Aalto University
School of Science
Degree Programme in Engineering Physics and Mathematics

Robust Portfolio Modelling for Allocating Resources to Decision Making Units

Mat-2.4108 Independent Research Project in Applied Mathematics
December 30, 2013

Juho Andelmin

This work was supervised by Prof. Ahti Salo and instructed by Dr. Juuso Liesiö.

This document can be stored and made available to the public on the open internet pages of Aalto University. All other rights are reserved.

Contents

1 Introduction

Portfolio decision analysis (PDA) aims to facilitate the process of selecting an optimal subset (or *portfolio*) from a large, discrete set of alternatives (e.g. projects) subject to various resource constraints, preferences and uncertainties (see, e.g. Salo et al., 2011). These resources can be either financial or non-monetary, such as human resources. PDA has been successfully applied in several case studies; for example, a resource allocation model for standardization activities in a major telecommunication company is developed in (Toppila et al., 2011). In this model, several standardization activities (projects) are evaluated in view of different funding levels, and expected sales are maximized in the presence of incomplete information and project interactions. Subsequently, decision recommendations are given on which standardization activities should either be strengthened, weakened or maintained at their current funding level.

Despite the success in numerous applications, most PDA models are best suited for portfolio selection problems where different alternatives (projects) are evaluated subject to a discrete set of resource funding levels. In most applications, the resulting model is a binary one, where projects are either fully funded or otherwise discarded, and the objective is to find an optimal portfolio subject to budget constraints (see, e.g. Liesiö et al., 2007, 2008). This approach is adequate for most situations; however, in many resource allocation problems it would be beneficial to adjust the funding levels continuously. Such resource allocation problems are typically encountered in organizations (both private and public) with a centralized decision-making environment, such as hospitals, universities and supermarket chains. In such organizations, several decision making units (DMUs) are operating under a central unit (decision maker, DM) with power to manage resources of those units (see, e.g. Korhonen and Syrjänen, 2004). The DM is assumed to be interested in allocating available resources such that the aggregate amount of outputs produced by the DMUs is maximized and/or the total consumption of inputs is minimized. The individual DMUs typically consume multiple inputs and produce multiple outputs, wherefore the allocation problem becomes a multi-criteria one that has no unique optimal solution.

In this paper, we present multiple objective linear programming (MOLP) models to support the DM allocate resources by taking into account the total consumption of inputs and overall production of outputs simultaneously. Additionally, we demonstrate how these models can be modified to incorporate the DM's preferences on the unit values of different inputs and outputs and identify those solutions that are non-dominated with regard to such preference information. Decision recommendations can subsequently be given on, for example, which ongoing projects should be terminated to free allocated resources and/or which new projects can be accepted within a given budget. Furthermore, we demonstrate how these models can be used to evaluate existing projects, thus providing managerial insight into where available resources should be (re-)allocated, i.e. which projects should receive additional resources and from which projects those resources should be taken from in order to improve the overall system performance.

The rest of this paper is structured as follows. Section 2 presents the general resource allocation model and demonstrates how to incorporate DM's preferences. Section 3 provides methods for estimating feasible allocations based on either current resource levels or expert opinions. Section 4 presents the computation of non-dominated portfolios. Section 5 presents an illustrative example, demonstrating how the developed models can be used in a real-life problem to facilitate the decision-making process. Section 6 concludes.

2 Resource allocation model

In what follows, we develop a general resource allocation model for organizations with a centralized decision making environment, where the decision maker (DM) aims to allocate available resources to several decision making units (DMUs) so that the total consumption of inputs is minimized and the the overall production of outputs is maximized. Such organizations can be private businesses, such as banks or supermarkets, or public institutions, such as universities or hospitals.

Let us assume that we have n DMUs operating under the DM, each consuming m different inputs and producing s different outputs. Let $x^j = (x_1^j, \ldots, x_m^j)^T \in \mathbb{R}^m$ denote the inputs and $y^j = (y_1^j, \ldots, y_s^j)^T \in \mathbb{R}^s$ the outputs of DMU j. Any combination of inputs and outputs $(x, y)^T \in \mathbb{R}^{m+s}$ is called a *mix*. The *production possibility set* P^j contains those mixes that are achievable for DMU j. Throughout this paper, we assume that the sets P^j, $j \in \{1, \ldots, n\}$, are convex.

Definition 1. $P^j \subset \mathbb{R}_+^{m+s}$ *is the production possibility set for DMU* j.

In the portfolio context, DMUs can be considered as ongoing projects or project proposals that can be combined to form project portfolios. These portfolios are formed by combining the inputs and outputs of the individual projects (DMUs). All such portfolios form the *portfolio possibility set.*

Definition 2. *Portfolio possibility set is*

$$P := \{(x, y)^T \in \mathbb{R}_+^{m+s} \mid x = \sum_{j=1}^{n} x^j, y = \sum_{j=1}^{n} y^j, (x^j, y^j)^T \in P^j\}.$$

The DM is ultimately interested in system-level performance with regard to the aggregate input-output mix $\sum_{j=1}^{n}(x^j, y^j)^T$. According to Definition 2, this mix forms a portfolio: $p = \sum_{j=1}^{n}(x^j, y^j)^T \in P$; thus, adjusting the allocation of resources among the existing projects can be considered, for instance, as a continuous portfolio management problem, in which the objective is to find the optimal allocation in view of the current portfolio.

