Osborne's Cartel Maintaining Rule Revisited⋆

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Abstract

We present a proportional reaction strategy in a repeated Cournot duopoly game with discounting. The strategy is based on increasing the total output quantity in proportion to deviations from a cartel point. Such a strategy was originally proposed by Osborne (1976) for a static oligopoly. We show that the resulting equilibrium is subgame perfect and weakly renegotiation proof when the possible deviations are bounded and the proportional reactions have sufficiently large slopes. The lower bounds for the slopes depend on the profit functions and discount factors in a simple way. A strategic explanation for conjectural variations equilibrium is discussed as well.

Key words: proportional strategy, conjectures, subgame perfection

1 Introduction

Tacit collusion in oligopolistic markets can be explained with subgame perfect equilibrium strategies in the framework of repeated games. In particular, there

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is a long tradition of studying strategies in which a firm's actions vary continuously as a response to its rivals' behavior. Following this tradition and the idea of Osborne's quota rule (Osborne, 1976) we consider linear strategies in which unilateral deviations from the cartel point are punished in proportion to the deviation. We call these strategies proportional strategies and show that they provide subgame perfect equilibrium in infinitely repeated game with discounting when restricting to sufficiently small deviations.

The main motivation for continuous strategies is that they are more plausible in many circumstances than discontinuous strategies, because with them small deviations lead only to small punishments rather than to the collapse of collusion (Friedman, 1968, 1973, 1976). Among continuous strategies, linear reaction strategies have raised particular interest because they are simple but their subgame perfection is a non-trivial question.

Previously Kalai and Stanford (1985) have shown that linear strategies give rise to an ε-perfect equilibrium when reaction times are short enough. Furthermore, using linear reaction functions leads to a subgame perfect equilibrium for a repeated duopoly when using the limit of the means evaluation criterion instead of discounting (Stanford, 1986a). Ehtamo and Hämäläinen (1993) consider linear reaction strategies in a continuous time natural resource model and study the credibility of these strategies. They call the corresponding equilibrium the incentive equilibrium, because, as they argue, punishing only slightly from small deviations is apt to encourage cooperation.

As we restrict to sufficiently small deviations we shall define the notion of local subgame perfection. Alós-Ferrer and Ania (2001) have previously introduced a related concept — the local Nash equilibrium for static games. As they argue, restricting to local deviations can be regarded as a form of boundedly rational behavior. However, since we are dealing with a dynamic game there is also another motivation for the local approach. Namely, the proportional strategy can be seen as a linearization of a more general equilibrium strategy. The local subgame perfection of the linearization then provides a necessary condition for the subgame perfection of the original strategy.

Our analysis owes to observations made by Osborne (1976) for static oligopoly settings. Osborne was inspired by the rather long lasting stability of the OPEC oil cartel and he suggested that maintaining the firms' market shares sustains the cartel in practice. In the literature this linear strategy has been called Osborne's rule or Osborne's quota rule, see Phlips (1988, Section 6.2) and Jacquemin and Slade (1989, Section 3.1.1) for discussion on this strategy. Osborne realized that the quota rule is also credible when deviations are sufficiently small.[1] We shall notice that the quota rule is a limiting case from

[1] By credibility Osborne refers to the property that the punishing firm is better off by following the linear punishment than by ignoring the deviation. In the literature

proportional strategies when Osborne's assumptions hold and the discount factors approach to one.

Linear reaction strategies as well as the proportional strategy are closely related to conjectural variations models that have been widely applied in the literature on imperfect competition. Indeed, the slope of the proportional scheme can be associated with a conjectural variation parameter. Moreover, conjectural variations can be interpreted as the slopes of firms' nonlinear reaction functions analogously as the proportional scheme can be seen as a linearization of a more general strategy. Hence, the results for proportional strategies are relevant for conjectural variations models. Namely, subgame perfection of proportional strategies could explain conjectural variations equilibrium as a result of rational behavior: Conjectural variations equilibrium can be maintained as an equilibrium with proportional strategies having the conjectural variations as their slopes.

There is a plethora of papers which analyze conjectural variations in dynamic settings. One part of this literature deals with dynamics that lead to particular conjectures (see, e.g., Friedman and Mezzetti, 2002, Jean-Marie and Tidball, 2006) and another part constructs equilibrium strategies of dynamic games that yield the same outcome as static conjectural variation models do (see, e.g., Cabral, 1995, Dockner, 1992). This paper is related to works such as Kalai and Stanford (1985) and Stanford (1986a) that aim to establish equilibrium properties for conjectured reaction functions, i.e., reaction functions corresponding to conjectural variations parameters. In this branch of literature subgame perfection of the strategies with discounted payoffs as the evaluation criteria poses the main challenge for which we propose a new resolution.

The proportional strategy that we analyze works only partially as a reaction function strategy. The deviations are punished in the manner of reaction functions but after a deviation the firm returns to cooperation. Hence, the strategy does not lead to a sequence of consecutive deviations from the cooperative outputs as reaction function models do. The reason for our formulation is that in infinitely repeated games with discounting the only subgame perfect reaction function strategies are those that prescribe the stage-game Nash equilibrium actions in each period, see Stanford (1986b) and Robson (1986).

This paper is structured as follows. In Section 2 we discuss the Osborne's rule for a static duopoly model. In Section 3 we define the proportional scheme with time delay and analyze its properties for bounded deviations. The subgame perfection is analyzed in Section 4 where implications for conjectural variations equilibria are also presented. In Section 5 we discuss the results.

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credibility often refers to subgame perfection. In this paper credibility is not used in this meaning.

2 Osborne's Rule for a Static Cournot Duopoly

In this section we go through the static Osborne's rule in an extent appropriate to our formulation. For simplicity we first consider a duopoly setting. Originally Osborne (1976) presented his quota rule in the context of more general oligopolies. The firms are indexed with i, $q = (q_1, q_2)$ denotes the pair of output quantities, and π_i is firm i's profit function. The subscript $-i$ refers to i's rival. When we use the index i without specifying its values, we are considering either of the two firms.

The first assumption we make for a profit function is the following:

(A1) π_i is differentiable, strictly concave, and strictly decreasing in q_{-i} when $q_i > 0$.

Since π_i is a strictly concave function, the maximum of π_i under convex constraints is unique. Our second assumption is that at the tacitly agreed cooperative point firm i's profit function is increasing with respect to q_i. Let q^λ denote the cooperative point, and let us assume that at this point the production quantities and the profits are positive. The second assumption can be written for π_i and q^λ as

(A2) $\partial \pi_i(q^\lambda)/\partial q_i > 0$.

Osborne's quota rule is based on the observation that when q^λ is a Pareto-optimal point and profit functions satisfy (A1) and (A2), then there is a joint tangent line to the contours of the profit functions. This tangent line is defined by

$$\nabla \pi_i(q^\lambda) \cdot (q - q^\lambda) = 0, \tag{1}$$

where the dot denotes the usual inner product. The joint tangency property is illustrated in Figure 1, where the solid contours are for π_1 and the dashed contours are for π_2. Although in Figure 1 the line also goes through the origin, this need not be the case in general. We will discuss this property below. When q^λ is not Pareto-optimal then the line given by (1) is tangential only to the contour of π_i at q^λ but not necessarily to the corresponding contour of π_{-i}.

