Copula Models for Dependence between Equity Index Returns

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# 1. Introduction

Standard correlation coefficient is adequate when describing linear dependency, or dependency between multivariate normally distributed variables (Fermanian and Scaillet 2002). Variables can however exhibit dependency, where standard correlation is insufficient in describing the dependency structure. An example of such case can be dependency structure between variables, which exhibit strong co-movement in large value changes, but weak co-movement in modest value changes. In context of equity indices, the large value changes could be caused by ‘external’ information such as unexpectedly good or bad information about the macro economical environment.

This research project presents a short description of copulas and their estimation techniques. In description of estimation, the paper restricts to bivariate case, because the empirical part concentrates only on the dependence of two equity indices. Copulas are attractive tools for modelling dependency structures in financial applications due to their following properties. Copulas are invariant under increasing transformations on the margins and the thus invariant on the popular scale invariant measures of association such as Kendall’s tau and Spearman’s rho (Fermanian and Scaillet 2002). The empirical part the study examines, using copulas, the statistical dependence structure of daily logarithmic returns of two stock equity indices.

The study begins with definition and formal representation of copulas. Fully parametric and semi-parametric copulas and their features are discussed in short. Non-parametric Deheuvels and Kernel copulas are presented and the frameworks for their estimation are presented.

The empirical part of this research project concentrates on the modelling of the dependency structure between historical returns of two equity indices. Marginal distributions are estimated parametrically using Gaussian distribution. Three archimedean parametric copulas are fitted and the fit is compared against a non-parametrically estimated copula. The goodness of fit of the different copulas is measured using discrete $L^2$ norm.

# 2. Copulas

Copulas break out from the traditional way of modelling multivariate dependency. The major difference between traditional multivariate distributions and copulas is that copulas do not require similar forms from the marginal distributions. Copulas estimate the multivariate distribution generally in two phases. In the first phase each of the marginal distributions is modelled individually. The second phase estimates a copula which summarizes the dependence structure based on the marginal distributions (Fermanian and Scaillet 2002). These phases may be also done simultaneously, as described later in context of the maximum likelihood estimation method.

The foundation of copulas lies in Sklar’s theorem (Sklar 1959). Let $X_1, \ldots, X_n$ be $n$ random variables and let $f_j(x_j)$ and $F_j(x_j)$ denote the respective probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of the variable $j$. According to Sklar’s theorem, if a joint distribution $H$ exists for these $n$ random variables, then there exists a copula $C$ which gives the joint distribution as a function of the marginal c.d.f.s. The joint distribution of the random variables is

$$H(x_1, \ldots, x_n) = \Pr(\{X_1 \leq x_1, \ldots, X_n \leq x_n\}) = C(F_1(x_1), \ldots, F_n(x_n)). \quad (1)$$
If the marginal distributions $F_j(x_j)$ are continuous, the copula is unique. Otherwise, the copula is unique within the range of values of the marginal distributions. Defining $u_j = F(x_j)$, the copula for the $n$ variables can be written as

$$C(u_1, \ldots, u_n) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))$$

(2)

where $F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)$ are the quasi-inverses of $F_1(u_1), \ldots, F_n(u_n)$ defined as

$$F_j^{-1}(u_j) = \inf \{ x | F_j(x) \geq u_j \}.$$  

(3)

The random variables are independent if and only if $C(F_1(x_1), \ldots, F_n(x_n)) = F_1(x_1) \cdots F_n(x_n)$ and accordingly, the copula representing independence is called the product copula. This means that the joint probability $\Pr[[X_1 \leq x_1], \ldots, [X_n \leq x_n]]$ is obtained as a product of the marginal probabilities. Lower and upper Fréchet copulas represent the highest negative and highest positive amount of dependence between the random variables. Product and Fréchet copulas are presented later in the study. Generally, an $n$-dimensional copula has the following properties (e.g. Fermanian and Scaillet 2002, Strelen and Nassaj 2007):