Without loss of generality, we can assume that the inputs are resources to be allocated, and the DM seeks to minimize the total resource consumption and simultaneously maximize the overall output production. Accordingly, the general multi-objective resource allocation problem can be formulated as the following multiple objective linear programming (MOLP) model with $m + s$ objectives.

$$\underset{x^j, y^j}{\text{v-max}} \quad \left(\sum_{j=1}^{n} y^j, -\sum_{j=1}^{n} x^j\right) \tag{1}$$

$$\text{s.t.} \quad (x^j, y^j)^T \in P^j \qquad j = 1, \ldots, n \tag{2}$$

$$\sum_{j=1}^{n} x^j \leq B. \tag{3}$$

Constraint (2) ensures that the input-output mixes of the individual DMUs remain within their respective production possibility sets in the allocation. Constraint (3) corresponds to the budget constraint, which limits the total amount of resources used in the allocation: the vector $B = (B_1, \ldots, B_m)^T$ determines an upper bound for each resource type.

Problem (1) - (3) has multiple objectives and no unique optimal solution. Instead, it has several *efficient* solutions to which attention can be focused on (see, e.g. Ehrgott, 2005).

Definition 3. *A portfolio $p = (x, y)^T \in P$ is efficient if and only if there does not exist another $p' = (x', y')^T \in P$ such that*

$$(-x', y')^T \geq (-x, y)^T \text{ and } (-x', y')^T \neq (-x, y)^T,$$

where the above inequalities hold component-wise. Moreover, the set of efficient portfolios is defined as

$$P_E := \{(x, y)^T \in P \mid \nexists (x', y')^T \in P \ s.t. \ (-x', y')^T \gneqq (-x, y)^T\}. \tag{4}$$

Instead of maximizing the aggregate output production and minimizing the total input consumption in the problem (1) - (3), any combination of input and output variables can be optimized. In that case, those inputs and outputs that are not selected as objective functions can be treated as constraints.

2.1 Preference modelling and dominance

Improvements in some inputs and/or outputs are typically considered more valuable than in others. Such preference information can be adopted by multiplying the inputs and outputs of some portfolio $p = (x, y)^T \in P$ by weight vectors $u = (u_1, \ldots, u_m) \in S_u$ and $v = (v_1, \ldots, v_s) \in S_v$ such that $I_u(x)$ denotes the weighted input and $O_v(y)$ the weighted output of the portfolio p, where

$$I_u(x) = \sum_{i=1}^{m} u_i x_i, \tag{5}$$

$$O_v(y) = \sum_{i=1}^{s} v_i y_i. \tag{6}$$

The sets S_u and S_v are called *information sets* that capture preferences of the decision maker; for instance, assuming that the DM values a unit increase in output $i = 1$ more than a unit increase in output $i = 2$ results in a linear constraint $v_1 \geq v_2$. Without loss of generality, the components of the weight vectors $u \in S_u$ and $v \in S_v$ can be scaled so that they sum to one, i.e. $S_u \subseteq \Delta^m$ and $S_v \subseteq \Delta^s$ where

$$\Delta^l = \{w \in \mathbb{R}^l_+ \mid \sum_{i=1}^{l} w_i = 1\}.$$

Unless the information sets S_u and S_v both contain only a single point, the weighted input and output values of the feasible portfolios change depending on the unit values chosen from those sets. Thus, instead of single point estimates, the weighted inputs $I_u(x)$ and weighted outputs $O_v(y)$ form value intervals containing the 'true' values. These value intervals typically overlap; nevertheless, it is possible to establish dominance relations by comparing the weighted input/output values using all feasible weight vectors $u \in S_u$ and $v \in S_v$.

Definition 4. *For any two portfolios $p = (x, y)^T$ and $p' = (x', y')^T$, p dominates p' with regard to information set $S = S_u \times S_v$, denoted by $p \succ_S p'$, if and only if*

$$\begin{aligned} I_u(x) &\leq I_u(x') \quad \textit{for all } u \in S_u, \\ O_v(y) &\geq O_v(y') \quad \textit{for all } v \in S_v, \end{aligned}$$

and one of the inequalities is strict for some $u \in S_u$ or $v \in S_v$.

A rational DM will never select a dominated portfolio from the portfolio possibility set P; hence, attention can be focused on the set of non-dominated portfolios when searching for the most preferred solution to the resource allocation problem.

Definition 5. *The set of non-dominated portfolios with regard to the information set $S = S_u \times S_v$ is*

$$P_N(S) := \{p \in P \mid p' \not\succ_S p \ \forall p' \in P\}. \tag{7}$$

Dominance structure between portfolios is dependent on the information set $S = S_u \times S_v$; the set of non-dominated portfolios $P_N(S)$ changes depending on the preference information. In the discrete case where the number of feasible portfolios is finite, loose preference statements typically result in a large number of non-dominated portfolios, whereas point estimates produce a unique optimal portfolio (see e.g. Liesiö et al., 2008). In the continuous case, accordingly, providing further preference information (stricter preference statements) produces a subset of the initial set of non-dominated portfolios.

