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ADVERSARIAL RISK ANALYSIS IN MILITARY COMBAT MODELING

Mat-2.4108 Independent Research Project in Applied Mathematics

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Abstract

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Abstract

Adversarial risk analysis (ARA) is an emerging field of study that combines statistical risk analysis and game theory to create methods for the analysis decision making in counterterrorism and corporate competition. ARA is concerned with problems in which there are two or more intelligent opponent who make decision with uncertain outcomes.

This study paper explores the possible applications for ARA in military combat modelling. Modelling the opponent's decision making has been studied before, but as far as we know, the ARA approach is a new one. The opponent's decision making has been modelled with game theory, wargames and expert opinions, but all of these methods have severe limitations.

With the ARA approach it becomes possible to model the decision making process of a rational adversary. The approach has its own limitations and difficulties that are mostly associated with determining and predicting the resources, costs, expectations and goals of the adversary, but it still has significant advantages over the other used methods.

In this study paper we discuss some of the most promising applications for ARA in combat modelling. We also present an example of how ARA can be applied to solve a combat modelling related problem.

Keywords: Adversarial risk analysis (ARA), Combat modelling and simulation

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Introduction

Adversarial risk analysis (ARA) combines statistical risk analysis and game theory to create new methods for the analysis of decision making in counterterrorism and corporate competition. ARA is concerned with problems in which there two or more intelligent opponents who make decisions with uncertain outcomes (Rios Insua *et al.*, 2009).

Traditional statistical risk analysis was developed to model risks in complex technological entities (such as nuclear power plants), insurance, finance, and other applications in which the loss is governed by chance (sometimes called Nature). ARA also seeks to model the (possibly malicious or self-interested) actions of intelligent actors, which means that ARA also needs a model for the decision-making of all the participants. This model can be based, for example, on classical game theory (Myerson, 1991) or more psychological considerations (Camerer, 2003).

Game theory alone is not ideal tool for describing human behavior. Minmax solutions can often lead to sub-optimal solutions, because in reality the opponent is not perfectly rational. The solutions may be too pessimistic, because by mitigating the worst possible scenario, one can end up avoiding better outcomes that may correspond better to the choices a human opponent would realistically make. Minmax solutions are also often difficult to compute in real world scenarios, and they often need to make heavy assumptions about common knowledge shared by the competitors (Kadane & Larkey, 1982 and Meng *et al.,* 2014).

ARA has many quite obvious uses in military organizations. A significant proportion of the more recent ARA literature focuses on counterterrorism and many of the results can be applied directly for military use. Zhuang and Bier (2007), for example, apply game theory to device strategies to allocate resources between protection from intentional attack and natural disasters. In addition to helping in decision making in relation to defending against terrorist threats, the same methods can be used to allocate resources between strategically important targets, or even to decide how much money should be allocated for different military projects or units. The finance literature based uses of ARA can be useful too, because military organizations are also acquiring numerous products and services from outside contractors.

This study paper does not, however, focus on exploring the ARA literature that could possibly benefit the military. Our aim is instead to discuss how ARA can be applied to combat simulation modeling or as a complement to it. There are numerous combat models and simulators for calculating the outcomes of battles and the losses sustained by the units and weapons systems, but most of these simulators do not model human decision making except on the most basic levels (Lappi, 2012, 14-20). Thus, ARA could be used to enhance these simulators by increasing the realism of the decision making process based on the simulation results or help them to predict the decision making of the opponent.

Another advantage of the ARA perspective in traditional combat modeling is that the effects and usefulness of military deception can be calculated. Game theory has been applied to calculating the benefits of deceit before (Reese, 1980), but its applications are still rare. This is partly because the classical game theory solution requires assumptions that both sides have common knowledge about each other's goals and resources, which is not realistic when modeling deceit. ARA is not similarly limited and it can even be applied to calculate the usefulness of decoys and dummy systems to estimate if they are worth the cost, which is very difficult for most combat simulation models to estimate.

Modeling adversarial risks

This section briefly describes how a situation in which there are adversaries whose actions affect each other's risks can be modeled. This section is mostly based on the article by Rios Insua *et al.* (2009) and for more comprehensive description on adversarial risk analysis (ARA) we refer to it. For a good overview on how the ARA approach compares to classical game theory, please see the paper by Banks *et al.* (2011).

Risk analysis

The simplest form of a non-adversarial risk management problem is a situation where the decision maker is faced with a single choice from a set of decisions, and each decision has uncertain costs associated with it. Uncertainties about costs may result because the outcome of the decision is uncertain, or because the costs associated with a particular outcome are uncertain, or both. This problem is presented as an influence diagram in Figure 1.

