System Analysis Laboratory
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Mat-2.108 Special assignment in applied mathematics

Evaluation of Constant Proportion Portfolio Insurance in Asset Allocation

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1 Introduction

When investing in risky assets, investors seek to hedge their wealth against market fluctuations. The investment strategy consists of allocating funds between risky and risk free assets. In this study, it is assumed that the investor can divide her funds in exactly two assets: the risk free asset (with growth rate of $r_f$) and the risky asset (with expected value $\mu$ and deviation $\sigma$). The purpose of the study is to investigate one asset allocation strategy with good hedging features, the Constant Proportion Portfolio Insurance (CPPI).

CPPI belongs to portfolio insurance strategies, which are characterised by Black & Perold (1992) in following manner: “Any rule that takes less risk at lower wealth levels and more risk at higher wealth levels is a candidate.” CPPI has several appealing properties, of which the most important one is its ability to insure the desired amount of invested funds, called guarantee or floor. This characteristic and ease of use has made the CPPI increasingly widespread tool among practitioners. Moreover, CPPI also performs rather well compared to other popular investment strategies: in an extensive simulation study comparing allocation strategies, Cesari & Cremonini (2003) found that CPPI is an effective and robust strategy.

This study presents first the theory of CPPI without trade restrictions and then discusses different CPPI strategies of which some include unsymmetric risk measures. They are evaluated and compared with a simpler allocation strategy, buy-and-hold (BH).

The study presents a theoretical framework, where the idea and notation of CPPI are presented followed by the characteristics of the strategy, first in continuous and then in discrete time. Also some examples of possible extensions to market dynamics are given. The choice of CPPI parameters is evaluated and the results of continuous and discrete CPPI are compared. Finally, the impact of some extensions is tested. In the conclusion part, some shortcomings of the study are given and some suggestions for future studies presented.
2 Theoretical framework

The performance of an asset strategy depends on many things, such as transaction costs or borrowing limits. With CPPI, we first consider a market with no transaction costs and restrictions, including borrowing. In this case, we speak of simple CPPI. The results of simple CPPI offer tools for CPPI parameter selection and the computational testing of simple CPPI strategies is fairly straightforward. However, when the assumptions of frictionless market or unlimited borrowing are violated, the theoretical (and computational) approach becomes more challenging. Here we first present the idea and notation of CPPI, followed by the theoretical framework of simple CPPI in continuous and discrete time. Also two risk measures, a benchmark strategy for CPPI and some extensions are discussed.

We assume that the investor holds an initial wealth of \( W_0 \) (and \( W_t \) at time \( t \)), which is divided among risky-free assets \( R_0 \) and risky assets \( E_0 \), called the exposure. In CPPI, the investor chooses an initial floor \( F_0 \), which is the minimum of total portfolio value at every time point. The excess of \( W_0 \) over \( F_0 \) is called the cushion, \( C_0 \). In CPPI, the investor balances the portfolio at time \( t \) by investing a certain multiple of \( C_t \) to risky assets:

\[
E_t = mC_t, \quad (1)
\]

where \( m > 1 \) is a fixed multiplier and \( C_t = W_t - F_t \). The rest of the wealth \( R_t = W_t - E_t \) is invested in risky-free assets or, in the case of negative \( R_t \), considered as a loan (we assume that the interest rate \( r_f \) is the same in both cases). The risky asset price at time \( t \) (stock) is denoted by \( S_t \) and riskless asset (bond) by \( B_t \).

2.1 Continuous version

In continuous time, trading takes place continuously and asset prices are modelled by differential equations

\[
\begin{align*}
\frac{dB_t}{B_t} &= r_t dt \\
\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t
\end{align*}
\]
and the floor by $F_t = F_0 e^{rt}$. The stock price is modelled as a geometric Brownian motion\(^1\), where $\mu$ denotes the expected stock growth rate and $\sigma$ the standard deviation. The portfolio is rebalanced continuously.