Adding further constraints on the feasible weights reduces the initial information set $S = S_u \times S_v$ to $\tilde{S} \subset S$. Consequently, it is possible that some of the portfolios that were non-dominated with regard to the information set S become dominated in view of the new set $\tilde{S}$; however, if a portfolio is dominated initially, it will remain dominated subject to the reduced information set $\tilde{S}$ as well. On the other hand, if no preference information exists, i.e. $S_u = \{u \in \mathbb{R}^m_+ \mid \sum_{i=1}^m u_i = 1\}$ and $S_v = \{v \in \mathbb{R}^s_+ \mid \sum_{i=1}^s v_i = 1\}$, the set of non-dominated portfolios $P_N(S)$ coincides with the set of efficient portfolios P_E, and can thus be calculated with the MOLP problem (1) - (2). These claims are formally stated in Theorem 1. All proofs are presented in Appendix A.

Theorem 1.

(i) *Let* $S = S_u \times S_v$ *and* $\tilde{S} = \tilde{S}_u \times \tilde{S}_v$ *be information sets such that* $\tilde{S} \subset S$ *and* $\operatorname{int}(S) \cap \tilde{S} \neq \emptyset$*. Then* $P_N(\tilde{S}) \subseteq P_N(S)$*.*

(ii) $P_N(\Delta^m \times \Delta^s) = P_E$

2.2 Establishing dominance relations

Both $I_u(x)$ and $O_v(x)$ are linear in u and v, respectively. Thus, dominance between any two portfolios p and p' can be established by comparing their weighted input and output values at the extreme points of the sets S_u and S_v. Let us denote these extreme points by $\operatorname{ext}(S_u) = \{u^1, \dots, u^{t_u}\}$ and $\operatorname{ext}(S_v) = \{v^1, \dots, v^{t_v}\}$ and introduce the following matrices

$$U_{\text{ext}} := \begin{pmatrix} u_1^1 & u_2^1 & \cdots & u_m^1 \\ u_1^2 & u_2^2 & \cdots & u_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{t_u} & u_2^{t_u} & \cdots & u_m^{t_u} \end{pmatrix} \in \mathbb{R}^{t_u \times m} \tag{8}$$

$$V_{\text{ext}} := \begin{pmatrix} v_1^1 & v_2^1 & \cdots & v_s^1 \\ v_1^2 & v_2^2 & \cdots & v_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{t_v} & v_2^{t_v} & \cdots & v_s^{t_v} \end{pmatrix} \in \mathbb{R}^{t_v \times s}. \tag{9}$$

It suffices to compare the values of the weighted inputs and outputs at every extreme point to establish dominance. Consequently, dominance relations can readily be determined with the help of the matrices U_{ext} and V_{ext}.

Lemma 1. *Let portfolios* $p, p' \in P$ *and information set* $S = S_u \times S_v$*. Then*

$$p \succ_S p' \Leftrightarrow \begin{cases} (U_{ext}x, -V_{ext}y) \leq (U_{ext}x', -V_{ext}y') \\ (U_{ext}x, -V_{ext}y) \neq (U_{ext}x', -V_{ext}y') \end{cases}$$

where the inequality $\leq$ holds component-wise.

With this notation, the set of non-dominated portfolios $P_N(S)$ with regard to the information set $S = S_u \times S_v$ can be obtained from the following $t_u + t_v$ objective MOLP problem

$$\underset{x^j, y^j}{\text{v-max}} \quad \left[V_{\text{ext}}\Big(\sum_{j=1}^{n} y^j\Big), -U_{\text{ext}}\Big(\sum_{j=1}^{n} x^j\Big) \right] \tag{10}$$

$$\text{s.t.} \quad (x^j, y^j)^T \in P^j \quad j = 1, \ldots, n \tag{11}$$

$$\sum_{j=1}^{n} x^j \leq B. \tag{12}$$

2.3 Decision recommendations for DMUs

Analyzing how the input consumption and output production of individual DMUs vary across different non-dominated solutions enables the DM to identify those DMUs that have the most potential and are worth expanding further. Additionally, the DM can evaluate if a certain low-performing DMU should be taken out of business completely by allowing the resources of that unit reach zero in the allocation. It is also possible to force the input/output values of some DMUs to remain unchanged in the process by imposing limitations on them. The developed model can hence be regarded as an interactive decision support that allows adjusting the resources of individual DMUs while focusing on the main goal of improving the overall system performance.

In the presence of incomplete preference information, solving the problem (10) - (12) results in a continuous set of non-dominated portfolios $P_N(S)$ that correspond to different production plans (i.e. input-output mixes of the DMUs); these plans are readily available (as decision variables) upon solving the problem. Consequently, the inputs and outputs of the individual DMUs form value intervals containing those input-output mixes that are achievable across the different non-dominated solutions. The minimum and maximum bounds of these intervals ($\min_{(x,y)^T \in P_N(S)} x_i^j$ and $\max_{(x,y)^T \in P_N(S)} x_i^j$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, s\}$ for inputs, similarly for outputs) are realized in the extreme points of the set of non-dominated portfolios; since a suitable solution algorithm, such as Benson's algorithm (Benson, 1998), produces these extreme points, the value interval bounds can readily be obtained by simple pairwise comparisons.

As an implication of Theorem 1, the lower bounds of the value intervals increase and the upper bounds decrease with the introduction of additional preference information. Let S be the information set containing the initial preferences, and let $\tilde{S} \subset S$ be the refined information set resulting from further preference statements. According to theorem 1, we have $P_N(\tilde{S}) \subseteq P_N(S)$; hence, the set of non-dominated portfolios can only become smaller. Consequently, the value intervals can never expand; instead, they can only shrink with additional preference information.