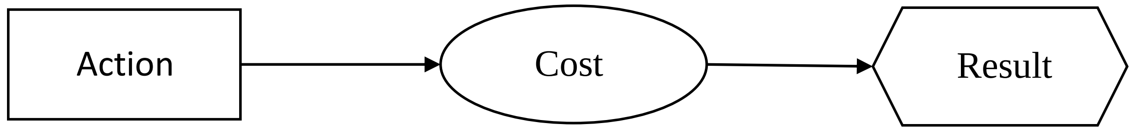

Figure 1. A simple influence diagram

An influence diagram is a directed acyclic graph with three kinds of nodes: rectangle shaped decision nodes, oval shaped uncertainty nodes, and hexagonal value nodes. Arrows pointing to value or uncertainty nodes indicate functional or probabilistic dependence, respectively. That means that the utility function at the value node depends on its immediately preceding nodes, and the probabilities associated with a chance node depend on the values of the immediately preceding nodes and are thus conditional on them. Arrows pointing into decision nodes indicate that the values of the nodes preceding the decision node are known at the time of the decision. (*cf.* Howard & Matheson, 2005)

The problem in Figure 1 represents a situation where the decision maker has to make a decision a from a set A of possible choices, represented by the rectangle. The cost c associated with this decision is uncertain and is modeled through density $\pi(c|a)$, represented by the oval node. The result is modeled by Von Neumann-Morgenstern utility function $u(c)$. The decision maker seeks the decision that maximizes the expected utility

$$\psi = \max_{a \in A}\left[\psi(a) = \int u(c)\pi(c|a)\, dc\right]. \tag{1}$$

In practice, the costs of a particular action are complex and depend on the outcome. The costs may often include fixed and random terms. For that reason, organizations will often perform a risk assessment to better identify the disruptive events, and their probabilities and associated costs. Figure 2 shows the influence diagram that has been extended to account for the disruptive hazards identified by the risk assessment and the additional costs they may cause.

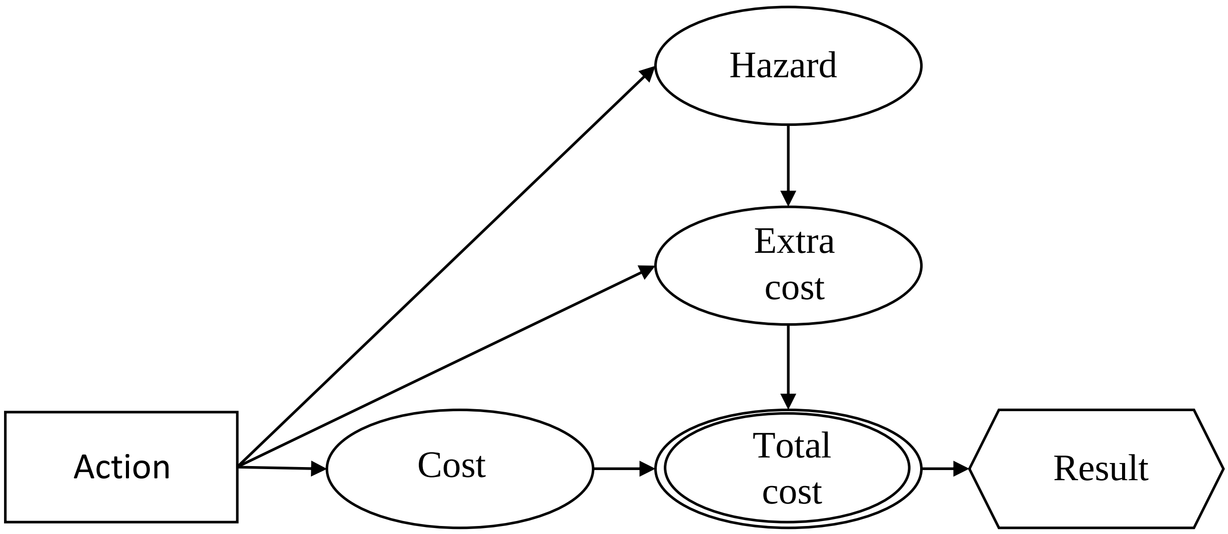

Figure 2. Influence diagram after risk assessment.

Adversarial risks

We now consider a situation in which there are two adversaries (Attacker and Defender), whose decisions affect the risks each face. Figure 3 shows how the influence diagram is extended to include the adversary in a symmetrical situation in which decisions of both parties affect the risks and costs the other faces, and both seek to maximize their own expected utilities. Even though the roles are symmetric in this example, this does not have to be the case. An asymmetrical scenario could also be modeled with an asymmetrical influence diagram.