The tangent line (1) takes the form

$$q_{-i} = L(q_i, \alpha_i^\lambda) = q_{-i}^\lambda + \alpha_i^\lambda \cdot (q_i - q_i^\lambda),$$

where

$$\alpha_i^\lambda = -\frac{\partial \pi_i(q^\lambda)/\partial q_i}{\partial \pi_i(q^\lambda)/\partial q_{-i}}. \tag{2}$$

Under (A1) and (A2) we have $\alpha_i^\lambda > 0$. We also assume that $q_i^\lambda > 0$, $i = 1, 2$, in which case (A1) gives $\partial \pi_i(q^\lambda)/\partial q_{-i} > 0$, and thus $\alpha_i^\lambda < \infty$.

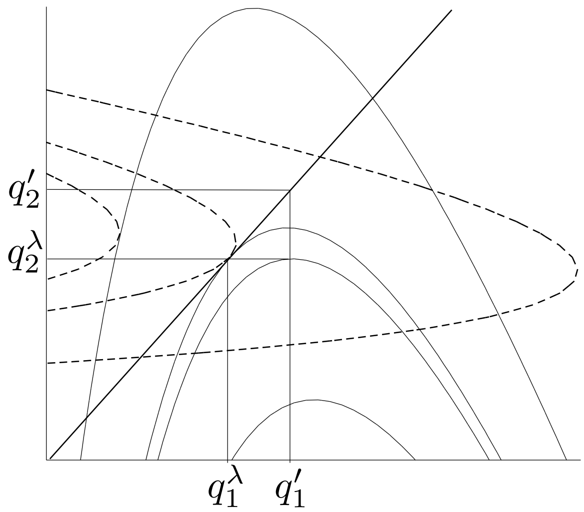

Figure 1. A Pareto-optimal point and a joint tangent line.

Osborne interpreted the joint tangent line strategically by assuming that firm i's rival responds to its actions according to the rule

$$q_{-i} = r_{-i}(q_i, \alpha_i^\lambda) = \max\{q_{-i}^\lambda, L(q_i, \alpha_i^\lambda)\}. \tag{3}$$

In other words, when firm i pushes its production from q_i^λ to increase its profit, the rival reacts by keeping the joint production on the tangent line, hence actually decreasing firm i's profit. For example, if firm 1 moves to q_1' as in Figure, then firm 2 raises its output to q_2', which decreases firm 1's profit. As Osborne showed, following this reaction rule suffices for an equilibrium when reactions are instantaneous. Note that it is not profitable for firm $-i$ to punish its rival from decreasing its output below q_i^λ.

Osborne showed the equilibrium property of linear reaction rules for n firms, which is not as obvious result as in the duopoly case. Spence (1978) extends the result by characterizing a more general class of reaction functions that give rise to efficient outcomes. The following theorem is a reformulation of Osborne's result, and it shows that α_i^λ is a lower bound for the slope α_i of L in r_{-i} such that q_i^λ becomes firm i's optimal choice. As we shall discuss in Section 4.2, α_i plays the same role as the conjectural variation parameter. The below result differs slightly from the Osborne's original work as we do not require the Pareto-optimality of q^λ. The proof is is given in Appendix.

Theorem 1. *Let us assume that $\alpha_i \geq \alpha_i^\lambda$ and assumptions (A1) and (A2) hold for π_i and q^λ. Then q_i^λ maximizes $\pi_i\left(q_i, r_{-i}(q_i, \alpha_i)\right)$.*

When profit functions and the slopes of their tangent lines satisfy the assumptions made in Theorem 1, then q^λ becomes the equilibrium outcome under the proportional reaction strategies. Osborne further showed that the tangent line defined by (1) has the *ray property* if q^λ is the joint profit maximum and, in addition to satisfying (A1) and (A2), the profits are of the form

$$\pi_i(q) = P(q)q_i - C_i(q_i), \tag{4}$$

for $i = 1, 2$. Here C_i is the cost function and P is the inverse demand function that satisfies $\partial P(q)/\partial q_1 = \partial P(q)/\partial q_2$. The ray property says that the common tangent line $q_{-i} = L(q_i, \alpha_i^\lambda)$, $i = 1, 2$, also passes through the origin, see Figure 1. The economic interpretation of this property is that by reacting according to the rule (3) the firms automatically preserve their market shares at $q_i^\lambda/(q_1^\lambda+q_2^\lambda)$, $i = 1, 2$. Or putting it in another way, by always reacting so that the market shares remain constant, the firms move along the joint tangent line and thus maintain cooperation.

Finally, Osborne discusses the *credibility* of the quota rule in his paper. The credibility refers to the property that the punishing firm is better off, at least for small deviations, by following the rule than by just ignoring the deviation. In this paper this property is tightened; credibility means that it is better to follow the punishment line than to choose any other output below it as the punishment. Note that by (A1), a deviating firm i would not mind if it is punished less than the proportional scheme $L(q_i, \alpha_i^\lambda)$ suggests. This property will be our main ingredient of the equilibrium strategy in dynamic setting.

In Figure 2 we see that for firm 1's deviation q_1' firm 2 would choose the punishment $q_2' = L(q_1', \alpha_1^\lambda)$ among all quantities below this output (vertical line segment from 0 to q_2'). Actually, firm 2 would prefer quantities above q_2', the optimal output being on the best response line (dark circle). For firm 1's outputs larger than q_1^L, e.g., for q_1'', firm 2 would rather choose an output below the punishment line. To be more specific, firm 2 would choose the point from the best response line. Hence, in case of large deviations firm 1 has no reason to believe that firm 2 would actually follow the proportional scheme, if also punishment outputs below it were possible; i.e., the proportional strategy is not credible for deviations larger than q_1^L. Note that q_1^L is the point in which firm 2's best response function (the decreasing line) crosses the punishment line. In the following section we discuss the credibility in the dynamic setting but the meaning will essentially remain the same.

Osborne's rule, as presented in this section, is static as the reaction to deviation is assumed to be instantaneous. The idea of punishing from deviations, however, implies the sequence of actions in the following order: first one of the firms deviates and then the other reacts. In the next section we formulate the proportional reaction strategy in a repeated game setting to account for the time delay between the observation of deviations and the resulting actions. As in the static setting we obtain lower bounds for the slopes of the proportional schemes such that cooperative play becomes the equilibrium outcome of the game.

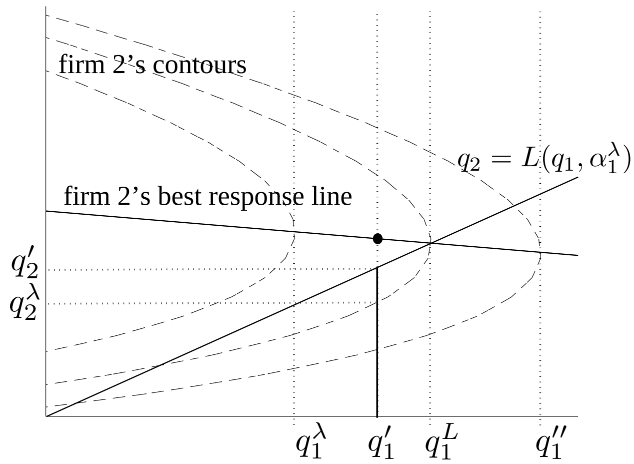

Figure 2. Credibility in a static setting. The further on the left the contour is, the higher the profit.