Corollary 1. *Let $S = S_u \times S_v$ and $\tilde{S} = \tilde{S}_v \times \tilde{S}_u$ be information sets such that $\tilde{S} \subset S$ and $\mathrm{int}(S) \cap \tilde{S} \neq \emptyset$. Then*

$$\begin{aligned}
\min_{(x,y)^T \in P_N(\tilde{S})} x_i^j &\geq \min_{(x,y)^T \in P_N(S)} x_i^j, \quad \forall i \in \{1,\ldots,m\}, j \in \{1,\ldots,s\} \\
\max_{(x,y)^T \in P_N(\tilde{S})} x_i^j &\leq \max_{(x,y)^T \in P_N(S)} x_i^j, \quad \forall i \in \{1,\ldots,m\}, j \in \{1,\ldots,s\} \\
\min_{(x,y)^T \in P_N(\tilde{S})} y_i^j &\geq \min_{(x,y)^T \in P_N(S)} y_i^j, \quad \forall i \in \{1,\ldots,m\}, j \in \{1,\ldots,s\} \\
\max_{(x,y)^T \in P_N(\tilde{S})} y_i^j &\leq \max_{(x,y)^T \in P_N(S)} y_i^j, \quad \forall i \in \{1,\ldots,m\}, j \in \{1,\ldots,s\}.
\end{aligned}$$

In view of Corollary 1, the DM can first select loose preference statements, calculate the set of non-dominated portfolios and estimate whether the results are satisfying or not in terms of individual DMUs' input consumption and output production. If the results are not satisfying, an option is to provide further preference information in form of additional weight constraints, thus reducing the value intervals of the input-output mixes. Additionally, imposing stricter limitations on the input and output values of some DMUs may also result in a smaller set of non-dominated portfolios and reduced value intervals.

3 Estimating the production possibility sets for DMUs

3.1 Estimating from observed data using Data Envelopment Analysis

Data Envelopment Analysis (DEA; Charnes et al., 1978) is a non-parametric method that allows constructing the production possibility set (PPS) based on linear combinations of the DMUs' observed (current) input-output mixes. Suitable linear combinations are specified by a set of feasible weight vectors that determines the shape of the PPS and also the underlying DEA model type. The border of the PPS contains the *efficient frontier* in which outputs cannot be increased without also increasing inputs. DMUs belonging to this frontier are characterized as *efficient*, whereas the relative efficiency of an inefficient DMU depends on its distance to the efficient frontier; this distance can be transformed into an efficiency score using different efficiency measures.

Specifically, we assume that the PPS is of the same form for each DMU $j \in \{1, \ldots, n\}$, defined as

$$P^j := \{(x^j, y^j)^T \in \mathbb{R}_+^{m+s} \mid x^j \geq \hat{X}\lambda^j, y^j \leq \hat{Y}\lambda^j, \lambda^j \in \Lambda^f\}, \tag{13}$$

where the matrices $\hat{X} = (\hat{x}^1, \ldots, \hat{x}^n) \in \mathbb{R}^{m\times n}$ and $\hat{Y} = (\hat{y}^1, \ldots, \hat{y}^n) \in \mathbb{R}^{s\times n}$ comprise the observed input-output mixes, and Λ^f denotes the set of feasible weight vectors which determines the shape of the set P^j. In this paper, we assume that Λ^f is a continuous set, defined as

$$\Lambda^f := \{\lambda \in \mathbb{R}_+^n \mid A\lambda \leq b\}, \tag{14}$$

where $A \in \mathbb{R}^{r\times n}$ and $b \in \mathbb{R}^r$ specify r linear constraints. Since the DMUs' observed input-output mixes belong to the PPS by default, it can be assumed that the unit vectors $e^i \in \Lambda^f$ for all $i \in \{1, \ldots, n\}$.

Modifying the set Λ^f produces different DEA models that correspond to different production possibility sets. The two most common models are CCR (Charnes et al., 1978) and BCC (Banker et al., 1984); for these models, the sets of feasible weight vectors are defined as

$$\begin{aligned} \Lambda^{CCR} &:= \{\lambda \in \mathbb{R}_+^n\}, \\ \Lambda^{BCC} &:= \{\lambda \in \mathbb{R}_+^n \mid \sum_{i=1}^n \lambda_i = 1\}. \end{aligned}$$

In the CCR model, all non-negative linear combinations of the DMUs' observed input-output mixes are feasible. CCR model evaluates all DMUs such that those DMUs having the same input-output mix ratio receive same efficiency scores, regardless of their size. This results in a *constant returns to scale* assumption (see, e.g. Charnes, 1994).