We denote the sets of possible actions of Attacker and Defender with A and D respectively. Their utility functions are $u_a(\cdot)$ and $u_d(\cdot)$. The sets containing their beliefs about different probabilities are P_a and P_d. As can be seen in the influence diagram in Figure 3, one of the nodes, Hazard, is common for both sides. This can represent the possible complications arising from risks common to both sides, such as weather for example. The other cost nodes are not common, and represent the random costs for both parties and they could be very different.

The expected utilities for both the Attacker and the Defender depend upon the actions of both. Specifically, by extending on (1), we get the utility the Attacker expects from choosing action $a \in A$ when the Defender chooses action $d \in D$

$$\psi_A(a,d) = \int u_A(c)\pi_A(c|a,d)\,dc, \qquad (2)$$

where $\pi_A(c|a,d) \epsilon P_A$ represents the Attacker's beliefs about his costs corresponding to the decision pair (a,d). It is noteworthy that these beliefs do not necessarily have to match reality, because we are only modeling the decision the Attacker makes. The expected utility for the Defender is analogous.

This representation of ARA matches normal form games, in which both players make the decisions simultaneously. One could also build an influence diagram that applies to sequential games, such as Stackelberg games, in which the players alternate making their moves. The ARA methodology can be applied to solving them too (*cf.* Banks *et al.*, 2011 and Rios & Rios Insua, 2012).

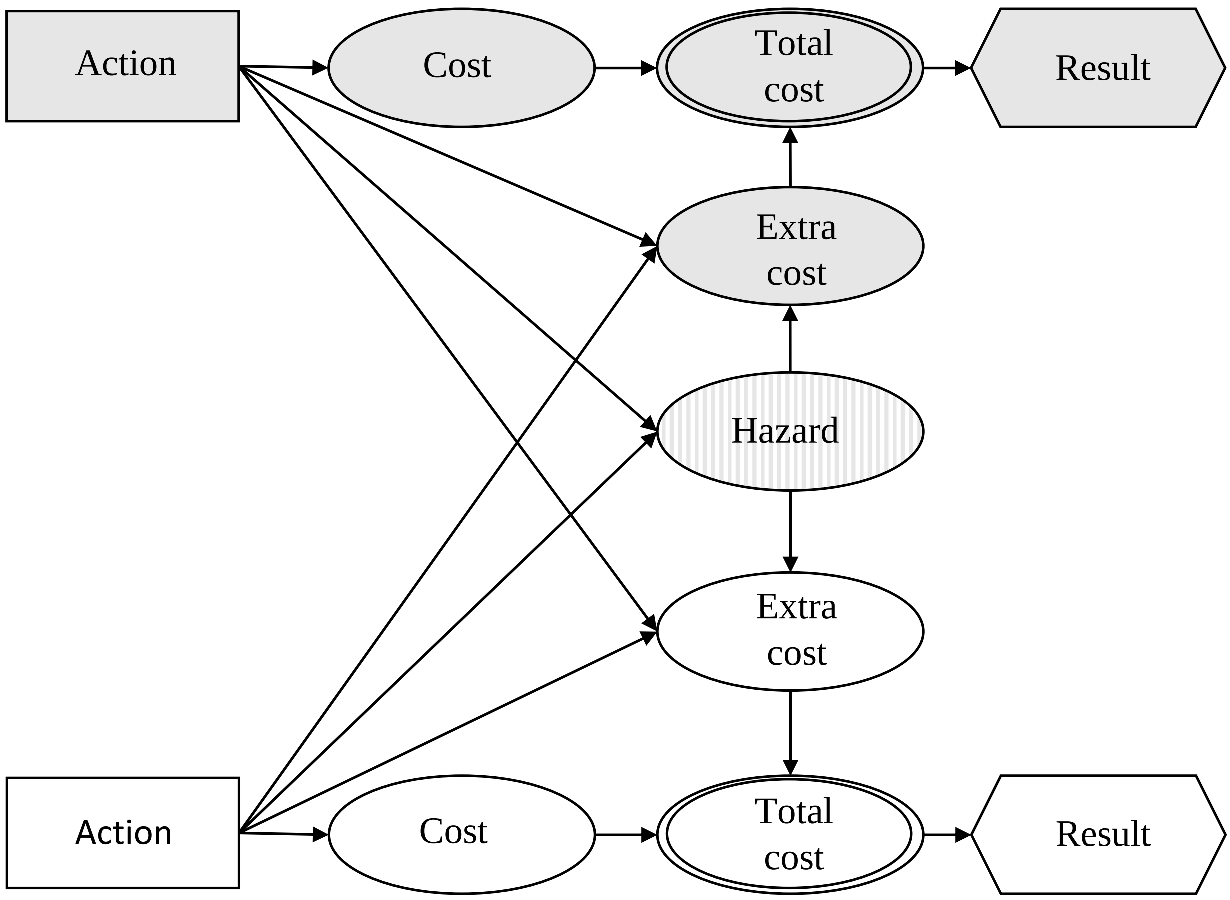

Figure 3. Influence diagram with an adversary

Bayesian framework for ARA

A problem like the one presented in Figure 3 can be readily solved using classical game theory if we assume that the costs and utility functions of both players are common knowledge. If the players do not possess correct and accurate information about the costs, resources, and goals of the adversary, (which is often the case in reality,) the Nash equilibrium solution does not exist.

