Arbitrage Pricing of Currency Derivatives

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1 Introduction

Financial markets can be roughly divided into two types of instruments. Basic instruments, called underlying instruments and the derivative instruments, often referred to as contingent claims. The price of underlying instruments is usually determined by the financial market and the payoff profile they generate is linear. Examples of underlying instruments are stocks, currencies and commodities.

there exists a wide variety of contingent claims on the other hand. As their name implies, their price dynamics are contingent on the underlying assets they refer to. Many standardized derivatives, such as futures and vanilla options, have prices quoted on the markets. But in addition to the standardized derivatives, there is a vast amount of non-standard derivatives, often called OTC (over-the-counter) derivatives.

Modeling of asset dynamics and valuing of contingent claims have posed challenges in the financial markets for a long time. Before the seminal paper by Black and Scholes [1973], there had not been a rigorous framework for calculating arbitrage-free prices. Black and Scholes [1973] laid the ground work for modeling asset dynamics and calculating no-arbitrage prices for simple contingent claims.

But as the ability to value contingent claims has evolved, so has the complexity of the contingent claims themselves. In this study, we introduce the basics of arbitrage pricing theory. The principles of arbitrage pricing theory are flexible enough that they are the still used today to value many of the present day complex contingent claims.


2 Mathematical tools

In the first section, we will briefly introduce selected mathematical tools needed in order to define the arbitrage pricing theory. The key tools are basic understanding of stochastic differential equations, and specifically martingales and Girsanov's theorem. The Itô formula, i.e. the "chain rule" for stochastic differentials is also needed and readers unfamiliar with it are referred to introductory books on stochastic calculus, e.g. Williams [1991].

Definition 2.1 A process $X_T$ is a martingale if the following holds:

1. $X_t$ is adapted to $\mathcal{F}_t$. This means that $\mathcal{F}_t$ includes all the information about $X$ up to time $t$.

2. $X_t$ is integrable, i.e. all the relevant expectations exist.

3. $\mathbb{E}[X_T|\mathcal{F}_t] = X_t$.

The above states that in regards to diffusion processes a martingale process $X_t$ is driftless, i.e. the expected value of the process $X$ at a future date $T$ given the information at current time $t$ is the current value of the process, $X_t$.

Let us also define the equivalent martingale measure:

Definition 2.2 Equivalent martingale measure $\mathbb{Q}$ is a probability measure on the space $\{\Omega, \mathcal{F}\}$ with the following properties:

1. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures ($\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$, $\forall A \in \mathcal{F}$),
2. Radon-Nikodym derivative $dQ/dP$ is square integrable with respect to $P$,

3. discounted asset price process $D(0, t)S$ is a $Q$-martingale, i.e. $E_Q(D(0, t)S^k_t|F_u)$, $\forall k \in [0, K]$ and $0 \leq u \leq t \leq T$.

Thus equivalent martingale measures $P$ and $Q$ share the same set of null probability (and equivalently probability one). In other words, an event that holds almost surely on $P$ also holds almost surely on $Q$.

When two measures are equivalent, one measure can be expressed in terms of the other, using the Radon-Nikodym derivative. The theory states that there exists a martingale process $\rho_t$ on $(\Omega, \mathcal{F}_t, P)$ so that

$$P^*(A) = \int_A \rho_t(\omega)dP(\omega), \quad A \in \mathcal{F}_t,$$

Which can equivalently be written as

$$\frac{dP^*}{dP} \bigg|_{\mathcal{F}_t} = \rho_t.$$

The process $\rho_t$ is called Radon-Nikodym derivative of $P^*$ with respect to $P$ restricted on $\mathcal{F}_t$.

The Girsanov theorem is based on the fact that the drift in a stochastic differential equation depends on the particular probability measure $P$ in the probability space $(\Omega, \mathcal{F}_t, P)$. It defines the principles of modifying the drift of a SDE in an intelligent way by defining a new probability measure via a suitable Random-Nikodym derivative. This way one can be greatly facilitate the calculation of expected value of a discounted payoff.

