Interpretation of Lagrange Multipliers in Nonlinear Pricing Problem

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Lagrange multiplier interpretations

- Shadow prices in economics
- Forces in mechanics
- Voltages in electric circuits
- New interpretation in nonlinear pricing problem
- Directed graph of flows between the buyer types

Diet problem by Stigler (1945)

Find minimum cost diet that satisfies some nutritional requirements defined by recommended dietary allowances.

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax \geq b, \;\; x \geq 0, \end{aligned}$$

where x is amounts of foods, A nutrient contents of foods, c costs, and b nutritional requirements.

Stigler's "optimal" diets

	August 1939		August 1945	
Food	Annual Quantity	Annual Cost	Annual Quantity	Annual Cost
Wheat Flour	370 lb.	$13.33	535 lb.	$34.43
Evaporated Milk	57 cans	3.84	—	—
Cabbage	111 lb.	4.11	107 lb.	5.23
Spinach	23 lb.	1.85	13 lb.	1.56
Dried Navy Beans	285 lb.	16.80	—	—
Pancake Flour	—	—	134 lb.	13.08
Beef Liver	—	—	25 lb.	5.48
Total Annual Cost		$39.93		$59.88
Total Daily Cost		$0.109		$0.135

Lagrange multipliers as shadow prices

Dual problem is revenue maximization so that artificial foods are competitive against the real foods in price.

$$
\begin{aligned}
\max \quad & b'\lambda \\
s.t. \quad & A'\lambda \leq c, \quad \lambda \geq 0,
\end{aligned}
$$

where the multipliers λ are interpreted as unit prices of nutrient pills and b the demand. Multipliers also carry sensitivity information.

Nonlinear pricing problem

Monopoly designs products with quality q and price p. Buyer types with utility $V_i(q) - p$, $i \in I = \{1, \ldots, n\}$.

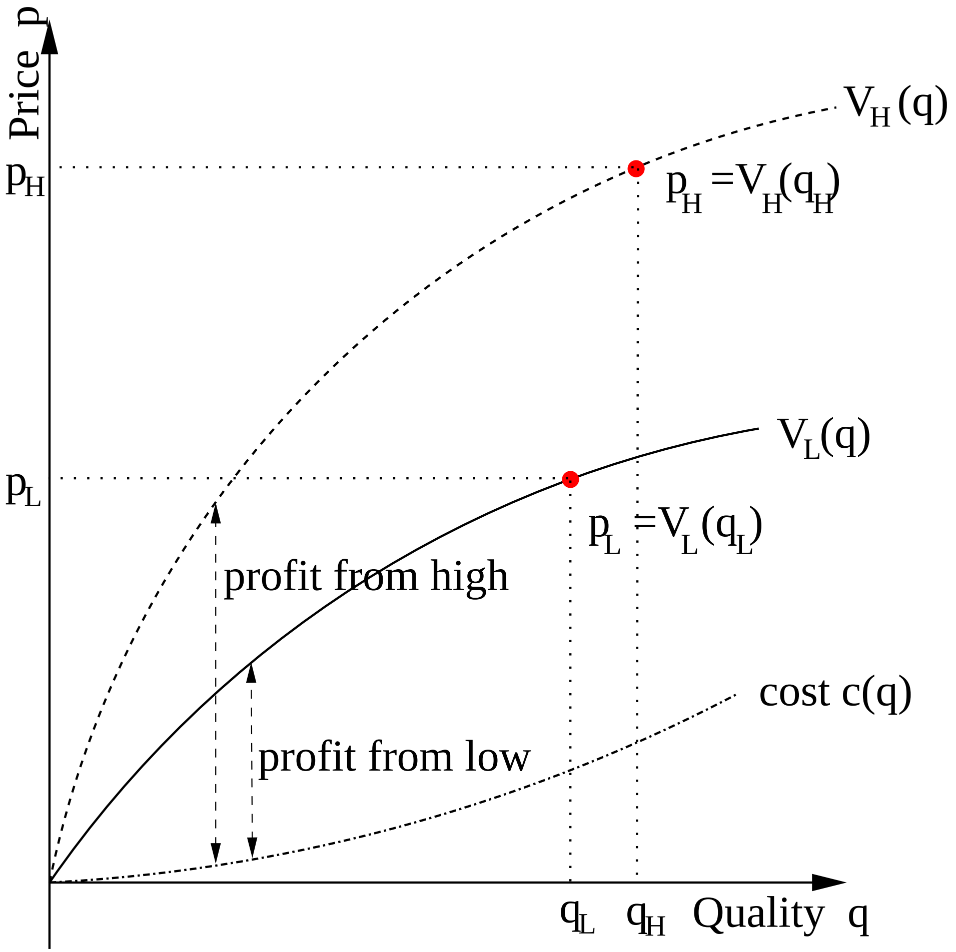
$$\max_{q,p} \quad \pi(q,p) = \sum_{i=1}^{n} f_i[p_i - c(q_i)]$$

