Simulation Analysis of Option Buying

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1 Introduction

Options were first traded on an organised exchange in 1973 [Hul]. Since then there has been a tremendous growth in the options market and options are now traded on many exchanges throughout the world. Options are also much traded over the counter in banks and other financial institutions. The pricing of options has been investigated even before options were traded on exchanges. Black and Scholes published the generally accepted theory on option pricing in 1973 and it was further generalised by Merton. The pricing formula of Black and Scholes gives an answer to the fair price of an option, but the formula is based on assumptions that are almost never fulfilled in real life. The formula does not give an answer to other related questions, such as "With what probability will I lose all my money?" etc. Simulation can help in loosening the assumptions of the formula and simulation also allows doing more detailed studies of the possible outcomes of an option.

In this study, the theory behind option pricing is described. From the basic theory of random walks and Wiener processes to the more general Ito processes. The generally accepted stock price process is also presented and some examples are given. Then the theory and assumptions behind the Black-Scholes equation are presented and the related pricing formula for options is analysed.

After describing the theory some simulations of different kinds of stocks and options are presented in detail. Some different possibilities of simulations are also examined and a few examples are shown.
2 Stock and option pricing theory
The theory for modeling stocks serves as a base for the pricing theory of all
derivatives. The basic underlying theory behind stock prices is simple random
walks and Wiener processes. These concepts will thus be introduced first, before
the actual stock price processes can be introduced.

2.1 Random walks and Wiener processes
In order to create a model for stocks in continuous time we will start with a simpler
discrete model. The first concepts to be introduced are special random functions of
time, called random walks and Wiener processes.

Suppose we have N periods of length $\Delta t$. We define the additive process $z$ by:

\begin{align*}
  z_{k+1} &= z_k + \varepsilon_k \sqrt{\Delta t} \quad (1) \\
  t_{k+1} &= t_k + \Delta t \quad (2)
\end{align*}

where $k = 0, 1, 2, \ldots, N$.

Here $\varepsilon_k$ is a normal random variable with mean 0 and variance 1. These random
variables are also supposed mutually uncorrelated: $E[\varepsilon_k, \varepsilon_j] = 0$, \quad $j \neq k$. The
process described by equation (1) is called a random walk [Lue]. The process is
normally started at 0, i.e. $z_0 = 0$. The realised path thereafter wanders according to
the values of the $\varepsilon_k$. An example of this where $N = 20$ is shown in Figure 1.
The difference between the random variables \( z_k - z_j \) is of special interest. The difference can be written as:

\[
\Delta z_k = \sum_{i=j}^{k-1} \varepsilon_i \sqrt{\Delta t}
\]  

(3)

For the expected value and variance of this difference we get:

\[
E[z_k - z_j] = 0
\]  

(4)

\[
\text{Var}[z_k - z_j] = E\left[ \sum_{i=j}^{k-1} \varepsilon_i \sqrt{\Delta t} \right]^2 = E\left[ \sum_{i=j}^{k-1} \varepsilon_i^2 \Delta t \right] = (k - j)\Delta t
\]  

(5)

The variance is thus exactly equal to the time difference between the points. This is also why \( \sqrt{\Delta t} \) was used in the original equation (1).

A Wiener process is then obtained by taking the limit as \( \Delta t \to 0 \) of a random walk process. For a Wiener process we thus get:
\[ dz = \varepsilon(t)\sqrt{dt} \quad (6) \]

where the \( \varepsilon(t) \)'s are uncorrelated.

### 2.2 Generalised Wiener processes and Ito processes

The Wiener process, also called *Brownian motion*, is a fundamental process that serves as a part of many different processes. Many of these different processes can be obtained from the so-called *generalised Wiener process*. The generalised Wiener process is of the following form:

\[ dx = adt + b dz \quad (7) \]

where \( dz \) is a Wiener process and \( a \) and \( b \) are constants.