**Definition 2.3** Girsanov theorem states that if $W_t$ is a $P$-Brownian motion and $\gamma_t$ a $\mathcal{F}$-previsible process that satisfies the boundedness condition

$$E^P(\exp(\frac{1}{2} \int_0^T \gamma_t^2 dt)) < \infty$$

then there exists a measure $Q$ such that
1. \( Q \) is equivalent to \( P \)

2. \( \frac{dQ}{dP} = \exp \left( - \int_0^T \gamma_t dW_t - \frac{1}{2} \int_0^T \gamma_t^2 dt \right) \)

3. \( \tilde{W}_t = W_t + \int_0^t \gamma_s ds \) is a \( Q \)-Brownian motion

and conversely within some regularity conditions

**Definition 2.4** The Girsanov converse

If \( W_t \) is a \( P \)-Brownian motion, and \( Q \) is a measure equivalent to \( P \), then there exists a \( \mathcal{F} \)-previsible process \( \gamma_t \) such that

\[ \tilde{W}_t = W_t + \int_0^t \gamma_s ds \]

is a \( Q \)-Brownian motion.

More formal treatment of Girsanov's theorem can be found in e.g. Musela and Rutkowski [1998].

**Definition 2.5** Martingale representation theorem

Suppose that \( M_t \) is a \( Q \)-martingale process, whose volatility \( \sigma_t \) satisfies the additional condition of being non-zero with probability one. Then if \( N_t \) is any other \( Q \)-martingale, there exists a \( \mathcal{F} \) previsible process \( \phi_t \) such that

\[ \int_0^T \phi_t^2 \sigma_t^2 dt \leq \inf \text{ with probability one, and } N_t \text{ can be written as} \]

\[ N_t = N_0 + \int_0^t \phi_s dM_s \]

or equivalently

\[ dN_t = \phi_s dM_s. \quad (2.1) \]
3 Key concepts of arbitrage pricing in continuous time

We review briefly some elementary concepts of arbitrage pricing theory, to show the general idea of pricing a contingent claim.

Let’s consider a financial market with two assets. Let $S_0(t) = B(t)$ be the risk free prototypical bank account asset and $S_1(t)$ the risky asset with the respective dynamics:

$$dB(t) = rB(t)dt,$$
$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t).$$

A trading strategy $\phi_t$ is specified with weights $\phi_t = w_i^t$ of the assets $S_i(t)$ at time $t$. The weights are assumed to be non-negative, and add up to one. The value process $V(\phi)$ of the trading strategy is defined as

$$V_t(\phi) = \sum_i w_i^t S_i(t).$$

A trading strategy is self financing, if

$$dV_t(\phi) = \sum_i w_i^t dS_i(t),$$

or equivalently,

$$V_t(w_i^t) = V_0(\phi) + \sum_i \int_0^t w_i^s S_i(s).$$

Intuitively, a trading strategy is self-financing if the value of the strategy changes only due to the changes in the values of the assets in the strategy. In other words, there are no inflows or outflows of capital occurring after the initial time $t = 0$. It is trivial to expand the above to include multiple risky assets.
The fundamental idea behind arbitrage pricing theory is that there are no arbitrage opportunities in the financial markets. An arbitrage opportunity arises, if one can construct a self-financing trading strategy such that:

\[ V_0(\phi) = 0 \]
\[ \mathbb{P}(V_T(\phi) \geq 0) = 1 \]
\[ \mathbb{P}(V_T(\phi) > 0) > 0 \]

That is, in order for the market to be arbitrage free, there cannot be a possibility to construct a trading strategy with an initial value of zero and a non-negative final value so that there is a positive probability for the trading strategy to have a greater than zero value at the final day.