For the investor, the essential decision is the choice of $m$ and $F_0$ (from now on denoted in the text by $F$). We limit the choice of floor to $F \in (0, W_0)$, as choosing zero would be “uninsured” portfolio and choosing $W_0$ “completely insured” portfolio. The choice of the multiplier is bounded from below ($m > 1$) but, theoretically, there is no upper bound. In practice, as shown by e.g. Black & Perold (1992), in order to ensure $W_t \geq F_t$ we set an upper bound of $m \leq \delta_{\text{max}}^{-1}$, where $\delta_{\text{max}}$ denotes the maximum possible (proportional) downward move of $S_t$ during the rebalancing interval. For example, if we expect at maximum 10% fall in the market during the rebalancing interval, we set $m \leq 0.1^{-1} = 10$.

The central results for (simple) continuous CPPI are (Mahayni et al. 2006):

\[
W_t = F_t + (W_0 - F_0) \left( \frac{S_t}{S_0} \right)^m e^{(1-m)(r-\frac{1}{2}\sigma^2)t} \tag{4}
\]

\[
E[W_t] = F_t + (W_0 - F_0) e^{(r+m(\mu-r))t} \tag{5}
\]

\[
Var[W_t] = (W_0 - F_0)^2 e^{2(r+m(\mu-r))t} (e^{m^2\sigma^2t} - 1) \tag{6}
\]

Some interesting results arise from (4)-(6). First, the current wealth depends on the chosen floor $F_0$, the multiplier $m$ and the current stock price $S_t$. This means that the wealth at time $T$ is indifferent of stock prices $t < T$, i.e. the simple CPPI is path independent. Perold & Sharpe (1988) state that the simple CPPI performs best in stable (low $\sigma$) and growing (high $\mu$) market. This is seen from Equations (5) and (6), for the expected wealth grows in $\mu$ and is independent of the stock variance $\sigma^2$, whereas the variance of wealth grows exponentially in $\sigma^2$. Moreover, Perold & Sharpe claim that CPPI is a \textit{convex allocation strategy}. That is, the payoff curve (the performance of the portfolio in terms of stock value) of CPPI is convex, which means that it tends to generate high payoffs in bull market and relatively low payoffs in flat market. This can be confirmed by differentiating (4) twice: $\frac{\partial^2}{\partial S_t^2} W_T = \frac{1}{S_0}(m-1) m \frac{S_t}{S_0} m^{-2}$, which is strictly positive with $m > 1$ and therefore $W_t$ is convex. In practice, convexity implies that the stocks are bought as they rise and sold as they depreciate.

\(^1\)This assumption is common and also the basis for binary tree generation in Section 2.2.
Based on the expected value and variance of $W_t$, the effect of parameter choice can be investigated. If $\mu > r_f$ (which is realistic), the expected wealth grows when $m$ grows according to Equation (5). This happens in the expense of exponentially growing variance, as can be seen from (6). The impact of $F$ can be seen with

$$E[W_t] \overset{(5)}{=} F_t + W_0 e^{(r + m(\mu - r))t} - F_0 e^{rt} e^{mt(\mu - r)} = CW_t + (1 - D)F_t < CW_t,$$

which shows that higher initial floor values imply lower expected wealth. Accordingly, the variance diminishes. The expected value and volatility with chosen parameter values ($t = 0.5, \mu = 9.8\%, \sigma = 23.2\%$) and $m = [1, 2, \ldots, 20], F_0 = [50, 55, \ldots, 95]$ are plotted in Figure 1.

![Expected value and volatility plots](image)

Figure 1: The expected wealth and its volatility ($t = 0.5, \mu = 9.8\%, \sigma = 23.2\%$) with respect to $m$ and $F$.

The wealth of the discrete CPPI (see Section 2.2) converges to continuous version in distribution, meaning that the limit of its expected value is (5) and the limit of variance
is (6) (for the proof, see Mahayni et al. 2006). Next we derive the distribution of $C_T$, which comes in use when different risk measures are defined.