A generalised Wiener process has the advantage that it has an analytic solution:

\[ x(t) = x(0) + at + bz(t) \quad (8) \]

By further generalising a generalised Wiener process, we arrive at an *Ito process*. An Ito process is thus described by the following equation:

\[ dx = a(x, t)dt + b(x, t)dz \quad (9) \]

where \( dz \) denotes a Wiener process and \( a \) and \( b \) are deterministic functions of \( x \) and \( t \). The discrete version of an Ito process is thus:

\[ \Delta x = a(x, t)\Delta t + b(x, t)\varepsilon\sqrt{\Delta t} \quad (10) \]

For \( (\Delta x)^2 \) we thus get:

\[ (\Delta x)^2 = b^2 \varepsilon^2 \Delta t + ab \varepsilon \Delta t \sqrt{\Delta t} + a^2 (\Delta t)^2 \quad (11) \]

When \( \Delta t \) is very small the higher order terms of \( \Delta t \) can be neglected. For the expected value and variance of \( (\Delta x)^2 \) we thus get:
\[
E[(\Delta x)^2] = b^2 \Delta t \quad (12)
\]
\[
\text{Var}((\Delta x)^2) = b^4 \text{Var}(\varepsilon^2 \Delta t) = b^4(\Delta t)^2 \quad (13)
\]

(remembering that \(\varepsilon \sim N(0,1)\)).

Since the variance of \((\Delta x)^2\) is of higher orders of \(\Delta t\), we can deduce that when \(\Delta t\) is very small, i.e. \(\Delta t \to dt\), the variance of \((\Delta x)^2\) diminishes and the value of \((\Delta x)^2\) approaches its expected value. When \(\Delta t \to dt\) we thus get:
\[
dx^2 = b^2 dt \quad (14)
\]

Let us now consider a continuously differentiable function of two variables \(G(x,t)\).

By taking the Taylor expansion of it to the second order we get:
\[
\Delta G \approx \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \left( \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t \right) \quad (15)
\]

By taking the limit \(\Delta t \to dt\), \(\Delta x \to dx\), together with the definition in equation (9) and the result of equation (14) we get:
\[
dG = \frac{\partial G}{\partial x} (adt + b dz) + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt \quad (16)
\]

By reordering equation (16), we get Ito's Lemma:
\[
dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} dz \quad (17)
\]

Thus, \(G\) is in itself also an Ito process.

### 2.3 A stock price process

Following this result let us consider a situation where the stock price follows geometric Brownian motion, as is often argued [Smi]:

\[
E[(\Delta x)^2] = b^2 \Delta t
\]
\[
\text{Var}((\Delta x)^2) = b^4 \text{Var}(\varepsilon^2 \Delta t) = b^4(\Delta t)^2
\]

(remembering that \(\varepsilon \sim N(0,1)\)).

Since the variance of \((\Delta x)^2\) is of higher orders of \(\Delta t\), we can deduce that when \(\Delta t\) is very small, i.e. \(\Delta t \to dt\), the variance of \((\Delta x)^2\) diminishes and the value of \((\Delta x)^2\) approaches its expected value. When \(\Delta t \to dt\) we thus get:
\[
dx^2 = b^2 dt
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Let us now consider a continuously differentiable function of two variables \(G(x,t)\).

By taking the Taylor expansion of it to the second order we get:
\[
\Delta G \approx \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \left( \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t \right)
\]

By taking the limit \(\Delta t \to dt\), \(\Delta x \to dx\), together with the definition in equation (9) and the result of equation (14) we get:
\[
dG = \frac{\partial G}{\partial x} (adt + b dz) + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt
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\[
dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} dz
\]

Thus, \(G\) is in itself also an Ito process.
\[ dS = \mu S dt + \sigma S dz \]  

(18)

S is here clearly an Ito process. Equation (18) is written in the so-called "standard form" [Lue]. It is also often written in the form:

\[ \frac{dS}{S} = \mu dt + \sigma dz \]  

(19)

The term \( \frac{dS}{S} \) can here be interpreted as the instantaneous rate of return for the stock. If we now define \( G = \ln S \), we then have the following relations:

\[ \frac{\partial G}{\partial S} = \frac{1}{S} \]  

