Modelling the Spatial Correlation of the Baltic Sea Surface Topography

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1 Introduction

Three Baltic Sea Level (BSL) campaigns have been organized in which the sea surface topography (SST) is examined. The aim of the campaigns has been to model the crust of the Northern Europe. [Pou00]. Based on the performed GPS measurements there evidently exists spatial correlation in the sea topography [Pou97]. In this work our aim is to describe the spatial variation of the Baltic Sea surface.

In describing the spatial distribution of the SST the GPS observations are, however, inadequate because the GPS can be used only at the tide gauges. Therefore the satellite altimetry is used instead to increase the spatial resolution of the observations.

The SST is analyzed by modelling the mean sea level by using regression analysis. By regression models the temporal average of SST is modeled. The residuals of the regression model are analyzed by using variograms [Man], [Hai] to describe the spatial correlation in the SST.

This paper is organized as follows. In chapter two SST and the needed measurement methods are defined. Chapter three presents the statistical tools that are used in chapter four in the analysis of the spatial correlation of the SST. Chapter five concludes the results.

2 Sea Surface Topography

Measuring sea surface height is far from trivial task: the reference coordinate system need to be selected and different tide components need to be taken into account. Additionally, the bias of the different measurement techniques remains as an open question and the spatial resolution of the GPS is poor on the open sea [Pou00].

2.1 Reference Coordinates

The most elementary reference coordinate system is the ellipsoid model, in which the SST is the height of the sea surface above a reference ellipsoid. The height typically refers to orthometric height, which is the distance between the measured location and the reference ellipsoid along a plumb line [Pou00]. However, an ellipsoid is far from accurate for our purpose. In this work, a geoid model is used instead of ellipsoid. It is an approximation of the equipotential surfaces of the gravitational field and it can offer as accurate
as 25 ppm estimate for the reference height \cite{Pou00}.

There is variety of different geoid models for different purposes. The geoids can be classified by the accuracy and the spatial range of validity. For instance, geoids such as OSU91A and EGM96 are global models offering of decimeter accuracy while FIN95 and NKG96 are local models covering only the Northern Europe offering even subcentimeter accuracy locally. \cite{Pou00}

Another property of a geoid model is how the tide effect is included in it. The Moon and the Sun have significant effect on the potential field of the Earth. They deform the crust causing so called permanent latitude dependent tide effect and they directly affect the potential field on the Earth causing long and short term periodic oscillations on the potential field. The geoid models are divided into three classes depending on how the Sola-Lunar effects have been taken into account: \cite{Pou96}

1. Non-tidal geoid
2. Zero geoid
3. Mean geoid

Non-tidal geoid models only the potential field of the Earth. Zero geoid includes the permanent tide effect and the mean geoid additionally takes the potential fields of the Sun and the Moon into account. Therefore the mean sea level obeys the shape of the mean geoid \cite{Pou95}. In this work the mean geoid is used. If necessary, the difference between non-tidal geoid and a mean geoid can be calculated by \cite[equation 1c]{Pou99b}

\[
\Delta N = (1 + k)(0.099 - 0.296\sin\phi)^2,
\]

where $\phi$ is the latitude of the location and $k$ is so called Loeve's constant. For discussion and proof of (1) see \cite{Pou96}. This transformation is especially needed if there are observations in relation to some non-tidal geoid.

### 2.2 Measuring Sea Surface Topography

SST can be measured by GPS observations at tide gauges. Tide gauge gives the height of the sea respect to a levelling reference benchmark and the coordinates of the levelling benchmark are obtained by GPS. The GPS measuring arrangements are illustrated in figure 1.
GPS gives the height in relation to a reference ellipsoid and SST can be obtained by

$$SST = h - \Delta h_1 - \Delta h_2 - \Delta h_3 - N - \Delta N,$$

where $h$ is the height of the GPS antenna above the reference ellipsoid, $\Delta h_1$ is the height of the GPS antenna, $\Delta h_2$ is the height of a levelling benchmark, $\Delta h_3$ is the height of the mean sea level during the measurement period, $N$ is the height of the non-tidal geoid above the ellipsoid and $\Delta N$ transforms the non-tidal heights to the mean heights as given in (1). [Pou99a]

The spatial resolution of the GPS is poor because it can be used only on the coast. Therefore, if needed, SST is to be extrapolated to cover the whole Baltic Sea.