Equation (3) implies that $\ln(\frac{S_t}{S_0}) \sim N((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$, which gives:

$$\ln(\frac{S_t}{S_0})^m \sim N((\mu - \frac{\sigma^2}{2})mt, m^2 \sigma^2 t).$$  \hspace{1cm} (7)

In order to find the distribution of $C_t$, we take first the logarithm of the cushion:

$$\ln C_t = \ln(W_t - F_t) = \ln((W_0 - F_0)(\frac{S_t}{S_0})^m e^{(1-m)(r-\frac{1}{2}m\sigma^2)t})$$

$$= \ln(W_0 - F_0) + \ln(\frac{S_t}{S_0})^m + (1 - m)(r - \frac{1}{2}m\sigma^2)t,$$

which together with Equation (7) implies that

$$\ln C_t \sim N(\ln(W_0 - F_0) + (1 - m)(r - \frac{1}{2}m\sigma^2)t + (\mu - \frac{\sigma^2}{2})mt, m^2 \sigma^2 t).$$ \hspace{1cm} (8)

In other words, the cushion follows a lognormal distribution.

### 2.1.1 CPPI strategies with risk measures

Using other risk measures than standard deviation of the terminal wealth is particularly useful with portfolio insurance strategies where the distribution of the wealth is unsymmetric (for example, it is bounded only from below). Measuring only the downside risk is logical, while the avoidance of loss is why investors turn to portfolio insurance strategies in the first place. When evaluating CPPI by different risk measures, the lognormality of the cushion (see Equation (8)) proves to be helpful. First, following the contingent portfolio programming (CPP) framework by Gustafsson & Salo (2005), we construct a mean-lower semi absolute deviation model (mean-LSAD) in order to investigate the behaviour of $m$ and $F$ when investors attitude towards risk changes. The LSAD measure is defined as the expected wealth of all wealths below the expected wealth. Figure 2 (upper part) presents an example with normally distributed terminal wealth. Formally LSAD is defined as:

$$\text{LSAD}[W_t] := \mathbb{E}[\mathbb{E}[W_t] - W_t] \quad W_t < \mathbb{E}[W_t].$$
It is easily seen that

\[
\text{LSAD}[W_t] = E\left[E[C_t + F_t] - (C_t + F_t)\right] \\
= E\left[E[C_t] - C_t\right] = \text{LSAD}[C_t],
\]

which allows us to use the derived lognormal distribution of Equation (8) in LSAD calculations. If we denote the lognormal probability distribution function of \(C_t\) with \(f_C(x)\), the LSAD is achieved with

\[
\text{LSAD}[W_t] = \text{LSAD}[C_t] = \int_0^{\mu_c} f(x)(\mu_c - x)dx,
\]

(9)

Where \(\mu_c\) is simply the difference of \(E[W_T]\) (from Equation 5) and \(F_t\). The mean-LSAD optimization problem is

\[
\max_{m,F_0}(E[W_t] - \lambda \text{LSAD}[C_t])
\]

(10)

where \(\lambda\) is the risk coefficient. In optimization several constraints, such as upper bounds for \(m\) and \(F_0\) can be used.

Figure 2: The LSAD and CVaR of arbitrary wealth distribution.

Another risk-measuring approach to simple CPPI is adapted from Kettunen & Salo (2005), where the Conditional Value-at-Risk (CVaR) is applied to portfolio programming. The idea is to maximize profits with a constraint that ensures a certain maximum for
expected value loss in the worst $\beta\%$ of cases. The optimization problem is simply