3 Osborne's Rule with Time Delay

In this section the duopoly game is played repeatedly infinitely many times and the firms observe all each others' previous actions. Furthermore, the firms maximize their discounted profits, i.e., firm i maximizes

$$\sum_{k=0}^{\infty} \delta_i^k \pi_i(q^k),$$

where k refers to the period and $\delta_i \in (0,1)$ is the firm's discount factor. The deviation from cooperation is observed immediately so that the firms can react to the deviations in the next period.

The proportional scheme works in the repeated setting as follows. After having observed that the other firm has deviated, i.e., exceeded the cooperative output, the firm punishes the deviator by choosing the output in the next period according to the proportional scheme. Simultaneous deviations are, however, neglected and the firms continue as if none of them had deviated. Moreover, the firms accept that their deviations are punished but the punishment output should not exceed the output given by the proportional scheme. The deviating firm should return to cooperation after a deviation, i.e., it should choose the cooperative output. If the punishment output has, however, exceeded the proportional scheme, the roles of the original deviator and the punisher are changed; i.e., exceeding the proportional scheme is interpreted as a deviation from the cooperative point q^λ.

We say that a firm has played *conventionally* if its behavior does not give a reason to punish it. The only reasons to punish firm i are that it has unilaterally exceeded the cooperative output q_i^λ, or it has punished the other firm too harshly, i.e., $q_i^k > L(q_{-i}^{k-1}, \alpha_{-i}) = q_i^\lambda + \alpha_{-i} \cdot (q_{-i}^{k-1} - q_{-i}^\lambda)$. Formally, the strategy is to choose $q_i^0 = q_i^\lambda$ and for $k \geq 1$ play as follows:

i) $q_i^k = q_i^\lambda$, if in period $k-1$ both firms have played conventionally, deviated simultaneously, or only firm i has not played conventionally.
ii) $q_i^k = L(q_{-i}^{k-1}, \alpha_{-i})$, if the other firm has not played conventionally in period $k-1$.

We denote the above strategy for firm i with slope α_{-i} by $\omega_i(\alpha_{-i})$. The strategy profile with both using proportional schemes is $\omega(\alpha) = (\omega_1(\alpha_2), \omega_2(\alpha_1))$ with $\alpha = (\alpha_1, \alpha_2)$. The slopes of the proportional schemes are common knowledge, i.e., both firms know each other's punishment scheme.

Ehtamo and Hämäläinen (1993) study a similar strategy as $\omega(\alpha)$ in a continuous time setting in maintaining Pareto-optimal solution as the outcome of a resource management game. However, according to their formulation of the proportional scheme, the players do not return to cooperation immediately after they have deviated. This is also the case with the linear reaction strategy studied by Kalai and Stanford (1985) and Stanford (1986a). Since the deviating players return to cooperation, $\omega_i(\alpha_{-i})$ works only partially as a reaction function strategy. The deviations are punished in the manner of reaction functions but after deviations firms return to cooperation unlike when using reaction function strategies.

The following Lemma shows that $q_i^k = q_i^\lambda$, for all k, is the optimal choice of actions for firm i when the other player uses $\omega_{-i}(\alpha_i)$ and the slope α_i is steep enough. Hence, the equilibrium outcome of the game with $\omega(\alpha)$ is *cooperative play*, i.e., $q^k = q^\lambda$ for all k. The proof of Lemma 1 is given in Appendix.

Lemma 1. *Let us assume that (A1) and (A2) hold for π_i. Then the optimal sequence of actions for firm i is to choose $q_i^k = q_i^\lambda$ for all $k \geq 0$, when the other firm follows $\omega_{-i}(\alpha_i)$ with $\alpha_i \geq \alpha_i^\lambda / \delta_i$.*

The lower bound that is obtained for the slope of the proportional scheme for firm $-i$ is the slope of the static equilibrium strategy divided by firm i's discount factor. This means that in presence of time lag, the deviator has to be punished stronger than the static equilibrium strategy would suggest. Furthermore, the lower bound depends only on the deviating firm's discount factor and the point q^λ. The larger the discount factor is, the less the equilibrium slope of the static setting has to be increased. This is natural since the smaller the discount factor is, the more the deviator should suffer from the current punishment.

By following $\omega_i(\alpha_{-i})$, with $\alpha_{-i} > 0$, firm i increases its output when the other firm has deviated. If we had $\alpha_{-i} < 0$, then following $\omega_i(\alpha_{-i})$ would decrease firm i's profits while deviating would be profitable for the other firm since by (A1) firm $-i$'s profits are decreasing with respect to q_i. It follows that $\omega(\alpha)$

cannot sustain cooperation when $\partial\pi_i(q^\lambda)/\partial q_i < 0$ for either of the firms since the lower bound $\alpha_i^\lambda/\delta_i$ depends on this partial derivative.

3.1 Credibility

It follows from Lemma 1 that the equilibrium outcome of the game is cooperative play, when the proportional schemes have sufficiently steep slopes. However, this does not guarantee that it would actually be optimal for the punishing firm to follow the proportional scheme, or the deviator to return to cooperation.

Following Osborne's original idea we say that $\omega(\alpha)$ is credible for firm i if it is optimal for firm i to follow $\omega_i(\alpha_{-i})$ when the deviator returns to cooperation and accepts punishments not above the line $L(q_i, \alpha_i)$; i.e., when firm $-i$ follows $\omega_{-i}(\alpha_i)$. Consequently, if $\omega(\alpha)$ is credible for the punishing firm, the other firm knows that the deviations are really punished according to the proportional scheme. The notion of credibility, in the sense it is used here, has also been discussed by Holahan (1978), Rothschild (1981), and Ehtamo and Hämäläinen (1993). However, in these papers a firm does not need to worry about the rival's future actions and credibility rather means that it is better to follow the proportional scheme than to do nothing.

In the rest of the paper we denote the interval of *acceptable deviations* by $I_i(\bar{q}_i) = (q_i^\lambda, \bar{q}_i]$, $\bar{q}_i > q_i^\lambda$, and the interval of *acceptable punishments* by $I_i^L(q_{-i}) = [0, L(q_{-i}, \alpha_{-i})]$. We need the interval $I_i(\bar{q})$ because credibility as well as subgame perfection will be obtained only when restricting to sufficiently small deviations, here $\bar{q}$ denotes the upper bound of acceptable deviations. Formally the credibility of $\omega(\alpha)$ for firm i is defined as follows.

Definition 1. $\omega(\alpha)$ is credible for firm i on $I_{-i}(\bar{q}_{-i})$ if it is optimal for firm i to follow $\omega_i(\alpha_{-i})$ after any unilateral deviation by firm $-i$ on $I_{-i}(\bar{q}_{-i})$, assuming that firm $-i$ follows $\omega_{-i}(\alpha_i)$ after the deviation.