Requiring arbitrage freeness has fundamental consequences for price dynamics in the markets. A key result in Harrison and Pliska [1981] is the connection between the economic concept of absence of arbitrage with the mathematical concept of existence of an equivalent martingale probability measure (the risk-neutral measure).

Let us now define an attainable contingent claim:

**Definition 3.1** A contingent claim is a square-integrable and positive random variable on \((\Omega, \mathcal{F}, \mathbb{P})\). A contingent claim \(H\) is attainable if there exists a self-financing trading strategy \(\phi\) such that \(V_T(\phi) = H\). Then \(\phi\) is said to generate \(H\), and \(\pi_t = V_t(\phi)\) is the price of \(H\) at time \(t\).

The following proposition by Harrison and Pliska [1981] provides a mathematical formulation of the unique no-arbitrage price associated with any attainable contingent claim.
Proposition 3.1 Assume there exists an equivalent martingale measure $\mathbb{P}$ and let $H$ be an attainable contingent claim. Then, for each time $t, 0 \leq t \leq T$, there exists a unique price $\pi_t$ associated with $H$:

$$\pi_t = \mathbb{E}(D(t,T)H|\mathcal{F}_t).$$

(3.1)

The above result generalizes the seminal result of contingent claim pricing by Black and Scholes [1973] in that it applies to pricing of any contingent claim, which, in particular, can be path dependent. In addition, the underlying asset price dynamics is quite general. It states that when the set of equivalent martingale measures is non-empty, it is then possible to derive a unique no-arbitrage price associated with any attainable contingent claim. Such a price is given by the expectation of the discounted claim payoff under the measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.

4 Pricing of a Contingent Claim

Let us now use the introduced tools of arbitrage pricing to price a contingent claim. In this example, the contingent claim of interest is a forward contract and an option and the risky asset is currency.

Let’s consider an investor interested in euro dominated investments wanting to agree on the value of one dollar in euros at some future date $T$. This can be accomplished with a static replicating strategy for the trade: the investor buys dollars and sells euros against them. The currencies yield different interest however. So the replicating strategy is the following:

- go long $e^{-rT}$ units of dollar bond
- go short $C_0e^{-sT}$ units of euro bond
where $C_0$ denotes the current exchange rate between dollar and euro. The initial value of this replicating strategy is zero, and at maturity $T$ the dollar holding will be one dollar as requested.

Here we will use a log-normal model for the currency exchange rate $C_t$, to agree with the Black-Scholes currency model. In addition we have two other processes to model: a euro denominated bond $B_t^€$ and a dollar denominated bond $B_t^\$.

So, to summarize, the three processes we need are:

- Euro bond $B_t^€ = e^{rt}$
- Dollar bond $B_t^\$ = $e^{st}$
- Exchange rate $C_t = C_0 e^{\mu t + \sigma W_t}$

for some $W_t$ a $\mathbb{P}$-Brownian motion and $r, s, \mu, \sigma$ constants.

Let us find the replicating strategy for our model. We have the euro bond $B_t^€$ as our riskless asset and we need a risky asset $S_t$ that is both euro denominated and tradeable. Neither of the Dollar bond $B_t^\$ and Exchange rate $C_t$ is such by itself, but combining them we have $S_t = C_t B_t^\$ as a natural risky asset. With the simple translation we have the basic Black-Scholes set up for our model.

Now, the discounted asset price is

$$Z_t = (B_t^€)^{-1} B_t^\$ C_t = C_0 e^{(\mu + s - r)t + \sigma W_t}.$$ 

Using Itô, the SDE for $Z_t$ becomes

$$dZ_t = (\mu + s - r + \frac{1}{2} \sigma^2) Z_t dt + \sigma Z_t dW_t.$$  \hspace{1cm} (4.1) \hspace{1cm} \hspace{1cm}