$$\max_{m,F_0} \quad E[W_t]$$

s.t. \quad CVaR \leq R

where CVaR is the difference between the expected value of $W_t$ and the expectation of $(1-\beta)$-tail of the probability distribution function of $W_t$, which we denote by $E[W_{t,1-\beta}]$. In other words, CVaR is the expected loss in the worst $(1-\beta)\%$ of the cases (see the example of Figure 2). In order to calculate CVaR, an upper bound for integration, the Value at Risk (VaR) has to be defined. VaR is the $W_t$ value that limits the distributions $(1-\beta)$-tail. As the only known distribution is that of $C_t$, we need to use the VaR of $C_t$ and denote it with $\alpha_{c,t}$. It is obtained with the help of cumulative distribution function $(F_{C,t})$: $\alpha_{c,t} = F_{C,t}^{-1}(1-\beta)$. With $\alpha_{c,t}$ as an upper bound for integration, we can calculate the expected wealth for $(1-\beta)$-level and by taking the difference from expected wealth, and the CVaR is obtained with:

$$CVaR = E[W_t] - \frac{1}{1-\beta} \int_0^{\alpha_{c,t}} f_C(x)(x + F_t)dx \quad \text{E}[W_{t,1-\beta}]$$

When computing (in Section 3.1) numerical values for the problems above, the integrals in (9) and (13) are achieved with the help of MATLAB\textvisiblespace\texttt{quad}-function.

### 2.2 Discrete version

In the discrete version of CPPI, the floor is discounted with $F_t = F_0(1 + r_f)^{ndt}$ and the price of risk-free asset with $B_{t+1} = B_t(1 + r_f)^{dt}$, where $n$ is the amount of discretization points and $dt$ the time interval. The development of wealth is dependent of the risky asset price $S_t$, which is a random process. In Section 2.1, the asset price was modelled through a geometric Brownian motion. The discrete version of the process is achieved through a binary pricing tree. The following binary tree construction follows the approach of Cox et al. (1979).

At every time point, there are two options for the stock price: go up by $(u-1)\%$ with probability $p$ and go down by $(d-1)\%$ with probability $1-p$. These parameters are
achieved with the help of $\mu$ and $\sigma$ of the stock:

$$u = e^{\sigma \sqrt{dt}}$$

$$d = u^{-1}$$

$$p = \frac{e^{\mu dt} - d}{u - d}.$$  \hfill (16)

At time $t$ there are $t + 1$ possible terminal prices and $2^t$ paths in the tree.

For the estimation of $\mu$ and $\sigma$, the mean and sample variance of monthly changes in OMX Helsinki CAP (former HEX) and MSCI Europe Index were investigated. The time span was from Jan 1998 to Dec 2006, a total of 106 months. The price change (monthly) and the annualised expected value and volatility of the index are achieved with

$$\Delta S_t = \frac{S_t - S_{t-1}}{S_{t-1}}$$  \hfill (17)

$$\hat{\mu} = \mathbb{E}[\Delta S_t] = \frac{12}{N} \sum_{i=1}^{N} \Delta S_t$$  \hfill (18)

$$\hat{\sigma} = \text{Var}[\Delta S_t] = \sqrt{\frac{12}{N-1} \sum_{i=1}^{N} (\Delta S_t - \hat{\mu})^2}$$  \hfill (19)

The corresponding figures for Helsinki CAP are $\hat{\mu}_H = \mu_H = 9.8\%$, $\sigma_H = 23.2\%$ and for MSCI Europe $\mu_E = 5.6\%$ and $\sigma_E = 17.9\%$. From now on, unless mentioned otherwise, the estimates $\mu_H$ and $\sigma_H$ are used in stock price modelling and the time interval is set to 2 weeks ($\Delta t = 24^{-1}$). From Equations (14)-(16), using the estimates above, it follows that in the tree the upward move is of size 4.85% and is taken with probability 0.53 (4.6% with 0.47 respectively). An example of such a pricing tree with $t = 3$ and $S_0 = 100$ is given in Figure 3.

The probabilities for terminal states are achieved from the binomial distribution, or simply multiplying all path probabilities leading to the corresponding end note. In the example, the terminal probabilities are: $P(S_3 = 115.25) = 0.53^3 \approx 0.15$, $P(S_3 = 104.85) \approx 0.40$, $P(S_3 = 95.39) \approx 0.35$ and $P(S_3 = 86.77) \approx 0.10$. We denote the terminal prices with $S_t = [S_{t,1}, \ldots, S_{t,n}]^T$ ($n = 2^t$) and the corresponding probabilities with $P_t = [P_{t,1}, \ldots, P_{t,n}]^T$.