1. The range of $C$ is the unit interval $[0,1]$. In the bivariate case this means $C : [0,1]^2 \to [0,1]$.
2. $C$ is grounded, i.e. $C(u_1, \ldots, u_n) = 0$ if at least one of the coordinates, or marginal c.d.f.s $u_j = F_j(x_j) = 0$ if at least one of the coordinates, or marginal c.d.f.s
3. $C(1, \ldots, u_j, \ldots, 1) = u_j$, if all other marginal c.d.f.s are equal to one, except the $j$-th one. If all other marginal c.d.f.s are equal to one, the copula equals to the value of the $j$-th marginal c.d.f. In bivariate case this means $C(u_1,1) = u_1$ and $C(1,u_2) = u_2$.
4. $C$ is $n$-increasing, i.e. for all vectors $a$ and $b$ in $[0,1]^n$ such that $a \leq b$, the volume $V([a,b])$ of the box $[a,b]$ is positive. This means that the copula function is an increasing function. In bivariate case this means that for every observation 1 and 2 from both variables $u_{11}, u_{12}, u_{21}, u_{22}$ such that $u_{11} \leq u_{12}$ and $u_{21} \leq u_{22}$,

$$C(u_{12}, u_{22}) - C(u_{12}, u_{21}) - C(u_{11}, u_{22}) + C(u_{11}, u_{21}) \geq 0.$$  

For definitions presented later in this study, the concept of copula density needs to be defined. Copula density $c$ related to copula $C$ can is the partial derivative of $C$ with respect to all $n$ variables. Density $c$ can be expressed as

$$c(u_1, \ldots, u_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n}.$$  

(4)

3. Parametric and Semi-Parametric Copulas

Parametric copula estimation aims in modelling the dependence structure with respect to known distribution forms, i.e., by assuming parametric form for the marginal distributions and the copula. For the copula the parameters describe the level of association between the variables. The methods for the copula estimation can be classified to two main ones, namely fully parametric
and semi-parametric methods. The two methods differ in the estimation and assumption for the form of their marginal distributions. Fully parametric copulas assume parametric form also for the marginal distributions whereas the semi-parametric copulas allow empirical distribution functional forms for the margins. The two commonly used methods are detailed in Genest et al. (1993) and Shi and Louis (1995). Below are listed three Archimedean parametric copula families and their properties in bivariate case. For a comprehensive list of different parametric copula families and their properties, see Nelsen (1998).

### 3.1. Archimedean Copulas

Three copulas are employed and estimated in the empirical part of the study. Clayton, Frank and Gumbel copulas are all Archimedean, that is, they can be stated in the form

\[ C(u, v) = \phi^{-1}(\phi(u) + \phi(v)) \]  \hspace{1cm} (5)

where \( \phi \) is a strictly increasing and smooth convex function, called the generator function.

**Clayton copula:**

\[ C(u, v; \theta) = \left( u^{-\theta} + v^{-\theta} - 1 \right)^{-\frac{1}{\theta}}, \quad \theta \in \{0, \infty\} \]  \hspace{1cm} (6)

When \( \theta \to 0 \), the copula implies independence between the margins. However, for no value of \( \theta \) does the Clayton copula attain the Fréchet lower bound. When \( \theta \to \infty \), the copula attains the Fréchet upper bound. Clayton copula implies higher dependence at left tails of the marginal distributions. In financial context this means higher dependence in bear markets. Clayton copula generator function is \( \phi(u) = \frac{u^{-\theta} - 1}{\theta} \) (Hu 2002).

**Frank copula:**

\[ C(u, v; \theta) = -\frac{1}{\theta} \log \left( 1 + \frac{e^{-\theta u} - 1}{e^{-\theta v} - 1} \right), \quad \theta \in (-\infty, \infty) \]  \hspace{1cm} (7)

The parameter \( \theta > 1 \) implies positive dependence and when \( \theta \to \infty \), Frank copula attains the Fréchet upper bound. When \( \theta \to 0 \), the copula implies independence between the margins. The parameter \( \theta < 0 \) implies negative dependence and when \( \theta \to -\infty \), Frank copula attains the Fréchet lower bound. Frank copula implies a symmetric dependence pattern on both tails. In financial context the dependence is symmetric in bear and bull markets. Frank copula generator function is \( \phi(u) = \log \left( \frac{1 - e^{-\theta u}}{1 - e^{-\theta}} \right) \) (Hu 2002).