(20)

\[ \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \]  

(21)

\[ \frac{\partial G}{\partial t} = 0 \]  

(22)

By inserting the relations (20) – (22) into Ito's lemma (17), we get the following equation for \( dG \):

\[ dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz \]  

(23)

It thus follows that \( G \) is a Wiener process and has an analytic solution. From the solution of \( G \) we can solve \( S(t) \):

\[ \ln(S(t)) = \ln(S(0)) + \left( \mu - \frac{\sigma^2}{2} \right) t + \varepsilon \sigma \sqrt{t} \]  

(24)

\[ S(t) = S(0) e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t}} \]  

(25)

\[ S(t) = S(0) e^{\ln(S(0)) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t}} \]  

(26)

\[ S(t) = e^{\ln(S(0)) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t}} \]  

(27)
S(t) is thus a normal variable exponentiated, i.e. it follows a lognormal distribution [Tho]. The expected value of S(t) is given by:

$$E[S(t)] = S(0)e^{\mu t}$$

and the variance can be shown to be given by [Hul]:

$$\text{Var}[S(t)] = S(0)^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1\right)$$

(28)

(29)

The distribution of S(t) thus looks something like Figure 2, in which the result of 10,000 simulations is shown. The paths of individual stock prices can be seen in Figure 3, where the paths of 50 stocks are shown.

*Figure 2: The distribution of the end price of a stock after one year, with starting price 100, $\mu = 0.1$ and $\sigma = 0.1$*
2.4 Options

An option is the right to, but not the obligation, to buy (or sell) an asset under some specified terms. Usually these terms include a specified price and a specified time during which the option is valid. There is a vast amount of different options covering assets such as stocks, stock indices, foreign currencies, debt instruments, etc. There are two basic types of options: A call option gives the holder the right to buy the underlying asset by a certain date for a certain price. A put option gives the holder the right to sell the underlying asset by a certain date for a certain price, the so-called exercise price. Each option has an expiration date (or maturity). There are then two different types of options; a European option can only be exercised on the expiration date whereas an American option can be exercised at any time up to the expiration date. European options are generally easier to analyse than what the American counterparts are, and many characteristics of American options are thus deduced from the analysis of European options.
An option gives the holder the right to do something, but the holder does not have to exercise this right. This fact distinguishes options from forwards and futures, where the holder is obliged to buy or sell something. The buyer of an option can thus at the most loose the price he paid for the option. The profit from buying call and put options are visualised in Figure 4 and Figure 5 respectively. The inverse is true for the trader selling the option; he can at the most gain the price of the option, whereas his losses can be considerable (selling a stock at a price $S$ when it's worth $K$, $K > S$, is here considered a loss). The value of a European call option can thus be defined as:

$$C = \max(S(T) - X, 0)$$  \hspace{1cm} (30)$$

where $S(T)$ is the stock price at the expiration date and $X$ is the exercise price.

Figure 4: The profit from buying a call option, option price = 5, strike price = 100
2.5 The Black-Scholes equation

The Black-Scholes (-Merton) equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend paying stock. It was first published in 1973 and has been widely accepted since. The equation is based on the following assumptions [Bla]:

1. The stock price follows geometric Brownian motion (see equation (18))
2. The short selling of securities is permitted
3. There are no transaction costs or taxes, all securities are perfectly divisible
4. There are no dividends during the life of the derivative
5. There are no arbitrage opportunities

Figure 5: The profit from buying a put option, option price = 5, strike price = 100
6. Security trading is continuous