After the wealth is divided between risky and risk free assets according to CPPI principle
Figure 3: A tree period binary price tree for $\mu = 9.8\%$, $\sigma = 23.2\%$ and $\Delta t = 24^{-1}$.

at every timestep, the expected terminal wealth and volatility are achieved with

$$E[W_T] = \mathbf{P}_T^T \mathbf{W}_T$$

$$Var[W_T] = \sum_{i=1}^{n} P_i(W_{T,i} - E[W_T])^2,$$

where $\mathbf{W}_T$ is the terminal wealth vector.

### 2.3 Buy and hold

A suitable benchmark strategy for the CPPI is buy-hold (BH), which represents the passive “do nothing” strategies. Basically the investor has to choose which proportion of her wealth she invests in risky assets and the rest is invested in bonds. After choosing the initial strategy, no trade occurs. In discrete time, the wealth of a BH investor at time $t$ with $\eta \in [0, 1]$ of wealth in risky assets is given by

$$W_t = \eta W_0 \frac{S_t}{S_0} + (1 - \eta) W_0 (1 + r)^t$$

The expected value and variance are given by Eqns. (20) and (21), where $\mathbf{W}_t$ is simply
Equation (22) with $S_t$ replaced by terminal price vector $S_t$.

We note that in continuous time, the expected return and volatility of the BH strategy become

$$
E[W_t] = (e^r - 1)(1 - \eta) + \eta \mu \quad (23)
$$

$$
\text{Var}[W_t] = \eta \sigma^2 \quad (24)
$$

and the CVaR is achieved with

$$
\text{CVaR} = E[W_t] - \frac{1}{1 - \beta} \int_{0}^{\alpha_{c,t}} f_C(x) dx, \quad (25)
$$

where $\alpha_{c,t}$ is the critical value for lower $(1 - \beta)$-tail of $\log\mathcal{N}(\mu - \frac{\sigma^2}{2}t, \sigma^2 t)$ distribution and $f_C(x)$ the corresponding probability function.

### 2.4 CPPI extensions

With restrictions to market dynamics, the results presented above no longer hold. For example, we can introduce a borrowing constraint $b > 1$ which as a multiplier of total wealth defines how much we are able to borrow. For example, if we set $b = 2$ we allow the proportion of risky assets to be at maximum twice the current wealth, which is covered by taking a loan. The formula for exposure becomes:

$$
E_t = \min(mC_t, bW_t) \quad (26)
$$

and the analytical treatment of the strategy becomes more complex. Another extension is to set an upper bound on the proportion of risky assets, i.e.

$$
\frac{E_t}{W_t} < \eta, \quad \eta \in (0, 1) \quad (27)
$$

Furthermore, Perold & Sharpe (1988) introduce a profit “lock in”, where the floor is raised in case of an upward move in the market:

$$
F_0 = W_0 \gamma, \quad F_t = \max(W_t \gamma I_{\Delta S_t > 0}, F_{t-1}), \quad \gamma \in (0, 1), \quad (28)
$$

where $I$ is the indicator function, which equals 1 when the price change is positive and 0 otherwise.
For example, if the investor wants to lock in 80% of total wealth every time she gains
wealth (asset price goes up), she chooses \( \gamma = 0.8 \) and updates floor everytime \( \Delta S_t \)
is positive. The lock-in is more robust strategy than the simple CPPI: in Cesari &
Cremonini the CPPI with lock-in was found to be the best strategy to choose if the
phase of market (bull, bear, no-trend etc.) is unknown.