**Gumbel copula:**

\[ C(u, v; \theta) = \exp \left( -\left( \log u \right)^{\theta} + \left( -\log v \right)^{\theta} \right)^{\frac{1}{\theta}}, \quad \theta \in [1, \infty) \]  \hspace{1cm} (8)

The parameter \( \theta = 1 \) implies independence between margins and with \( \theta \to \infty \) Gumbel copula attains the Fréchet upper bound. For no value of \( \theta \) does the Gumbel copula attain the Fréchet lower bound. Gumbel copula implies higher dependence at right tails of the marginal distributions. In financial context, the dependence is higher in bull markets. Gumbel copula generator function is \( \phi(u) = \left( -\log(u) \right)^{\frac{1}{\theta}} \) (Hu 2002).
In presence of the parametric assumption for the copula, the parameter can be estimated using the maximum likelihood method or the inference functions for margins method.

### 3.2. Maximum Likelihood (ML) Method

The estimation method is presented in the bivariate case. Extension to higher number of variables is straightforward. Let \( X_1, X_2 \) denote two random variables and \( (x_1, x_2) \) observation vectors of size \( T \) from the respective variables. The observation pairs are \( (x_{1t}, x_{2t}), t = 1, \ldots, T \).

The log-likelihood of the joint distribution can be expressed as

\[
\ell(a_1, a_2; \theta) = \sum_{t=1}^{T} \ln f_1(x_{1t}; a_1) + \sum_{t=1}^{T} \ln f_2(x_{2t}; a_2) + \sum_{t=1}^{T} \ln c(F_1(x_{1t}; a_1), F_2(x_{2t}; a_2); \theta) \tag{9}
\]

where \( a_1 \) and \( a_2 \) are the marginal distribution parameter vectors and \( \theta \) the copula parameter vector. Maximum likelihood estimates can be obtained by maximizing equation (8) with respect to parameter vectors \( a_1, a_2 \) and \( \theta \). The maximum likelihood method can be computationally expensive with high sample size \( T \), because it requires joint estimation of both the margins and the dependence structure.

Under the usual regularity conditions (Shao 1999), the maximum likelihood estimator exists and it is consistent and asymptotically efficient:

\[
-\sqrt{T}(\hat{\theta}_{MLE} - \theta) \rightarrow N(0, \mathcal{I}^{-1}(\theta)) \tag{10}
\]

where \( \theta \) is the true value and \( \mathcal{I}(\theta) \) is Fisher’s information matrix (Cherubini et al. 2004).

### 3.3. Inference Functions for Margins (IMF) Method

As stated earlier, the principle of copula framework is to split the estimation into two phases. Referring to equation (9), the marginal distribution parameters can be estimated first

\[
a_i' = \arg \max \ell(a_i) = \arg \max \sum_{t=1}^{T} \ln f_i(x_{it}; a_i), \quad i = 1, 2. \tag{11}
\]

Following the marginal distribution parameter optimization the copula parameter can be estimated given the margin estimates

\[
\hat{\theta} = \arg \max \ell(\theta) = \arg \max \sum_{t=1}^{T} \ln c(F_1(x_{1t}; a_1'), F_2(x_{2t}; a_2'); \theta). \tag{12}
\]

The method presented above is called the inference functions for margins method and it assumes a parametric form for the margins. The same approach can be utilized as well without assuming parametric form for the margins. Durrelman et al.(2000) present a semi-parametric IMF, even though they call it the canonical inference functions for margins method. The semi-parametric IMF transforms the data \( (x_{1t}, x_{2t}) \) into uniform variates \( (\hat{u}_{1t}, \hat{u}_{2t}) \) using the empirical distributions and then estimates the copula parameter as

\[
\hat{\theta} = \arg \max \sum_{t=1}^{T} \ln c(\hat{u}_{1t}, \hat{u}_{2t}; \theta). \tag{13}
\]
Joe (1997) proves that the IMF estimator verifies under regularity conditions, the property of asymptotic normality:

\[ -\sqrt{T} (\hat{\theta}_{\text{IMF}} - \theta) \rightarrow N(0, \Psi^{-1}(\theta)) \]  

(14)

where \( \theta \) is the true value and \( \Psi(\theta) \) is Godambe’s information matrix (Cherubini et al. 2004).

4. Non-Parametric Copulas

Non-parametric estimation aims in modelling the dependence structure consistently to the empirical evidence. The non-parametric methods converge in certain probabilistic sense to the underlying dependence structure. There are some non-parametric methods for copula estimation, for example empirical and kernel copula estimation. In addition, some non-parametric copulas are presented in the literature in connection with the parametric families. The margins of these copulas can be estimated in a parametric or non-parametric way, but the copula itself is not parameterized. Examples of these type of copulas are product and Fréchet copulas.