7. The risk-free rate of interest, $r$, is constant and the same for all maturities

Suppose then that we have a function $f$ that describes the price of an option or other derivative contingent on $S$. If $f$ is a function of $S$ and $t$ (as can be expected) according to Ito's lemma (equation (17)) $f$ must satisfy:

$$df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$  \hspace{1cm} (31)$$

We now try to form a portfolio where the Wiener process can be eliminated. One such portfolio is, as will be shown, a portfolio with one unit of the considered derivative and $-\frac{\partial f}{\partial S}$ units of the stock. The value of the portfolio is thus:

$$\Pi = f - \frac{\partial f}{\partial S} S$$ \hspace{1cm} (32)$$

The change in value of the portfolio over a short period of time $dt$ is then:

$$d\Pi = df - \frac{\partial f}{\partial S} dS$$ \hspace{1cm} (33)$$

Using equations (17) and (31) this is equal to:

$$d\Pi = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz - \frac{\partial f}{\partial S} (\mu S dt + \sigma S dz)$$  \hspace{1cm} (34)$$

$$d\Pi = \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$ \hspace{1cm} (35)$$

Since this equation does not contain the factor $dz$, and thus no random factors, it is riskless during the time $dt$. It can then be argued that according to the arbitrage principle the portfolio must thus earn the same interest as the other risk free assets, i.e. the risk free interest $r$. It thus follows that:

$$d\Pi = r\Pi dt$$ \hspace{1cm} (36)$$
Using equations (35) and (31) it follows that:

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt = r \left( f - \frac{\partial f}{\partial S} \right) dt$$

By reordering, we get:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf$$  \hspace{1cm} (37)

Equation (37) is the Black-Scholes differential equation [Hul]. It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The kind of derivative to use depends on the boundary conditions.

The above Black-Scholes (BS) equation (37) does not contain the growth factor μ however. The evaluation of a stock in the BS equation is only driven by its volatility and, the growth rate of a riskless security. This can be justified, by supposing that the portfolio could be readjusted often enough, by purchasing options when the price is less than the BS price and selling them when the market price is above that. Thompson further argues that if all traders act in this fashion, according to the BS equation, this will in fact drive the market to the BS price, whether the equation is correct or not [Tho].

Let us now regard the price of a European call option. The value of the option at the exercise time is given by equation (30). The value at a time before the exercise time must be discounted, since the money could be invested in a riskless security instead. Since the price of the stock at the exercise time is random, the expected value should be taken in order to get a fair evaluation of the option price. We thus get in accordance with the Black-Scholes equation:

$$C_{BS} = e^{-rT} \mathbb{E}[\text{Max}(S(T) - X, 0)]$$  \hspace{1cm} (38)
\[ C_{BS} = e^{-r} \frac{1}{\sqrt{2\pi \sigma^2 T}} \int_{\ln \left( \frac{X}{S(0)} \right)}^{\infty} (S(0) - X) e^{-\frac{1}{2\sigma^2 T} (x - \frac{1}{T} \sigma^2 T)^2} \, dz \] \quad (39)

Now the riskless rate \( r \) is used instead of \( \mu \). The equation (39) can be shown to get the form:

\[ C_{BS} = e^{-r} \left( e^{\mu} S(0) N(d_1) - X N(d_2) \right) \] \quad (40)

where

\[ d_1 = \frac{\ln(S(0) / X) + \left( r + \sigma^2 / 2 \right) T}{\sigma \sqrt{T}} \] \quad (41)

\[ d_2 = \frac{\ln(S(0) / X) + \left( r - \sigma^2 / 2 \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \] \quad (42)

For a proof of this see [Smi] or [Hul]. When the price is estimated with a different interest rate than the riskless one, e.g. \( \mu \) equation (40) takes the following form:

\[ C_{A} = e^{-\mu r} \left( e^{\mu} S(0) N(d_1) - X N(d_2) \right) \]

where

\[ d_1 = \frac{\ln(S(0) / X) + \left( \mu + \sigma^2 / 2 \right) T}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

If we now use the pricing formula of equation (40), with the same variables as in the first stock example: \( S(0) = 100, \mu = 10\%, \sigma = 0.1, T = 1 \) year, combined with the riskless rate of 5\%, we get the prices displayed in Table 1.