The dynamic resetting of floor or multiplier can change the strategy dramatically. For
example (from Perold & Sharpe 1988), assume that the investor (with \( m = 2 \)) resets the
floor at every time point at 80% of the wealth in risky assets: \( E_t = m(W_t - 0.8W_t) =
0.4W_t \), which leads to constant proportion (40/60, stock/bill) strategy. It is, contrary to
CPPI, a concave allocation strategy and therefore suitable for very different markets than
CPPI. The same concavity occurs in an example of Wilcox (2000) with parameter values
\( F = -100\% \) and \( m = 0.5 \). Other example is introducing resetting rules for the multiplier
(e.g. \( m_t = f(C_t) \)), which can lead to an option-based portfolio insurance strategy (OBPI)
that, again, has different characteristics from CPPI.

3 Results

In order to compare different strategies, we assume that the initial wealth is \( W_0 = 100 \),
risk-free interest \( r_f = 4\% \) and the asset price development is assumed to be uncorrelated
with \( r_f \). The rebalancing interval is two weeks (\( dt = 24^{-1} \)) and the time span six months
(\( t = \frac{1}{2} \)). The asset price models the Helsinki CAP (see Sec. 2.2) and for the discrete
version, we use the constructed binary pricing tree model.

We want to compare simple CPPI strategies with arbitrary \((m,F)\)-pairs and therefore
realistic bounds for control variables must be set. From Figure 1 we see that with low
floor values, the volatility in continuous time exceeds 100% when \( m \) is over 5 and even
with floor values close to initial wealth \( W_0 = 100 \), volatility tends to reach excessive
high values with high enough \( m \). Based on the figure, we restrict \( m \leq 10 \) (although the
theoretical bound is \( 1/d = 1/0.046 = 21.65 \)) and the minimum floor is set to 50.
Table 1: The results of chosen portfolio allocation strategies.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>a) BH</th>
<th>b) BH</th>
<th>c) CPPI</th>
<th>d) CPPI</th>
</tr>
</thead>
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<td>(\eta = 0.9)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(m)</td>
<td>-</td>
<td>-</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>(F)</td>
<td>-</td>
<td>-</td>
<td>80</td>
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</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>e) CPPI-L</th>
<th>f) CPPI-L</th>
<th>g) CPPI-C</th>
<th>h) CPPI-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>(\lambda = 0.45)</td>
<td>(\lambda = 0.75)</td>
<td>(R = 15)</td>
<td>(R = 30)</td>
</tr>
<tr>
<td>(m)</td>
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<td>1.05</td>
<td>2.17</td>
<td>3.34</td>
</tr>
<tr>
<td>(F)</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

### 3.1 The continuous CPPI model results

We first compare the strategies in the continuous case. We examined eight different strategies (with two parameter set each): buy-hold with parameter \(\eta\), CPPI with chosen \((m,F)\), CPPI achieved from optimised mean-LSAD model with parameter \(\lambda\) (CPPI-L) and CPPI with CVaR constraint \(R\) (CPPI-C). The significance level \(\beta\) is 0.95. The strategies and corresponding parameters are presented in Table 1 and the results are plotted in Figure 4.

Figure 4 shows only minor differences between \(\sigma\) and CVaR as a risk measure. For example, the strategy g dominates c in CVaR-sense, but in traditional \(\mu-\sigma\)-sense domination does not apply. Interestingly freely chosen CPPI strategies c and d do not belong into efficient frontier in either case. Another remark is that BH strategies are efficient with both risk measures. Altogether this indicates that BH might offer good results for a conservative investor and, when seeking for higher profits than the BH can offer, the investor should choose the CPPI parameters with care.

