ADJUSTMENT OF AN AFFINE CONTRACT WITH FIXED-POINT ITERATION

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ABSTRACT. We study a principal-agent game where the principal commits to an affine contract. We suppose that the principal has incomplete information but he can adjust the contract according to the myopically behaving agent's reactions when the game is played repeatedly. The adjustment process can be considered as a learning model. We derive convergence conditions for fixed-point iteration as an adjustment scheme and study a related continuous time process. The analysis is based on parameterizing the problem such that we obtain a degree zero homogeneous system of equations, where the nonlinear mapping satisfies Walras' law.

1. INTRODUCTION

Contract design problems and their applications have been widely studied in the literature of game theory and economics in principal-agent framework. In a principal-agent model the principal offers the agents one or several contracts that the agents can either accept or reject. This setting has several variations depending on the principal's information on the agents' types and actions. See Macho-Stadler and Pérez-Castrillo (2001), Salanié (1997) for textbooks on contract design with asymmetric information.

Games where the principal moves first and has incomplete information on the agents' preferences, but can observe their actions, are called mechanism design or adverse selection models. These problems can be analyzed as Bayesian games where the principal's task is to design a set of incentive compatible contracts, sometimes called mechanisms, given the beliefs over the agents' possible types. If the principal observes the agents' actions imperfectly the model is known as a hidden knowledge or a moral hazard problem.

In this paper we consider a two-player game, where the principal commits to a contract that is an affine mapping of the agent's actions. We suppose that the principal has incomplete information but instead of Bayesian approach we assume that the game with the same players is played repeatedly and the principal can adjust the contract according to the observations on the agent's behavior. We show how the complete information equilibrium can be reached by adjusting the contract with fixed-point iteration when supposing that the agent acts myopically.

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In the literature on repeated adverse selection models the focus has been on commitment and renegotiation issues and the analysis is usually based on the type parameterization. In a repeated principal agent game where the principal knows the agent's utility function except for one parameter there is no need for adjustment because the principal knows the agent's utility function completely after the first round. In some cases, however, we may not have a type parameterization and adjustment becomes the only choice.

In the recent literature of game theory, processes where the players adjust their strategies have been studied in the framework of learning models, see Fudenberg and Levine (1999). A simple example of a learning scheme is the Cournot process in which the players use their best responses sequentially to the opponents' latest moves. Ehtamo et al. have recently studied adjustment of a linear wage contract in a simple principal-agent setting by using a three-phase procedure, Ehtamo et al. (2002). Adjustment processes have also been studied in the stability analysis of Walrasian equilibria and we shall briefly discuss how the adjustment of an affine contract presented here is related to the stability of Walrasian equilibrium.

The contents of the paper are as follows. In Section 2 we present the principal-agent game with complete information and discuss the existence of solution for the principal's contract design problem. In Section 3 we parameterize the game such that the contract design problem can be formulated as a system of equations to be solved. Furthermore, we study the properties of the parameterized problem. The results of sections 2 and 3 are based on concavity properties of the agent's utility function.

In Section 4 we show how fixed-point iteration can be used in adjusting an affine contract when the two-player game defined in Section 2 is played repeatedly and the principal is supposed to have incomplete information. We derive new convergence conditions for fixed-point iteration from the properties of the parameterization of Section 3. In Section 5 we discuss the corresponding continuous time adjustment process and show the similarities of the contract design problem and price adjustment in exchange economies. In sections 4 and 5 we assume that the principal's optimum is such that the agent always participates the game when the contract passes through that point. In Section 6 we briefly characterize processes that work for finding the principal's optimum in the case where the agent does not always participate the game.

2. A Contract Design Game and the Complete Information Solution

In this section we define the principal-agent game and derive conditions for the existence of an affine solution for a contract design game with complete information. Although we adopt principal-agent terminology the words principal and agent do not refer to any specific agency problem.

We suppose that there are two utility maximizing players, a principal and an agent with utility functions v, $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, respectively. The principal's decision variable is $y \in \mathbb{R}^m$ and the agent's is $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$, $\mathbb{R}^n$ and in their product space.

In a game of incomplete information the principal does not know the agent's utility function. Usually it is assumed that the agent's utility function is determined by a type parameter $\theta \in \{\theta_1, \ldots, \theta_N\}$ that is unknown for the principal, who, however, has a probability distribution over the possible values of the parameter. The agent knows his own type. There are two interpretations for using type parameters. One is that there is a population of agents whose types are drawn from a probability distribution. Alternatively we could assume that there is only a single agent whose type is random from the principal's view. In this paper we shall not use type parameterization but our approach is close to the latter one because we shall assume that the principal meets the same or similar agent when the game is played repeatedly.

In the contract design game the principal offers the agent a menu of contracts $\gamma_i(x)$, $i = 1, \ldots, N$, and commits to make his decision according to the contract that the agent chooses. The agent may also reject all the contracts in which case he obtains his reservation utility $\bar{u}$. After the agent chooses to sign a contract γ_i he makes a decision x' and the principal implements the contract, i.e., chooses the action $y' = \gamma_i(x')$, and the game ends. The principal's problem is to design the menu of contracts that maximizes his expected utility such that the agent obtains at least his reservation utility and chooses the contract intended for his type.

There is a wide variety of applications for contract design games. For example, the principal could be a seller who offers a buyer a price tariff that specifies the prices y of the goods for any amounts x to be bought. Nonlinear pricing with a multi-product monopoly has been studied, e.g., in Roberts (1979), Mirman and Sibley (1980), Spence (1980).

In this paper we shall assume that the principal observes perfectly the agent's move. Furthermore, in this and the next section we assume that the principal has complete information, that is to say he knows the agent's utility function or type. We shall show under mild technical assumptions that the principal gets his optimum with a single affine contract. Affine contracts are often considered as simple to monitor and implement. Compared to discontinuous contracts, e.g., various threshold contracts, affine contracts have the advantage that the risk of losses becomes low for small deviations from the principal's optimum. In a buyer-seller game an affine price tariff could be implemented by specifying the unit prices plus possible fixed prices for each good to be sold.

