Data Envelopment Analysis for Portfolios

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1 Introduction

Efficiency analysis examines the efficiency of decision-making units (DMUs) by using their observed input and output values. Typically, a DMU is an individual unit consuming some inputs and producing some outputs. For instance, DMUs can be public departments, such as hospitals (e.g., Hynninen et al. 2011) and universities, or private companies. A DMU is considered efficient if greater outputs cannot be produced with fewer inputs. Typical inputs are monetary and human resources, whereas outputs can be, for example, treated patients by hospitals or scientific publications by universities.

Data envelopment analysis (DEA; e.g., Cooper et al. 2000) is a non-parametric method for efficiency analysis. DEA includes different kinds of model types, which determine the set of possible input–output combinations. These combinations, i.e, portfolios, form the production possibility set. All efficient combinations are in the boundary of the production possibility set called the efficient frontier. The most central tools offered by DEA are efficiency scores that measure how close the DMU is to the efficient frontier. The efficiency scores can be calculated by using linear programming (LP) or integer linear programming (ILP), and thus they offer a useful method for comparing the efficiency of DMUs.

The oldest and most typical DEA models are CCR (for Charnes, Cooper and Rhodes; 1978) and BCC (for Banker, Charnes and Cooper; 1984). Different models score the DMUs differently. For example, the BCC model takes the scale of different input and output values into account, unlike CCR. That is, the CCR model has constant returns to scale, whereas the BCC model has variable returns to scale. In practise, this usually means that the largest and sometimes the smallest DMUs have significantly better BCC efficiency scores than CCR scores.

In addition to standard DEA, the efficiency of DMUs can be compared by using ratio-based efficiency analysis (REA; e.g., Salo and Punkka 2011), which also includes the standard CCR-DEA efficiency score. REA offers methods, for example, for calculating ranking intervals for individual DMUs and dominance relations among them. These usually characterize efficiencies better than the efficiency scores.

In this study, we extend Green and Cook's (2004) DEA theory for more general production possibility sets. Especially, we consider DMU combinations in production possibility sets whose shape somehow reflects possible portfolios. We introduce one specified production possibility set for continuous portfolios and one for discrete portfolios and formulate the corresponding DEA models. In addition, we formulate the linear programming dual in the general continuous case, and show how preference information about relative values of different inputs and outputs can be added to the model. Furthermore, we develop theory for generalizing the idea of dominance in ratio-based efficiency analysis for portfolio production possibility sets.

2 Generalization of the input-oriented DEA

2.1 Input-oriented DEA

Let the set of DMUs be indexed by $j \in J = \{1, ..., m\}$, each consuming s different inputs and producing n different outputs. The ith input value of DMU j is denoted by x_i^j and the kth output value by y_k^j. The input and output values of DMU j form vectors $x^j \in \mathbb{R}_+^s$ and $y^j \in \mathbb{R}_+^n$. Here $\mathbb{R}_+^s = \{x \in \mathbb{R}^s \mid x \geq 0, x \neq 0\}$, where $\geq$ holds componentwise. The vectors of input and output values form input and output matrices denoted by $X := [x^1, ..., x^m] \in \mathbb{R}_+^{s \times m}$ and $Y := [y^1, ..., y^m] \in \mathbb{R}_+^{n \times m}$, respectively.

DMUs are compared to (linear) combinations of DMUs' input and output values. The combinations that are included in the comparison, is determined by the type of the DEA model. The model type is denoted by f and the set of feasible DMU combination weights by Λ^f.

Definition 1. *The set of feasible DMU weights for DEA model type f is*

$$\Lambda^f := \{\lambda \in \mathbb{F}_+^m \mid A^f \lambda \leq a^f\},$$

where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{Z}$, and $A^f \in \mathbb{R}^{r \times m}$ and $a^f \in \mathbb{R}^r$ are a matrix and a vector determining r linear constraints such that $e^j \in \Lambda^f$ for all $j \in J$, where

$$e^j = [\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0]^T \in \mathbb{R}^m.$$

The set Λ^f determines which consumption–production combinations can be achieved. This attainable set is called the production possibility set. Different DEA models make different assumptions of the production possibility set through the set Λ^f.

Definition 2. *The production possibility set for DEA model type f is*

$$T^f := \{(x, y) \in \mathbb{R}_+^s \times \mathbb{R}_+^n \mid x \geq X\lambda, y \leq Y\lambda \text{ for some } \lambda \in \Lambda^f\}.$$

Inequalities are used in Definition 2 instead of equalities in order to make also clearly inefficient consumption–production combinations possible. When we want to highlight the sort of the set $\mathbb{F}$ in Definition 1, we call the set Λ^f connected if $\mathbb{F} = \mathbb{R}$ and discrete if $\mathbb{F} = \mathbb{Z}$. The assumptions made in the definition of Λ^f ensure that calculations remain (I)LP problems and $(x^j, y^j) \in T^f$ for all $j \in J$. The most typical model types are CCR (for Charnes, Cooper and Rhodes) and BCC (for Banker, Charnes and Cooper), where the set Λ^f is defined as follows:

$$\begin{aligned} \Lambda^{\mathrm{CCR}} &:= \{\lambda \in \mathbb{R}^m_+\}, \\ \Lambda^{\mathrm{BCC}} &:= \{\lambda \in [0,1]^m \mid \sum_{j=1}^{m} \lambda_j = 1\}. \end{aligned}$$

That is, the CCR model includes all non-negative linear combinations of DMUs into the production possibility set, whereas the BCC model includes only all convex combinations of DMUs. The CCR and BCC models are the extremities of DEA models in the connected case, since Λ^{CCR} and Λ^{BCC} are the largest and the smallest connected Λ^f that are possible by Definition 1.

The production possibility set can be divided into efficient and inefficient points. A point is considered efficient if none of the inputs and outputs can be improved without worsening some other input or output. All the efficient points are included in the boundary of the production possibility set, which is called the efficient frontier. It also contains all 'weakly efficient' points.

Definition 3. *The efficient frontier of the production possibility set T^f is*

$$Ef(T^f) := \{(x, y) \in T^f \mid \nexists (x', y') \in T^f \text{ s.t. } x' < x, y' > y\}.$$

In DEA, the efficiency of a DMU is considered in a relation to the efficient frontier. In an input-oriented DEA model, the efficiency score is calculated by examining how much the input values would have to decrease for the DMU to belong to the efficient frontier.

Definition 4. *Input based f-efficiency score of DMU j is*

$$\hat{E}^f(x^j, y^j) := \min_{\alpha \in \mathbb{R}} \{\alpha \mid \exists (x, y) \in T^f \text{ s.t. } \alpha x^j \geq x, y^j \leq y\}.$$

f-efficiency scores can be solved by using linear programming, since combining Definition 2 with Definition 4 gives

$$\hat{E}^f(x^j, y^j) = \min_{\alpha \in \mathbb{R}} \{\alpha \mid \alpha x^j \geq X\lambda, y^j \leq Y\lambda, \lambda \in \Lambda^f\}. \quad (1)$$

Clearly, $\hat{E}^f(x^j, y^j) \leq 1$, since $e^j \in \Lambda^f$. On the other hand, $\hat{E}^f(x^j, y^j) > 0$ because $\Lambda^f \subset \mathbb{R}_+^m$ and $x^k \in \mathbb{R}_+^s$ for all $k \in J$. When using (1), the efficiency score can be calculated for individual DMUs. In addition, we can easily extend it to comprising all combination DMUs in the production possibility set; The f-efficiency score of any point $(x, y) \in T^f$ can be calculated as the optimal value of the following (I)LP problem:

$$\begin{aligned} \hat{E}^f(x, y) = &\min_{\alpha,\lambda} \alpha \\ \text{s.t. } x\alpha - X\lambda &\geq 0 \\ Y\lambda &\geq y \\ \alpha \in \mathbb{R}, \lambda &\in \Lambda^f. \end{aligned} \tag{2}$$

For all points (x, y) in the efficient frontier, the efficiency score $\hat{E}^f(x, y)$ is equal to 1. The optimization problem (2) is often called the envelopment form of the corresponding DEA model (e.g., Cooper et al. 2000).