<table>
<thead>
<tr>
<th>( X )</th>
<th>102</th>
<th>104</th>
<th>106</th>
<th>108</th>
<th>110</th>
<th>112</th>
<th>114</th>
<th>116</th>
<th>118</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{BS} )</td>
<td>5,59</td>
<td>4,53</td>
<td>3,60</td>
<td>2,82</td>
<td>2,17</td>
<td>1,65</td>
<td>1,23</td>
<td>0,90</td>
<td>0,65</td>
<td>0,46</td>
</tr>
</tbody>
</table>

Table 1: Black-Scholes prices for a one year option on a stock with \( \mu = 10\%, r = 5\%, \sigma = 0.1 \)
The BS price-formula is however far from perfect. Of the seven assumptions, several can be assumed not to hold in the real world. The risk-free rate \( r \), for example, is not constant, and Merton has developed the appropriate expansion for the case of non-constant risk-free rate \( r \), among with other expansions [Mer1]. Many of the other assumptions can also be relaxed, as discussed in [Smi]. Empirical research has also shown that the BS formula is somewhat flawed, for a discussion see [Hul].

How then are options sold and bought if both the seller and the buyer follow Black-Scholes prices? One reason for the trade of an option is that the buyer and seller have different expectations on the stock. The seller often follows the BS price, but they buyer might have reasons to assume that the stock will not follow the general trend. This might be due to information that the buyer has that the market in general has not for example. Thompson argues that buyers can also use different mechanisms than Brownian motion to model the stock, which can give completely different expectations than what the BS price gives [Tho].

If the seller follows the BS price with a small commission, say interest \( \nu = 10\% \) (the riskless rate is 5\%), the buyer might expect the stock to have a higher growth rate, for example \( \mu = 15\% \). We would then get the BS prices shown in Table 2. Here we clearly see that both the seller and buyer would be likely to make a deal.

<table>
<thead>
<tr>
<th>( X )</th>
<th>102</th>
<th>104</th>
<th>106</th>
<th>108</th>
<th>110</th>
<th>112</th>
<th>114</th>
<th>116</th>
<th>118</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{\text{Seller}} )</td>
<td>7,17</td>
<td>6,16</td>
<td>5,26</td>
<td>4,45</td>
<td>3,74</td>
<td>3,13</td>
<td>2,59</td>
<td>2,13</td>
<td>1,75</td>
<td>1,42</td>
</tr>
<tr>
<td>( C_{\text{Buyer}} )</td>
<td>8,58</td>
<td>7,47</td>
<td>6,46</td>
<td>5,55</td>
<td>4,73</td>
<td>4,00</td>
<td>3,37</td>
<td>2,81</td>
<td>2,33</td>
<td>1,92</td>
</tr>
</tbody>
</table>

*Table 2: Black-Scholes prices for a 6-month option with rates 10\% and 15\% respectively and \( \sigma = 0,2 \)
3 Simulation

In order to simulate stocks and option prices MATLAB® was used. Simple Matlab functions were made that effectively perform Monte Carlo simulations on the stock price evolution.

3.1 Stock price simulation

3.1.1 Simulation of lognormal stocks

In order to simulate a stock price that follows geometric Brownian motion, according to equation (18), the Matlab function logstock was created. The function is included in Appendix 1.

This function however only gives one possible outcome. In order to visualise the distribution of the end price of the stock the pricesim function, shown in Appendix 3 was created. The function allows us to model the price of both options and stocks, and two different models for the behaviour of the stocks are allowed. The results of a simulation using the geometric Brownian motion model for the stocks, was shown in Figure 2, where 10 000 simulations where used.

3.1.2 Simulation of stocks with bear jumps

Simulation does also make it easy to step outside the realm of geometric Brownian motion. One such change would be to add the possibility of bear jumps to the stock as Thompson suggests [Tho]. Thompson proposes the use of bear jumps in such a way that there is a 10% drop with the probability 0.08 and a 20% drop with the probability 0.015 monthly.
Modelling stocks with both geometric Brownian motion and jumps has also been proposed by Merton, but he uses Poisson jumps. Here we will however restrict the study to the jumps suggested by Thompson. The stock price process can thus be written in the following form [Mer2]:

\[
\frac{dS}{S} = (\mu - \lambda)dt + \sigma dz + dq
\]  