The contract design problem is defined as follows. Find a contract $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ that maximizes $v(x_\gamma, \gamma(x_\gamma))$ over all feasible contract mappings γ, where x_γ solves

$$\max_x u(x, \gamma(x)) \tag{1}$$

and

$$u(x_\gamma, \gamma(x_\gamma)) \geq \bar{u}. \tag{2}$$

Equation (2) is called the agent's participation constraint. We next show that there is an affine contract of the form

$$\gamma(x) = y_0 + Lx, \tag{3}$$

where L is a linear mapping ($m \times n$ matrix) from $\mathbb{R}^n$ into $\mathbb{R}^m$, and y_0 is a fixed vector in $\mathbb{R}^m$, that solves the contract design problem.

Suppose that x^*, y^* solves

$$\max_{(x,y)\in D} \quad v(x,y),$$

where D is the set of points that satisfy the agent's participation constraint $u(x,y) \geq \bar{u}$. Because the pair x^*, y^* is the best outcome the principal can hope to get in the game, we can restrict our attention to those affine contracts that pass through x^*, y^*, i.e., $y^* = \gamma(x^*)$. Thus for any contract of the form (3) we should have $y_0 = y^* - Lx^*$. Hence, we can without loss of generality assume that the principal gives his contracts in the form

$$\gamma(x) = y^* + L(x - x^*). \tag{4}$$

Note that since the contract goes through x^*, y^*, which belongs to D, the agent will get at least his reservation utility when accepting the contract. Thus he will always participate the game. We shall assume throughout this paper that there are no constraints other than the participation constraint for the principal's and the agent's decisions.

The affine contract design problem can now be defined as follows. Find a contract of the form (4), i.e., find an $m \times n$ matrix L such that x^*, y^* solves

$$\max_{x,y} \quad u(x,y) \tag{5}$$

$$\text{s.t.} \quad y = y^* + L(x - x^*). \tag{6}$$

The contract γ that solves the above problem is called a contract at x^*, y^* and the pair (x^*, γ) is a Nash equilibrium for the game.

Let us suppose that the agent's objective function u is concave and let us denote the set of all subgradients of u at x^*, y^*, i.e., the subdifferential of u at x^*, y^*, by $\partial u(x^*, y^*)$. By a subgradient at x^*, y^* we mean a pair $(\xi_x, \xi_y) \in \mathbb{R}^{n+m}$ that satisfies

$$u(x^*, y^*) - u(x,y) \geq \xi_x^T(x - x^*) + \xi_y^T(y - y^*) \quad \forall (x,y) \in \mathbb{R}^{n+m}.$$

The subdifferential of a concave function u is a non-empty set and if the function is differentiable at x^*, y^* then the subdifferential is a singleton and equals the gradient of u at x^*, y^*.

For given L necessary and sufficient optimality condition for (5), (6) at x^*, y^* is that

$$\partial u(x^*, y^*) \cap \{(\xi_x, \xi_y) \in \mathbb{R}^{n+m} \mid \xi_x + L^T \xi_y = 0\} \neq \emptyset. \tag{7}$$

Geometrical interpretation of condition (7) is that there is a subgradient of u at x^*, y^* that is normal to the affine set defined by (6).

Now, suppose that $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$ is such that $\xi_y \neq 0$. Then there is a contract of the form (6) at x^*, y^*. One possible L satisfying (7) is given by

$$L = -\xi_y \xi_x^T / \|\xi_y\|^2. \tag{8}$$

Usually there are also other affine contracts than the one defined by (8). Suppose, e.g., that the dimension of x equals the dimension of y, i.e., $m = n$, and suppose $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$ is such that all components of ξ_y are nonzero. Then we can choose L to be a diagonal matrix. In a multi-product buyer-seller situation the corresponding tariff can be specified by giving a unit price for each good to be sold.

We collect the essential of the above discussion to the following.

Theorem 2.1. *If u is concave and has a subgradient ξ_x, ξ_y at x^*, y^* such that $\xi_y \neq 0$, then there is a solution to the affine contract design problem. Furthermore, a mapping of the form (4) is a contract at x^*, y^* if and only if the x, y points satisfying (6) belong to the hyperplane*

$$\{(x, y) \in \mathbb{R}^{n+m} \mid \xi_x^T(x - x^*) + \xi_y^T(y - y^*) = 0\} \tag{9}$$

for some $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$.

The latter part of the theorem is equivalent to the necessary and sufficient optimality condition (7). Also note that if u is differentiable the requirement $\xi_y \neq 0$ becomes $\nabla_y u(x^*, y^*) \neq 0$, which means that the agent's utility is sensitive for the changes of y around x^*, y^*.

The rather general formulation of the contract design problem in this way is inspired by some early papers in the field of control theory and differential games. Affine contract design problems, or affine incentive design problems as they are called in these papers, and their relation to incentive problems in economics is discussed in Ho et al. (1982). For mathematical analysis of affine incentive design problems in dynamic game settings of complete information see Başar (1984), Ehtamo and Hämäläinen (1993).

3. Parameterization of the Problem

Theorem 2.1 suggests us to parameterize the principal's problem. Let us denote the subgradients of u appearing in (9) by parameter vectors p_x and p_y, and denote the column vector composed of p_x and p_y by p. The contract design problem can then be formulated as follows: Find $p \in \mathbb{R}^{n+m}$, $p_y \neq 0$, such that x^*, y^* solves

$$\begin{aligned} &\max_{x,y} \;\; u(x, y) \\ &\text{s.t.} \;\; y = y^* + L(p)(x - x^*), \end{aligned} \tag{10}$$

where the matrix $L(p)$ is chosen such that the contract defines an affine subset on the hyperplane

$$p_x^T(x - x^*) + p_y^T(y - y^*) = 0. \tag{11}$$

An appropriate parameterization for L is given by

$$L(p) = -p_y p_x^T / \|p_y\|^2, \tag{12}$$

for $p_y \neq 0$. Because the contract is chosen to satisfy (11), L becomes, regardless of its explicit form, degree zero homogeneous, i.e., $L(\alpha p) = L(p)$ for $\alpha \neq 0$. This is because αp defines the same hyperplane as p.