ARA solves this problem by using a Bayesian strategy to express uncertainty about the adversary's decision. If we examine the problem from the Attacker's point of view, the uncertainty means that the Defender's decision is a random variable as presented in Figure 4. To solve this problem, the Attacker needs more than just $\pi_A(c|a,d)\epsilon P_A$ and $u_A(c)$. The Attacker also needs $p_A(d)$, which is the probability that the Defender chooses defense d as estimated by the Attacker. To find that, he is assumed to use mirroring to form an estimate of both the Defender's utility function $u_D(c)$ and the Defender's costs $\pi_D(c|a,d)$. That means that the Attacker assumes the Defender is acting rationally and is using a similar strategy to predict the actions of the Attacker.

If the Attacker tried to estimate the Defender's utility function and cost function by assuming the Defender is doing the exact same thing that he is doing, the Attacker would need to think what the Defender thinks he thinks. To avoid infinite regress, the chain is usually cut there and the Attacker just forms an educated guess about the Defender's thoughts about the Attacker's estimated utilities and costs. Obviously the thinking could be taken even further, but it usually not a realistic way to resolve the problem at hand.

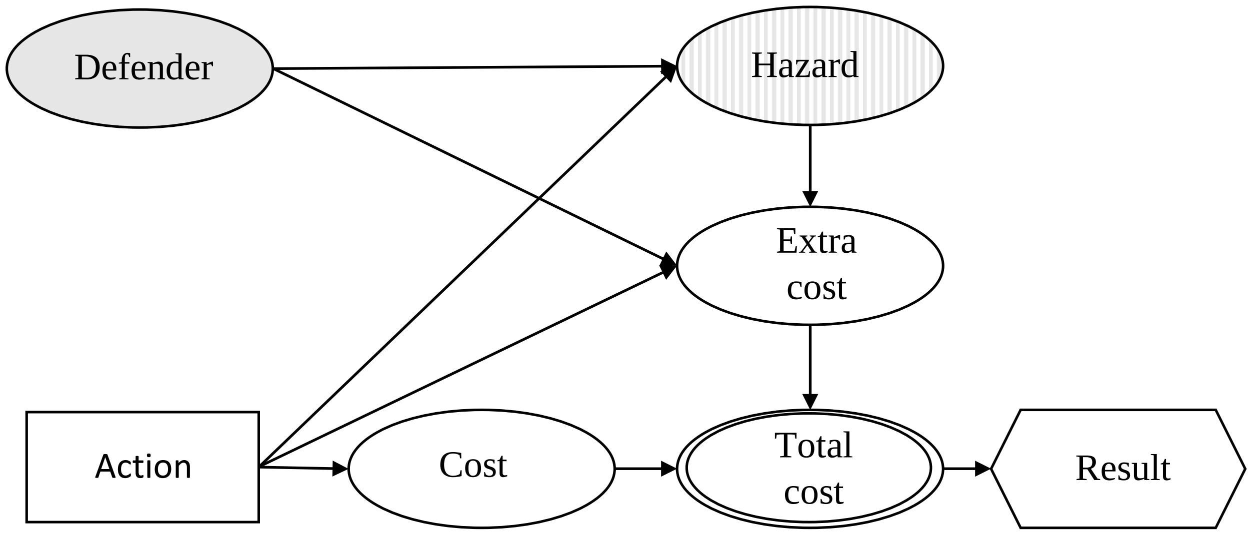

Figure 4: Influence diagram from the Attacker's point of view.

The ARA methodology is quite similar to Bayesian level-k thinking. The way of modeling opponents thoughts presented here resembles most closely level-2 thinking. Rothschild *et al.* (2012) have taken the approach further and applied actual level-k thinking to the ARA approach. Their methodology is not without drawbacks as the level-k approach requires some additional assumptions and the problems become intractable even faster as their complexity increases. Possibly the greatest advantage gained from the level-k thinking approach is the ability to easily perceive how the level of adversary's thinking affects the optimal decision.

Applying ARA to military combat modeling

In this section we present some ideas for applying ARA to military combat modeling problems and modeling process. These ideas are still mostly untested, and they are offered mostly as suggestions for interesting future research subjects.