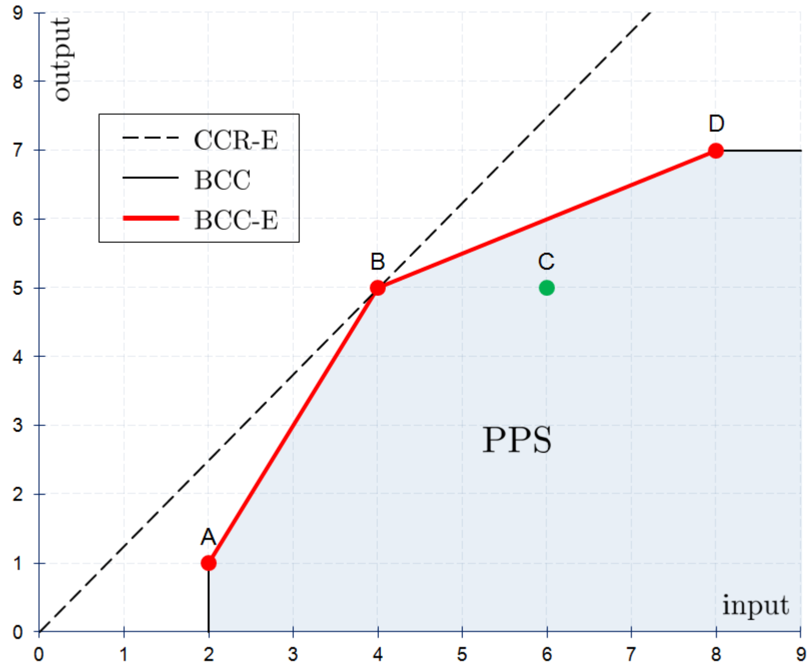

Figure 1: Illustration of the PPS based on both CCR and BCC models. The letter 'E' denotes the efficient frontier of the PPS in the corresponding model.

In the BCC model, on the other hand, only convex combinations of the DMUs' observed input-output mixes are possible. This implies *variable returns to scale* assumption, where the size of a DMU has an impact on its efficiency. Specifically, BCC evaluates the efficiency of a DMU in relation to those DMUs with similar size. This also implies that the smallest and the largest DMUs always receive high efficiency scores.

Figure 1 presents an illustrative example where four DMUs (A, B, C and D), each consuming one input and producing one output, are used to estimate production possibility sets based on both CCR and BCC models. It can be seen that DMUs A, B and D are efficient in the BCC model, whereas only DMU B remains efficient in the CCR model. On the other hand, DMU C is inefficient in both models; however, its efficiency score (with regard to most efficiency measures) is smaller in the CCR model than in the BCC model due to its larger distance to the border of the PPS. More generally, since the PPS in the BCC model is a subset of the corresponding PPS in the CCR model, the relative efficiency of a DMU in the CCR model is always smaller than or equal to that in the BCC model.

Efficient frontier of the PPS contains all the input-output combinations that are efficient, i.e. those input-output mixes whose outputs cannot be increased without also increasing inputs. Accordingly, the efficient frontier P_E^j for each DMU $j \in \{1, \ldots, n\}$ can be defined as

$$P_E^j := \{(x^j, y^j)^T \in P^j \mid \nexists (x', y')^T \in P^j \ s.t. \ (-x', y')^T \gneqq (-x^j, y^j)^T\}. \quad (15)$$

As can be seen in Figure 1, the efficient frontier in the CCR model coincides with the border of the PPS. The efficient frontier in the BCC model, however, does not contain the whole border of the corresponding PPS. Border points that are not contained in the efficient frontier are typically called *weakly efficient* (see, e.g. Charnes, 1994).

3.1.1 Measuring efficiency

DEA provides means for estimating the efficiencies of individual DMUs with regard to the efficient frontier. Specifically, different efficiency measures can be used to calculate efficiency scores that may help the DM focus its attention to those units with most potential. The *input-oriented* efficiency score θ^j of DMU j can be obtained by solving the following linear programming (LP) problem

$$\theta^j = \min_{\theta \in \mathbb{R}} \{(\theta \hat{x}^j, \hat{y}^j)^T \in P^j\}. \quad (16)$$

Accordingly, the input-oriented efficiency score is acquired by reducing the inputs as much as possible while keeping the outputs constant until a point belonging to the border of the set P^j is reached. Similarly, the *output-oriented* efficiency score σ^j of DMU j is obtained by increasing the outputs while retaining constant input values:

$$\sigma^j = \max_{\sigma \in \mathbb{R}} \{(\hat{x}^j, \sigma \hat{y}^j)^T \in P^j\}. \quad (17)$$

In order to make the input- and output-oriented efficiency scores comparable, the inverse value $1/\sigma^j$ of the output-oriented efficiency score of DMU j is usually reported, wherefore both orientations produce efficiency scores in the range $[0, 1]$. Thus, the DMUs located in the (weakly) efficient frontier receive an efficiency score 1, and the score of an inefficient DMU depends on its distance to the (weakly) efficient frontier.

Additionally, other efficiency measures can also be used, such as additive, slacks based (SBM) or hybrid measures (see, e.g. Charnes, 1994). However, they are not considered in this report, but could be incorporated in the models with only minor modifications.

Obviously, the underlying DEA model type affects the efficiency scores of the DMUs. Nevertheless, these scores provide insight into the performances of the individual units in relation to the other DMUs. In some cases, it is reasonable to assume that the relative efficiencies of the DMUs remain unchanged after resource allocation. Golany and Tamir (1995) suggest this kind of approach and claim that it is likely for those DMUs that were inefficient in the past to remain inefficient, despite the best efforts of the decision maker. Constraints that preserve the efficiency scores with regard to input- and output-oriented efficiency measures can be included in the resource allocation model by replacing the constraint (2) with

$$(\theta^j x^j, \sigma^j y^j)^T \in P^j, \quad j = 1, \ldots, n \tag{18}$$

where θ^j and σ^j are calculated with (16) and (17), respectively.