The credibility of $\omega(\alpha)$ for firm i means two things: Within the acceptable range of punishment outputs $I_i^L(q_{-i})$, where q_{-i} is the deviation in the prior period, it is optimal for firm i to choose the output given by the proportional rule, and it is better to follow $\omega_i(\alpha_{-i})$ than to choose the maximal deviation and then to be punished. In particular, credibility requires that the firm that has first deviated punishes the other firm for its unfair punishment outputs strongly enough. The condition for the optimality of the proportional scheme within the range of acceptable punishment outputs $I_i^L(q_{-i})$ is

$$\pi_i\left(L(q_{-i}, \alpha_{-i}), q_{-i}^\lambda\right) = \max_{x \in I_i^L(q_{-i})} \pi_i(x, q_{-i}^\lambda). \tag{5}$$

Let us further define

$$q_{-i}^L = \sup \left\{ x \in \mathbb{R} : (5) \text{ holds for all } q_{-i} \in [q_{-i}^\lambda, x] \right\},$$

which gives an upper bound for the largest interval where $\omega(\alpha)$ is credible for firm i. When the deviation exceeds q_i^L the punishing firm would rather choose a smaller quantity than the one given by the proportional scheme. Hence, the greatest upper bound for the credibility interval is obtained from the crossing point of the best response function and the proportional strategy. This is as in the static setting of Section 2, see also Figure 2.

The following lemma shows that $\omega(\alpha)$ is credible for firm i when the deviations do not exceed q_{-i}^L and the punishment outputs that exceed $L(q_{-i}, \alpha_{-i})$ are treated as deviations. The proof is given in Appendix and technically it is close to that of Lemma 1.

Lemma 2. *Let us assume that* π_i *satisfies (A1) and (A2), and* $q_{-i}^L > q_{-i}^\lambda$. *Then the strategy* $\omega(\alpha)$ *is credible for firm* i *on* $I_{-i}(q_{-i}^L)$ *when* $\alpha_i \geq \alpha_i^\lambda/\delta_i$ *and* $\alpha_{-i} > 0$.

We can observe that the farther the best response function R_i crosses the line $q_i = L(q_{-i}, \alpha_{-i})$, the greater q_i^L becomes. Furthermore, when the slope α_{-i} decreases, the upper bound increases. In particular, the slope may decrease as δ_i increases. Finally, we point out that it may happen that firm's Cournot quantity is less than q_i^L, i.e., in some cases even quite large deviations can be punished credibly with the proportional scheme.

3.2 Re-establishing Cooperation

In addition to credibility, it should be optimal for the deviator to return to cooperation when the retaliation follows the proportional scheme. In that case the proportional scheme prevents further deviations from cooperative play. More formally this property is defined as follows.

Definition 2. $\omega(\alpha)$ returns firm i to cooperation on $I_i(\bar{q}_i)$, if it is optimal for the firm to follow $\omega_i(\alpha_{-i})$ after any of its own unilateral deviations on $I_i(\bar{q}_i)$, assuming that firm $-i$ follows $\omega_{-i}(\alpha_i)$.

The strategy profile $\omega(\alpha)$ can be shown to return a firm to cooperation for bounded deviations when the firm's marginal profit is decreasing with respect to the other firm's output. We can formulate this condition for π_i as follows:

(A3) $\partial \pi_i(q_i^\lambda, q_{-i})/\partial q_i$ is decreasing with respect to q_{-i}.

In the specific case when π_i is of the form (4), the assumption (A3) holds when $\partial P(q_i^\lambda, q_{-i})/\partial q_i$ and $P(q_i^\lambda, q_{-i})$ are decreasing with respect to q_{-i}.

The result on re-establishing cooperation is formulated in the following lemma, the proof of which is partly based on Lemma 1 and is given in Appendix.

Lemma 3. *Let us assume that π_i satisfies (A1)–(A3), and $\alpha_i \geq \alpha_i^\lambda/\delta_i$. Then $\omega(\alpha)$ returns firm i to cooperation on $I_i(q_i^+)$ with*

$$q_i^+ = \sup\left\{x \in \mathbb{R} : \partial\pi_i\left(q_i^\lambda, L(q_i, \alpha_i)\right)/\partial q_i \geq 0 \;\forall q_i \in [q_i^\lambda, x]\right\},$$

assuming that $q_i^+ > q_i^\lambda$.

The explanation for the upper bound q_i^+ is that if the deviation q_i^k is too large, it becomes optimal for firm i to choose $q_i^{k+1} < q_i^\lambda$. This happens because by decreasing the output quantity in the period after the deviation, the firm can compensate the punishment, which loses its effect as a sufficient threat to prevent further deviations.

4 Subgame Perfection for Bounded Deviations

We have observed that for sufficiently small deviations the proportional scheme is credible, i.e., it is optimal for the punishing firm to follow the proportional strategy. Moreover, it returns firms to cooperation for bounded deviations when marginal profits are decreasing, i.e., it prevents further deviations from cooperation after sufficiently small unilateral deviations. When these properties hold simultaneously the strategy profile $\omega(\alpha)$ is a subgame perfect equilibrium (SPE) for bounded deviations, which means that if the deviations have been and will be small enough during the history of the play, it is optimal for both firms to follow $\omega(\alpha)$ when they know that the other firm will follow it, too.

Within the range of deviations where the equilibrium is subgame perfect, the strategy profile is also a weakly renegotiation proof equilibrium (WRPE), which means that in addition to subgame perfection none of the continuation payoffs of $\omega(\alpha)$ is Pareto dominated by any other continuation payoff of $\omega(\alpha)$ (Farrell and Maskin, 1989). Continuation payoffs are the discounted profits that the firms obtain when they follow $\omega(\alpha)$ beginning from a given history of the play. Hence, weak renegotiation proofness can be interpreted as the result of the firms negotiating the original agreement anew in any contingencies.

The following theorem summarizes the assumptions on the profit functions and the resulting properties of $\omega(\alpha)$. Here we denote $q_i^\alpha = \min\{q_i^L, q_i^+\}$.

Theorem 2. *Let us assume that (A1)–(A3) hold for both profit functions and* $\alpha_i \geq \alpha_i^\lambda/\delta_i$ *for* $i = 1, 2$. *Then* $\omega(\alpha)$ *is a WRPE, when the unilateral deviations do not exceed* q_i^α *for* $i = 1, 2$.

Proof. If none of the firms has deviated from q_i^λ, i.e., $q_i^{k-1} \leq q_i^\lambda$, $i = 1, 2$, or they have both deviated simultaneously from q_i^λ, i.e., $q_i^{k-1} > q_i^\lambda$, $i = 1, 2$, then by Lemma 1 it is optimal for the players to follow $\omega(\alpha)$. If firm i has deviated in period $k - 1$ while the other firm has played conventionally and $q_i^{k-1} \leq q_i^+$, then by Lemma 3 it is optimal for firm i to return to cooperation as suggested by $\omega_i(\alpha_{-i})$. On the other hand, by Lemma 2 it is optimal for the punishing firm to choose the output according to $\omega_i(\alpha_{-i})$ when $q_{-i} \leq q_{-i}^L$. Thus, $\omega(\alpha)$ is a SPE.