It is possible to use satellite measurement techniques to have higher spatial resolution. Poutanen [Pou00] has calculated SST from the satellite measurements that were completed between years 1991 and 1998 by using CERSAT ERS-1 and ERS-2 data. ERS2 data is used in this work.

A satellite probes its height above the sea surface and during a 35 days period the ground tracks of an ERS satellite cover the whole Baltic sea. Thus, the measured data is divided into sets covering 35 days of data. [Pou00]

The satellite altimetry is sensitive to the magnetic phenomena and the wavy sea causes remarkable errors. These sources of errors can be eliminated and the SST can be calculated by

$$SST = h_s - H_m + s_{sb} + e_{tide} + i,$$

where $h_s$ is the height of the satellite above the reference ellipsoid, $H_m$ is the average of the observations over time horizon of one second, $i$ contains the correction terms to eliminate ionospheric errors, $s_{sb}$
eliminates the errors caused by the wavy sea by taking the significant wave height into account. $\epsilon_{\text{tide}}$ is the difference between the reference ellipsoid and a mean geoid. [Pou00]

Other sources of error are small islands and shorelines. To reduce errors the Baltic Sea is divided into a grid consisting of $0.5^\circ \times 0.5^\circ$ squares and median of the SST is calculated over each square.

3 Statistical Models

Consider now the median filtered SST data or some other data observed on a two dimensional surface, as well. Denote this data by $(\lambda_i, \phi_i, h_i)$, where $(\lambda_i, \phi_i)$ are the coordinates of the location of observation $h_i$. Imagine that $h_i$ is an realization of random variable $H_i$. Our aim is to model the correlation between $H_i$ and $H_j$ in terms of the distance between locations $i$ and $j$. However, before analyzing the correlation one needs to remove the trends in the data.

3.1 Regression Models

There may exist some trends in the observations, like South-North slope in the SST [Pou00] that can be explained by using regression models:

$$H_i = \beta_2 \lambda_i + \beta_1 \phi_i + \beta_0 + \varepsilon N i,$$

or in more general form

$$H = \beta X + \varepsilon,$$

where $H = [H_1, \ldots, H_n]^T$, $X$ is a matrix containing the explaining variables on the rows and $\beta$ is a vector containing the regression parameters that are estimated from the measured data. $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_n]^T$ are random variables.

If the regression parameters are estimated ordinary least square (OLS), the estimates of the regression parameters are

$$\hat{\beta} = (X^T X)^{-1} X^T H.$$

OLS estimators are obtained by minimizing the square sum of the residuals $\varepsilon^T \varepsilon$, where $\varepsilon = h - \hat{\beta} X$ and $h = [h_1, \ldots, h_n]^T$. OLS estimates can be used if the residuals $\varepsilon_i$ are uncorrelated, have constant variance
and constant mean: [Pin, Chapter 2]

\[
\begin{align*}
\text{Var}[\varepsilon_i] &= \sigma^2 \\
E[\varepsilon_i] &= 0 \\
E[\varepsilon_i \varepsilon_j] &= \sigma^2 I
\end{align*}
\]

If these assumptions are not satisfied then the generalized least squares (GLS) is used. The GLS estimates of the regression parameters are [Pin, Appendix 6.3]

\[
\hat{\beta} = (X^TV^{-1}X)^{-1}X^TV^{-1}H,
\]

where \( V = E[\varepsilon \varepsilon^T] \) is the covariance matrix of the residuals.

### 3.2 Mantel’s Test

After the regression model is estimated, the existence of the spatial correlation of the residual data is a relevant question. To examine it, construct a new set of difference data \( c_{ij} = |\hat{\varepsilon}_i - \hat{\varepsilon}_j| \) and calculate the distance between location \( i \) and \( j \) \( d_{ij} \). Now, the spatial correlation can be described by analyzing \( (c_{ij}, d_{ij}) \) pairs. If there is no spatial correlation then all the differences of the observations are located on a \((c, d)\)-plane around some line \( c = \text{constant} \). If spatial correlation exists then some trend is visible on the \((c, d)\)-plane.