$$s.t. \quad V_i(q_i) - p_i \geq 0, \forall i \in I \qquad (IR)$$

$$V_i(q_i) - p_i \geq V_i(q_j) - p_j, \forall i, j \in I, j \neq i, \quad (IC)$$

where $\pi(q,p)$ is firm's profit, f_i weight of buyer i, and $c(x)$ cost of producing product with quality q.

Illustrative example



KKT optimality conditions

$$f_i + \sum_{k \neq i} \lambda^{ki} = \sum_{j \neq i} \lambda^{ij}, \ \forall i \qquad (1)$$

$$f_i \nabla c(q_i^*) + \sum_{k \neq i} \lambda^{ki} \nabla u_k(q_i^*) = \sum_{j \neq i} \lambda^{ij} \nabla u_i(q_i^*), \qquad (2)$$

$$\lambda^{ij}(u_i(q_j^*) - u_i(q_i^*) + p_i^* - p_j^*) = 0, \ \forall i \neq j \qquad (3)$$

$$\lambda^{i0}(p_i^* - u_i(q_i^*)) = 0, \ \forall i \qquad (4)$$

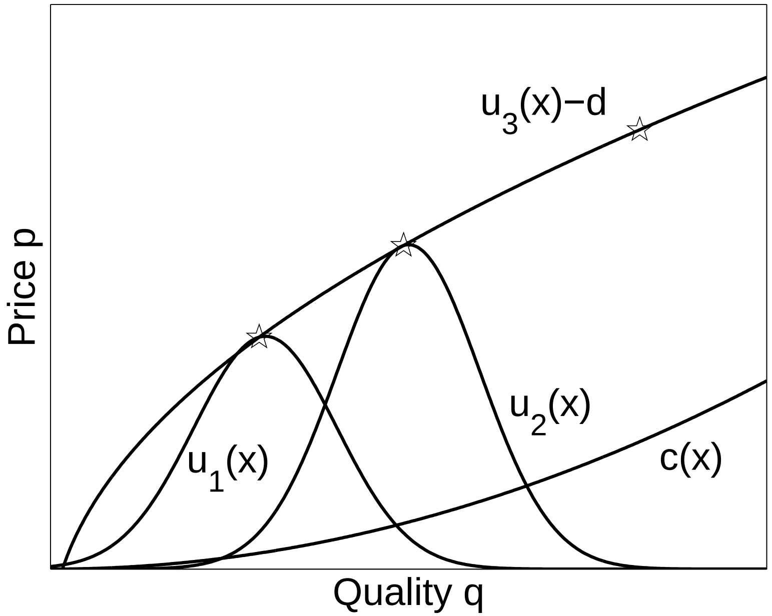
$$\lambda^{ij}, \lambda^{i0} \geq 0, \ \forall i, j \qquad (5)$$

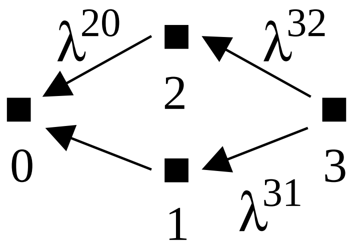
where $i, j, k \in I$, and λ^{ij} are the Lagrange multipliers of the IR and IC constraints.