Distribution of resources

Like we mentioned in the introduction section, a significant proportion of ARA literature is focused on preventing terrorist threats, and more specifically on how should limited resources be allocated to best combat these threats (*cf.* Pat-Cornell &

Guikema, 2002, Zhuang & Bier 2007 and Kroshl *et al.*, 2015). Resource allocation is a problem that military faces always, not just when combating terrorism. The methods developed for allocating resources against terrorist threats are easy to apply to combat environment, because assessing the effects of the adversary's decision is one of the biggest challenges in ARA. Combat modeling has been developed for decades to calculate those effects i.e. the results of combat.

This is probably the easiest way to benefit from ARA in the context of combat modeling. Studying the ARA research related to counter terrorism and applying the developed methods to combat situations by calculating the related risks using combat modeling tools has the potential to reap large rewards with only relatively minor investments.

Modeling decision making

Currently many combat models do not include algorithms that are capable of modeling the human thought process behind tactical or higher level decisions (Washburn & Kress, 2009, 111-130). Depending on the model practically all higher level decisions above soldier or platoon level are made by the operator. As a result, the time required to create a scenario is usually significantly longer than the time required to calculate the results (Lappi, 2012).

The fact that the user defines the actions taken by the troops in the modeled scenario is not always a hindrance. It also offers many benefits, including the ability to fine tune the scenario to match events of a historical battle (Lappi *et al.*, 2014) or to easily examine and change the actions taken by the troops. The reliance on user making all the significant decisions can, however, become problematic when a really large number of scenarios is required, which is often the case when data farming more complex cases (Lappi *et al.*, 2015). Currently many aspects of the scenario can modified by just changing some simple parameters, but that approach requires a high level of operator expertise and is still somewhat limited in its applications.

Including a model that would allow the units inside the simulation make simple tactically sensible decisions would widen the range of problems that can be analyzed using data farming. The ARA methodology could well be used as a basis for such decision making algorithm. Use of ARA over some other game theory based decision method is made more attractive by the fact that the ARA framework makes it possible to better take into account the uncertainties and possible misinformation that are present on the battlefield.

However, there are some limitations to applying ARA to modeling decision making. Because the method is so calculation intensive, it is not ideal for modeling low level or continuous decision making. The most effective way to apply ARA would be to limit the choice to between a few possible strategies. Including too many different and separate choices will cause the problem to become intractable.

Simulating larger chains of events

The ARA methodology can also be applied to aid in modeling military operations that are too large to simulate as single scenario. The scale can become an issue if the number of units involved is too large, or the operation takes place over such a long timeframe that it starts to take too many different possible paths based on the events. Kangas and Lappi (2006) present how methods of probabilistic risk analysis can be used in conjunction with stochastic combat modeling to analyze larger chains of events. The ARA approach could be used to build on those results and take the analysis one step further. In addition to predicting the success chances of larger operations, it would also be possible to predict those of the adversary’s choices that can affect the path of events.

Practically any combat model can be used with ARA methodology as long as it can be used to calculate the probabilities for each side winning the battle and the expected losses on both sides. This includes practically all stochastic combat models and even some of the deterministic ones. The combat model should be chosen first and foremost to fit the problem at hand. Sometimes the best choice is a platform level Monte Carlo simulation, and sometimes it is a high level attrition model like FATHM (Brown & Washburn, 2000).

In some cases, it can be possible to use ARA to model these larger chains of events without having to rely on an actual stochastic combat modeling software like Sandis as Kangas and Lappi (2006) did. There are also alternative, lighter stochastic computational models that can be used to predict the outcome of a duel between two platoon sized forces (Lappi *et al.*, 2012; Åkesson, 2012; Roponen, 2013). They can be used to significantly cut down the calculation time required for solving all the success probabilities and expected losses in different stages of the chain, not to mention the time savings from not having to create a complete model scenario, which is, as stated earlier, a time consuming process. The use of the lighter duel simulation methods could even be automatized to a certain degree, because they require much fewer input parameters.

Modeling the effectiveness of military deceit

Using deceit to gain upper hand against an adversary is absolutely integral part of military tactics and strategy. At the same time, the effects of deceit are very difficult to predict and simulate using existing operational analysis and combat modeling software.

Because the effects are difficult to reduce to mathematical formulas, modeling the effects of deceit relies usually on expert opinions, which in the context of combat modeling usually means that it is left to the operator of the software. A common alternative is to use wargames to model the uncertainties associated with human decision making, but that approach is not without its problems either (Washburn & Kress, 2009, 111-130).

The ARA approach could be used to help measure the effects of deceit tactics have on the decision making of the adversary. The ability to model the effects of the adversary's altered perceptions should at least be useful when used as a complement to the elicitation of expert opinions. Mathematical equations are, after all, immune to effects of optimistic thinking.