Weak renegotiation proofness holds because the deviator's losses increase as the deviation gets larger whereas the other firm's profit increases. Hence, when comparing the continuation payoffs, one of the firms is always worse off and one of them is better off after unilateral deviations. Therefore, no continuation payoffs of $\omega(\alpha)$ dominate any other. □

According to Theorem 2, deviations have to be punished in proportion to the α_i^λ obtained from the static Osborne's rule times the inverse of the discount factor δ_i. The slope of the static equilibrium strategy is obtained as the limit of the lower bound of the proportional scheme when the discount factor goes to one. In particular, the line of constant market shares is the limit of both firms' proportional schemes when the joint profit maximizing point is to be supported as the equilibrium outcome. Obtaining the static case in the limit is natural, since large discount factor could be interpreted as an implication of an ability to react rapidly to rivals' output changes. See Kalai and Stanford (1985) for another approach to consider reaction times.

One interpretation of Theorem 2 is that the equilibrium is subgame perfect even though large deviations were possible but the firms trust that the other firm will not make such deviations intentionally. As a just married couple would believe their relationship to last forever without unexpectedly serious deviations, the firms would believe in a similar way. Behind such a belief there might well be a local continuous mechanism that sustains cooperation under small unintentional errors. Intuitively, large deviations would signal breaking of cooperation and launching yet another type of mechanism. Indeed, if we have a strategy profile that is subgame perfect for all deviations, then switching to this strategy profile after deviations that exceed q_i^α, $i = 1, 2$, and using proportional strategies for smaller deviations, is a possible way to sustain q^λ as SPE outcome for all deviations. For example, switching to Cournot quantities after deviations larger than q_i^α, $i = 1, 2$, would work as a way to obtain subgame perfection for all deviations. Cournot-trigger is known to be SPE when $\pi_i(q^\lambda)$, $i = 1, 2$, are greater than profits at the Cournot-point and the discount factors are large enough. It can be shown that the combination of a proportional strategy and a trigger strategy would be subgame perfect for exactly the same discount factors as the trigger strategy.

Let us now go to the characterization of those cooperative points that can be supported as WRPE outcomes for bounded deviations with proportional schemes. If q^λ is a WRPE outcome for bounded deviations under $\omega(\alpha)$, we say that it can be supported locally with $\omega(\alpha)$. More formally this property is defined below.

Definition 3. q^λ is locally supportable as a WRPE outcome with proportional strategies if there are $\alpha > 0$ and intervals of deviations and punishments such that $\omega(\alpha)$ is WRPE when restricting the possible deviations and punishments to these intervals.

Note that, as punishments depend continuously on deviations, they are bounded whenever deviations are bounded. Hence, we could simply define local subgame perfection by requiring the deviations to be bounded. However, even though deviations should not exceed q_i^α, the punishment outputs may exceed this upper bound, i.e., it may happen that $L(q_{-i}^\alpha, \alpha_{-i}) > q_i^\alpha$. The above definition emphasizes that deviations and punishments may have different upper bounds.

The following lemma shows that the upper bound of allowed deviations is larger than q_i^λ when the proportional scheme has a positive slope and π_i is continuously differentiable. Hence, when both firms' profit functions and proportional schemes satisfy these conditions, then for any discount factors $\delta_i \in (0,1)$, $i = 1,2$, there are intervals of deviations on which $\omega(\alpha)$ is a WRPE. The proof of the lemma is given in Appendix.

Lemma 4. *Let us assume that π_i, $i = 1,2$, satisfy (A2) and (A3), are continuously differentiable, and $\alpha_i > 0$ for $i = 1,2$. Then $q_i^\alpha > q_i^\lambda$ for $i = 1,2$.*

The set of supportable cooperative points depends on assumptions (A2) and (A3). Namely, whenever these conditions hold for profit functions at q^λ, the result of Lemma 4 is valid. Hence, $\omega(\alpha)$ supports q^λ as a locally WRPE outcome if the firms' marginal profits are decreasing with respect to each others outputs, $\partial\pi_i(q^\lambda)/\partial q_i > 0$, for $i = 1,2$, and the slopes α_i, $i = 1,2$, are steep enough. Recall that the positivity of the partial derivative is required for α^λ to be positive.

The set of points on which (A2) holds is actually the part of q_1, q_2-plane that is below the best response functions, i.e., $q_i^\lambda < R_i(q_{-i}^\lambda)$, for $i = 1,2$. This is shown in the following lemma, the proof of which is presented in Appendix.

Lemma 5. *When π_i satisfies (A1), then (A2) is equivalent to $q_i^\lambda < R_i(q_{-i}^\lambda)$.*

The above result is intuitive. Namely, when the firm's cooperative output is below the best response to the other firm's output, the firm would like to increase the output, which means that (A2) holds.

By combining lemmas 4 and 5, and Theorem 2 we obtain the following "folk theorem" for locally supportable points. Here (A3) is assumed to hold for all quantity pairs, which means that at any output level the firms' marginal profits are decreasing with respect to rival's output.

Theorem 3. *Let us assume that both profit functions are continuously differentiable, satisfy (A1), and $\partial\pi_i(q)/\partial q_i$ is decreasing with respect to q_{-i} for all $q_i > 0$ and $i = 1, 2$. Then any q^λ with $0 < q_i^\lambda < R_i(q_{-i}^\lambda)$, for $i = 1, 2$, is locally supportable as a WRPE outcome with proportional strategies.*

It follows from Theorem 3 that the set of locally supportable outcomes is nonempty when the Cournot quantities are positive. Namely, at the Cournot point the firms' best response functions cross and at least the points that both prefer to the Cournot point are locally supportable. Moreover, all Pareto-optimal points, except for the firms' global optima, belong to the set of locally supportable points. The global optima cannot necessarily be supported because at these points $q_i^\lambda = 0$ for either of the firms, and then α_i^λ may become infinitely large.

4.1 Example: Symmetric Duopoly with Quadratic Profits

This example illustrates how the upper bound of the allowed deviations is determined. Let us assume that $\delta_i = \delta$ and $\pi_i(q) = (a - q_i - q_{-i})q_i$ for $i = 1, 2$, which satisfy (A1). The profits are of this form when the inverse demand function P and the cost functions C_1 and C_2 are linear. Here $a - q_1 - q_2$ is assumed to be positive so that profits are positive.

The slope of the tangent line for π_i at the cooperative point q^λ, as defined in (2), is $\alpha_i^\lambda = (a - 2q_1^\lambda - q_{-i}^\lambda)/q_i^\lambda$. We can see that when q_{-i}^λ is kept fixed and q_i^λ is increased α_i^λ decreases, which means that firm i's deviations become easier to prevent with $\omega(\alpha)$. As q_i^λ goes to zero the slope α_i^λ becomes infinitely large, i.e., the deviations become more difficult to punish. In particular, no proportional scheme prevents firm i' deviations from a point in which it produces nothing.