In addition to the efficiency score, the solution (α^*, λ^*) to the LP problem (2) includes a point that is in the efficient frontier. We call this point the reference point. For an inefficient DMU, the reference point is also a feasible DMU combination that achieves at least the same output with fewer inputs. Thus, also the reference points are usually interesting to consider.

Definition 5. *Let (α^*, λ^*) be the solution to the LP problem (2) for the point $(x, y) \in T^f$. The reference point of (x, y) is*

$$Rp(x, y) := (X\lambda^*, Y\lambda^*) \in Ef(T^f).$$

Different DEA models treat DMUs with small and large scale differently. This property is often called the returns to scale of the model (e.g, Cooper et al. 2000). The type of the returns to scale is determined by the (in)equalities in the definition of the set Λ^f, and it is reflected in the shape of the efficient frontier. For example, the CCR model has constant returns to scale because it has no constraints in the definition of Λ^{CCR}. As a consequence, the CCR efficient frontier is straight in radial directions, and thus the scale of DMUs does not affect the efficiency scores. The BCC model, instead, has variable returns to scale because of the equality constraint, and its efficient frontier becomes 'curved' around some DMU points. Thus, the BCC model compares DMU combinations in the same scale, and also the most efficient small-scale and large-scale DMUs get good efficiency scores. The returns to scale can also be non-increasing or non-decreasing when defining Λ^f such that only either inequality $\leq$ or inequality $\geq$ holds for λ. In the non-increasing case, the variable scale occurs only in large-scale DMUs, whereas in the non-decreasing case, only in small-scale DMUs.

In the CCR and BCC cases, it is also typical to deal with another kind of optimization problem for calculating the efficiency score, called the ratio form. In the ratio form, efficiency is considered as a quotient, total output divided by total input, which is more intuitive way of thinking efficiency. In the ratio form, the inputs and outputs are weighted instead of the DMUs, and these weights are chosen such that the output–input ratio is maximized. In practise, the ratio form is the linear programming dual of the envelopment form with a change of variables. Thus, the efficiency score is the same in the both forms. For instance, the ratio forms of the CCR and BCC models (e.g., Cooper et al. 2000) are

$$\begin{aligned} &\max_{v,u} \frac{u^T y}{v^T x} \\ &\text{s.t. } \frac{u^T y^k}{v^T x^k} \leq 1, \quad \forall k \in J \\ &v, u \geq 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\max_{v,u,u_0} \frac{u^T y - u_0}{v^T x} \\ &\text{s.t. } \frac{u^T y^k - u_0}{v^T x^k} \leq 1, \quad \forall k \in J \\ &v, u \geq 0, u_0 \in \mathbb{R}, \end{aligned} \tag{4}$$

respectively.

2.2 Portfolio production possibility sets

In the portfolio framework, DMUs are considered as 'project proposals' that can be combined to form a portfolio. Usually, this portfolio forming has different kinds of constraints for the chosen projects, for example budget constraints. These constraints should be reflected in the production possibility set such that the set of feasible DMU weights Λ^f includes exactly all suitable combinations of DMUs. Thus, the CCR and BCC models are not the most convenient DEA models for measuring the efficiencies of DMUs in relation to portfolios.

Typically, a portfolio can include several individual DMUs as a whole, but only a limited number of each DMU. For example, Green and Cook (2004) consider production possibility sets for $\Lambda^{\mathrm{K}} := \{\lambda \in [0,1]^m\}$ ('K' for Koopmans (1977)) and $\Lambda^{\mathrm{FCH}} := \{\lambda \in \{0,1\}^m\}$ ('FCH' for Free Coordination Hull). They also mention more general production possibility sets, where a fixed number of 'copies' of each DMU are allowed and total number of DMUs in the portfolio is limited. Two that kinds of sets are considered in this study, 'CP' for Continuous Portfolio and 'DP' for Discrete Portfolio:

$$\begin{aligned}\Lambda^{CP} &:= \{\lambda \in \mathbb{R}^m_+ \mid \sum_{j=1}^{m} \lambda_j \le b,\ \lambda_j \le c_j,\ \forall j \in J\},\\ \Lambda^{DP} &:= \{\lambda \in \mathbb{Z}^m_+ \mid \sum_{j=1}^{m} \lambda_j \le b,\ \lambda_j \le c_j,\ \forall j \in J\},\end{aligned}$$

where $b \ge 1$ and $c_j \ge 1$ for all $j \in J$ to ensure that all individual DMUs belong to the production possibility set. The sets Λ^{CP} and Λ^{DP} satisfy Definition 1, and thus the problem (2) is a LP problem when $f = \mathrm{CP}$, or an ILP problem when $f = \mathrm{DP}$. Furthermore, Definition 1 also accepts adding constraints for the number of selected DMUs from a specified subset $L \subset J$, i.e., $\sum_{j \in L} \lambda_j \le c_L$.

CP models always have non-increasing returns to scale, since λ is limited only by constraints with the inequality $\le$. The CP efficiency score of a point (x, y) is always between its CCR and BCC efficiency scores when (x, y) belongs to the corresponding production possibility sets. This is since $\Lambda^{\mathrm{BCC}} \subset \Lambda^{\mathrm{CP}} \subset \Lambda^{\mathrm{CCR}}$, wherefore $T^{\mathrm{BCC}} \subset T^{\mathrm{CP}} \subset T^{\mathrm{CCR}}$ by Definition 2, and thus $\hat{E}^{\mathrm{BCC}}(x, y) \ge \hat{E}^{\mathrm{CP}}(x, y) \ge \hat{E}^{\mathrm{CCR}}(x, y)$ by (2). If the constraints in a CP model are not strict, i.e., the constants $b, c_1, \ldots, c_m$ are large enough, it does not really differ from the CCR model. In addition, if $b, c_1, \ldots, c_m \to \infty$, then $T^{\mathrm{CP}} \to T^{\mathrm{CCR}}$.

An example of the CP production possibility set is presented in Figure 1. In this simple case, four DMUs, $A\,(2, 3), B\,(4, 4), C\,(7, 6)$ and $D\,(9, 7)$, consume one input and produce one output. Each of the DMUs has an upper bound $c_j = 1,\ j = A, B, C, D$, and the sum bound of DMUs is $b = 3$. The figure includes the CP efficient frontier and composite DMUs obtained from the individual DMUs by summing their input and output values, respectively. In addition, the corresponding CCR and BCC efficient frontiers are presented in the figure. The CP, CCR and BCC production possibility sets consist of the corresponding efficient frontier and the domain under it. DMUs A, C and D are BCC efficient, whereas only the DMU A is CP and CCR efficient. In addition, the composite DMUs $A + B$, $A + B + C$, $A + C + D$ and $B + C + D$ are CP efficient.