Now we want to find an equivalent martingale measure $Q$ under which $Z_t$ is a martingale. We let $\gamma_t = \gamma = (\mu + s - r + \frac{1}{2} \sigma^2)/\sigma$. Then, as $\gamma_t$ is obviously adapted to $\mathcal{F}_t$ and satisfies the boundness condition, Girsanov’s
Theorem states that there exists a measure \( Q \) such that \( W_t = \tilde{W}_t + \gamma t \) is \( Q \)-Brownian motion. Substituting this to (4.1) we have

\[
dZ_t = \sigma Z_t d\tilde{W}_t.
\]

Solving this we get

\[
Z_t = C_0 e^{-\frac{1}{2} \sigma^2 t + \sigma \tilde{W}_t}.
\]

and thus

\[
C_t = C_0 e^{(r-s-\frac{1}{2} \sigma^2)t + \sigma \tilde{W}_t}.
\]

Now, let us define the conditional expectation process \( E_t = \mathbb{E}_Q^Q(B - 1 T X | \mathcal{F}_t) \).

We can make a replicating strategy of \( X_t \) by holding

- \( \phi_t \) units of the dollar cash bond
- \( E_t - \phi_t Z_t \) units of euro cash bond

The euro value of the replicating strategy at time \( t \) is \( V_t = \phi_t S_t + (E_t - \phi_t Z_t)B_t = B_t E_t \) and from (3.1) it is easy to see that \( X_t \) is attainable.

Let us now calculate a price of a forward contract and a call option on the exchange rate. For a forward contract, our payoff at time \( T \) is

\[
X = C_T - F,
\]

where \( F \) is the forward exchange rate. The price of the payoff at time \( t \) is

\[
\pi_t = B_t \mathbb{E}_Q(B^{-T}X | \mathcal{F}_t) = e^{-r(T-t)} \mathbb{E}_Q(C_T - F | \mathcal{F}_t).
\]

The no-arbitrage forward price at time zero for the contract is

\[
F = \mathbb{E}_Q(C_T) = \mathbb{E}_Q(C_0 e^{(r-s-\frac{1}{2} \sigma^2)T + \sigma \tilde{W}_T}) = C_0 e^{(r-s)T}.
\]
A call option on the exchange rate gives the holder of the option the right but not the obligation to buy a dollar at time $T$ in the future for the price of $k$ euros. The call option payout at time $T$ is

$$X = (C_T - k)^+,$$

where $k$ is called the strike of the option. The price of the payoff at time $t$ is again $\pi_t = B_t\mathbb{E}_Q(B_{-T}X|\mathcal{F}_t)$ and since we know $C_T$ is log-normally distributed, we can use the following probabilistic result:

**Proposition 4.1** If $Z$ is a normal $N(0, 1)$ random variable and $F, \hat{\sigma},$ and $k$ are constants, then

$$\mathbb{E}\left((Fe^{\hat{\sigma}Z - \frac{1}{2}\hat{\sigma}^2} - k)^+\right) = F\Phi\left(\frac{\log F + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}\right) - k\Phi\left(\frac{\log F - \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}\right)$$

Now, we have the forward price $F = \mathbb{E}_Q(C_T)$ and the value of $C_T$ can be written as $Fe^{\hat{\sigma}Z - \frac{1}{2}\hat{\sigma}^2}$, where $\hat{\sigma}^2$ is the variance of $\log C_T = \sigma^2 T$ and $Z = N(0, 1)$ is normal under $Q$ then the option price at time zero is

$$\pi_0 = e^{-rT}\left\{F\Phi\left(\frac{\log F + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - k\Phi\left(\frac{\log F - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\}$$

5 Conclusions

In this study, we have introduced the reader to the arbitrage pricing theorem. In this study we used the basic steps of arbitrage pricing to price simple contingent claims, forwards and vanilla call options when the underlying asset was an exchange rate. But we would like to point out, that the same principles apply even when pricing more complicated, and possibly path dependent, contingent claims. And with slight modification that can be found
for example in Oksendal [2005] the same model can be used when there are multiple underlying assets. Such needs arise especially when pricing contingent claims on interest rates.


References