For notational reasons reindex the new data to have \((d_i, c_i)\), where \( i = 1, \ldots, n^2 \) and \( n \) is the number of observations. Now, Pearson’s correlation \( R \) can be estimated for the data to describe the linear trend in the difference data

\[
\hat{R} = \frac{\sum_{i=1}^{n^2} (d_i - \bar{d})(c_i - \bar{c})}{\sqrt{\sum_{i=1}^{n^2} (d_i - \bar{d})^2 \sum_{i=1}^{n^2} (c_i - \bar{c})^2}},
\]

where \( \bar{c} \) and \( \bar{d} \) are the averages of the \( c \) and \( d \).

Pearson’s correlation describes how similar are the observations that are measured near to each other in the original data. If the correlation is close to zero, then the differences in the measured quantity do not depend on the distance between the locations. If the correlation is large enough, then there exists correlation between the difference of the measured quantity and the distance of the locations.

We do not have reliable distribution for \( H_i \) and thus we need to examine the significance of the spatial correlation by using statistical simulation, namely Mantel’s test.
We need to test the hypothesis:

\[ H_0 : R = 0, H_1 : R \neq 0 \]

Assume \( H_0 \), i.e., that there exists no spatial correlation in the residual data. The quantities \( \varepsilon_i \) could have been occurred at any location on the \((\lambda, \phi)\) plane. This implies that \( \varepsilon_i \) can be reordered randomly in the data set \((\lambda_i, \phi_i, \varepsilon_i)\) and still the randomized data \((\lambda_i, \phi_i, \varepsilon_j)\) could be observed with probability equal to probability of observing the original data. Now, the Pearson’s correlation can be estimated for the reordered data. The reordering can be repeated \( N \) times to produce the randomized correlations \( \hat{R}_1, \ldots, \hat{R}_N \). By using the simulated correlations the experimental distribution of \( R \hat{f}(R) \) can be estimated.

By using the simulated distribution of the correlation the p-value of the observed correlation \( \hat{R} \) can be calculated by

\[ p = \int_{\hat{R}}^{\infty} \hat{f}(R) dR. \]

Now, if \( p \) is sufficiently small, say \( p < \alpha \), the hypothesis \( H_0 \) is false with confidence level \( \alpha \) and \( H_1 \), spatial correlation exists, is accepted.

### 3.3 Variogram

The spatial correlation can be described by using variograms. Assume first that the distribution of the residual data is stationary, i.e., it’s expected value and variance are constant over \((\lambda, \phi)\)-plane and the correlation of the residuals depends only on the distance between the locations:

\[
\begin{align*}
E[\varepsilon_i] &= E[\varepsilon_j], \quad (6) \\
Var[\varepsilon_i] &= Var[\varepsilon_j] = \sigma^2, \quad (7) \\
\rho(\varepsilon_i, \varepsilon_j) &= \rho(d_{ij}). \quad (8)
\end{align*}
\]

Under these assumptions we may define the variogram

\[ \gamma = E[0.5(\varepsilon_i - \varepsilon_j)^2] = 0.5Var[(\varepsilon_i - \varepsilon_j)] \]

for which we have that [Man]

\[ \gamma(d) = \sigma^2(1 - \rho(d)) . \quad (10) \]
Variogram can be modelled, for instance, by a parametric model $\gamma(d) = \gamma(d; \lambda)$, for which there are many suggested forms in literature. For instance, Gaussian model,

$$\gamma(d; [c, S, a]) = c + (S - c)(1 - e^{-\frac{|d|^2}{a^2}})$$

is a reasonable choice for practical purposes. In figure 3.3 a gaussian model of a variogram is sketched.

By a gaussian model we can define three concepts: nugget effect, sill and the range of influence. Nugget effect is $c = \gamma(0; \lambda)$, sill $S = \lim_{d \to \infty} \gamma(h; \lambda)$ and the range of influence is the distance $d^*$ at which the variogram $\gamma(d^*; \lambda) = 0.95S$. Note that

$$S = a^2$$ (11)

if $\lim_{d \to \infty} \rho(d) = 0$. Nugget effect expresses the repeatability of the measurements, that is $c = \text{Var}[h_i]$.\[Hai\], [Man]

The parameters $\lambda$ can be estimated by GLS. Unfortunately, it is not easy to estimate the correlation matrix of the residuals consistently and thus another approach would be more suitable.