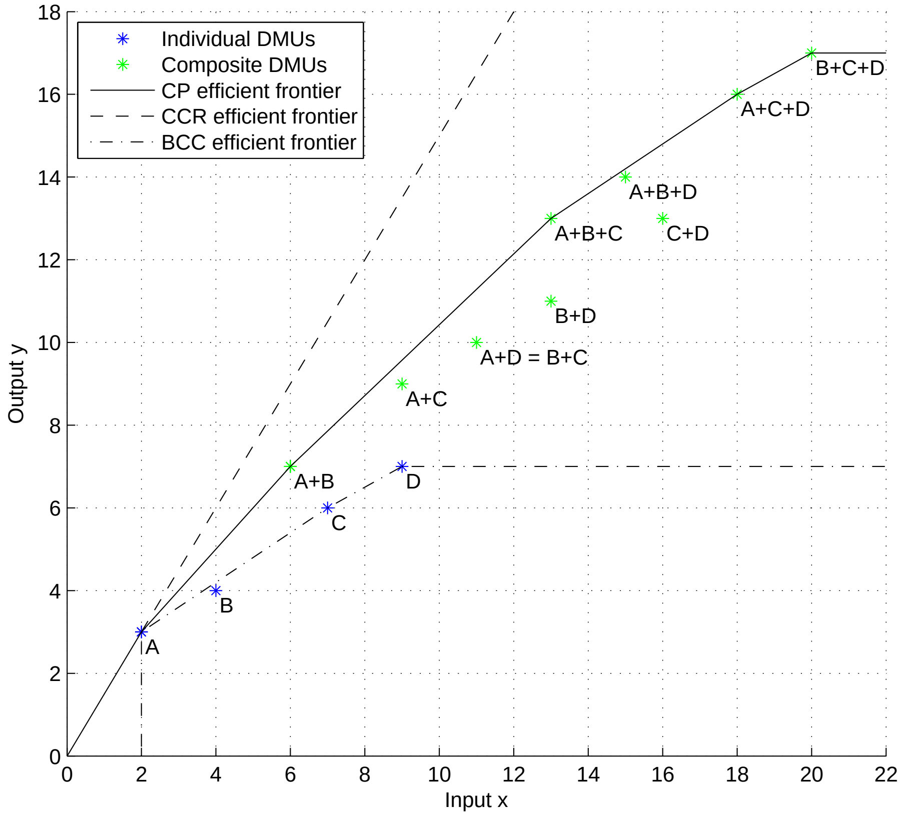

Figure 1: CP production possibility set - one input and one output

3 Dual representation

In this section, we formulate the ratio form for a general DEA model. This is done by using Lagrangean dual formulation instead of the standard LP dual because it can be formulated for more general DEA model all at once. In the ratio form, inputs and outputs are weighted instead of DMUs, which offers a different perspective for DEA; The ratio form is not computationally attractive for solving the efficiencies, but it offers a more intuitive interpretation for efficiency. It also makes it possible to use preference information about relative values of different inputs and outputs, and develop ratio-based efficiency analysis in the portfolio framework.

3.1 Ratio form

As in the CCR case, the linear programming duality makes it possible to calculate the f-efficiency score alternatively as a problem in which the output–input ratio is maximized. In what follows, we assume that Λ^f is connected, since the optimum value of the Lagrangean dual in discrete cases is not necessarily equal to the optimum value of the original LP problem.

Theorem 1. *Let Λ^f be a connected set. Then, the optimal value $\hat{E}^f$ of the minimization problem (2) is equal to the optimal value of the following maximization problem:*

$$\begin{aligned} &\max_{v,u,u_0} \frac{u^T y - u_0}{v^T x} \\ &s.t.\ \frac{u^T Y\lambda - u_0}{v^T X\lambda} \leq 1, \quad \forall \lambda \in \Lambda^f \\ &\qquad v, u \geq 0, u_0 \in \mathbb{R}. \end{aligned} \tag{5}$$

Proof. First, the Lagrangean dual formulation (e.g., Bertsimas and Tsitsiklis 1997) gives

$$\hat{E}^f = \max_{p\geq 0} \min_{\alpha\in\mathbb{R},\lambda\in\Lambda^f} \left[\alpha + p^T\left(b - A\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}\right)\right],$$

$$\text{where} \quad p = \begin{bmatrix}\nu\\ \mu\end{bmatrix}, A = \begin{bmatrix}x & -X\\ 0 & Y\end{bmatrix}, b = \begin{bmatrix}0\\ y\end{bmatrix}$$

are the dual vector, the constraint matrix and the constraint vector in the problem (2), respectively. By calculating the matrix products above, we get

$$\begin{aligned} \hat{E}^f &= \max_{\nu,\mu\geq 0} \min_{\alpha\in\mathbb{R},\lambda\in\Lambda^f} \left[\alpha + [\ \nu^T \quad \mu^T\ ]\left(\begin{bmatrix}0\\ y\end{bmatrix} - \begin{bmatrix}x & -X\\ 0 & Y\end{bmatrix}\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}\right)\right] \\ &= \max_{\nu,\mu\geq 0} \min_{\alpha\in\mathbb{R},\lambda\in\Lambda^f} \left[\alpha + 0 - \nu^T x\alpha + \nu^T X\lambda + \mu^T y - 0 - \mu^T Y\lambda\right] \\ &= \max_{\nu,\mu\geq 0} \left[\mu^T y + \min_{\alpha\in\mathbb{R}} [(1-\nu^T x)\alpha] + \min_{\lambda\in\Lambda^f} [\nu^T X\lambda - \mu^T Y\lambda]\right]. \end{aligned} \tag{6}$$

If $1 - \nu^T x \neq 0$ in the optimum, then $\min_{\alpha\in\mathbb{R}}[(1-\nu^T x)\alpha] = -\infty$, which clearly does not maximize (6). Thus, we get a new equality constraint $\nu^T x = 1$. The another minimization can be removed by introducing a new variable $\mu_0 := \min_{\lambda\in\Lambda^f}[\nu^T X\lambda - \mu^T Y\lambda]$. Because μ_0 is defined as a minimum value, we get a new inequality constraint: $\mu_0 \leq \nu^T X\lambda - \mu^T Y\lambda,\ \forall\lambda \in \Lambda^f$. After these changes we have

$$\begin{aligned} \hat{E}^f = \max_{\nu,\mu,\mu_0} & \ \mu^T y + \mu_0 \\ \text{s.t. } & \ \nu^T x = 1 \\ \nu^T X\lambda - \mu^T Y\lambda & \geq \mu_0, \quad \forall \lambda \in \Lambda^f \\ \nu, \mu & \geq 0, \mu_0 \in \mathbb{R}. \end{aligned} \tag{7}$$

Finally, we make a change of variables $\nu = \frac{v}{v^T x}$, $\mu = \frac{u}{v^T x}$ and $\mu_0 = \frac{-u_0}{v^T x}$. This scaling makes the first constraint hold automatically, and thus it can be removed. The change of variables leads to

$$\begin{aligned} \hat{E}^f = \max_{v,u,u_0} & \ \frac{u^T y - u_0}{v^T x} \\ \text{s.t. } v^T X\lambda - u^T Y\lambda & \geq -u_0, \quad \forall \lambda \in \Lambda^f \\ v, u & \geq 0, u_0 \in \mathbb{R}. \end{aligned} \tag{8}$$

This is the same as the problem (5), which can be seen by adding $u^T Y\lambda$ and dividing by $v^T X\lambda$ in both sides of the inequality constraint. □

In the problem (5) in Theorem 1, we have to assume that the denominators $v^T x$ and $v^T X\lambda$ are nonzero. If these assumptions do not hold, (5) can always be replaced with (8) or (7). The problem of the form (7) is often called the multiplier form of the DEA model (e.g., Cooper et al. 2000).

In (5), we are looking for the best feasible weights for inputs and outputs to maximize the ratio of total output and total input. In addition to input and output weight vectors v and u, the ratio includes a scalar weight u_0 for shifting the total output. The inequality constraint means that the shifted output–input ratio is scaled so that it is less or equal to 1 for all input–output combinations $(X\lambda, Y\lambda)$, where $\lambda \in \Lambda^f$.

The problem (5) has infinitely many constraints, but it is sufficient to deal only with the constraints in which λ is an extreme point of the set Λ^f.