Let us assume that q^λ is the joint profits maximizing point, i.e., $q^\lambda = (a/4, a/4)$. Note that assumptions (A2) and (A3) are satisfied at this point. The slopes of the tangent lines are $\alpha_i^\lambda = 1$, $i = 1, 2$. Hence, we should have $\alpha_i \geq 1/\delta$, $i = 1, 2$, for $\omega(\alpha)$ to be a SPE. Let us take $\alpha_i = 1/\delta$ for $i = 1, 2$. Then the proportional scheme for firm i is

$$L(q_{-i}, \alpha^{\lambda}_{-i}) = (1 - 1/\delta)a/4 + q_{-i}/\delta.$$

Now q^L_{-i} is obtained at the intersection of the best response function

$$R_i(q_{-i}) = (a - q_{-i})/2$$

and the line of punishment outputs. The intersection point is at $q^L_{-i} = q^L_i = (\delta + 1)a/(2\delta + 4)$. Furthermore, the upper bound q^+_i of Lemma 3 is obtained from

$$\partial\pi_i \left(q^{\lambda}_i, L(q_i, \alpha_i)\right)/\partial q_i = a - 2q^{\lambda}_i - L(q_i, \alpha_i) \geq 0,$$

which gives $q^+_i = (1 - \delta/2)a$. Noticing that $q^+_i \geq q^L_i$ we have $q^{\alpha}_i = q^L_i$.

Now we can see that the more patients the firms are, the more tolerant they become to deviations, i.e., the larger q^{α}_i, $i = 1, 2$, become. Furthermore, as $\delta \to 1$ we have $q^{\alpha}_i \to a/3$, which equals the firms' Cournot quantities.

4.2 Conjectural Variations Equilibria

In this section we discuss a possible way to detect empirically whether an observed market situation can be interpreted as a collusion with proportional strategies as the supporting mechanisms. This discussion is based on the observation that proportional strategies are linked to conjectural variations models. The main idea of these models is that each firm believes that its rival's choices depend on the firm's output. [2] Hence, firm i is assumed to have a conjecture on its rival's reactions around q^{λ}. This behavioral assumption is captured in the conjectural variation parameter $\nu_i = dq_{-i}(q^{\lambda}_i)/dq_i$.

As the static Osborne's rule, conjectural variations models assume that the response for deviations is instantaneous. Indeed, in the static setting the slope of the punishment line $L(q_i, \alpha_i)$ plays exactly the same role as a constant conjectural variation ν_i. Hence, α_i can be identified with ν_i. In conjectural variations literature L is called the conjectured reaction function. Instead of associating ν_i with the conjectured reaction function we associate it with a proportional strategy. Consequently, proportional strategies in repeated game give a rational justification for conjectural variations models; when the conjectural variations are large enough, a conjectural variations equilibrium corresponds to a locally subgame perfect equilibrium under proportional strategies. More specifically, a conjectural variations equilibrium can be interpreted as a locally

[2] In the most general formulation the choices depend on both firms' outputs.

SPE (or WRPE to be specific) with the slopes of proportional schemes equaling the conjectural variations. Recall that the assumptions (A1)–(A4) should hold.

We say that conjectural variations that lead to a locally SPE are strategically consistent. More formally this consistency is defined as follows.

Definition 4. Firm i's conjectural variation is *strategically consistent* if $\nu_i \geq \alpha_i^\lambda/\delta_i$. If q^λ is below the reaction functions and ν_1 and ν_2 are strategically consistent, then the conjectural variations equilibrium is strategically consistent.

Other dynamic interpretations of conjectural variations and various consistency concepts have been discussed, e.g., in Figuières et al. (2004, Chapters 2 and 3).

Let us assume that firm i's profit function is of the form (4) and $P(q) = P(\sum_j q_j)$. With the conjectural variation ν_i, the necessary conditions for firm i's static profit maximization problem can be written as

$$1 - C_i'(q_i^\lambda)/P(q^\lambda) = 1/|\eta|(1+\nu_i), \tag{6}$$

where $|\eta| = -[\partial q_i/q_i]/[\partial P(q)/P(q)]$, i.e., $|\eta|$ is the absolute value of the elasticity of demand. The left hand side of (6) is the firm's Lerner index and we denote it by LE_i. This index measures the competitiveness of an oligopolistic market; the larger the Lerner index is, the less competitive the market is.

Note that as we are not considering a static oligopoly game, condition (6) need not hold for $\nu_i = \alpha_i$. However, we can derive another relationship for ν_i, LE_i, and η to obtain strategic consistency. Namely, it can be seen that $\alpha_i^\lambda = LE_i/|\eta| - 1$. Hence, strategic consistency, i.e., $\nu_i \geq \alpha_i^\lambda/\delta_i$, requires that $\delta_i\nu_i \geq LE_i/|\eta| - 1$. By using this inequality we can estimate the lowest strategically consistent conjectures for given discount factors, demand elasticities, and Lerner indices. Let us also observe that (A2) can be written equivalently as $LE_i > 1/|\eta|$, when $P(q^\lambda) > 0$. Hence, empirically observed Lerner indices that are greater than $1/|\eta|$ could be due to tacit collusion with proportional strategies.

5 Discussion

We have shown that for sufficiently regular profit functions, using proportional reaction strategies sustains cooperation as a subgame perfect outcome when the deviations are small enough. Using the strategies is also Nash equilibrium for all deviations. The slopes of the proportional strategies and the ranges of acceptable deviations depend on the profit functions and discount factors in a simple way. Moreover, the cooperative point that is to be supported as the equilibrium outcome should be in the region below the firms' best response functions.

Local subgame perfection could be seen as one form of bounded rationality similarly as ε-equilibria. The firms are behaving optimally only when they omit their rival's possibility of making large deviations. Nevertheless, the main motivation for the local nature of the results of this paper is that a linear strategy can be obtained as a linearization from a more general nonlinear equilibrium strategy. Since linearization is reasonably accurate only in the neighborhood of the cartel point, it is natural that the properties of a linear strategy are also local. Local equilibrium properties of linear strategies were first analyzed by Osborne (1976) in a static setting.

The linearization idea appears also in the conjectural variations literature, where the conjectural variations are explained as slopes of linearized reactions. Indeed, the results of this paper give new motivation for both Osborne's model and conjectural variations. Osborne's quota rule is obtained as a limiting case from the proportional equilibrium strategies in repeated game as discount factors approach to one. For conjectural variations equilibrium these strategies give a possible explanation: With large enough conjectures the corresponding equilibrium becomes locally subgame perfect with proportional strategies, where the slopes equal the conjectural variations.

The local results of this paper show that using linear strategies provides a simple way to punish small unintentional errors or trembles so that collusion can be re-established. Moreover, in a duopoly setting the equilibrium is weakly renegotiation proof, which means that there is no need to renegotiate the cooperation anew after small deviations. To obtain subgame perfection for all deviations the proportional scheme can be combined with other equilibrium strategies, e.g., with trigger strategies such that only large deviations lead to collapse of the cartel and launch the trigger. In the spirit of forward induction, deviations can be interpreted as strategic signals: A small deviation signals an accidental error but a large deviation is a signal of breaking the collusion.