**Product copula:** \( C(u, v) = uv \)  

(15)

Product copula implies complete independence between the marginal distributions.

**Fréchet copulas:**

\[
\begin{align*}
C_{\text{Lower}}(u, v) &= \max([u + v - 1], 0) \\
C_{\text{Upper}}(u, v) &= \min[u, v]
\end{align*}
\]  

(16)

The Fréchet copulas correspond to Fréchet bounds in the bivariate case, being thus related to a type of perfect dependence. The copula \( C_{\text{Lower}} \) puts all mass to the diagonal connecting \((0,1)\) and \((1,0)\) and corresponds to a perfect negative dependence. The copula \( C_{\text{Upper}} \) puts all mass to the diagonal connecting \((0,0)\) and \((1,1)\) and corresponds to a perfect positive dependence (de Melo Mendes and de Souza 2004). In the context of this study, \( C_{\text{Lower}} \) corresponds to perfect reverse co-movement and \( C_{\text{Upper}} \) to perfect co-movement of the equity indeces.

4.1. Empirical Copula Estimation

This rank statistic based method was introduced by Deheuvels (1979, 1981a,b) and the copula is often referred to accordingly as the Deheuvels copula. Deheuvels copula is discontinuous, that is, it describes the dependence structure only in certain points of a \([0,1]^d\) lattice. The Deheuvels copula utilizes the entire information included in the observations and thus one can say that it is only a different representation for the data and not actually a model.

Let \( X_1, X_2 \) denote two random variables and \((x_1, x_2)\) observation vectors of size \( T \) from the respective variables. The observation pairs are \((x_{1t}, x_{2t}), t = 1, \ldots, T\), also called the order statistic of the sample. Order statistic is thus the real order of observations. Let \((r_{1t}, r_{2t}), t = 1, \ldots, T\) denote the rank statistic of the sample, which is the observed real sample arranged into an ascending order. The empirical copula is the function.
\[ \mathcal{C}\left( \frac{t_1}{T}, \frac{t_2}{T} \right) = \frac{1}{T} \sum_{n=1}^{T} I(r_n \leq t_1) \cdot I(r_n \leq t_2) \]  

(17)

where \( I \) is an indicator function taking a value of 1, when its argument condition is satisfied. This means that a decision has been taken about the lattice size \( n \). Then \( n \) even threshold values are taken from the rank vector and the number of real observed values, which are less or equal to the value, is evaluated until the threshold. The copula is achieved by multiplying the vectors achieved in the process and the copula values in the lattice are scaled by dividing the frequencies by the samples size \( T \).

4.2. Kernel Copula Estimation

Kernel-based estimation for copulas is presented in Cherubini et al (2004) and Scaillet and Fermanian (2002). The method produces a smooth differentiable reconstruction of the copula function without assuming any a priori parametric structure between the margins. The kernel estimation performs, both for the margins and for the copula, well on the center of the distribution, but poorly on the tails and boundaries. Thus the estimated margins and copula need to be tail and boundary adjusted (Xi Chen and Huang 2007).

5. Measures of Association

The usual descriptive measures of association between variables, Kendall’s tau, Spearman’s rho and Blomqvist’s beta can be measured in connection with copulas, directly from the copulas themselves. All are rank statistic based measures and can be determined directly from the copulas.

Kendall’s Tau: \( \tau_{X_1, X_2} = 1 - 4 \int_{0}^{1} \frac{1}{\partial \mathcal{C}(u_1, u_2)} \frac{\partial \mathcal{C}(u_1, u_2)}{\partial u_1} du_1 du_2, \quad \tau \in [-1,1] \)  

(18)

Kendall’s tau describes the probabilities of concordance and discordance between two independent pairs \( (X_{11}, X_{12}) \) and \( (X_{21}, X_{22}) \), of observations 1 and 2 from each variable. The two observations are said to be concordant if \( (X_{11} - X_{12})(X_{21} - X_{22}) > 0 \) and discordant if \( (X_{11} - X_{12})(X_{21} - X_{22}) < 0 \). Kendall’s tau measures the probability

\[ \tau_{X_1, X_2} = \Pr\{(X_{11} - X_{12})(X_{21} - X_{22}) > 0\} - \Pr\{(X_{11} - X_{12})(X_{21} - X_{22}) < 0\} \]  

(19)

The measure value -1 implies perfect reverse correlation, i.e. in the case of this paper, adverse behaviour in the returns of two indices. Respectively, measure value 1 implies perfect co-movement of the two index returns.