Corollary 1. *Let Λ^f be connected and bounded. Then, in the inequality constraint of the problem (5), $\forall \lambda \in \Lambda^f$ can be replaced with $\forall \lambda \in ext(\Lambda^f)$.*

Proof. Assume that the constraint holds for all $\lambda' \in \text{ext}(\Lambda^f)$ and let $\lambda \in \Lambda^f \setminus \text{ext}(\Lambda^f)$. Then, λ can be represented as a convex combination of some extreme points:

$$\lambda = \sum_{k=1}^{K} t_k \lambda^k, \quad \lambda^k \in \text{ext}(\Lambda^f), \quad t_k > 0, \quad \sum_{k=1}^{K} t_k = 1, \quad k = 1, \ldots, K.$$

Because the constraint holds for all extreme points, we have

$$\frac{u^T Y \lambda^k - u_0}{v^T X \lambda^k} \leq 1 \quad \text{or} \quad u^T Y \lambda^k - u_0 \leq v^T X \lambda^k, \quad k = 1, \ldots, K.$$

By multiplying the latter inequalities with t_k and summing them over k from 1 to K, we get

$$\begin{aligned} \sum_{k=1}^{K} (t_k u^T Y \lambda^k - t_k u_0) &\leq \sum_{k=1}^{K} t_k v^T X \lambda^k \\ u^T Y \sum_{k=1}^{K} t_k \lambda^k - u_0 \sum_{k=1}^{K} t_k &\leq v^T X \sum_{k=1}^{K} t_k \lambda^k \\ u^T Y \lambda - u_0 &\leq v^T X \lambda \\ \frac{u^T Y \lambda - u_0}{v^T X \lambda} &\leq 1. \end{aligned}$$

Thus, the inequality constraint also holds for all $\lambda \in \Lambda^f \setminus \mathrm{ext}(\Lambda^f)$ resulting that it is sufficient to assume the constraint only in the extreme points. □

Corollary 1 reduces the number of constraints in the problem (5) to a finite number when the set Λ^f is bounded. For example, $|\mathrm{ext}(\Lambda^{\mathrm{BCC}})| = m$ and $|\mathrm{ext}(\Lambda^{\mathrm{CP}})| = \mathcal{O}(2^m)$. Even if the number of constraints is finite in the CP case, it increases rapidly when the number of DMUs increases. Thus, Corollary 1 does not in the general case result in computationally attractive optimization models.

By Corollary 1, the only difference between the general problem (5) and the BCC problem (4) is, in which points their constraints hold. Because the constraint of (5) holds in $\mathrm{ext}(\Lambda^f)$ and the constraint of (4) in the DMU points, (5) can be seen as a BCC problem with the points $\{(X\lambda, Y\lambda) \mid \lambda \in \mathrm{ext}(\Lambda^f)\}$ as DMUs. That is, T^f over J is equivalent to T^{BCC} over $\mathrm{ext}(\Lambda^f)$. This is a generalization of Green and Cook's result (2004) in which they show that T^{K} over J is equivalent to T^{BCC} over 2^J (the power set of J).

If the set Λ^f is unbounded, it can be approximated by a sequence of bounded sets. Thus, the argumentation in the proof of Corollary 1 also holds for unbounded Λ^f when the points that are 'arbitrarily far' from the origin are considered as extreme points.

After introducing Corollary 1, it is quite straightforward to see that Theorem 1 also contains the DEA models CCR and BCC as its special cases. It can be reduced to the CCR and BCC cases in the following way. When f = CCR, we can choose $\lambda = te^k,\ t > 0,\ k = 1, \ldots, m$. Then, the inequality constraint changes as follows:

$$\frac{u^T Y te^k - u_0}{v^T X te^k} \leq 1, \quad t > 0, \quad k = 1, \ldots, m$$
$$\frac{tu^T y^k - u_0}{tv^T x^k} \leq 1, \quad t > 0, \quad k = 1, \ldots, m$$
$$\frac{u^T y^k - \frac{1}{t} u_0}{v^T x^k} \leq 1, \quad t > 0, \quad k = 1, \ldots, m.$$

When $t \to 0$, we see that $u_0 \geq 0$. When $t \to \infty$, instead, we get

$$\frac{u^T y^k}{v^T x^k} \leq 1, \quad k = 1, \ldots, m.$$

Finally, because the only constraint left containing u_0 is $u_0 \geq 0$, and the optimization problem is the maximization

$$\max_{v,u,u_0} \frac{u^T y - u_0}{v^T x},$$

u_0 must be equal to 0 in the optimum. Thus, we ended up to the ratio form of the standard CCR model (3). The BCC model (4), instead, can be derived directly from (5) by choosing the extreme points of Λ^{BCC}: $\lambda = e^k,\ k = 1, \ldots, m$.

In general, Theorem 1 does not hold for discrete Λ^f. However, the problem (5) gives a lower bound for (2), since the minimum value of an ILP problem is always greater or equal to the optimum value of its Lagrangean dual relaxation (e.g., Bertsimas and Tsitsiklis 1997).

3.2 Preference information in ratio form

The ratio form (5) allows us to include preference information about the relative values of inputs and outputs in the model. Because it consists of weighted sums of inputs and outputs, these weights can easily be limited with linear constraints (e.g., Salo and Punkka 2011). For example, if one unit of output 1 is considered to be at least as valuable as a unit of output 2 but not more valuable than three units of output 2, then the constraints $u_2 \leq u_1 \leq 3u_2$ must hold. There is no need to limit the variable u_0 in this context, since it corresponds to the constraints of Λ^f in the envelopment form.

Definition 6. *The set of feasible input and output weights are*

$$\begin{aligned} S_v &= \{v \in \mathbb{R}^s \,|\, v \neq 0, v \geq 0, A_v v \leq 0\}, \\ S_u &= \{u \in \mathbb{R}^n \,|\, u \neq 0, u \geq 0, A_u u \leq 0\}, \end{aligned}$$

respectively, where A_v and A_u are coefficient matrices determining how valuable different amounts of inputs and outputs are.

After introducing the sets of feasible input and output weights in Definition 6, the ratio form (5) can be written as follows:

$$
\begin{aligned}
&\max_{v,u,u_0} \frac{u^T y - u_0}{v^T x} \\
&\text{s.t.}\ \frac{u^T Y\lambda - u_0}{v^T X\lambda} \leq 1, \quad \forall \lambda \in \Lambda^f \\
&A_v v \leq 0, A_u u \leq 0 \\
&v, u \geq 0, u_0 \in \mathbb{R}.
\end{aligned} \tag{9}
$$

The problem (9) is difficult to solve, since it is not a LP problem. However, it can be converted back into its envelopment form which is a LP problem.

Corollary 2. *The optimum value of the problem (9) is equal to the optimum value of the following minimization problem:*

$$
\begin{aligned}
&\min_{\alpha,\lambda,\sigma,\tau} \alpha \\
&s.t.\ x\alpha - X\lambda + A_v^T\sigma \geq 0 \\
&Y\lambda + A_u^T\tau \geq y \\
&\alpha \in \mathbb{R}, \lambda \in \Lambda^f, \sigma, \tau \geq 0.
\end{aligned} \tag{10}
$$

Proof. The claim is easier to show by starting from the problem (10). By following the same steps for the optimum value α^* as in the proof of Theorem 1, we get:

$$
\begin{aligned}
\alpha^* &= \max_{\nu,\mu\geq 0} \min_{\alpha,\lambda,\sigma,\tau} \left[\alpha + \begin{bmatrix} \nu^T & \mu^T \end{bmatrix} \left(\begin{bmatrix} 0 \\ y \end{bmatrix} - \begin{bmatrix} x & -X & A_v^T & 0 \\ 0 & Y & 0 & A_u^T \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda \\ \sigma \\ \tau \end{bmatrix} \right)\right] \\
&= \max_{\nu,\mu\geq 0} \min_{\alpha,\lambda,\sigma,\tau} \left[\alpha - \nu^T x\alpha + \nu^T X\lambda - \nu^T A_v^T\sigma + \mu^T y - \mu^T Y\lambda - \mu^T A_u^T\tau\right] \\
&= \max_{\nu,\mu\geq 0} \left[\mu^T y + \min_{\alpha\in\mathbb{R}} [(1-\nu^T x)\alpha] + \min_{\lambda\in\Lambda^f} [\nu^T X\lambda - \mu^T Y\lambda] + \min_{\sigma\geq 0} [-\nu^T A_v^T\sigma] \right. \\
&\quad \left. + \min_{\tau\geq 0} [-\mu^T A_u^T\tau]\right].
\end{aligned} \tag{11}
$$