Appendix: Auxiliary Proofs

In this appendix $I_i^\lambda = [0, q_i^\lambda]$ denotes the interval of acceptable outputs for firm i, given that the other firm has played conventionally. Recall that $I_i(\bar{q}_i) = (q_i^\lambda, \bar{q}_i]$ and $I_i^L(q_{-i}) = [0, L(q_{-i}, \alpha_{-i})]$ is firm i's set of acceptable punishment outputs after a deviation q_{-i} by the other firm.

It is worth noticing that all the proofs go through with assuming concavity of π_i instead of strict concavity.

Proof of Theorem 1: Let us first observe that the maximization problem

$$\max_{q_i \geq 0} \pi_i\left(q_i, r_{-i}(q_i, \alpha_i)\right),$$

where $r_{-i}(q_i, \alpha_i) = \max\{q^\lambda_{-i}, L(q_i, \alpha_i)\}$, has exactly the same solution as the convex optimization problem

$$\begin{aligned} &\max \pi_i(q) \\ &\text{s.t. } q \in \left\{\hat{q} \in \mathbb{R}^2 : \hat{q}_i \geq 0,\ \hat{q}_{-i} \geq \max\{q^\lambda_{-i}, L(\hat{q}_i, \alpha_i)\}\right\}. \end{aligned}$$

This is because π_i is decreasing with respect to q_{-i} so that at the optimum we have $q_i = \max\{q^\lambda_{-i}, L(q_i, \alpha_i)\}$ even though inequality was allowed. The necessary and sufficient condition of this problem at q^λ is that the below variational inequality holds for all feasible q:

$$\nabla \pi_i(q^\lambda) \cdot (q - q^\lambda) \leq 0, \tag{7}$$

i.e.,

$$[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q^\lambda_i) + [\partial \pi_i(q^\lambda)/\partial q_{-i}](q_{-i} - q^\lambda_{-i}) \leq 0.$$

For $q_i > q^\lambda_i$ condition (7) holds when

$$[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q^\lambda_i) + \alpha_i[\partial \pi_i(q^\lambda)/\partial q_{-i}](q_i - q^\lambda_i) \leq 0,$$

because now $q_{-i} \geq L(q_i, \alpha_i)$. By (A1) and (A2) $\alpha_i \geq \alpha^\lambda_i > 0$, and by (A1) $\partial \pi_i(q^\lambda)/\partial q_{-i} < 0$. Thus,

$$\alpha_i \cdot (q_i - q^\lambda_i) \cdot \partial \pi_i(q^\lambda)/\partial q_{-i} \leq \alpha^\lambda_i \cdot (q_i - q^\lambda_i) \cdot \partial \pi_i(q^\lambda)/\partial,$$

which gives the optimality condition.

For $q_i \leq q^\lambda_i$ we have $q_{-i} \geq q^\lambda_{-i}$. Hence, the left hand side of condition (7) is less than zero if $[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q^\lambda_i)$ is less than zero. It follows from (A2) that π_i is decreasing with respect to q_i. Hence, we obtain the inequality $[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q^\lambda_i) \leq 0$, i.e., (7) holds for $q_i \leq q^\lambda_i$. □

Proof of Lemma 1: Let $\{q^k\}_k$ be a sequence of output quantity pairs and $\Pi_i(\{q^k\}_k) = \sum_k \delta^k_i \pi_i(q^k)$. The assumption is that $q^0_{-i} = q^\lambda_{-i}$, and we need to show that then it is optimal for firm i to choose $q^k_i = q^\lambda_i$ for all $k \geq 0$, which means that $q^k = q^\lambda$ for all k is the optimal choice of output quantity pairs for firm i.

As in Theorem 1, the choice of the output can be written as a convex optimization problem

$$\begin{aligned} &\max \Pi_i(\{q^k\}_k) \\ &\text{s.t. } \{q^k\}_k \in F(\alpha_i), \end{aligned}$$

where

$$\begin{aligned} F(\alpha_i) = \Big\{ &\{q^k\}_k : q^k_i \geq 0\ \forall k \geq 0,\ q^0_{-i} = q^\lambda_{-i}, \\ &q^k_{-i} \geq \max\{q^\lambda_{-i}, L(q^{k-1}_i, \alpha_i)\}\ \forall k \geq 1 \Big\}. \end{aligned}$$

We assume that in the above optimization problem $\{q^k\}_k \in l_\infty \times l_\infty$, which is a Banach-space with the norm $\|\{q^k\}_k\| = \max_k |q_1^k| + \max_k |q_2^k|$. Hence, the sequence $\{q^k\}_k$ should be bounded, which is not too restrictive assumption since choosing large outputs usually causes losses for the firms. Moreover, the results that are based on this lemma require that the outputs stay within certain ranges.

Analogously to the static case, the sufficient and necessary condition for the optimality of $\{q^\lambda\}_k$ is that the variational inequality

$$\nabla\Pi_i(\{q^\lambda\}_k)(\{q^k\}_k - \{q^\lambda\}) \leq 0 \tag{8}$$

holds for all $\{q^k\}_k \in F(\alpha_i)$, see, e.g., Ekeland and Témam (1976, Proposition 2.1 in Chapter II). Here $\nabla\Pi_i(\{q^\lambda\}_k)$ denotes the Fréchet-differential of Π_i at $\{q^\lambda\}_k$. It can be seen that

$$\nabla\Pi_i(\{q^\lambda\}_k)\{q^k\}_k = \sum_k \delta_i^k \nabla\pi_i(q^\lambda) \cdot q^k.$$

In the following we denote $I_+ = \{k : q_i^k > q_i^\lambda\}$ and $I_- = \{k : q_i^k \leq q_i^\lambda\}$. Now for $k \in I_-$ we have $q_{-i}^k \geq q_{-i}^\lambda$ and for $k \in I_+$ we have $q_{-i}^k \geq L(q_i, \alpha)$. The latter yields $q_{-i}^{k+1} - q_{-i}^\lambda \geq \alpha_i(q_i^k - q_i^\lambda)$ for $k \in I_+$. Let us now consider (8) in more detail:

$$\begin{aligned}
\nabla\Pi_i(\{q^\lambda\}_k)(\{q^k\}_k - \{q^\lambda\}) &= \sum_k \delta_i^k \nabla\pi_i(q^\lambda) \cdot (q^k - q^\lambda) \\
&= \sum_k \delta_i^k [(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + (q_{-i}^k - q_{-i}^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i}] \\
&\leq \sum_{k\in I_+} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_+} \alpha_i \delta_i^{k+1}(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} \\
&\quad + \sum_{k\in I_-} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_-} \delta_i^k (q_{-i}^k - q_{-i}^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} \\
&\leq \sum_{k\in I_+} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_+} (\alpha_i^\lambda/\delta_i)\delta_i^{k+1}(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} = 0,
\end{aligned} \tag{9}$$

where the last equality follows by plugging α_i^λ from (2) in the equation. The second last inequality holds because $\alpha_i \geq \alpha_i^\lambda/\delta_i$ and the terms of the sum with $k \in I_-$ are negative. The latter is true because $\partial\pi_i(q^\lambda)/\partial q_{-i} < 0$ by (A1), and $\partial\pi_i(q^\lambda)/\partial q_i > 0$ by (A2). The above deduction shows the optimality of the sequence $q_i^k = q_i^\lambda$ for all k. □

