdar, M., 1992. Chaotic tâtonnement. Economic Theory 2 (4), 437–445.

Day, R. H., Pianigiani, G., 1991. Statistical dynamics and economics. Journal of Economic Behavior and Organization 16 (1–2), 37–84.

Goeree, J. K., Hommes, C., Weddepohl, C., 1998. Stability and complex dynamics in a discrete tâtonnement model. Journal of Economic Behavior and Organization 33 (3), 395–410.

Hildenbrand, W., 1983. On the "law of demand". Econometrica 51 (4), 997–1020.

Hildenbrand, W., Kirman, A. P., 1988. Equilibrium Analysis: Variations on the Themes by Edgeworth and Walras. North-Holland, Amsterdam.

Mas-Colell, A., Whinston, M. D., Green, J. R., 1995. Microeconomic Theory. Oxford Univ. Press, New York.

Mukherji, A., 1999. A simple example of complex dynamics. Economic Theory 14, 741–749.

Rockafellar, R. T., Wets, R. J.-B., 1998. Variational Analysis. Springer, Berlin.

Saari, D. G., 1985. Iterative price mechanisms. Econometrica 53 (5), 1117–1132.

Saari, D. G., 1995. Mathematical complexity of simple economics. Notices of the AMS 42 (2), 222–230.

Samuelson, P. A., 1947. Foundations of Economic Analysis. Harvard University Press, Cambridge, MA.

Scarf, H., 1960. Some examples of global instability of competitive equilibrium. International Economic Review 1, 157–172.

Shannon, C., Zame, W., 2002. Quadratic concavity and determinacy of equilibrium. Econometrica 70 (2), 631–662.

Spivak, M., 1979. A Comprehensive Introduction to Differential Geometry, Vol. 4. Publish or Perish, Massachusettes.

Tuinstra, J., 2000. A discrete and symmetric price adjustment process on the simplex. Journal of Economic Dynamics and Control 24 (5–7), 881–907.

Uzawa, H., 1960. Walras' tâtonnement in the theory of exchange. Review of Economic Studies 27, 182–194.