If the vector $-\nu^T A_v^T$ has a strictly negative component in the optimum, then $\min_{\sigma\geq 0} [-\nu^T A_v^T\sigma] = -\infty$, which clearly does not maximize (11). Thus, $-\nu^T A_v^T \geq 0^T$ and

$-\nu^T A_v^T \sigma = 0$ in the optimum. The inequality $-\nu^T A_v^T \geq 0^T$ is equivalent to $A_v \nu \leq 0$, which is equivalent to $A_v v \leq 0$, where v is chosen such that $\nu = \frac{v}{v^T x}$. Analogously, from the term $\min_{\tau \geq 0} [-\mu^T A_u^T \tau]$ we get $A_u u \leq 0$ when $\mu = \frac{u}{v^T x}$. The rest of the proof continues in the same way as in Theorem 1, which leads to (9). □

In practise, the efficiency scores with preference information can easily be solved using the LP problem (10). It can also be used when Λ^f is discrete, but its interpretation is not as clear as in the connected case, since the equivalence between (2) and (5) does not necessarily hold.

4 Portfolio dominance

In this section, we formulate a more general concept of efficiency in portfolio production possibility sets, and generalize the efficiency ratio from the previous section. This allows us to develop ratio-based efficiency analysis in the portfolio framework proceeding in the same way as Salo and Punkka (2011) with standard REA. Especially, we introduce the concept of portfolio dominance, which is a generalization of the DMU dominance analyzed by Salo and Punkka.

When dealing with all feasible DMU combinations, we call an individual point of the production possibility set a portfolio. Also the DMU points are called portfolios.

Definition 7. *A portfolio is any point* $p = (x, y) \in T^f$.

Thus, the efficient portfolio frontier is determined by Definition 3, and the efficiency score of a portfolio can be calculated using (2) or (10) depending on if preference information about inputs and outputs is included. Furthermore, Definition 5 determines the reference portfolio for any portfolio.

We continue considering the same feasible weights (v, u, u_0) as in the problem (9). That is, the weights v and u can contain preference information about relative values of inputs and outputs, whereas the scalar weight u_0 is free.

Definition 8. *The general set of feasible weights is*

$$S := \{(v, u, u_0) \in S_v \times S_u \times \mathbb{R}\}.$$

In this context, we could also introduce some constraints for the scalar weight u_0. This would be sensible if the new constraints somehow reflected the shape of the production possibility set. In the CCR case, for example, the weight u_0 is always 0 in the optimum of (9), and thus $u_0 = 0$ is a natural constraint. Adding this constraint leads to standard ratio-based efficiency analysis (e.g., Salo and Punkka 2011).

After introducing the set of feasible weights, the shifted output–input ratio maximized in the problem (9) can be represented as a function of $(x, y) \in T^f$ and $(v, u, u_0) \in S$.

Definition 9. *Let $p = (x, y) \in T^f$, $(v, u, u_0) \in S$ and $v^T x > 0$. The generalized efficiency ratio $E : T^f \times S \to \mathbb{R}$ is*

$$E_p(v, u, u_0) = \frac{u^T y - u_0}{v^T x}.$$

However, since u_0 is free, this ratio can be negative or zero in some points of S, which has no sensible meaning. Especially, the efficiency ratios of two different portfolios are not directly comparable when they have negative values. Therefore, we restrict the domain S to such points in which the generalized efficiency ratio is positive.

Definition 10. *The set of feasible weights for portfolio $p \in T^f$ is*

$$S_p := \{(v, u, u_0) \in S \mid E_p(v, u, u_0) > 0\}.$$

When considering the generalized efficiency ratio $E_p(v, u, u_0)$ restricted to S_p, it achieves any feasible values of the shifted output–input ratio in which the shifted output is positive.

The efficiency score $\hat{E}(p)$ of the portfolio p calculated with (9) or (10) is the maximum value of $E_p(v, u, u_0)$ over S_p. The generalized efficiency ratio includes more information about portfolio efficiencies than the corresponding efficiency score, since it notices all feasible weights, not only those weights in which the ratio achieves its maximum.

The generalized efficiency ratio allows us to determine a dominance relation between two portfolios. In this case, the concept of dominance is sensible only when both portfolios have positive efficiency ratios.

Definition 11. *A portfolio $p \in T^f$ dominates another portfolio $q \in T^f$ if and only if*

$$\begin{aligned} E_p(v, u, u_0) &\geq E_q(v, u, u_0) \qquad \text{for all } (v, u, u_0) \in S_p \cap S_q, \\ E_p(v, u, u_0) &> E_q(v, u, u_0) \qquad \text{for some } (v, u, u_0) \in S_p \cap S_q. \end{aligned}$$

This dominance is denoted by $p \succ q$.

If $p \succ q$, the efficiency ratio of portfolio p is at least as high as that of portfolio q for all feasible weights, and moreover, there exist some weights for which its efficiency ratio is strictly higher. By definition, the relation $\succ$ is clearly an irreflexive and asymmetric binary relation in T^f. In addition, if $p \succ q$, then $S_q \subset S_p$, which makes also the transitivity of $\succ$ easy to see. Thus, $\succ$ is a strict partial order.

Definition 11 catches only the most obvious dominance relations among portfolios because the scalar weight u_0 is not limited more than necessary. For example, if portfolio q has higher output values than portfolio p, we cannot have $p \succ q$, no matter how much higher input values q has. When adding more constraints for u_0, the set of feasible weights S becomes smaller, and thus new dominances may appear. In our case, when the weight u_0 is not limited, Definition 11 can be reduced to a form in which outputs and inputs are considered independently. Before proving this, we need the following lemma.

Lemma 1. *Let* $x, x' > 0, y, y' \geq 0$.

If $y < y'$ *or* $x > x'$, *then* $\dfrac{y - u_0}{x} < \dfrac{y' - u_0}{x'}$ *for some* $u_0 \in \{u_0 \in \mathbb{R} \mid y - u_0 > 0, y' - u_0 > 0\}$.

Proof. First, we prove the case $y < y'$. If $x \geq x'$, we can choose $u_0 = 0$, and the claim clearly follows. Then, assume $x < x'$ and choose $u_0 = \frac{2x'y - xy'}{2x' - x}$. This leads to

$$\frac{y - u_0}{x} = \frac{1}{x}\frac{(2x' - x)y - (2x'y - xy')}{2x' - x} = \frac{1}{x}\frac{-xy + xy'}{2x' - x} = \frac{y' - y}{2x' - x} > 0$$

and

$$\frac{y' - u_0}{x'} = \frac{1}{x'}\frac{(2x' - x)y' - (2x'y - xy')}{2x' - x} = \frac{1}{x'}\frac{2x'y' - 2x'y}{2x' - x} = 2\,\frac{y' - y}{2x' - x}.$$

Thus, $\frac{y - u_0}{x} < \frac{y' - u_0}{x'}$ and $u_0 \in \{u_0 \in \mathbb{R} \mid y - u_0 > 0, y' - u_0 > 0\}$.