Let $S(p) \subset \mathbb{R}^{n+m}$ denote the set of solutions to (10) for given p. Then the contract design problem above is to find p so that

$$(x^*, y^*) \in S(p). \tag{13}$$

Theorem 2.1 gives conditions for the existence of a solution for (13), and due to the degree zero homogeneity of L, there is at least a ray of solutions if the conditions of Theorem 2.1 hold. One can always obtain a system with a unique solution from a homogeneous system that has a unique ray of solutions, e.g., by setting one of the components of p to a nonzero constant and dropping the corresponding equation from the system. However, in this paper we do not have any need to do so. Furthermore, (13) may also hold for some other p's than the subgradient directions and therefore we do not necessarily have a unique ray of solutions.

Note that if $(x(p), y(p)) \in S(p)$, then $x(p)$ is the agent's reaction for the contract parameterized by p. Furthermore, as a solution set of a convex optimization problem, $S(p)$ if non-empty, is compact and convex set. The other properties of S are summarized in the following theorem, which readily follows from Theorem A.1 and Corollary A.1 presented in Appendix A.

Theorem 3.1. *If u is concave, L is continuous at p and $S(p) \neq \emptyset$, then the set-valued mapping S is closed at p. If u is strictly concave and D is compact, then S is single-valued and continuous at p, $p_y \neq 0$.*

Notice that the compactness of a level set, e.g., the set D, of a concave function is equivalent with the compactness of all the level sets, see, e.g., Corollary 8.7.1 in Rockafellar (1970). Obviously strongly concave functions satisfy conditions of Theorem 3.1. Strong concavity of u means that $-\partial u$ is a strongly monotone mapping, i.e., there is a constant $\sigma > 0$ such that

$$(\xi_{x_1} - \xi_{x_0})^T(x_0 - x_1) + (\xi_{y_1} - \xi_{y_0})^T(y_0 - y_1) \geq \sigma(\|x_1 - x_0\|^2 + \|y_1 - y_0\|^2)$$

for all (x_1, y_1),(x_0, y_0) and $(\xi_{x_i}, \xi_{y_i}) \in \partial u(x_i, y_i)$, $i = 0, 1$. If u is twice continuously differentiable, then strong concavity is equivalent with the negative semidefiniteness of $\nabla^2 u(x, y) + \sigma I$ for every x, y pair, where $\nabla^2 u$ denotes the Hessian of u with respect to x and y and I is an identity matrix. We shall see in the following section that strong concavity is essential for the convergence of fixed-point adjustment.

A two-dimensional example of points $x(p), y(p)$ with $L(p)$ defined by (12) is presented in Figure 1. In the figure the agent's optimum with a given p and the corresponding contract line (dashed line) is a point where the contract line is tangent to one of the contours (dotted lines) of u. The solid line represents the locus of all $x(p), y(p)$ points. In the figure K denotes the (negative) cone of solutions of (13). The opposite directions are solutions as well.

4. Adjustment with Fixed-Point Iteration

In a game of incomplete information the explicit form of u is unknown for the principal. As explained earlier, the principal-agent game with incomplete information is usually formulated as a Bayesian game by supposing that the principal knows the form of u except for one parameter and has a probability distribution over the possible values of that parameter.

Instead of Bayesian approach we assume that the game of incomplete information is played repeatedly and the principal can adjust the affine contract according to observations on the agent's actions, i.e., the principal is committed to a contract only for one period at a time and faces the same agent in each round. Note that when u is unknown to the principal he cannot necessarily have prior knowledge about the agent's participation constraint; hence he cannot use D when solving for (x^*, y^*) as in Section

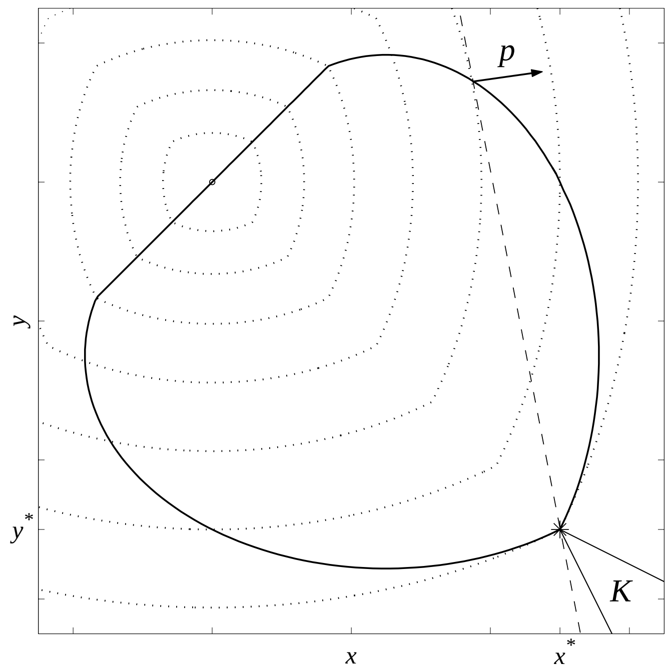

FIGURE 1. Two-dimensional illustration: the agent's contours and reactions.

2. We shall therefore assume here that the principal knows that his global optimum (x^*, y^*), solving $\max v(x,y)$ over $\mathbb{R}^{n+m}$, belongs to D. Thus the agent will always participate the game, recall the discussion in Section 2 below (4). For example, this can happen when it is common knowledge that the agent does not have a participation constraint at all. In Section 6 we study the general case where the principal does not know D but can find the best point in D through adjustments.

The basic idea of the adjustment approach is that the principal tries to find a solution p so that (13) holds. An appropriate method for this task is fixed-point iteration

$$p^{k+1} = p^k + \mu d(p^k), \tag{14}$$

where d denotes the mapping

$$d(p) = \begin{pmatrix} x(p) - x^* \\ y(p) - y^* \end{pmatrix},$$

$(x(p), y(p)) \in S(p)$, and $\mu \neq 0$ is a fixed parameter. The advantage of fixed-point iteration is that it can be implemented in a repeated game where the principal does not know u. This is because the agent's response $x(p)$ is sufficient information for updating p by (14). Note that $y(p)$ is defined through the affine contract given $x(p)$.

An interpretation for the above adjustment scheme is that it describes a learning model, where the iteration specifies the principal's learning rule. Furthermore, the agent is assumed to be myopic in the sense that he does not consider outcomes of other games than the current one. An explanation for myopic behavior is that the agent's discount factor is small compared to the speed at which the learning rule converges. Another argument for myopic learning comes from matching models where there are a great number of players and in each period the players match their strategies with different opponents. Since the same players are unlikely to meet anew they tend to play myopically. In the principal-agent game there could be a large number of similar agents and in each round one agent is chosen randomly to play the game.