(43)

where \(dq\) is the random process generating the jumps and \(\lambda\) is the average jump ratio. The term \((\mu - \lambda)\) thus displays the average growth rate. With the current jumps proposed by Thompson \(\lambda\) equals:

\[
\lambda = 0,08 \times (-10\%) + 0,015 \times (-20\%) = -1,1\%
\]

This is however, the monthly rate, so the yearly rate would equal: \(\lambda = -13,2\%\). The efficient \(\mu\) is thus:

\[
\mu = \mu_c + \lambda = \mu_c - 13,2\%
\]

So if we want the average growth rate \(\mu\) to be 10\%, \(\mu_c\) has to be set to 23,2\%.

In order to simulate stocks with bear jumps the functions \texttt{bearstock} was created (see Appendix 2). Simulating a stock with bear jumps over a period of 10 years with the parameters \(\mu = 23,2\%, \sigma = 0,1\) we get the distribution displayed in Figure 6. The average given by the function \texttt{pricesim}: 292,09 is close to the expected value calculated for the corresponding geometric Brownian motion model:

\[
E[S(T)] = S(0)e^{\mu T} = 100 \times e^{0,1 \times 10} \approx 271,83
\]

This expected value should however not match the simulated price perfectly, since the simulated equation does not follow geometric Brownian motion. The expected value of this model including bear jumps is much more complex, and beyond the scope of this study.
3.2 Simulating options

When simulating options the pricing of options has to be used, i.e. equation (30). The exercise price thus plays an important role. Several things concerning the pricing of options can be visualised with simulations. The average price and the standard deviation can be calculated as well as showing the distribution of the resulting value of the option.

Let us now simulate the option value for the same stock as in Table 1 at say $x = 102$. The average result given by the simulation is 9.67 (from 20 000 simulations). This is much higher than the value given by the BS equation: 5.59 (see Table 1). That is natural however, since the BS equation uses the riskfree rate $r$ (5%) and we in the simulation used the growth rate $\mu$ (10%). If we instead simulate with the
riskfree rate of 5%, the simulation average is 5.63 which seems to be in accordance with the BS price.

The value distribution for the above simulation is shown in Figure 7. From the figure, we can see what the BS equation does not tell, namely that 46% of the time the option turns out to be worthless.

Figure 7: The simulated value distribution of a one year option, with $r = 5\%$, $\sigma = 0.1$, $S(0) = 100$ and $x = 102$

We can now use any model for the stocks. If we use the model with bear jumps, the results differ somewhat from the prices predicted by the BS equation. If we for example run simulations on a 6-month option where the stock has bear jumps with
the same parameters as were used in Table 2 ($\mu = 10\%$, $\sigma = 0,2$) we get the results displayed in Table 3 (all simulations where done 20 000 times).

<table>
<thead>
<tr>
<th>X</th>
<th>102</th>
<th>104</th>
<th>106</th>
<th>108</th>
<th>110</th>
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<td>$C_{BS}$</td>
<td>7,17</td>
<td>6,16</td>
<td>5,26</td>
<td>4,45</td>
<td>3,74</td>
<td>3,13</td>
<td>2,59</td>
<td>2,13</td>
<td>1,75</td>
<td>1,42</td>
</tr>
<tr>
<td>Sim Avg</td>
<td>7,77</td>
<td>6,80</td>
<td>5,98</td>
<td>5,17</td>
<td>4,39</td>
<td>3,85</td>
<td>3,24</td>
<td>2,77</td>
<td>2,32</td>
<td>1,97</td>
</tr>
</tbody>
</table>

*Table 3: The calculated BS price and the simulated average for 6-month options with bear jumps, average growth rate 10% and $\sigma = 0,2$*

If we take a closer look at the option value for an option with exercise price at, say 110, we get the distribution shown in Figure 8. From the figure, we see that even though the simulation average is 4,39 and the BS price is 3,74, the option will be worthless as often as 62% of the time.

*Figure 8: The value distribution of a 6-month option with bear jumps, strike price = 110, average growth =10% and $\sigma = 0,2$*
4 Conclusions

In this study, the pricing theory behind stocks and options has been described. The mathematical models generally used for modelling stocks have been described in detail and finally the derivation of the Black-Scholes equation has also been described.