Then, we prove the case $x > x'$. If $y \leq y'$, we can choose any $u_0 < 0$, and the claim clearly follows. Then, assume $y > y'$ and choose $u_0 = \frac{xy' - x'y - xx'}{x - x'}$. This leads to

$$\frac{y - u_0}{x} = \frac{1}{x}\frac{(x - x')y - (xy' - x'y - xx')}{x - x'} = \frac{1}{x}\frac{xy - xy' + xx'}{x - x'} = \frac{y - y' + x'}{x - x'} > 0$$

and

$$\frac{y' - u_0}{x'} = \frac{1}{x'}\frac{(x - x')y' - (xy' - x'y - xx')}{x - x'} = \frac{1}{x'}\frac{-x'y' + x'y + xx'}{x - x'} = \frac{y - y' + x}{x - x'}.$$

Thus, $\frac{y - u_0}{x} < \frac{y' - u_0}{x'}$ and $u_0 \in \{u_0 \in \mathbb{R} \mid y - u_0 > 0, y' - u_0 > 0\}$. □

Theorem 2. *Let $p = (x^p, y^p)$, $q = (x^q, y^q) \in T^f$. Then, $p \succ q$ is equivalent to the conjunction of the following three conditions:*

$$u^T y^p \geq u^T y^q \quad \text{for all } u \in S_u, \tag{12}$$
$$v^T x^p \leq v^T x^q \quad \text{for all } v \in S_v, \tag{13}$$
$$u^T y^p > u^T y^q \quad \text{or} \quad v^T x^p < v^T x^q \quad \text{for some } (v, u) \in S_v \times S_u. \tag{14}$$

Proof. First, assume that (12), (13) or (14) does not hold. This implies that either $u^T y^p = u^T y^q$ and $v^T x^p = v^T x^q$ for all $(v, u) \in S_v \times S_u$, or there exists $(v, u) \in S_v \times S_u$ such that $u^T y^p < u^T y^q$ or $v^T x^p > v^T x^q$. In the former case, $\frac{u^T y^p - u_0}{v^T x^p} = \frac{u^T y^q - u_0}{v^T x^q}$ for all $(v, u, u_0) \in S_p \cap S_q$, and thus $p \not\succ q$. In the latter case, we apply Lemma 1 for $x = v^T x^p$, $y = u^T y^p$, $x' = v^T x^q$ and $y' = u^T y^q$, which gives that there exists $u_0 \in \{u_0 \in \mathbb{R} \mid u^T y^p - u_0 > 0, u^T y^q - u_0 > 0\}$ such that $\frac{u^T y^p - u_0}{v^T x^p} < \frac{u^T y^q - u_0}{v^T x^q}$. Since $(v, u, u_0) \in S_p \cap S_q$, we have $p \not\succ q$.

Then, assume that (12), (13) and (14) hold. (12) and (13) clearly imply that $\frac{u^T y^p - u_0}{v^T x^p} \geq \frac{u^T y^q - u_0}{v^T x^q}$ for all $(v, u, u_0) \in S_p \cap S_q$. On the other hand, (14) gives a point $(v, u) \in S_v \times S_u$ for which the strict inequality $\frac{u^T y^p - u_0}{v^T x^p} > \frac{u^T y^q - u_0}{v^T x^q}$ clearly holds when $u_0 < 0$. Thus, $p \succ q$. □

By Theorem 2, the dominance between two portfolios in Definition 11 can easily be determined by linear programming.

Corollary 3. *Let $p = (x^p, y^p)$, $q = (x^q, y^q) \in T^f$. Then, $p \succ q$ is equivalent to the conjunction of the following three conditions:*

$$\min_{u \in S_u} u^T (y^p - y^q) \geq 0, \tag{15}$$
$$\max_{v \in S_v} v^T (x^p - x^q) \leq 0, \tag{16}$$
$$\max_{u \in S_u} u^T (y^p - y^q) > 0 \text{ or } \min_{v \in S_v} v^T (x^p - x^q) < 0. \tag{17}$$

First, if (15) and (16) hold and at least another of them has a strict inequality, then p dominates q. Second, if (15) or (16) does not hold, p does not dominate q. Third, if the equality holds in both (15) and (16), then the condition (17) determines the dominance. Furthermore, if we want to compute the whole dominance structure for a finite number of portfolios, for example, for the DMU points, the asymmetricity and transitivity properties can be applied. Thus, we do not need to compare all actual pairs of portfolios.

5 Application to analysis of healthcare units

In this section, we apply our portfolio DEA model to efficiency analysis of healthcare units. We use the same data as Hynninen et al. (2011) in their analysis. The data consists of real observed data about healthcare units in Finland provided by THL (National Institute for Health and Welfare).

The part of the data set we are considering includes 33 DMUs that have three inputs and one output. The inputs are 'number of dentists', 'number of dental hygienists' and 'number of dental assistants', whereas the output is 'weighted sum of operations completed' in which the operations are weighted by cost factors to take account varying expenses between operation types. DMUs are categorized based on the output variable 'weighted sum of operations completed' such that DMUs in category A have output more than 200 000, in category C less than 100 000, and category B contains the rest of the DMUs. The DMU C12 has been left out from the analysis, since its input values are 0. The data is presented in Appendix 1.

First, we compare efficiency scores calculated using different DEA models. We have chosen five different CP models in our analysis, and these are compared with each others and the CCR and BCC models. In these CP models, every DMU has the same upper bound $c_j = c,\ j = 1, \ldots, 33$. The upper bounds c and the sum bounds b of the CP models are presented in Table 1.

Table 1: Different CP models

Bound	CP1	CP2	CP3	CP4	CP5
c	1	1	1	2	1
b	1	1.1	2	2	∞

The only difference between the model CP1 and the BCC model is that CP1 has an inequality $\leq$ instead of the equality in the sum bound constraint. Mainly for this reason, the model CP1 also occurs in the literature with the name NIRS for non-increasing returns to scale (e.g., Green and Cook 2004). The model CP5, instead, is the same as the model 'K' mentioned in section 2.2.

The efficiency scores have been calculated both without and with preference information about relative values of the input weights. When dealing with preference information, we have used the same weight restrictions as Hynninen et al. (2011), but we do not take a stance on if these restrictions are sensible. The restriction constraints are

$$1 \le \frac{\text{dentists}}{\text{assistants}} \le 5, \quad 1 \le \frac{\text{dentists}}{\text{hygienists}} \le 5 \quad \text{and} \quad 0.5 \le \frac{\text{hygienists}}{\text{assistants}} \le 5. \tag{18}$$

The efficiency scores of the DMUs in different models are presented in Appendices 2 and 3. These efficiency scores mainly behave in the same way in both restricted and non-restricted case. The only significant difference is that some DMUs are efficient in the non-restricted case and inefficient in the restricted case. This shows that the chosen restriction constraints have effect on the results.

For simplicity, we compare the efficiency scores of different models only in the restricted case because it has fewer efficient DMUs. To be more illustrative, these efficiency scores are also presented as percents (rounded down) in Table 2. Three DMUs, B2, B8 and C2, are efficient in every model, and these all have middle scale. In addition, DMUs A1, A2 and C17 are BCC efficient, and A1 and A2 CP1 efficient. The results give a good example that variable returns to scale (e.g., BCC) gives higher efficiency scores for small-scale DMUs than non-increasing returns to scale (e.g., CP): BCC scores are equal to CP1 scores for DMUs with large scale (A1–B2), but clearly higher for small-scale DMUs (C3–C17).

When increasing the upper bounds b and c and moving from CP1 to CP5, only the efficiency scores of the largest DMUs change. This is because all CP models have non-increasing returns to scale, and thus only large-scale DMUs are scored differently by these models. In the model CP2, the upper bound b (1.1) is only a little higher than in CP1 (1.0), but even this little change makes the largest DMUs A1 and A2 inefficient.