4.1. **Convergence Analysis.** The convergence analysis of this section is based on properties of the parameterized problem (13). We shall show essentially that the strong concavity of the agent's utility function is required for the convergence of (14).

Because L is degree zero homogeneous, S and d are homogeneous, too. Moreover, $d(p)$ is perpendicular to p, i.e., $d(p)^T p = 0$, because the contract satisfies (11). This property is known as Walras' law and it generally holds for excess demand functions of exchange economies. In the following lemmas we give general convergence conditions and characterize the convergence properties of fixed-point iteration in a problem of finding a solution for a system of equations, where the nonlinear mapping satisfies Walras' law together with an additional condition. The proofs of the lemmas can be found in Appendix B.

Lemma 4.1. *Let the continuous mapping* $F : B(p^*, r) \to \mathbb{R}^N$, $B(p^*, r) = \{p \in \mathbb{R}^N \mid \|p - p^*\| < r\}$, $r > 0$, *satisfy the following conditions:*

1. $F(p)^T p = 0 \ \forall p \in B(p^*, r)$,
2. $\|F(p)\|^2 \le 2F(p)^T p^* \ \forall p \in B(p^*, r)$.

Then fixed-point iteration $p^{k+1} = p^k + \mu F(p^k)$, $\mu \neq 0$, *converges to a solution of* $F(p) = 0$ *when* $p^0 \in B(p^*, r)$. *Moreover* p^* *is a solution.*

Lemma 4.2. *Let conditions 1 and 2 of Lemma 4.1 hold for* F *and let fixed-point iteration* $p^{k+1} = p^k + \mu F(p^k)$ *converge to a solution* $\bar{p}$ *that satisfies*

$$\|F(p)\|^2 \le 2\alpha F(p)^T \bar{p} \quad \forall p \in B(p^*, r) \tag{15}$$

for some $\alpha > 0$. *Then the iteration converges monotonically, i.e.,* $\|p^{k+1} - \bar{p}\| < \|p^k - \bar{p}\|$.

The latter lemma tells that the convergence is monotonic if the sequence of parameters p converges to a solution, e.g., to p^* that satisfies the second condition in Lemma 4.1. This can be guaranteed in some specific cases as will be seen in the example of Section 4.2.

Using Lemmas 4.1 and 4.2 we obtain the following convergence theorem for the adjustment of an affine contract using fixed-point iteration.

Theorem 4.1. *If* u *is strongly concave and assumptions of Theorem 2.1 hold, then fixed-point iteration (14), for* $p_y^0 \neq 0$, *either converges to a solution of (13) or* $p_y^k = 0$ *for some* k. *If the iteration converges to a subgradient direction of* u *at* x^*, y^*, *then it converges monotonically.*

Proof. As a strongly concave function u satisfies assumptions of Theorem 3.1 and it follows that d is continuous when $p_y \neq 0$. Clearly condition 1 of Lemma 4.1 holds for all p with $p_y \neq 0$. Therefore we need to show only that condition 2 holds. Without loss of generality we can choose $\mu = 1$. Let $(x, y) \in S(p)$, $(\xi_{x^*}, \xi_{y^*}) \in \partial u(x^*, y^*)$ as in Theorem 2.1 and $(\xi_x, \xi_y) \in \partial u(x, y)$. From strong concavity we have

$$(\xi_x - \xi_{x^*})^T(x^* - x) + (\xi_y - \xi_{y^*})^T(y^* - y) \geq \sigma(\|x - x^*\|^2 + \|y - y^*\|^2), \tag{16}$$

where $\sigma > 0$. By plugging the contract in place of y we get from the second term on the left-hand side of (16)

$$\xi_y^T L(p)(x^* - x) - \xi_{y^*}^T(y^* - y),$$

and hence the left-hand side of (16) equals

$$(\xi_x + L(p)^T\xi_y)^T(x^* - x) - \xi_{x^*}^T(x^* - x) - \xi_{y^*}^T(y - y^*).$$

From the optimality condition it follows that $\xi_x + L(p)^T\xi_y = 0$; hence the left-hand side of (16) is equal to

$$\xi_{x^*}^T(x - x^*) + \xi_{y^*}^T(y - y^*) = d(p)^T p^*$$

where $p^* \neq 0$ denotes a vector that is composed of ξ_{x^*} and ξ_{y^*}. Note that the right-hand side of (16) is equal to $\sigma\|d(p)\|^2$. Thus, condition 2 holds for d when $p_y \neq 0$. It follows from Lemma 4.1 that if $p_y^k \neq 0$ $\forall k = 1, 2, \ldots$, then the iteration converges. Otherwise $p_y^k = 0$ for some k. Because condition 2 holds when p^* is any subgradient direction, Lemma 4.2 implies monotonic convergence. □

Theorem 4.1 shows that fixed-point iteration is an appropriate method when computing the solution for the contract design problem. Moreover, in a repeated game the iteration describes a convergent learning rule for the principal and we may say that the equilibrium of the game is stable. Note that it is obvious from Lemma 4.1 and Theorem 4.1 that when the initial parameter vector p^0 is chosen close enough to a solution the iteration does not stall.

4.2. **Example.** The purpose of this example is to illustrate the geometrical ideas of the convergence analysis. Let us assume that the agent's utility function is

$$u(x, y) = \min\{-x^2/2 - y^2, -x^2 - y^2/2\},$$

and the principal's optimum is achieved at $x^* = 1$, $y^* = -1$. As a minimum of two strongly concave functions u is also strongly concave but it is not differentiable for all x, y. Contours of u are illustrated in Figure 1. Because the example is only two-dimensional (11) defines the contract uniquely by (12).