In general the solving of optimization problems caused numerical problems and sometime the optimum was not found even after 2000 evaluation rounds. Even the changing of the initial guess for the fmincon optimization routine of MATLAB did change the optimization results sometimes. Therefore focusing on single optima points can be misleading.
Figure 4: $\mu$-$\sigma$ and $\mu$-CVaR diagram of chosen strategies, for legend see Table 1

With mean-LSAD optimization, a closer look to objective function values with different $\lambda$ shows (see Figure 5) that with low $\lambda$, the investor is willing to choose low floor values and high multipliers. Respectively, with high $\lambda$, the investor favours high floor values and low multipliers. Interestingly, with high $\lambda$ and low enough $m$, the floor choice hardly makes any difference (the contour line is straight). When $\lambda = 0.6$, the contour lines show that there exists a more complicated relationship between the multiplier and floor and there seems to be some kind of global minimum in $m \approx 16$. However, with $m < 10$ (as we set before), the behaviour of contour lines concurs with $\lambda = 0.9$.

The evaluation of CVaR values (with $m \in \{1, 2, \ldots, 10\}$ and $F_0 \in \{50, 55, \ldots, 95\}$) shows some interesting results. Keeping both parameters constant yields the iso-lines of Figure 6. When interpreting the lines, in the case of constant floor the corresponding multiplier values increase when moving right. Respectively, when $m$ is kept constant, the value of $F$ decreases while moving right. The lines show that the floor should always be set to
its minimum, while the iso-line of $F = 60$ dominates every other $F$. I.e. after choosing the wanted CVaR, the corresponding multiplier value can be found from the highest (i.e. lowest $F$) iso-line. This is not seen immediately when looking the iso-lines with fixed $m$ in the right side figure. Then the choice of $m$ (and corresponding $F$) seems to be case-spective, depending on the wanted value of CVaR. For example the ($m,F$) corresponding to CVaR= 20 would be (approx.) (8,80) (in the initial part of the iso-line $m = 8$), while CVaR= 25 would imply choosing (4,50) which is the end of iso-line $F = 80$. Adding infinite amount of iso-lines, however, would create an efficient line created by iso-line maxima (where $F$ is at minimum i.e. 50), which is marked in Figure 6 with a dotted line. Therefore the figures concur with each other, and also with the optimization results of the CVaR strategies in Table 1: in the optima strategies (resulting from solving the problem (11)) the floor is at its minimum in either case.
3.2 The discrete CPPI model results

The above results in continuous time do not automatically apply for real-world investing. Similar $\sigma$-CVaR approach can however be done, for all key figures can be calculated for the discrete case as well. The expected value and volatility for arbitrary wealth vectors are already given in equations (20) & (21), and the CVaR is achieved (among other techniques) with the help of MATLAB's quantile-function. It gives the critical value corresponding to $(1 - \beta)$-tail of the empirical distribution of data, i.e. the value at risk (VaR). The expectation of the tail below VaR is therefore

$$\mathbb{E}[W_{t,1-\beta}] = \frac{\sum_i (P_i W_i)}{\sum_i P_i} \quad \forall i : W_i \leq VaR$$

and it follows that $CVaR = \mathbb{E}[W_t] - \mathbb{E}[W_{t,1-\beta}]$. To illustrate the difference between the continuous and discrete strategies, we use the exactly same CPPI-strategies as in Table 1 and calculate the terminal wealths and key figures with the discrete CPPI. The results are presented in Figure 7.
Figure 7: $\mu$-$\sigma$ and $\mu$-CVaR diagram of chosen discrete strategies, for legend see Table 1.

The results show that the continuous and discrete CPPI strategies yield significantly different (absolute) results. There are also some qualitative differences, i.e. some inefficient strategies in continuous time are efficient in discrete time (see e.g. strategy e) and vice versa. The differences are not discussed here further, though one quite general conclusion can be made. The (statistically significant) correlation coefficient of volatilities of continuous and discrete results is 0.98 and between CVaR’s 0.87, so clearly the results of continuous time give results that apply (as suggestive) to the discrete time as well.

3.3 Possible extensions

In order to verify that simple restrictions to market dynamics can change the parameter choice drastically, we calculate the expected wealth, volatility and CVaR in discrete time for two CPPI strategies with and without these extensions. The strategies are $m = 4$, $F_0 = 80$ and $m = 8$, $F_0 = 60$, and the extensions include a borrowing constraint of 20%
of the current wealth (i.e. \( b = 1.2 \) in Equation (26)) and a constraint for the proportion of risky assets, which is set to 80% (\( \eta = 0.8 \) in Equation 27). The results are plotted in Figure 8.