There thus exists a large theoretical framework for evaluating the prices of options and other derivatives. It is however somewhat flawed and the formulas do not always work in practice. It has been shown that with simulations it is easy to analyse situations that differ from the norm. Stock price models that are difficult to handle theoretically can be easy to analyse with simulation. The information that can be found through a simulation analysis can often be much deeper than what the theories can give. A good example of this is the Black-Scholes price, which only gives the expected price, versus the value distribution that can easily be achieved through simulation.

This study could easily be deepened into several different directions. Different models for the stock prices could be analysed, including correlations between stocks, as stocks often are correlated. Using real data to produce predictions through simulation that then could be compared would also be field of further study. There are still lots of unexplored possibilities of simulation.
References


Appendix 1.

function [S, t]=logstock(s0, mu, si, n);
% s0 is the starting price, mu the growth rate, si the standard '
% deviation and the total time

% This simulation is going in steps of 1 week at a time and n is
% given in years
h=52;
dt=1/h;

S(1)=s0;
t(1)=0;
for j=1:1:n*h

    % The following value of the stock is evaluated
    S(j+1)=S(j)*exp((mu-si^2/2)*dt+si*sqrt(dt)*randn(1));
    t(j+1)=j;
end;
Appendix 2.

function [S, t]=bearstock(s0, mu, si, n);
% s0 is the staring price, mu the growth rate, si the standard
% deviation and the total time

% This simulation is going in steps of 1 month at a time and n is
given in years
h=12;
dt=1/h;

S(1)=s0;
t(1)=0;
for j=1:1:n*h

% The depth of the possible bear jump is calculated according
% to the assigned probabilities

    prob = rand(1);
    if prob < 0.015
        q = -0.20;
    elseif prob < 0.095
        q = -0.10;
    else
        q = 0;
    end;

% The following value of the stock is evaluated
S(j+1)=S(j)*exp((mu-si^2/2)*dt+si*sqrt(dt)*randn(1)+q);
t(j+1)=j;
end;
function [distr, ubar, m, st]= pricesim(s0, x, mu, si, T, n2, stock, option, bars);
% The parameters are:
% s0 is the starting price,
% x the exercise price of the option,
% mu is the growth rate,
% si is the standard deviation,
% T the time for each stock or option,
% n2 the number of options to be simulated,
% stock tells whether to use lognormal or stocks with bear jumps,
% option tells whether to simulate options or stocks,
% bars tells how many bars are wanted for the distribution

% The endprices are simulated n2 times
for j=1:1:n2
   if strcmp(stock,'log')
      [S, t] = logstock(s0, mu, si, T);
      disc = mu;
   elseif strcmp(stock,'bear')
      [S, t] = bearstock(s0, mu, si, T);
      disc = mu-0.132;
   else
      error('stock must be of either type log or bear')
   end;
   if strcmp(option,'y')
      % The values of the options are discounted
      % if the bear jump model is used the options are discounted
      % by the average growth rate, and not the maximum
      E(j)=exp(-disc*T)*max(S(length(S))-x, 0);
   elseif strcmp(option,'n')
      E(j)=S(length(S));
   else
      error('option must be either y or n')
   end;
end;
m = mean(E);
st = std(E);

% The upper limits for each bar in the bar diagram are
% calculated and stored in the ubar variable
if option == 'y'
   minE = - max(E)/(bars -1);
elseif option == 'n'
   minE = min(E);
end;
diff = (max(E)-minE)/bars;
for j=1:1:bars
   ubar(j)=round(1000*(minE+diff*(j-1)))/1000;
end;
% The proportion of values in each bar are calculated
% and stored in the distr variable
distr=zeros(1,bars);
for j=1:1:n2
    i=bars;
    while E(j) < ubar(i)
        i=i-1;
    end;
    distr(i)=distr(i)+1;
end;
for j=1:1:bars
    distr(j)=distr(j)/n2;
end;