The graph of S consists of four parts given below:

	(p_x, p_y)-region	Image under S
I:	$1/2 \leq p_x/p_y \leq 2$	$y = x,\ -1/3 \leq x \leq 1/3$
II:	$\lvert p_x/p_y \rvert \leq 1/2$	$(y+1/2)^2 + (x-1/2)^2/2 = 3/8,$ $x \leq 1,\ y \leq -1/3$
III:	$2 \leq \lvert p_x/p_y \rvert$	$(y+1/2)^2/2 + (x-1/2)^2 = 3/8,$ $1/3 \leq x,\ -1 \leq y,\ x \neq 1,\ y \neq 0$
IV:	$-2 \leq p_x/p_y \leq -1/2$	$x^*, y^*.$

The first three parts in x, y-plane are marked in Figure 2. Part IV is the point $(1, -1)$. Notice that S and d are not defined when $p_y = 0$ and therefore there is a discontinuity at $x = 1$, $y = 0$. The discontinuity is not, however, illustrated in the figure.

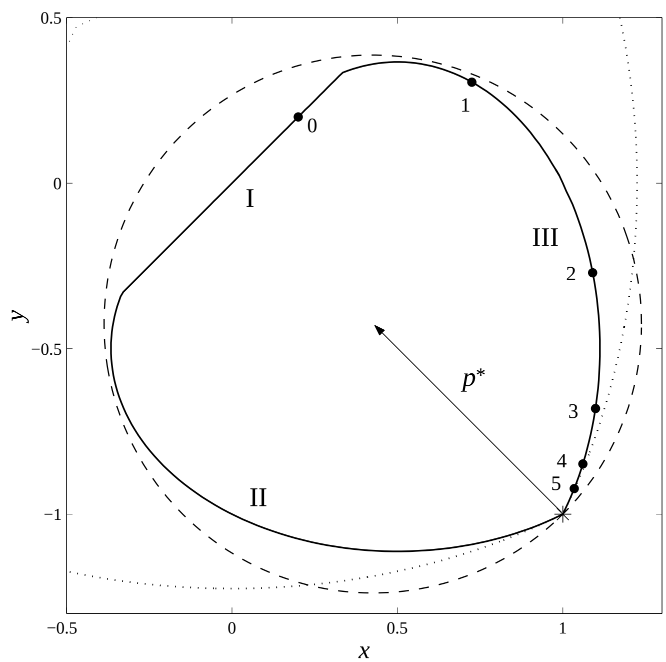

FIGURE 2. Two-dimensional illustration of the second convergence condition and an iteration.

The first convergence condition of Lemma 4.1, namely that $d(p)$ is perpendicular to p, was explained in the second paragraph of Section 4.1 and it is illustrated in Figure 1. The geometric interpretation of the second convergence condition in Lemma 4.1 for d is that the image of $\{(p_x, p_y) \in \mathbb{R}^{n+m} \mid p_y \neq 0\}$ under d is contained in a ball. This can be seen by writing the condition in an equivalent form

$$\|d(p) - p^*\|^2 \leq \|p^*\|^2,$$

i.e., the ball is centered at p^* and has radius $\|p^*\|$. This condition is now satisfied and the dashed line in Figure 2 illustrates one appropriate ball when the origin is transformed to $(1, -1)$.

In general, strong concavity of u implies that the level set $\{(x, y) \in \mathbb{R}^{n+m} \mid u(x, y) \geq u(x^*, y^*)\}$ is contained in a sufficiently large ball that goes through x^*, y^*. The region inside the dotted line in Figure 2 belongs to the aforementioned level set. Because the points $(x(p), y(p))$ are inside the level set and $d(p)$ is the difference vector of (x^*, y^*) and this point, d is inside the ball obtained from the one that contains the level set by transforming the origin to x^*, y^*. Notice that the condition 2 of Lemma 4.1 has both global and local interpretation. Globally the condition implies that $d(p)$ belongs to a compact set and in the vicinity of p^* it means that d is not too flat.

In this example y is one-dimensional and therefore all the solutions of (13) are subgradient directions. Hence, fixed-point iteration converges monotonically, see Theorem 4.1. The first six (x, y) points of an iteration with $\mu = 1$ and initial parameter vector $p^0 = [3 \;\; 2]^T$ are illustrated with numbered dots in Figure 2.

5. Continuous Time Process

In this section we focus on a continuous time process for adjusting the affine contract. Continuous time approach is commonly used in the stability analysis of Walrasian equilibrium and it turns out that the famous stability result by Arrow et al. (1959) is related to the adjustment of an affine contract.

In this section we denote by $p(t)$ the parameter vector at time t and we suppose that p is differentiable with respect to t and denote its derivative by $\dot{p}$. If we set $p(t^k) = p^k$, $p(t^{k+1}) = p^{k+1}$ and assume that $\mu = t^{k+1} - t^k$, we get the process

$$\dot{p} = d(p) \tag{17}$$

as a limit from (14) when $\mu \to 0$. This process can not be implemented in a repeated game but it works as an idealization for the discrete time process where the principal reacts arbitrarily fast. Hence, the process describes a continuous time learning model.

The following lemma gives convergence conditions for a continuous time adjustment process for a system that satisfies Walras' law. The lemma is a modification of the stability theorem for Walrasian equilibrium by Arrow et al. (1959). The formulation and proof follow the presentation of Theorem 3.E.1 in Takayama (1974) with the difference that we do not require that there is a unique ray of solutions. A similar result for exchange economies with multiple equilibria is given in Arrow and Hurwicz (1960).

Notice that in Lemma 5.1 we need to assume existence of a solution for the equation, which was not assumed in Lemma 4.1. However, the condition that is required in addition to Walras' law is less stringent than condition 2 of Lemma 4.1. The proof is presented in Appendix B.

Lemma 5.1. *Let $K \subset \Omega$, $K \neq \emptyset$ be the set of solutions of $F(p) = 0$ and let the continuous mapping $F : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, satisfy Walras' law and*

$$F(p)^T p^* > 0 \quad \forall p \in \Omega \setminus K \tag{18}$$

for some $p^* \in K$. *Then the process* $\dot{p} = F(p)$ *converges monotonically to a solution of* $F(p) = 0$ *when* $p(0) \in B(p^*, r) \subset \Omega$.

The geometric interpretation of (18) is that points $F(p)$, $p \in \Omega \setminus K$, are on the half-space defined by the hyperplane with normal p^*. Clearly F has this property if it satisfies condition 2 of Lemma 4.1. Using Lemma 5.1 we can prove the following convergence theorem for the continuous time adjustment process (17).

Theorem 5.1. *If assumptions of Theorem 2.1 and Theorem 3.1 hold, then the process (17), for* $p_y(0) \neq 0$, *either converges to a solution or stalls at a point in which* $p_y = 0$.