3.2 Estimating the PPS based on expert opinions

In some situations, such as establishing a new organization or managing the resources of future projects, it might be difficult (or impossible) to estimate the PPS based solely on DMUs' observed input-output mixes. On the one hand, such information might not exist; on the other hand, constructing the PPS this way might not produce accurate enough estimates of the DMUs' growth potential. For example, some DMUs might be able to expand their output production beyond the estimated PPS when receiving additional resources. On the other hand, the estimated PPS may overestimate the true growth potential of certain DMUs, especially when no other DMU with similar size exists.

In order to overcome these issues, the PPS for each DMU can be constructed based on information elicited from all the experts involved in the portfolio management process (see, e.g. Toppila et al., 2011). This can be executed by organizing a decision conference and requesting the experts to provide point estimates for DMUs' possible output values with pre-determined resource (input) consumption levels. These levels should be selected so that they provide a rich representation over the possible allocations that the DMUs might be able to obtain. In case there are multiple point estimates (by various experts) for the output values of each resource consumption level, a weighted average of these points, for instance, can be used as a basis for further adjustment. The elicitation process is iterated as long until a consensus about the output values is achieved. The estimated input-output mixes are subsequently used to construct the production possibility set.

Specifically, let $\hat{x}^j(k)$, $k \in \{1, \ldots, q\}$ denote the pre-determined resource consumption levels and $\hat{y}^j(k)$, $k \in \{1, \ldots, q\}$ the corresponding output value estimates for DMU j. Furthermore, let $\hat{X}^j = (\hat{x}^j(1), \ldots, \hat{x}^j(q)) \in \mathbb{R}^{m \times q}$ and $\hat{Y}^j = (\hat{y}^j(1), \ldots, \hat{y}^j(q)) \in \mathbb{R}^{s \times q}$ denote the matrices comprising the q estimated inputs and outputs for DMU j, respectively. With this notation, the PPS for each DMU $j \in \{1, \ldots, n\}$ can be constructed, for example, as convex combinations of the estimated input-output mixes:

$$P^j = \big\{(x^j, y^j)^T \in \mathbb{R}_+^{m+s} \mid x^j = \hat{X}^j \lambda^j, y^j = \hat{Y}^j \lambda^j, \lambda^j \in \mathbb{R}_+^q, \sum_{i=1}^{q} \lambda_i^j = 1\big\}. \quad (19)$$

Figure 2 illustrates the estimated PPS for some DMU j consuming one input x^j and producing one output y^j. In this case, four possible input-output mixes $(x^j(k), y^j(k))$, $k \in \{1, \ldots, 4\}$ are used to estimate the PPS P^j. As implied by (19), P^j coincides with the convex hull of these input-output mixes.

4 Computation of non-dominated portfolios

The set of non-dominated portfolios $P_N(S)$ with regard to the information set $S = S_u \times S_v$ can be obtained by solving the MOLP problem (10) - (12). Before a suitable MOLP algorithm can be applied, some initial preparations are required. First step is to find the extreme points U_{ext} and V_{ext} of the information sets S_u and S_v, respectively; these extreme points can be solved using a suitable vertex enumeration algorithm. In case the PPS is estimated using DEA, the input- and/or output-oriented efficiency scores can be calculated with (16) and (17), respectively, and possibly included in the model. These calculations are LP-problems that are easily solved using any LP routine.

According to Theorem 1, the set of efficient portfolios P_E coincides with the set of non-dominated portfolios when no preference statements have been given, i.e. $P_E = P_N(S_u \times S_v)$ when $S_u = \Delta^m$ and $S_v = \Delta^s$. Consequently, we

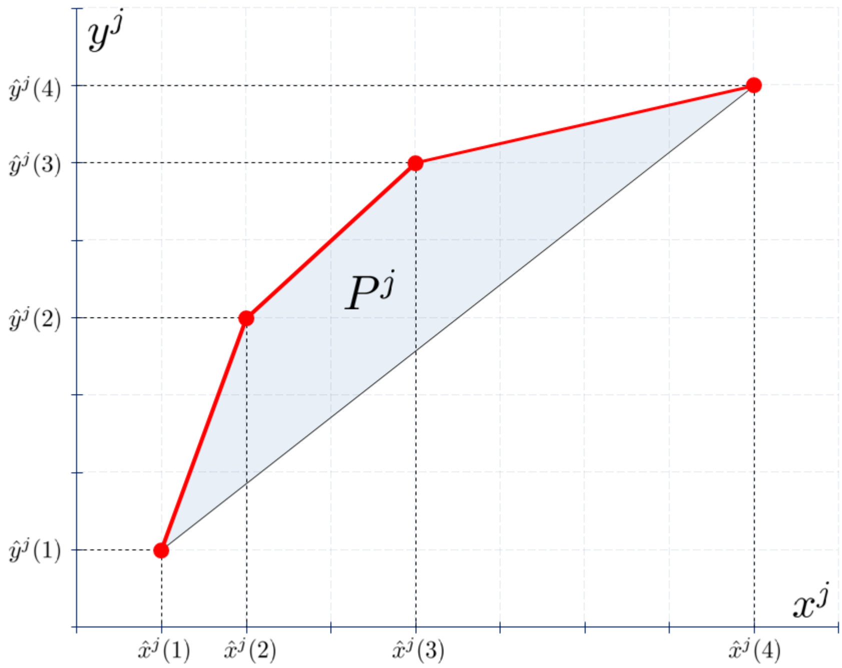

Figure 2: Illustration of the PPS for some DMU j based on (19).