Multipliers are flows between the buyer types

- Graph consisting of nodes and directed arcs
- Nodes are the buyer types and the zero bundle
- Arcs are the active constraints
- Multipliers give the magnitude of flows

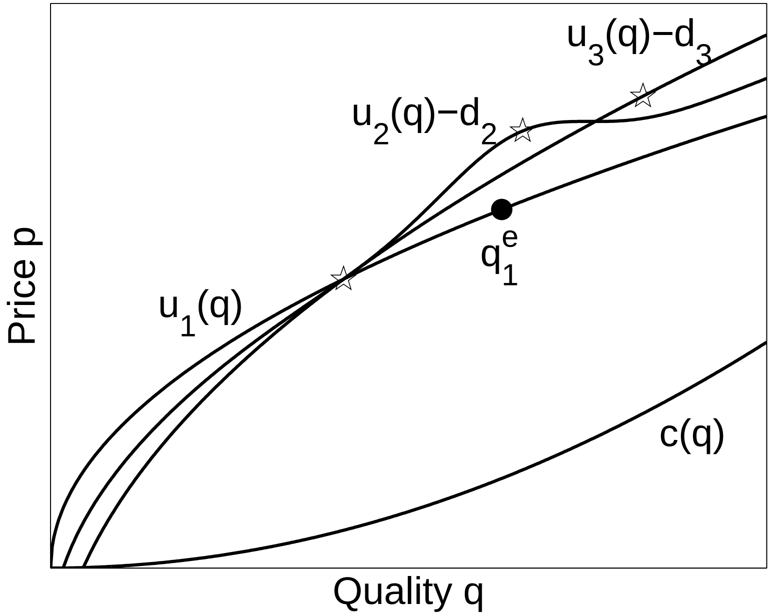
Numerical example

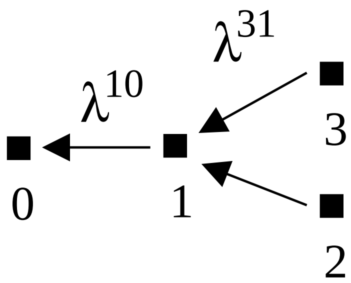




- Active constraints: λ^{32},λ^{31},λ^{20},λ^{10}
- Types 1,2 get zero utility
- Type 3 gets utility $d > 0$
- Flow conservation:
  $f_2 + \lambda^{32} = \lambda^{20}$
  $f_3 = \lambda^{32} + \lambda^{31}$

Numerical example 2

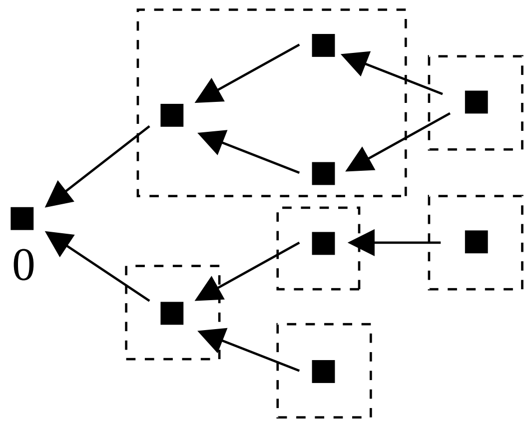




- Active constraints: λ^{31},λ^{21},λ^{10}
- Type 1 gets zero utility
- Types 2,3 get utilities d_2, d_3
- Values known: $\lambda^{31} = f_3$, $\lambda^{21} = f_2$, $\lambda^{10} = f_1 + f_2 + f_3$
- Redundant: $\lambda^{32} = \lambda^{30} = \lambda^{23} = \lambda^{20} = \lambda^{13} = \lambda^{12} = 0$

How to improve computation

- Choosing the set of active constraints adaptively
- Lagrange multipliers are known exactly in most cases
- Solving the parts of the digraph in parallel



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