The only DMU whose efficiency score remarkably differs in the models CP2–CP5, is the largest DMU A1. Its CP2 efficiency score 0.9668 is 38 % better than its CP5 score 0.699. The model CP4 can score a DMU better or worse than CP5: A1 has better CP4 score (0.7085) than CP5 score (0.699), whereas A2 has better CP5 score (0.9755) than CP4 score (0.9737). Anyway, CP4 and CP5 scores do not significantly differ from the corresponding CCR scores.

Table 2: Efficiency scores (%) in different models with weight restrictions

DMU	BCC	CP1	CP2	CP3	CP4	CP5	CCR
A1	100	100	96	79	70	69	68
A2	100	100	99	98	97	97	96
A3	72	72	71	70	69	70	69
A4	78	78	78	77	77	77	77
A5	82	82	81	80	80	80	80
A6	58	58	58	58	57	58	57
A7	66	66	66	65	65	65	65
B1	99	99	99	98	98	98	98
B2	100	100	100	100	100	100	100
B3	88	88	88	88	88	88	88
B4	99	99	98	98	98	98	98
B5	96	96	96	96	96	96	96
B6	74	73	73	73	73	73	73
B7	81	81	81	81	81	81	81
B8	100	100	100	100	100	100	100
B9	81	81	81	81	81	81	81
B10	88	87	87	87	87	87	87
C1	84	83	83	83	83	83	83
C2	100	100	100	100	100	100	100
C3	84	82	82	82	82	82	82
C4	77	75	75	75	75	75	75
C5	83	80	80	80	80	80	80
C6	85	82	82	82	82	82	82
C7	78	75	75	75	75	75	75
C8	92	90	90	90	90	90	90
C9	63	59	59	59	59	59	59
C10	83	78	78	78	78	78	78
C11	90	84	84	84	84	84	84
C13	77	71	71	71	71	71	71
C14	91	82	82	82	82	82	82
C15	86	75	75	75	75	75	75
C16	95	80	80	80	80	80	80
C17	100	87	87	87	87	87	87

Then, we compare reference portfolios in the models CP1–CP5 and CCR. For simplicity, this is done in the non-restricted case. Most DMUs have no differences in their reference portfolios with different models, but the largest DMUs do. The preference portfolios of four largest DMUs, A1–A4, are presented in Table 3. The effect of the constraints in different CP models can be seen, for example, by considering the reference portfolios of the DMU A1. In the CCR model, the reference portfolio of A1 consists of 3.22 copies of the DMU B2 and 1.35 copies of the DMU B4. When moving to the model CP5 by adding the constraint that a portfolio cannot contain more than one copy of each DMU, the reference portfolio of A1 gets three new DMUs, A2, B8 and C2. The DMUs B2, B4 and B8 are included in this portfolio as a whole.

Table 3: Reference portfolios of four largest DMUs when there are no weight restrictions

Model	A1	A2
CCR	3.22B2+1.35B4	1.79B2+0.61B4
CP5	0.69A2+B2+B4+B8+0.64C2	B2+B4+0.28B8+0.66C2
CP4	1.80A2+0.04A5+0.17B4	0.29A2+1.28B2+0.44B4
CP3	0.28A1+A2+0.15A5+0.57B2	0.36A2+B2+0.50B4+0.14C2
CP2	0.89A1+0.21A2	0.93A2+0.13B2+0.04B4
CP1	A1	A2
Model	A3	A4
CCR	1.33B2+0.87B4	0.80B2+0.95B4
CP5	B2+B4+0.16B8+0.26C2	0.80B2+0.95B4
CP4	0.14A2+1.08B2+0.78B4	0.80B2+0.95B4
CP3	0.16A2+B2+0.80B4+0.04C2	0.80B2+0.95B4
CP2	0.75A2+0.04A5+0.31B4	0.45A2+0.03A5+0.63B4
CP1	0.76A2+0.12A5+0.12B4	0.46A2+0.10A5+0.44B4

In the models, CP1–CP4, instead, the total number of DMU copies is limited, and these limits are achieved in the corresponding optimums. Thus, the reference portfolios of A1 mainly consist of the largest DMUs in these models. When the constraints in the model get tighter, fewer small DMUs are included in the reference portfolio. In the model CP1, the reference portfolio of A1 finally contains only the DMU A1 itself. That is, A1 is efficient in this model.

Finally, we compare the number of dominating DMUs determined by Definition 11 and the standard REA (e.g., Salo and Punkka 2011). In the former case, the scalar weight u_0 is free, whereas the latter case includes the restriction $u_0 = 0$. The number of dominated DMUs has been calculated for each DMU both without and with preference information (18). The results are presented in Appendix 4.

The results show how the free-scalar-weight (FSW) case accepts only the most obvious dominance relations, whereas REA dominances occur quite a lot; When there is no preference information, REA includes 25 dominated DMUs, whereas only 7 DMUs are dominated in the FSW case. When including the preference information, more dominance relations appear, and the corresponding numbers are 30 and 14. In addition, the scale of DMUs affects differently the REA and FSW dominances, which can be seen, especially, when the preference information is included; In the REA case, DMUs in large (A1–A7) and small scale (C4–C17) have significantly more dominating DMUs than in middle scale, whereas the scale has no remarkable effect in the FSW case.

6 Conclusions

In this study, we have extended DEA theory for more general production possibility sets. The efficiency scores can be defined and calculated as a (I)LP problem for many kinds of DEA models using the envelopment form. The solution to this problem also suggests a strictly more efficient DMU combination for an inefficient DMU or portfolio. In the case of connected production possibility set, DEA models also have a ratio form that allows us to access input and output weights. Through the ratio form, preference information about relative values of inputs and outputs can easily be added as linear constraints also in the envelopment form of the model. In addition, we have considered general DEA efficiency in the REA framework by defining generalized efficiency ratio and introducing the concept of dominance.

We have mainly focused on the connected production possibility sets, since the ratio form is not sensible in discrete cases because it is not equivalent to the envelopment form. Thus, it would be of interest to examine the DEA theory more carefully for discrete production possibility sets. For example, we have seen that preference information about relative values of inputs and outputs can also be included in the discrete case, but its effects has not been analyzed. Also, discrete portfolio DEA models possibly have observable correspondences with portfolio decision analysis (e.g, Liesiö et al. 2008).

In addition, the concept of dominance could possibly be extended to reflect the shape of the production possibility set by restricting the free scalar weight. This is since the non-restricted weight catches only the most obvious dominance relations, whereas the zero weight totally ignores different scales of DMUs. Especially, restricting the weight correctly would lead to extend REA in the BCC case. However, restricting the weight sensibly seems to be difficult, particularly, if solving the dominance is wanted to remain a LP problem. Finally, it would be of interest to examine how the results change when building the theory on output-oriented DEA instead of input-oriented.

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Appendices

- Appendix 1: Data set
- Appendix 2: Efficiency scores without preference information
- Appendix 3: Efficiency scores with preference information
- Appendix 4: Number of dominating DMUs

Appendix 1: Data set

Input and output values of each DMU; 'Dentists': number of dentists, 'Hygienists': number of dental hygienists, 'Assistants': number of dental assistants, 'W. Sum': weighted sum of operations completed.

	Input			Output
DMU	Dentists	Hygienists	Assistants	W. Sum
A1	153	72.2	325.3	720093
A2	61	29.3	97.8	383983
A3	74.3	33.5	125.4	335847
A4	52.8	22.1	83.1	255905
A5	51.5	15	82	242502
A6	69	22	99	232321
A7	46.4	19.4	104.2	206113
B1	28.2	16.4	45	186201
B2	25.4	13.5	42.4	171424
B3	24.8	14	41	147874
B4	22.5	7	27.5	124524
B5	20.9	7.9	29.8	121078
B6	23.5	12	38	114151
B7	19.6	9.3	34.7	105407
B8	17	7	21	101510
B9	20.9	8	27.3	101301
B10	20.1	5.7	29.9	100976
C1	20.3	5.3	23.3	91062
C2	11.1	9.6	28	90271
C3	16.5	6	30.5	85070
C4	18.5	6	32.5	84606
C5	16.8	5.8	26.4	80949
C6	16	6.6	22.3	80322
C7	17.2	6	28.3	78929
C8	12.3	8.6	19.2	74428
C9	20	7	31	71949
C10	14.2	7	17	66338
C11	13.8	5.8	15.1	65953
C13	15	4	21	60085
C14	9.7	5.8	12.5	49912
C15	10.5	5	13	48061
C16	8	2	18	39455
C17	5.4	4.2	9.5	33247

Appendix 2: Efficiency scores without preference information

Efficiency scores of each DMU in different DEA models without preference information.