can focus solely on computing the non-dominated portfolios. The resource allocation model (10) - (12) is a continuous $t_u + t_v$ objective MOLP problem, thus having an infinite number of non-dominated solutions. Fortunately, a suitable solution algorithm, such as Benson's algorithm (Benson, 1998) or multi-objective Simplex (Ehrgott, 2005), produces all the extreme points over the set of non-dominated portfolios. Subsequently, all non-dominated portfolios can be constructed as convex combinations of these extreme points, and the DM can select the most preferred portfolio based on its preferences.

The problem (10) - (12) has a total of $n^2 + n(s + m)$ decision variables when the production possibility sets are estimated from observed data by (13); therefore, solving all the non-dominated portfolios becomes computationally demanding when the number of DMUs (n) increases. Problems start emerging especially with the multi-objective Simplex algorithm: its computation time increases rapidly with the number of decision variables. Benson's algorithm, on the other hand, operates in objective space, and its running time is not as dependent on the amount of decision variables (see, e.g. Hamel et al., 2013).

Unlike the multi-objective Simplex, Benson's algorithm can efficiently find all the non-dominated extreme points of a MOLP problem having a substantial amount of decision variables and constraints. However, the performance of the Benson's algorithm depends heavily on the number of objective functions in the problem. Fortunately, this number is relatively small in many practical applications; additionally, in larger problems the number of objective functions can typically be reduced by transforming some of the objectives to constraints instead, and possibly introducing limitations on their values.

In problems with four or more objective functions and several hundred decision variables, Benson's algorithm may take too much time, and the algorithm typically returns several non-dominated solutions that are very close to each other (see, e.g. Shao and Ehrgott, 2008). In such cases, one can use a modified version of Benson's algorithm that approximates the true non-dominated set (Shao and Ehrgott, 2008). The modified algorithm produces ϵ-*non-dominated* solutions: for every true non-dominated point, there exists, within an Euclidean distance less than $\epsilon \in \mathbb{R}_+$, an ϵ-*non-dominated* point in the approximated set. Since the approximation error ϵ is absolute rather than relative, the non-dominated extreme points of the approximated set are close to the actual non-dominated set. However, the modified algorithm produces less non-dominated extreme points when the value of ϵ increases, implying that some of the true non-dominated extreme points are not accounted for. This is likely not a major issue, because the number of non-dominated points is typically overwhelming, even with the modified algorithm with small enough values of ϵ.

5 Illustrative example

To illustrate our framework, we consider a real-life data set consisting of $n = 25$ supermarkets (DMUs) belonging to a Finnish supermarket chain (Korhonen and Syrjänen, 2004). We assume that the DMUs are operating under the same decision maker (DM). Each DMU consumes $m = 2$ inputs, man-hours x_1 and floor area x_2, and produces $s = 2$ outputs, sales y_1 and profit y_2. The DMUs' observed (current) input-output mixes and the corresponding output-oriented BCC efficiency scores are presented in table 1.

Suppose that the DM is planning on hiring new personnel, and is aiming to allocate available man-hours among the DMUs so that the total output production (i.e. sales and profit) is maximized. The amount of new employees to be hired corresponds at most to a 15% increase in total man-hours (budget constraint). Additionally, it is possible to re-allocate the current resources among the DMUs, but with the condition that the output production of

Table 1: Observed input and output values of 25 supermarkets belonging to a Finnish supermarket chain. (Korhonen and Syrjänen, 2004)

Supermarket	Man-hours	Floor area	Sales	Profit	BCC-O
	(10^3 h)	(10^3 m^2)	(10^6 Mk)	(10^6 Mk)	Efficiency
j	$\hat{x}_1^j$	$\hat{x}_2^j$	$\hat{y}_1^j$	$\hat{y}_2^j$	$1/\sigma^j$
1	79.1	4.99	115.3	1.71	0.821
2	60.1	3.30	75.2	1.81	0.772
3	126.7	8.12	225.5	10.39	1
4	153.9	6.70	185.6	10.42	1
5	65.7	4.74	84.5	2.36	0.769
6	76.8	4.08	103.3	4.35	0.806
7	50.2	2.53	78.8	0.16	1
8	44.8	2.47	59.3	1.30	1
9	48.1	2.32	65.7	1.49	1
10	89.7	4.91	163.2	6.26	1
11	56.9	2.24	70.7	2.80	1
12	112.6	5.42	142.6	2.75	0.824
13	106.9	6.28	127.8	2.70	0.673
14	54.9	3.14	62.4	1.42	0.736
15	48.8	4.43	55.2	1.38	0.803
16	59.2	3.98	95.9	0.74	0.978
17	74.5	5.32	121.6	3.06	0.930
18	94.6	3.69	107.0	2.98	0.817
19	47.0	3.00	65.4	0.62	0.969
20	54.6	3.87	71.0	0.01	0.804
21	90.1	3.31	81.2	5.12	0.858
22	95.2	4.25	128.3	3.89	0.876
23	80.1	3.79	135.0	4.73	1
24	68.7	2.99	98.9	1.86	0.973
25	62.3	3.10	66.7	7.41	1
Total	**1 901.5**	**102.94**	**2 586.1**	**81.69**	

each DMU must not decrease. Furthermore, for obvious reasons, it could be difficult (or impossible) to (re-)allocate floor area among the DMUs in a reasonable manner. Hence, the floor area of each DMU is assumed to remain constant in the allocation.