Spearman’s Rho: \( \rho_{X_1, X_2} = 12 \int_{0}^{1} \mathcal{C}(u_1, u_2) du_1 du_2 - 3, \quad \rho \in [-1,1] \)  

(20)

Let \( (X_{11}, X_{12}) \), \( (X_{21}, X_{22}) \) and \( (X_{31}, X_{32}) \) be three independent random vectors from the common joint distribution function \( H \), whose margins are \( F_1 \) and \( F_2 \) linked by copula \( C \). Spearman’s rho is proportional to the probability of concordance minus the probability of discordance for two vectors \( (X_{11}, X_{12}) \) and \( (X_{21}, X_{22}) \). The first vector has a distribution \( H \).
The components of the second vector are independent and thus it has a distribution $F_1 \cdot F_2$. Therefore Spearman’s rho is defined as (Emberchts et al. 2001)

$$\rho_{X_1,X_2} = 3 \Pr[(X_{11} - X_{21})(X_{12} - X_{22}) > 0] - \Pr[(X_{11} - X_{21})(X_{12} - X_{22}) < 0]$$

(21)

Blomqvist’s Beta: $\beta_{X_1,X_2} = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1$, $\beta \in [-1, 1]$  

(22)

Blomqvist’s beta is referred to also as medial correlation coefficient. It as well measures the concordance – discordance probability expressed in formula (24), but is more inexpensive computationally. Let $(x_{11}, x_{21}), \ldots, (x_{1T}, x_{2T})$ be $T$ observations from joint distribution $F$. A two-way contingency table is employed using sample medians as cutting points. Let $n_1$ denote the number of observations belonging to the first or third quadrant and $n_2$ the number of observations belonging to the second or fourth quadrant. The simple dependence measure is based on the contingency table expressed as (Schmid and Schmidt 2006)

$$\beta_{X_1,X_2} = \frac{n_1 - n_2}{n_1 + n_2}$$

(23)

For a pair of random variables, the population version of $\beta_{X_1,X_2}$ takes the form

$$\beta_{X_1,X_2} = \Pr[(X_1 - \bar{x}_1)(X_2 - \bar{x}_2) > 0] - \Pr[(X_1 - \bar{x}_1)(X_2 - \bar{x}_2) < 0]$$

(24)

where $\bar{x}_i$ are the medians of the random variables. The beta values -1, 0 and 1 imply perfect reverse correlation, independence and perfect correlation, respectively.

Other dependence measures applicable in context of copulas are rank correlation coefficients and upper and lower tail dependencies, which are presented in de Melo Mendes and de Souza (2004).

6. Copula Fit Measures

Copulas have many proposed functional forms, though no generalized rules exist for selection between them. This study employs discrete $L^2$ norm in assessing the goodness-of-fit following Durrleman et al. (2000). The $L^2$ norm evaluates the distance between two copulas. Generally, the smallest distance to the reference copula implies the best fit. As reference this paper uses the Deheuvels copula. The discrete $L^2$ norm between the Deheuvels and the comparison copula is defined as

$$\mathcal{T}_2(C_{\text{Deheuvels}}, C) = \left|C_{\text{Deheuvels}} - C\right|_{L^2} = \left(\sum_{T_{10} \leq 1} \sum_{T_{20} \geq 1} \left[ C_{\text{Deheuvels}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) - C\left(\frac{t_1}{T}, \frac{t_2}{T}\right)\right]^2\right)^{\frac{1}{2}}$$

(25)

where $C_{\text{Deheuvels}}$ denotes the Deheuvels reference copula and $C$ the compared copula.

Other consistent goodness-of-fit measures found in the literature are based on Kolmogorov and Anderson-Darling distances and their modifications (Malevergne and Sornette 2003) and a divergence measure initially introduced by Diks et. al (1996) and later applied by Panchenko (2005).
7. Dependence between two Equity Indices

Data for the empirical part of the study consists of closing values of CAC40 and DAX indices from the time period 28 August 2001 until 21 November 2007 giving a total of 1600 data points on the return data. To illustrate the data, figure 1 represents the relative price movements of each index. The initial level of each index has been rescaled to 100 to facilitate the comparison of relative performance over the period. Closing values are then converted into daily logarithmic returns which are respectively plotted. The relative price movements figure reveals that the indices have developed very similarly, almost hand-in-hand. The return plot implies that the return variance has changed over time. Especially, until around day 500 the variance has been significantly higher than during the rest of the observation period. Neither the time series properties nor the time varying variance are taken into account in this study. Especially, if the objective would have been simulation from the copula, the returns should have been modelled with time series methods and only the filtered index return dependencies modelled using copulas (Hu 2002).