Examples of situations that could be relatively easily modeled with ARA include cases in which the adversary is deliberately misinformed about the strength of the opposing forces. This can be achieved for example by hiding troop movements and employing dummy units or decoys. ARA can be then used to estimate the effect of the deceit on adversary's decision making and whether that effect is beneficial or not. An example of such estimation process is given in the next section.

Supporting decision making

Possibly the most important reason for military combat modeling is its use for supporting strategic, tactical or technical decision making process (Tolk, 2012, 55-78). However, it is not an easy task to translate the results of combat models to actual decisions or recommendations (Davis & Blumenthal, 1991).

The ARA framework could be applied to translate the data, produced by the mathematical models, to answers to more concrete questions such as what will happen if we do not allocate more troops to a specific airfield, or where is the enemy likely to attack if we do X? Obviously, problems like this are mathematically and computationally very difficult, but they are not impossible to answer if sufficient time and effort is invested into developing the methods. (What is prohibitively hard to calculate today may not be so five years from now, thanks to the rapid increase in computer power.)

One possible way of using ARA to translate simulation results to an easily applicable form is to perform a portfolio analysis on the possible strategies being considered. Similar methods have already been used in assessing cost-efficiencies of different weapon system combinations (Kangaspunta *et al.*, 2012). ARA could be used in this instance to predict the most likely responses of the adversary and calculate the expected utilities gained from choosing each strategy under different conditions. The applicability of this approach would again rely on streamlining or automatizing the process so that it would provide accurate results and save time compared to manually assessing the need for use of combat simulation and analyzing the results it provides.

Example of applying ARA to a tactical problem

To demonstrate how ARA can be applied in practice, we use it here to solve a relatively simple tactical problem. The methodology can be applied to significantly more complex situations. The purpose of this section is just to give a general idea of the methodology.

The problem

Let us examine a situation where there are two sides: the Defender and the Attacker. The Defender is moving in more troops to protect a valuable target, and the Attacker has an opportunity to decide whether he will try and take the valuable target or use is his troops somewhere else.

To make the problem easier to understand and calculate, we have reduced the choices of the Defender and the Attacker to simple binary questions. The Defender decides whether to hide the movements of his reinforcements from the Attacker. The Attacker will decide, after observing the perceived strength of the defender, whether he should commence the attack against the target or not. If the Defender decides to hide his reinforcements, the Attacker will not know about them, and thus make the decision, whether to attack or not, using incomplete information. We will solve the problem from the point of view of the Defender. Figure 5 shows the influence diagram from the Defender's point of view.

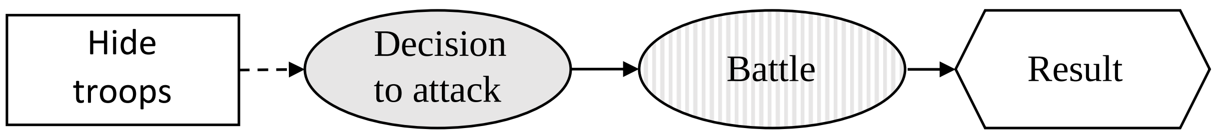

Figure 5. Influence diagram of the example case from the point of view of the Defender.

Let us denote the options the Defender and the Attacker as $D = \{0,1\}$ and $A = \{0,1\}$ respectively. The only uncertainty in this case is the outcome of the battle S (say, success or failure for the Defender to keep hold of the target). The utility functions over the costs are $u_D(c_D, c_A)$ and $u_A(c_D, c_A)$, with costs dependent on the actions of the Attacker.

Solving the problem requires assessing the probabilities over the costs, conditional on (a, d, S); and about S, conditional on (a, d). The Attacker and the Defender have in this case different assessments: for example, for success, these are $p_D(S = 1|a, d)$ and

$p_A(S = 1|a, d)$, respectively. The Attacker's assessment of the success of the assault is likely different from the Defender's because the Attacker is not even aware of any choices being made by the Defender, as can be seen in Figure 6. The expected utility, for the Attacker, obtained with (a, d) is

$$\begin{aligned}\psi_A(a,d) = p_A(S=0|a,d) \sum_{c_A} \sum_{c_D} [u_A(c_A,c_D)\pi_A(c_A,c_D|a,d,S=0)] \\ + p_A(S=1|a,d) \sum_{c_A} \sum_{c_D} [u_A(c_A,c_D)\pi_A(c_A,c_D|a,d,S=1)].\end{aligned} \quad (3)$$

The Defender's expected utility is similar.