Cressie and Hawkins have derived a model of the variogram [Hai, Ch. 6.3.]

$$\gamma_{CH}(d) = f(n_d) \frac{1}{n_d} \sum_{i,j \in [d-d, d+d]} (\sqrt{|h_i - h_j|})^4,$$ (12)

where $f(n_h) = 0.457 + 0.494n^{-1}_h + 0.045n^{-2}_h$. This model is nonparametric and thus it is quite a robust.

4 Spatial Correlation in the SST of the Baltic Sea

Here we present the results of the analysis of the spatial correlation of the SST. Before any analysis the data was scaled such that the values of the coordinates of the locations and the observations varied
between -1 and 1. This scaling offers higher level of numerical stability. Also some filtering of the outliers was used. All the observations below -30 cm and above 30 cm were removed.

We analyzed all the data sets between Summer 1995 and 1998. However, some data sets needed to be censored. The reasons to censoring were that either the icy sea caused too much noise to the data and some data sets contained only a few observations. We did not include the sounds between Denmark and Sweden to our analysis because the data is such a noisy [Pou00].

The working procedure was iterative. First, we used the OLS estimates in the regression analysis and then the variogram model was estimated. By using the variogram a more accurate model for the correlation of the residuals was obtained and it was used in the GLS estimation. These phases were repeated until the regression parameters saturated. The saturation condition was that in the iteration the norm of the difference of two successive regression parameter vectors is less than 0.01. In regression analysis we applied a simple model

$$h_i = \hat{\beta}_0 + \hat{\beta}_1 \lambda_i + \hat{\beta}_2 \phi_i + \hat{\epsilon}_i.$$  \hfill (13)

Based on the regression analysis the slope of the Baltic Sea is evident. In figures 3 and 4 the histograms of the slope in the North-South and West-East direction are presented.

We used the Mantel’s test to test the existence of the spatial correlation. The test was completed for the
residual data produced by the regression model with the OLS estimates. First the test was completed as suggested in chapter 3 and p-values ranged from 0.04 to 0.1. Thus, it seemed that the spatial correlation might be weak. However, if the test was performed by replacing the distances $d$ by the inverses $1/d$ as Manley suggests [Man] the p-values were less than 0.01 in all of the analyzed cases. This difference is explained by the fact that the correlation is clear at the small distances and the inverse of the distance alleviates the distinctions at small distances.

For the residual data, the Cressie and Hawkins variogram model was estimated. In the figure 5 the variograms between April and December 1996 are presented. The gaussian model is fitted in the Cressie and Hawkins model and it was used in determining the correlation matrix $V$. By using the equation (10) and the variograms, we can estimate the model for the spatial correlation. We selected the sill of the estimated Gaussian model as the estimate of the variance $\sigma^2$, see equation (11). The spatial correlation is shown in the figure 6 for the SST between April and December 1996. The fluctuations in the variograms are most likely due to seasonal and and other random variations in the weather conditions and environment in general.

In figure 7 the histogram of range of effect is presented. The results show that there exists clear positive correlation at the distances less than 150-300 km.
Figure 5: The variograms between April and December 1996
Figure 6: The spatial correlation between April and December 1996
5 Conclusions

There are two main conclusions that can be made based on the analysis presented in this paper: the Baltic Sea has slope and there exist clear spatial correlation in the SST even if the slope is eliminated.

The validity of the results seems to be credible when comparing them to the results presented by Poutanen [Pou00]. Poutanen has shown that due to the S-N slope in the Baltic Sea the northernmost parts of the Baltic were about 30 centimeters above the coast of Poland in 1997. This corresponds to 3 cm per degree on average which is close our results in figure 3.

This paper has only given rough overview on how the SST should be modeled by using variograms and regression models. Therefore, there exists need for further research. The distribution of the range of influence, slope and the variance of the residual data should be analyzed. Another point of interest would be the temporal variation of the SST because the seasons do likely effect on the SST.
References