![Figure 8: \( \mu-\sigma \) and \( \mu-CVaR \) diagram of chosen discrete strategies with extensions, for legend see Table 2](image)

The results show that with low \( m \) values (strategies a, c and e), the extensions hardly make a difference. With higher \( m \), however, the differences are considerable. Both restrictions lower the expected wealth significantly, but the risk is reduced respectively.
4 Conclusions

This study presented some tools for evaluating CPPI in asset allocation. The continuous and discrete version were compared and it was found that they do not yield identical results. Alongside with the traditional Markowitz-framework, some risk measures were considered. We think that because of the insurance nature of CPPI, it is pertinent to use downside risk measures instead of standard deviation or other symmetric risk measures when defining the optimal parameters. We also noted that changes in market dynamics can change the setting quite a bit, so the simple CPPI results should not be transferred into more complicated market without careful consideration. One general result considering the CPPI evaluation was that all differences among optimization strategies, time discretization or restrictions are amplified with the multiplier. That is, the higher $m$, the larger differences occur.

When defining CPPI parameters, optimization of the mean-LSAD objective function or expected wealth with CVaR restrictions appears to function quite well. The problem of LSAD is defining a numerical value to $\lambda$, which reflects the investors attitude towards risk. In that sense, the CVaR approach seems more practical. In the study, we evaluated different strategies mostly by solving the optimization problems numerically, which at times suffered from numerical problems. However, it might be possible to find an analytical solution to presented problems, which could lead to deterministic rules for parameter choice, e.g. derive an explicit formula depending on parameters $R$ and $\beta$ in the CVaR case. Although this approach is theoretically appealing, its practical use might be remote.

There are also some questions that may be interesting for future studies. First, the sensitivity analysis (with respect to $\mu$ and $\sigma$ of the stock) was not contained in the study. The formulas for expected wealth and variance (in continuous time) included several exponential terms with $\mu$ and $\sigma$, which suggests that the results might change drastically with different market parameters.

Moreover, with many extensions, the solution of presented (and other) optimization problems can become computationally demanding. For example the terminal wealth is path-dependent when market dynamics become more complicated. If we consider an
example of 12 timepoints, which yields $2^{12} = 4096$ different price paths of length 12, the terminal wealth calculations include nearly 50 000 points. When $t$ grows, the amount rises quickly to levels where computations are impossible to accomplish.

The analytical treatment of the discrete time CPPI was ignored. However, while the stock price can be modelled as a function of a binary random variable, it should be possible to define an explicit formula for the terminal wealth. This would ease the computational problems as $t$ grows larger.

Third, the impact of different extensions and market restrictions was only superficially covered. This is interesting, as there are restrictions (such as the examples presented in the study) in investment strategy planning. This could also include the consideration of different resetting strategies, i.e., when does the balancing of portfolio take place. This can be a question of choosing a suitable time interval, but it also includes alternatives such as rebalancing whenever the stock price variation exceeds a certain level (e.g. 5%).

Finally, the CPPI could be benchmarked with more sophisticated strategies. For example, consider an investor who seeks to find an optimal investment strategy without deterministic allocation rules. That is, to adjust the proportion of risky assets at every time point arbitrarily (or with chosen restrictions), maximising the expected terminal wealth. Such a dynamic optimization problem could be approached by Contingent Portfolio Programming framework (Gustafsson & Salo), which suits well for few rebalancing timepoints (e.g. strategy for 2 years and rebalancing twice a year). However, if the amount of timepoints becomes larger (larger time span and/or more frequent rebalancing), the amount of decision variables ($2^t \cdot t$) can pose serious challenges to numerical problem solving. Still, especially for fairly inactive investors, the CPP-based approach offers an interesting alternative to CPPI.
References