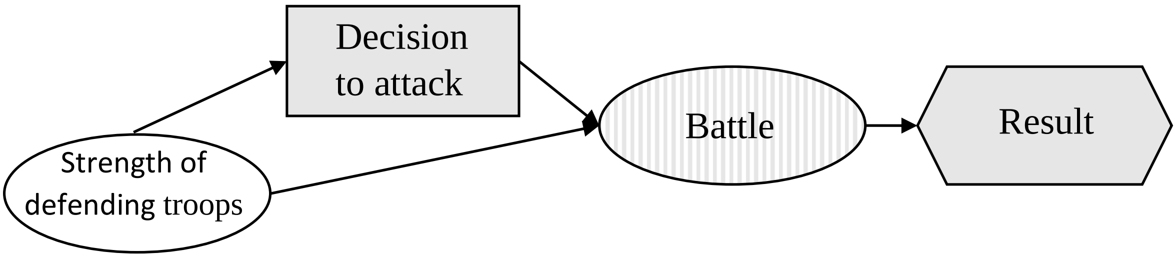

Figure 6. Influence diagram of the example case from the point of view of the Attacker.

We now solve the game from the Defender's point of view. The Defender has 15 men defending the target and has 15 more men coming in as reinforcements. He has the opportunity to hide the presence of the reinforcements from the attacker. The Defender estimates that the Attacker has at least 20 men but no more than 35, and he thinks that the most likely number is 30, so he fits a triangular distribution as seen in Figure 7.

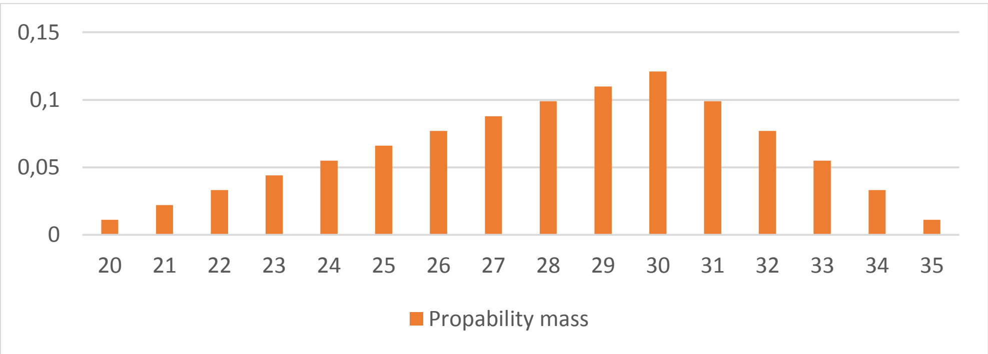

Figure 7: Defender's estimation of Attacker's strength as a probability mass distribution.

Using the strength estimates of both forces, the Defender can use, for example, a stochastic combat model to calculate p_D and π_D. The Defender also determines that the utility gained from the situation follows function

$$u_D(S, a, c_A, c_D) = 35S + 5a + 0.1c_A - c_D, \quad (4)$$

where $S = 1$ corresponds to the situation where the Defender manages to hold the target area, $a = 1$ corresponds to the situation where the Attacker decides to attack, instead of using the troops elsewhere, and c_A and c_D are the Attacker's and the Defender's losses respectively.

It is not enough to solve the problem for the Defender to know p_D, π_D and u_D. To calculate the expected utility from decision d, he first needs to estimate $p_D(a|d)$. To do that, he needs to solve the problem from the viewpoint of the Attacker. He assumes that the Attacker is also an expected utility maximizer. The problem is presented from the Attacker's point of view in Figure 6.

The Defender estimates that the Attacker thinks the Defender has 13 to 17 men, and finds all values equally likely, and will not find out about the reinforcements if the Defender decides to hide their movement. If the Defender decides not to hide the reinforcements, he estimates the Attacker will think the Defender has 28 to 32 men, and find all values within that range equally likely. Using those strengths for his estimates he can use the same stochastic combat model used to solve p_D and π_D to calculate p_A and π_A.

The Defender estimates that the Attackers utility function is similar to his own but is uncertain how high the attacker values the target and the loss of opportunity to use the troops elsewhere. Thus he estimates that the Attacker's utility function is

$$u_A(S, a, c_A, c_D) = -(35 + U_1)S - (5 + U_2)a + 0.1c_D - c_A, \tag{5}$$

where U_1 and U_2 are uniform on $[-5, 5]$.

Let us look at solving the problem step by step. To solve the problem, the Defender will:

1. Calculate the success probabilities and expected losses for both sides for all the possible combinations of strengths of both sides as perceived by the attacker.
2. Calculate the Attackers expected utilities ψ_A for attacking and not attacking for all possible strengths of the Attacker's force taking into account the uncertainties with u_A.
3. Compare the expected utilities to get an estimate for the probability of an attack for each possible strength of the attacker as seen in Figure 8.
4. Use probability of an attack with a specific strength of the attacker (Figure 8) and the probability for each of those strengths (Figure 7) to calculate $p_D(a|d)$.
5. Calculate ψ_D for all possible values of a.
6. Use $p_D(a|d)$ to calculate the d with best expected utility.