Proof of Lemma 2: Because $q_i^\alpha \leq q_i^L$, it is optimal to choose $q_i^1 = L(q_{-i}^0, \alpha_{-i})$ within the acceptable range of punishments $I_i^L(q_{-i}^0)$, when $q_{-i}^0 \in I_{-i}(q_{-i}^L)$ and $q_i^0 \in I_i^\lambda$. Hence, we only need to show that it is not optimal to make the maximal deviation from $\omega_i(\alpha_{-i})$ and then to be punished. This is the case because

$$\max_{\{q^k\}\in F(\alpha_i)} \Pi_i\left(\{q^k\}_k\right) = \pi_i^\lambda/(1-\delta_i) \leq \pi_i\left(L(q_{-i}^0, \alpha_{-i}), q_{-i}^\lambda\right) + \delta_i\pi_i^\lambda/(1-\delta_i),$$

where the first equality follows from Lemma 1, and the inequality holds because

$$\pi_i\left(L(q_{-i}^0, \alpha_{-i}), q_{-i}^\lambda\right) \geq \pi_i^\lambda$$

by the choice $q^0_{-i} \in I_{-i}(q^L_{-i})$. Note that by making an unreasonably large punishment the firm cannot exceed the profits that maximize $\Pi_i\left(\{q^k\}_k\right)$ subject to $\{q^k\} \in F(\alpha_i)$ Thus, $\omega(\alpha)$ is credible for firm i. $\square$

Proof of Lemma 3:

Let us denote $\tilde{q}_{-i} = L(q^0_i, \alpha_i)$ and $q' = (q^\lambda_i, \tilde{q}_{-i})$. It is optimal to choose $q^k_i = q^\lambda_i$ for all $k \geq 1$, if the variational inequality

$$\nabla\pi_i(q') \cdot (q^1 - q') + \sum_{k\geq 2} \delta^{k-1}_i \nabla\pi_i(q^\lambda) \cdot (q^k - q^\lambda) \leq 0 \tag{10}$$

holds for all feasible sequences $\{q^k\}_k$, similarly as in the proof of Lemma 1. This condition can be written as:

$$S_1 + S_2 \leq 0,$$

where S_1 contains the terms that include q^1_i and q^1_{-i}, and S_2 contains the rest of the sum. As in the necessary condition (8), we have:

$$\begin{aligned} S_2 \leq & \sum_{k\in I_+} \delta^{k-1}_i (q^k_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_i] + \sum_{k\in I_+} \alpha_i \delta^k_i (q^k_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_{-i}] \\ & + \sum_{k\in I_-} \delta^{k-1}_i (q^k_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_i] + \sum_{k\in I_-} \delta^{k-1}_i (q^k_{-i} - q^\lambda_{-i})[\partial\pi_i(q^\lambda)/\partial q_{-i}], \end{aligned}$$

where $I_+ = \{k \geq 2 : q^k_i > q^\lambda_i\}$ and $I_- = \{k \geq 2 : q^k_i \geq q^\lambda_i\}$. As in the case $q^0_i = q^\lambda_i$ in the proof of Lemma 1, it can be shown that $S_2 \leq 0$ when $\alpha_i \geq \alpha^\lambda_i/\delta_i$.

Hence, to obtain (10) we need to show that $S_1 \leq 0$. Let us first note that

$$\nabla\pi_i(q') \cdot (q^1 - q') = [\partial\pi_i(q')/\partial q_i](q^1_i - q^\lambda_i)$$

because $q^1_{-i} = \tilde{q}_{-i}$ according to $\omega_{-i}(\alpha_i)$. From this and the proportional scheme we obtain

$$S_1 = \begin{cases} [\partial\pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) + \alpha_i\delta_i(q^1_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_{-i}] & \text{if } q^1_i > q^\lambda_i, \\ [\partial\pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) & \text{if } q^1_i \leq q^\lambda_i. \end{cases}$$

Let us assume that $q^1_i > q^\lambda_i$. By the assumption (A3) we have $\partial\pi_i(q')/\partial q_i \leq \partial\pi_i(q^\lambda)/\partial q_i$ and hence

$$[\partial\pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) \leq [\partial\pi_i(q^\lambda)/\partial q_i](q^1_i - q^\lambda_i).$$

It follows that

$$\begin{gathered} [\partial\pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) + \alpha_i\delta_i(q^1_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_{-i}] \leq \\ [\partial\pi_i(q^\lambda)/\partial q_i](q^1_i - q^\lambda_i) + \alpha_i\delta_i(q^1_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_{-i}] \leq \\ [\partial\pi_i(q^\lambda)/\partial q_i](q^1_i - q^\lambda_i) + \alpha^\lambda_i(q^1_i - q^\lambda_i)[\partial\pi_i(q^\lambda)/\partial q_{-i}] = 0 \end{gathered}$$

where the last equality is obtained by plugging α^λ_i in the equation. Hence, $S_1 \leq 0$ and consequently (10) holds.

Because $[\partial \pi_i(q')/\partial q_i] \geq 0$ by $q_i^0 \leq q_i^+$, we have $S_1 \leq 0$ also for $q_i^1 \leq q_i^\lambda$. Thus, (10) holds for all feasible sequences. □

Proof of Lemma 4: The continuity of $\partial \pi_i(q_i^\lambda, q_{-i})/\partial q_i$, (A2), and (A3) imply that $q_i^+ > q_i^\lambda$. By (A2) we know that π_i is growing at q^λ with respect to its first argument. It then follows from the continuity of the derivative that there is $\tilde{q}_i > q_i^\lambda$ such that for all $q_i^1, q_i^2 \in [q_i^\lambda, \tilde{q}_i]$, with $q_i^1 \geq q_i^2$, we have $\pi_i(q_i^1, q_{-i}^\lambda) \geq \pi_i(q_i^2, q_{-i}^\lambda)$, i.e., π_i is growing on $[q_i^\lambda, \tilde{q}_i]$ with respect to q_i.

Since $\alpha_i > 0$, there is $\hat{q}_{-i} > q_{-i}^\lambda$ such that $\tilde{q}_i = L(\hat{q}_{-i}, \alpha_{-i})$, i.e., L maps $[q_{-i}^\lambda, \hat{q}_{-i}]$ into $[q_i^\lambda, \tilde{q}_i]$. Because π_i is growing on $[q_i^\lambda, \tilde{q}_i]$ it follows that for all $q_{-i}^0 \in I_i(\hat{q}_{-i})$ it is optimal to choose $q_i^1 = L(q_{-i}^0, \alpha_i)$. Thus, $q_{-i}^L \geq \hat{q}_{-i} > q_{-i}^\lambda$, and we have $q_i^\alpha > q_i^\lambda$. □

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