	DEA model						
DMU	BCC	CP1	CP2	CP3	CP4	CP5	CCR
A1	1	1	0.9727	0.8243	0.7536	0.7457	0.733
A2	1	1	0.9981	0.9847	0.9811	0.9833	0.9734
A3	0.7435	0.7435	0.7372	0.721	0.7201	0.7205	0.717
A4	0.8177	0.8177	0.8085	0.7911	0.7911	0.7911	0.7911
A5	1	1	0.9904	0.9211	0.9077	0.8956	0.882
A6	0.6895	0.6895	0.6758	0.6162	0.6037	0.6135	0.6037
A7	0.7362	0.7362	0.7336	0.7254	0.7254	0.7254	0.7254
B1	1	1	0.9989	0.9978	0.9978	0.9978	0.9978
B2	1	1	1	1	1	1	1
B3	0.8884	0.8873	0.8873	0.8873	0.8873	0.8873	0.8873
B4	1	1	1	1	1	1	1
B5	0.9849	0.9814	0.9814	0.9814	0.9814	0.9814	0.9814
B6	0.7451	0.7329	0.7329	0.7329	0.7329	0.7329	0.7329
B7	0.8464	0.8358	0.8358	0.8358	0.8358	0.8358	0.8358
B8	1	1	1	1	1	1	1
B9	0.83	0.8227	0.8227	0.8227	0.8227	0.8227	0.8227
B10	0.9853	0.9853	0.9797	0.9684	0.9684	0.9684	0.9684
C1	0.9912	0.9562	0.9562	0.9562	0.9562	0.9562	0.9562
C2	1	1	1	1	1	1	1
C3	0.8978	0.8853	0.8853	0.8853	0.8853	0.8853	0.8853
C4	0.8261	0.8156	0.8156	0.8156	0.8156	0.8156	0.8156
C5	0.857	0.842	0.842	0.842	0.842	0.842	0.842
C6	0.8554	0.8268	0.8268	0.8268	0.8268	0.8268	0.8268
C7	0.8136	0.7991	0.7991	0.7991	0.7991	0.7991	0.7991
C8	0.9454	0.9231	0.9231	0.9231	0.9231	0.9231	0.9231
C9	0.6426	0.6258	0.6258	0.6258	0.6258	0.6258	0.6258
C10	0.8867	0.8073	0.8073	0.8073	0.8073	0.8073	0.8073
C11	0.994	0.9036	0.9036	0.9036	0.9036	0.9036	0.9036
C13	0.9068	0.8211	0.8211	0.8211	0.8211	0.8211	0.8211
C14	0.9846	0.8481	0.8481	0.8481	0.8481	0.8481	0.8481
C15	0.9428	0.7659	0.7659	0.7659	0.7659	0.7659	0.7659
C16	1	1	1	1	1	1	1
C17	1	0.893	0.893	0.893	0.893	0.893	0.893

Appendix 3: Efficiency scores with preference information

Efficiency scores of each DMU in different DEA models with preference information.

	DEA model						
DMU	BCC	CP1	CP2	CP3	CP4	CP5	CCR
A1	1	1	0.9668	0.7921	0.7085	0.699	0.6857
A2	1	1	0.9974	0.9802	0.9737	0.9755	0.9662
A3	0.7203	0.7203	0.718	0.7057	0.6986	0.7019	0.6986
A4	0.789	0.789	0.7857	0.7741	0.7728	0.7741	0.7728
A5	0.8236	0.8236	0.8199	0.8097	0.8085	0.8097	0.8085
A6	0.5892	0.5892	0.5865	0.5803	0.5796	0.5803	0.5796
A7	0.6645	0.6645	0.6613	0.6581	0.6575	0.6581	0.6575
B1	0.9997	0.9997	0.993	0.9896	0.9896	0.9896	0.9896
B2	1	1	1	1	1	1	1
B3	0.8852	0.8834	0.8834	0.8834	0.8834	0.8834	0.8834
B4	0.9946	0.9946	0.987	0.987	0.987	0.987	0.987
B5	0.9657	0.964	0.964	0.964	0.964	0.964	0.964
B6	0.7401	0.7324	0.7324	0.7324	0.7324	0.7324	0.7324
B7	0.819	0.8129	0.8129	0.8129	0.8129	0.8129	0.8129
B8	1	1	1	1	1	1	1
B9	0.8192	0.8176	0.8176	0.8176	0.8176	0.8176	0.8176
B10	0.8832	0.8782	0.8782	0.8782	0.8782	0.8782	0.8782
C1	0.8475	0.8353	0.8353	0.8353	0.8353	0.8353	0.8353
C2	1	1	1	1	1	1	1
C3	0.8454	0.8221	0.8221	0.8221	0.8221	0.8221	0.8221
C4	0.7763	0.7543	0.7543	0.7543	0.7543	0.7543	0.7543
C5	0.8311	0.8025	0.8025	0.8025	0.8025	0.8025	0.8025
C6	0.8506	0.8224	0.8224	0.8224	0.8224	0.8224	0.8224
C7	0.7857	0.7559	0.7559	0.7559	0.7559	0.7559	0.7559
C8	0.9299	0.9017	0.9017	0.9017	0.9017	0.9017	0.9017
C9	0.6316	0.599	0.599	0.599	0.599	0.599	0.599
C10	0.8322	0.7819	0.7819	0.7819	0.7819	0.7819	0.7819
C11	0.908	0.8479	0.8479	0.8479	0.8479	0.8479	0.8479
C13	0.7787	0.7158	0.7158	0.7158	0.7158	0.7158	0.7158
C14	0.9156	0.8297	0.8297	0.8297	0.8297	0.8297	0.8297
C15	0.8674	0.7595	0.7595	0.7595	0.7595	0.7595	0.7595
C16	0.9573	0.8018	0.8018	0.8018	0.8018	0.8018	0.8018
C17	1	0.8709	0.8709	0.8709	0.8709	0.8709	0.8709

Appendix 4: Number of dominating DMUs

Number of dominating DMUs for each in four different cases; 'REA': REA, without preference information; 'FSW': free scalar weight, without preference information; 'REA - PI': REA, with preference information; 'FSW - PI': free scalar weight, with preference information.

	Dominance model			
DMU	REA	FSW	REA - PI	FSW - PI
A1	16	0	28	0
A2	0	0	2	0
A3	17	1	26	1
A4	8	0	17	0
A5	2	0	14	0
A6	20	1	31	3
A7	19	0	30	1
B1	0	0	1	0
B2	0	0	0	0
B3	4	0	7	0
B4	0	0	1	0
B5	0	0	2	0
B6	10	2	23	2
B7	6	0	9	0
B8	0	0	0	0
B9	5	1	9	1
B10	1	0	7	1
C1	1	0	6	1
C2	0	0	0	0
C3	3	0	10	2
C4	6	1	22	4
C5	4	0	15	1
C6	5	0	9	0
C7	7	1	21	4
C8	3	0	6	0
C9	20	5	30	10
C10	7	0	15	0
C11	2	0	6	0
C13	5	0	25	1
C14	5	0	9	0
C15	8	0	17	1
C16	0	0	18	0
C17	3	0	9	0