Proof. From Theorem 2.1 it follows that the set of solutions K of $d(p) = 0$, is non-empty and clearly condition 1 of Lemma 4.1 holds. Moreover from Theorem 3.1 we know that d is continuous when $p_y \neq 0$. Now let us suppose that p is not a solution of (13) and let $(x, y) \in S(p)$. Let $(\xi_{x^*}, \xi_{y^*}) \in \partial u(x^*, y^*)$ as in Theorem 2.1 and $(\xi_x, \xi_y) \in \partial u(x, y)$. From strict concavity of u we get

$$
\begin{aligned}
&(\xi_x - \xi_{x^*})^T(x^* - x) + (\xi_y - \xi_{y^*})^T(y^* - y) = \\
&(\xi_x + L(p)^T\xi_y)^T(x^* - x) - \xi_{x^*}^T(x^* - x) - \xi_{y^*}^T(y^* - y) = \\
&\xi_{x^*}^T(x - x^*) + \xi_{y^*}^T(y - y^*) = d(p)^T p^* > 0,
\end{aligned}
$$

where $p^* \neq 0$ is composed of ξ_{x^*} and ξ_{y^*}. Hence the conditions of Lemma 5.1 are satisfied for $F = d$ with $\Omega = \mathbb{R}^{n+m} \setminus \{p \in \mathbb{R}^{n+m} \mid p_y \neq 0\}$. If we get during the process a point $p(t)$ such that $p_y(t) = 0$, the iteration stalls since d is not defined at such a point.

□

Compared to Theorem 4.1 the convergence conditions in Theorem 5.1 are weaker because instead of strong concavity only strict concavity is required. Similarly as for the discrete time adjustment, the process does not stall when initial parameter vector $p(0)$ is chosen close enough to a solution.

The essential properties of d are its homogeneity and that it satisfies Walras' law and these properties are also typical for excess demand functions of exchange economies. According to the well-known theorem by Sonnenschein (1973), Mantel (1974), and Debreu (1974) any continuous function that satisfies Walras' law for $p \geq 0$ is an excess demand function for some economy.

Our two-player model is, indeed, similar to an exchange economy with only one consumer and a Walrasian auctioneer. This can be seen from (10) and (11) by making the following alternative interpretation. The variables x and y represent the amounts of commodities, p_x and p_y are the corresponding prices, (x^*, y^*) is the consumer's initial bundle, (11) gives a budget constraint for the consumer, and $x(p), y(p)$ are the amounts that the consumer is willing to buy for given prices. Furthermore, process (17) describes a price adjustment scheme that drives the excess demand of commodities, $d(p)$, to zero. The principal acts as an auctioneer who sets the prices according to the excess demand but no trade occurs until an equilibrium is reached, i.e., (17) is a tâtonnement process.

The analog to exchange economies leads us to ask whether fixed-point iteration works in finding a Walrasian equilibrium. The convergence conditions for fixed-point iteration are, however, more stringent than those required for the convergence of the continuous time process. Particularly the second condition of Lemma 4.1 does not necessarily hold for excess demand functions of exchange economies. The reason for this is that when d is an excess demand mapping it usually has the following property: when price of some commodity goes to zero, the excess demand for that commodity grows infinitely large given that the other prices are fixed. Therefore, d cannot be inside any ball and condition 2 of Lemma 4.1 does not hold. Though we cannot generalize the convergence result as such to exchange economies, it is possible that the condition holds for some class of economies.

6. Processes for Finding the Principal's Optimum

So far we have assumed that the agent will always participate the game during the process, i.e., the principal's global optimum (x^*, y^*) belongs to D, recall the discussion in the beginning of Section 4. If (x^*, y^*) does not belong to D, the agent does not necessarily participate the game. In this case the contract design problem is to find the optimal point for v over D and a corresponding contract at that point. In this section we characterize procedures for finding the principal's optimum assuming that a contract can be found at any given $(x^*, y^*) \in D$, e.g., with the adjustment process described previously in this paper.

Let $w = (\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$, we call w a reference point, and let $S(p, w)$ denote the set of solutions to

$$\begin{aligned} &\max_{x,y} \;\; u(x, y) \\ &\text{s.t.} \;\; y = \bar{y} + L(p)(x - \bar{x}), \end{aligned} \tag{19}$$

for given p and w, where $L(p)$ is as in (10). Now the contract design problem is to find w^* such that $v(w^*) = \max_{w \in D} v(w)$ and $p^* \in G(w^*)$, where $G(w) = \{p \mid w \in S(p, w)\}$. We assume now that given any $w \in D$, a parameter vector p that defines the contract at w, i.e., $p \in G(w)$, can be found. Namely, given that u is strongly concave, and for all $w \in D$ there is $\xi_y \neq 0$, $(\xi_x, \xi_y) \in \partial u(w)$, the convergence result of Theorem 4.1 holds for any reference point $w \in D$. In addition, we make the following assumptions for the adjustment of reference point w:

1. The reference point $w^k \in D$ is updated only if a strictly better point $w^{k+1} \in D$ can be found.
2. If it holds that $w^k \in D$ and $w^k \neq w^*$, then $w^{k+1} \in D$ such that $v(w^{k+1}) > v(w^k)$ can be found.

The first assumption is natural because there is no reason for the principal to expect less utility from the future rounds than from the previous ones. Furthermore, it is rather easy to generate better reference points, so that the second assumption is reasonable, too. For example when the current reference point is in the interior of D, one can find a better reference point that is also in the interior of D by taking an appropriate step to an improving direction of v at the current reference point. It is also worth noticing that if the reference point is taken outside of D, an affine contract going through that point will usually have points in common with D. Nevertheless, the agent will reject the contract at some stage of the updating procedure. In that case some of the agent's previous choices can be taken as a new reference point.

We further need the concept of complete process. We say that a process is incomplete if the sequence of reference points converges to $\bar{w}$ but there is a non-trivial process that starts from $\bar{w}$ and satisfies assumptions 1 and 2. By a non-trivial process we mean a sequence $\{\bar{w}^k\}_k$ generated by the process with $\bar{w}^0 = \bar{w}$, such that $\bar{w}^k \neq \bar{w}$ at least for some k. Process that is not incomplete is called complete. Similar concept of a complete process in the price adjustment framework has been used in Smale (1976).