Before the estimation of the margin distributions the margins are examined visually with scatter plot and plotting the index return histograms on the respective axes. The scatter plot implies strong linear dependence. By calculating Spearman’s correlation coefficient, a value of 0.88 is high, but actually surprisingly low in light of the closing value plot. Both the margin histograms and the scatter plot imply slightly heavier lower tails for the marginal distributions. That is, large losses have been more probable for both indices than large gains.

Figure 1: Rescaled Index Closing Values and Index Returns
Based on the daily returns above the marginal distributions are estimated for both indices. The margins are assumed to be normally distributed. Table 1 presents the margin parameters and figure 3 the estimated normal p.d.f.s and the margin histograms. Based on the sample mean and standard deviation, the two margins are extremely close to each other. The difference between the index return samples comes in skewness and kurtosis. CDAX returns have slightly higher skewness whereas CAC40 returns have slightly higher kurtosis. Figure 3 where the histograms are plotted with higher number of bins, supports the earlier perception of heavier left tails. The figure also reveals that the distribution of CAC40 returns is more symmetric. Comparing the fitted normal p.d.f.s against the histograms with higher number of bins raises some concern whether the normal assumption captures the distribution properties, especially kurtosis, correctly.

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<tr>
<th>Index</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
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<td>CAC40</td>
<td>6.940E-05</td>
<td>1.389E-02</td>
<td>-0.069</td>
<td>6.891</td>
</tr>
<tr>
<td>CDAX</td>
<td>2.761E-04</td>
<td>1.413E-02</td>
<td>-0.145</td>
<td>6.373</td>
</tr>
</tbody>
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Figure 2: Marginal distribution histograms and joint scatter plot
The parametric Archimedean copula estimation was done in two phases as the margin parameters were estimated first. The normal c.d.f.s were utilized in the estimation of the joint c.d.f which was done by employing the inference functions for margins method. That is, the copula parameters were determined employing the maximum likelihood method described in formula (13). For the IMF estimation, a Matlab copula toolbox written by Patton (2008) was utilized. After the numerical optimization of copula parameters, they were employed in calculation of the copula surfaces. All copulas were calculated using frequency-based margins for discrete L^2-norm fit comparison as described in formula (25). For studying the fit of the different copulas, the empirical Deheuvels copula was estimated and compared against the other estimated copulas using the L^2 norm. Best fit was selected based on the smallest distance. As the values for the measures of association are known for product and Fréchet copulas, they were also calculated for reference. The copula estimation results are presented in table 2 and copula contours are presented in figures 4 and 5.

Based on the L^2-norm the Clayton copula fits best to the data. The negative log likelihood for the Clayton copula was the second smallest after the Gumbel copula, though the difference is shown only in the third decimal. Visual examination of figure 4 reveals that the Clayton copula contours follow the Deheuvels copula relatively well. As stated, the Clayton copula implies higher dependence at the left tails of the marginal distributions. Thus and with the margins being slightly heavier left-tailed, the Clayton copula best captures the dependence structure between the index returns. Clayton copula is also the closest to the Deheuvels copula with regards to Spearman’s rho and Blomqvist’s beta.

Gumbel copula produced the smallest negative log likelihood but the L^2 norm was higher than that of the Clayton copula. The numerical optimization produced a large dependence parameter for the Gumbel copula. This is evident both in visual examination of the Gumbel copula in figure
4 and in examination of Spearman’s rho and Blomqvist’s beta. Visually, comparing figures 4 and 5, the Gumbel copula is very close to the Fréchet upper copula and both association measures are close to one implying perfect positive dependence between the margins. As described the Gumbel copula captures better the margins’ right tail dependence and as the margins were heavier on the left tails, Gumbel copula did not fit as well as the Clayton copula.