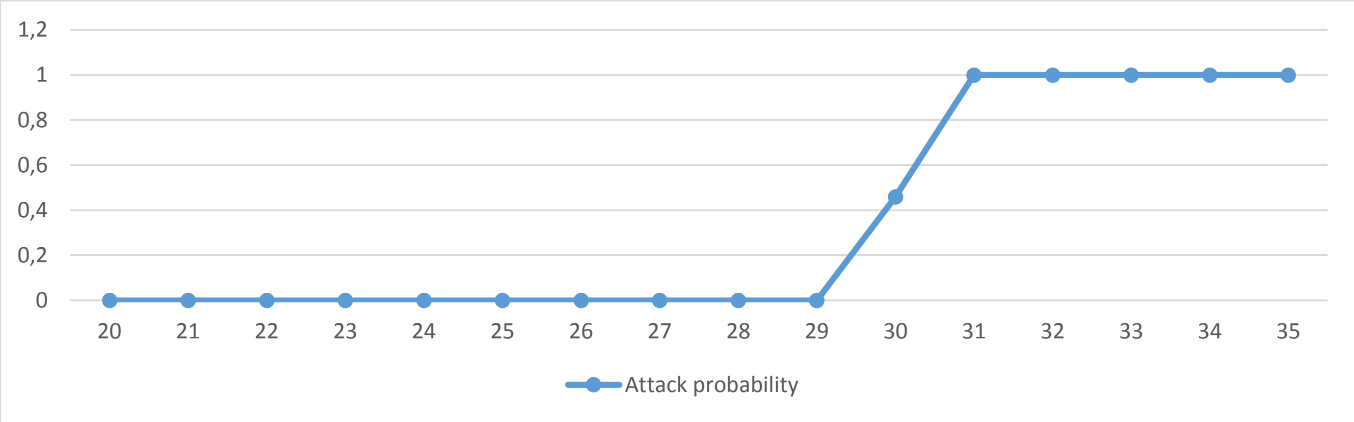

Figure 8: Probability of an attack as a function of the Attacker's strength, when the Defender has not hidden the reinforcements.

We used the approximative method for simulating a duel between two forces presented by Roponen (2013) to calculate p_D, π_D, p_A and π_A, because the program code was easily available and it calculated the results fast and with sufficient accuracy. This program code was also written to loop through all the possible strengths of both sides and calculate the expected utilities to find out the attack probabilities $p_D(a|d)$. The attack probability when the Defender hides the reinforcements $p_D(a|d = 1) = 1$, and when he hides the reinforcements $p_D(a|d = 0) \approx 0.33$. Then the expected utilities of the Defender were calculated from p_D and π_D as

$$\psi_D(a, d = 1) \approx 13.44, \text{ and } \psi_D(a, d = 0) \approx 27.88. \quad (6)$$

Thus the Defender decides that hiding the reinforcements is not in his best interests.

The result may seem counterintuitive, but it makes sense, if one takes into account that (I) the Defender values the safety of the defended target more than the losses sustained by the Attacker and (II) the fact that the Defender does not necessarily outnumber the Attacker even with the reinforcements at hand.

Discussion

Adversarial risk analysis (ARA) is still a relatively new research area that has been becoming more prominent in the context of counter terrorism and corporate competition. However, we feel that ARA has much to offer for military combat modeling too. It is able to combine the statistical approach of risk analysis, already used in combat modeling, with a game-theoretical perspective that can be used to help predict the actions of one's opponents.

We have listed some possible uses for the ARA approach in the context of military combat modeling, operations analysis, and decision making in general. The ideas presented in this paper are still tentative, and they would require more research and work before they can be fully implemented. Real problems are extremely complex and solving them using the ARA methodology is very difficult. Actual battles can easily involve thousands of possible decisions, and the uncertainties associated with the goals and resources are significant. However, we feel that this should not stop us into looking at the new possibilities ARA has to offer.

We also presented a relatively simple example using the ARA methodology in tandem with stochastic combat modeling. Most of the numerous calculations required for solving the ARA side problem were relatively simple to implement into program code, and there already exist numerous tools for calculating the outcomes of battles (Kangas, 2005). It seems that it is possible to build software tools for solving numerous comparable problems by formulating valid approaches for solving, how the opponent's utilities affect his decision making. This, however, is not an easy task.

Possibilities for future research are abundant as one of the primary aims of this paper was to find some such possibilities. At the moment, the avenue that in our opinion shows the most promise in advancing the research into possibilities of ARA, would be to tackle a more complicated combat simulation problem than the one presented in this paper to push the boundaries of what is possible using ARA with stochastic combat modeling.

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