The following theorem shows that a complete adjustment processes, which satisfies the above assumptions, converges and there is a subsequence of contracts converging to the solution of the contract design problem.

Theorem 6.1. *When v is strictly concave, then for any complete process, which begins from $w^0 \in D$ and satisfies assumptions 1 and 2, the sequence of reference points converges to w^*. Furthermore, when u is strictly concave, and for all $w^k \in D$ there is $(\xi_x^k, \xi_y^k) \in \partial u(w^k)$ such that $\xi_y^k \neq 0$, then there is a subsequence of parameters $p^k \in G(w^k)$ converging to $p^* \in G(w^*)$ given that $G(w^*) \neq \emptyset$.*

Proof. Let us first observe that due to assumption 2 and the assumption that $w^0 \in D$ the sequence of reference points $\{w^k\}_k$, obtained during the process, belongs to D. Let us now show that this sequence converges to the principal's optimum over D, i.e., to point w^*. There are two possibilities: either $w^k = w^*$ when $k > N$, or for every k we have $v(w^{k+1}) > v(w^k)$. In both cases v is a Lyapunov function for the subsequence. Hence the subsequence converges and the limit is in D, which is a closed set because u is continuous. Notice also that v is strictly concave and w^* is its unique maximizer over D, so that it is an appropriate Lyapunov function. From the completeness assumption it follows that the limit is w^*.

It follows from the continuity of S, see Corollary A.1, that $G(w)$ is a closed mapping. Because of homogeneity of S we may choose a bounded sequence $\{p^k\}_k$, $p^k \in G(w^k)$, where $w^k \in D$, e.g., we may set $\|p^k\| = 1$. This sequence has a convergent subsequence and from the closedness of G it follows that the limit is in $G(w^*)$ when $G(w^*) \neq \emptyset$. Hence we have the result. □

In view of Theorem 6.1 iteration (14) can be started using any reference point $w \in D$, and as we discussed it is rather easy to generate reference points. When during an iteration a point $(x(p), y(p))$ is encountered, giving the principal a better outcome than the current reference point, it can be taken as a new reference point in iteration (14), which can be continued from the current parameter vector.

7. Conclusion

In this paper we have presented a new adjustment approach for an affine contract design problem. When a principal-agent game with incomplete information is played repeatedly, the principal can adjust his contract according to agent's previous move. The adjustment procedure is based on parameterizing the problem appropriately and updating the parameters with fixed-point iteration.

The parameterization of the contract design problem results to a degree zero homogeneous system of equations, where the mapping satisfies Walras' law. We showed that the iteration converges when an additional condition, the condition 2 of Lemma 4.1, holds for the system. As a result we obtained a convergence result for a principal-agent game where the agent has a strongly concave utility function. In addition to fixed-point iteration we have studied a related continuous time adjustment process.

The idea of using linear constraints in coordinating decision makers to a desired outcome has been used in the context of Walrasian tâtonnement and recently in negotiation analysis. In Ehtamo et al. (1999) a method of finding a Pareto-optimal solution for a two-party negotiation is formulated as a problem of searching for a joint tangent hyperplane for the parties utility functions. The problem results to a degree zero homogeneous system of equations that satisfies Walras' law. The search of the joint tangent hyperplane is done interactively between a mediator, using the method, and the parties. In this framework fixed-point iteration has been used successfully in numerical experiments The convergence results that we have presented in this paper may prove useful for this kind of adjustment of hyperplane constraints in finding Pareto optimal solutions or more generally adjustment of linear budget constraints for exchange economies.

Appendix A. Continuity Properties of the Optimal Set Mapping

Here $S(p,w) \subset \mathbb{R}^{n+m}$ denotes the set of solutions for (19) for given p and $w = (\bar{x}, \bar{y})$. The following theorem characterizes the continuity of S with respect to p and w.

Theorem A.1. *If u is concave, $L(p)$ is continuous at p and $S(p,w) \neq \emptyset$, then the set-valued mapping S is closed at (p,w).*

Proof. Let us assume that $S(p,w) \neq \emptyset$, $p^k \to p$, $w^k \to w$ and $(x^k, y^k) \to (x,y)$, where $(x^k,y^k) \in S(p^k,w^k) \neq \emptyset$. We denote the set of feasible points of problem (19), i.e., the set of points satisfying the linear contract, with $C(p,w)$ and the normal cone of the feasible set, $\{(\xi_x,\xi_y) \in \mathbb{R}^{n+m} \mid \xi_x + L(p)^T\xi_y = 0\}$, with $N(p)$. Let us first note that $C(p,w)$ is a closed mapping with respect to (p,w), because L is continuous.

According to sufficient optimality conditions $(x^k,y^k) \in S(p^k,w^k)$ if and only if $(x^k,y^k) \in C(p^k,w^k)$ and

$$\partial u(x^k,y^k) \cap N(p^k) \neq \emptyset.$$

From continuity of L it follows that N is a closed mapping. Concavity of u implies upper hemi-continuity of ∂u. Moreover,

$$\cup_{(x,y)\in\{(x^k,y^k)\}_k} \partial u(x,y)$$

is bounded, see, e.g., Prop. 6.2.1 and 6.2.2 in Section 6 of Hiriart-Urruty and Lemaréchal (1993). Hence, there is a convergent sequence $\{\xi_i\}_i$ such that

$$\xi_i \in \partial u(x^{k_i}, y^{k_i}) \cap N(p^{k_i}).$$

It follows that $\lim_{i\to\infty} \xi_i \in \partial u(x, y) \cap N(p)$. Because C is a closed mapping we have $(x, y) \in C(p, w)$. Thus, sufficient optimality conditions are satisfied and $(x, y) \in S(p, w)$.

□

Corollary A.1. *If u is strictly concave with compact level sets and $L(p)$ is continuous at p, then S is single-valued and continuous at (p, w), $w \in D$.*

Proof. From the compactness of the level sets we know that D is compact and $S(p, w) \neq \emptyset$. The latter follows from Weierstrass theorem. Strict concavity of u implies that $S(p, w)$ is a singleton. Furthermore, since D is compact there is $\bar{w} = (\bar{x}, \bar{y})$ such that $u(\bar{x}, \bar{y}) = \max_{(x,y)\in D} u(x, y)$ and clearly $S(p, w) \subset \{(x, y) \in \mathbb{R}^{n+m} \mid u(x, y) \geq u(\bar{x}, \bar{y})\}$ so that S is a closed mapping into a compact set. Hence S is upper hemi-continuous, see, e.g., Prop. 11.9 (c) in Border (1985).