For Clayton and Gumbel copulas the parameter estimates were reasonable, but with Frank copula the estimation difficulties emerged. The numerical optimization routine converged producing a large parameter of 13739. Even though the value is acceptably in the parameter range of the Frank copula, it led in the later stage to computational accuracy problems with the calculation of the Frank copula lattice. As stated, for Frank copula the parameter value \( \theta > 0 \) implies positive dependence and this is clearly the case with the data. A maximum value of the parameter, which still didn’t cause problems with the lattice calculation, was iterated manually and a value of 37.5 was utilized for calculation of the Frank copula. Even with that value the copula appears quite similar to the Fréchet upper copula. From the association measures especially the Spearman’s rho receives a very high value. Frank copula captures symmetric dependence on the margins’ tails. As the margins in this case were slightly heavier on the left tails, Frank copula had inferior performance compared to Clayton copula.

The calculation of the association measures performed relatively well for Spearman’s rho and Blomqvist’s beta. Both measures imply correctly, given the slight error due to computational accuracy, the marginal distribution independence for product copula, perfect positive dependence for Fréchet upper copula and perfect negative dependence for Fréchet lower copula. However, as seen in table 2, the calculation of Kendall’s tau ‘broke down’ and Kendall’s tau implies perfect positive dependence for all copulas. The calculation was performed according to equation (18), but due to computational accuracy or too small number of lattice points the calculation did not perform correctly.

<table>
<thead>
<tr>
<th>Copula</th>
<th>Parameter ( \Theta )</th>
<th>ML</th>
<th>( L_1 ) Norm</th>
<th>Spearman’s Rho</th>
<th>Blomqvist's Beta</th>
<th>Kendall’s tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>4.90</td>
<td>-17.50</td>
<td>1.73</td>
<td>0.87</td>
<td>0.74</td>
<td>1.00</td>
</tr>
<tr>
<td>Gumbel</td>
<td>87.29</td>
<td>-17.50</td>
<td>3.22</td>
<td>0.99</td>
<td>0.98</td>
<td>1.00</td>
</tr>
<tr>
<td>Frank</td>
<td>37.5(^1)</td>
<td>-9.5275(^2)</td>
<td>2.77</td>
<td>0.97</td>
<td>0.90</td>
<td>1.00</td>
</tr>
<tr>
<td>Product</td>
<td></td>
<td></td>
<td>-0.02</td>
<td>-0.01</td>
<td></td>
<td>1.00</td>
</tr>
<tr>
<td>Fréchet lower</td>
<td></td>
<td></td>
<td>-1.00</td>
<td>-1.00</td>
<td></td>
<td>1.00</td>
</tr>
<tr>
<td>Fréchet upper</td>
<td></td>
<td></td>
<td>0.99</td>
<td>0.99</td>
<td></td>
<td>1.00</td>
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<tr>
<td>Empirical</td>
<td></td>
<td></td>
<td>0.87</td>
<td>0.75</td>
<td></td>
<td>1.00</td>
</tr>
</tbody>
</table>

1 Maximum parameter value enabling copula surface calculation
2 Optimization produced log likelihood corresponding to parameter value 13739

Table 2: Copula estimation results
Figure 4: Empirical and parametric archimedean copula contours

Figure 5: Empirical and non-parametric copula contours
8. Conclusions

The study defined copulas as dependence modelling tools. The estimation framework for parametric, semi-parametric and non-parametric copulas was presented. The Archimedean Clayton, Gumbel and Frank copulas were fitted to the data consisting of two equity index closing values and the best fit was determined between these copulas. The estimation conducted in the study performed only relatively well as problems were encountered both with Frank copula estimation and the calculation of Kendall’s tau. Based on the $L_2$-norm the Clayton copula was found to best fit the data utilized in this study. The result is however only indicative based on the limited number of parametric copulas fitted to the data and due to the aforementioned problems.

The copula models estimated in this study can be utilized mostly in simulations. The models are simplified as the time series properties are not taken into account. However, simulations from the model can be utilized in replicating efficient frontiers, pricing derivatives which have no closed form for the price, estimating joint risks etc. The framework for the simulations is presented in Embrechts et al. (2001).

Copula modes provide a powerful tool for dependence modelling. Copula models provide a more robust and accurate way to model joint dependency compared to other n-variate joint distributions which require same form of the marginal distributions. They capture various types of dependency structures. Copulas have especial significance to simulations, which play an important role in the financial industry.
9. References