Continuity follows from upper hemi-continuity and single-valuedness, see prop 11.9 (d) in Border (1985).

□

Appendix B. Proofs of the Lemmas

Proof of lemma 4.1. If function F satisfies the conditions 1 and 2, then they hold also for μF with $\mu \neq 0$. Hence, without loss of generality we can prove the convergence with $\mu = 1$. Let p^* be as in condition 2, then

$$\|p^{k+1} - p^*\|^2 = \|p^k + F(p^k) - p^*\|^2 =$$
$$\|F(p^k)\|^2 - 2F(p^k)^T p^* + \|p^k - p^*\|^2 \leq \|p^k - p^*\|^2.$$

Note that $p^k \in B(p^*, r)$ $\forall k = 0, 1, \ldots$, when $p^0 \in B(p^*, r)$. Therefore the sequence $\{\|p^k - p^*\|\}_k$ converges and the sequence $\{\|p^k\|\}_k$ is bounded. From condition 1 it follows that

$$\|p^k\|^2 = \|p^0\|^2 + \sum_{i=0}^{k-1} \|F(p^i)\|^2,$$

so that $\{\|p^k\|\}_k$ is a growing and bounded sequence and hence convergent. From the iteration formula we have

$$p^k = p^0 + \sum_{i=0}^{k-1} F(p^i).$$

Hence $\|p^0 + \sum_{i=0}^{k-1} F(p^i)\|$ converges, too. From triangular inequality we get

$$\|p^0 + \sum_{i=0}^{k+l} F(p^i)\| \geq \left| \|p^0 + \sum_{i=0}^{k} F(p^i)\| - \| \sum_{i=k+1}^{k+l} F(p^i)\| \right|$$

and it follows that

$$\|p^{k+l} - p^k\| = \| \sum_{i=k+1}^{k+l} F(p^i)\| \to 0, \qquad (20)$$

when $k \to \infty$ and $l \geq 1$. Thus $\{p^k\}_k$ is a Cauchy sequence and hence convergent; let $\bar{p}$ denote its limit point. Moreover, from (20) we get by setting $l = 1$ that $\|F(p^k)\| \to 0$, and from the continuity of F we have $F(\bar{p}) = 0$.

We can construct a sequence of solutions converging to p^* by taking neighborhoods $B(p^*, r^k)$ with $r \geq r^0 > r^1 > ... > r^k \to 0$. There is a solution $\bar{p}^k$ in each of these neighborhoods, and $\bar{p}^k \to p^*$ since $r^k \to 0$. From the continuity of F we have $F(p^*) = 0$. □

Proof of lemma 4.2. If (15) holds for $\alpha > 0$ then it holds for any $\bar{\alpha} > \alpha$. Specifically, we can choose $\bar{\alpha} > 0$ such that (15) holds for $p^* = \bar{\alpha}\bar{p} - 2\bar{p}$ instead of $\alpha\bar{p}$. Moreover we can take α such that $\|F(p)\|^2 < 2\alpha F(p)^T\bar{p}$ if p is not a solution. Similarly as in Lemma 4.1 we can deduce that $\|p^{k+1} - p^*\|^2 < \|p^k - p^*\|^2$, and $\|p^{k+1} - \alpha\bar{p}\|^2 < \|p^k - \alpha\bar{p}\|^2$ when p^k is not a solution. From parallelogram law we get

$$\|p^k - \alpha\bar{p}\|^2 + \|p^k - p^*\|^2 = 2\|p^k - \bar{p}\|^2 + 2(\alpha - 1)\|\bar{p}\|^2.$$

By rearranging the terms we have

$$\begin{aligned} 2\|p^k - \bar{p}\|^2 &= 2(\alpha - 1)\|\bar{p}\|^2 - \|p^k - \alpha\bar{p}\|^2 - \|p^k - p^*\|^2 \\ &> 2(\alpha - 1)\|\bar{p}\|^2 - \|p^{k+1} - \alpha\bar{p}\|^2 - \|p^{k+1} - p^*\|^2 = 2\|p^{k+1} - \bar{p}\|, \end{aligned}$$

and hence $\{p^k\}_k$ converges monotonically to $\bar{p}$. □

Proof of lemma 5.1. From the first condition it follows that $\|p(t)\| = \|p(0)\|$, because

$$d\|p(t)\|^2/dt = 2p(t)^T\dot{p} = 2p(t)^T F(p) = 0,$$

i.e., $\|p(t)\|$ is constant. Let us choose p^* such that $\|p^*\| = \|p(0)\|$, and differentiate $D(t) = \|p(t) - p^*\|^2$, where p^* is as required in the assumptions:

$$\begin{aligned} dD(t)/dt &= d\|p(t) - p^*\|^2/dt = 2\dot{p}^T(p(t) - p^*) \\ &= 2F(p)^T(p(t) - p^*) = -2F(p)^T p^* < 0, \end{aligned} \tag{21}$$

when $p(t)$ is not a solution. Hence, $p(t) \in \Omega$ and $p(t)$ moves monotonically towards p^*. We need to show that $p(t)$ is not bounded away from K, i.e., the process converges to a solution. Let us suppose that $p(t)$ is bounded away from K, i.e., there is $\varepsilon > 0$ such that

$$p(t) \in S = B(p^*, r) \setminus \{p \mid \|p - \bar{p}\| < \varepsilon \text{ for some } \bar{p} \in K\}$$

for all t. Note that $S \neq \emptyset$ because $p^* \in K$. From continuity of F it follows that $f(p) = -2F(p)^T p^*$ is a continuous function. From Weierstrass theorem we know that $f(p)$ achieves its maximum $-\delta < 0$ in the compact set S. Hence, we have $dD/dt = -2F(p(t))^T p^* \leq -\delta < 0$. Integrating both sides from 0 to t and rearranging the terms we get

$$D(t) < D(0) - \delta t.$$

For t large enough we have $D(t) < 0$, which is a contradiction with nonnegativity of the norm. □

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