Essentially, the DM strives to maximize the total output production by allocating available man-hours among the DMUs while retaining the floor area of each DMU constant. Suppose that, in addition to this, the DM wishes to perform a benefit-cost analysis to evaluate how much profit and sales can be achieved when the amount of hired man-hours varies between 0% - 15%. Such an analysis could be performed by solving the problem multiple times with different budget constraints. However, an easier method is to introduce the total consumption of man-hours as an objective to be minimized, in addition to maximizing the aggregate sales and profit. Solving the resource allocation problem with these objectives produces all the non-dominated solutions with varying budget constraints in the specified range. The production possibility set for each DMU is estimated from the observed data based on the BCC model, i.e. $\Lambda^f = \Lambda^{BCC} = \{\lambda \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i = 1\}$ in (13). These assumptions lead to the following problem formulation

$$\begin{aligned}
\operatorname*{v-max}_{x^j, y^j, \lambda^j} \quad & \left(\sum_{j=1}^{25} y^j, -\sum_{j=1}^{25} x_1^j \right) && (20)\\
\text{s.t.} \quad & (x^j, y^j)^T \in P^j = \big\{ (x^j, y^j)^T \in \mathbb{R}^{2+2}_+ \mid x^j \geq \hat{X}\lambda^j, y^j \leq \hat{Y}\lambda^j, \\
& \qquad \lambda^j \in \Lambda^{BCC} \big\}, \quad j = 1, \ldots, 25 && (21)\\
& y^j \geq \hat{y}^j, \quad j = 1, \ldots, 25 && (22)\\
& x_2^j = \hat{x}_2^j, \quad j = 1, \ldots, 25 && (23)\\
& \sum_{j=1}^{25} \hat{x}_1^j \leq \sum_{j=1}^{25} x_1^j \leq 1.15 \sum_{j=1}^{25} \hat{x}_1^j. && (24)
\end{aligned}$$

Solving the problem (20) - (24) with the exact Benson's algorithm takes approximately 70 seconds using a laptop with 2.1 GHz dual-core processor and 2 GB RAM. The algorithm returns 87 non-dominated extreme solutions that can be achieved with 644 different production plans (i.e. input/output combinations of the DMUs).

Figure 3 presents the overall sales $y_1^\Sigma = \sum_{j=1}^n y_1^j$ and profit $y_2^\Sigma = \sum_{j=1}^n y_2^j$ that correspond to the non-dominated extreme solutions when the amount of extra man-hours is allowed to vary between 0% - 15%. The two frontiers

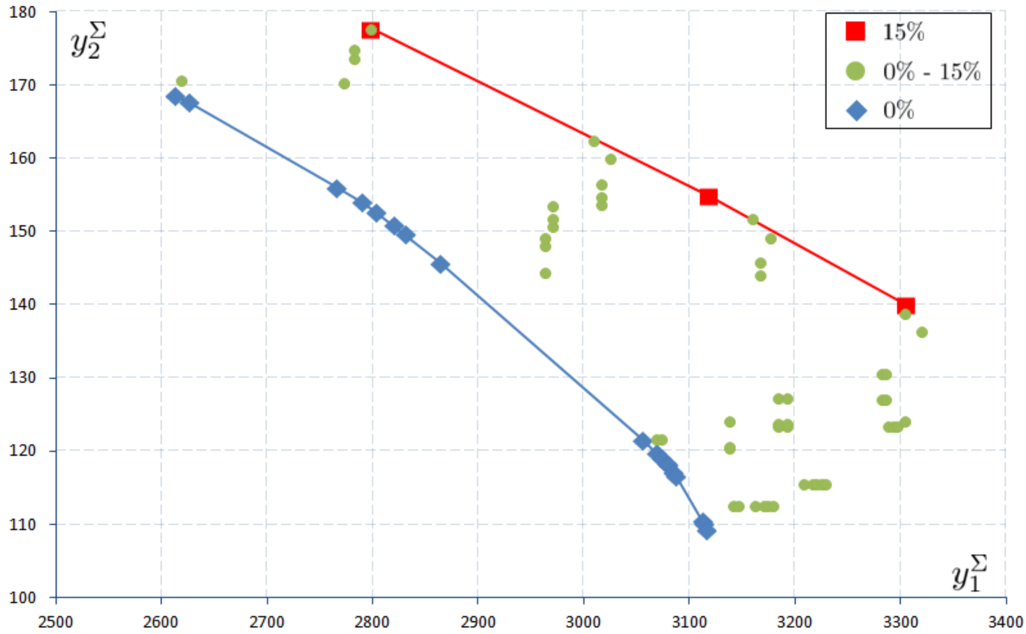

Figure 3: Non-dominated solutions with varying budget constraints that correspond to 0% - 15% increases in total man-hours.

highlighted in the figure correspond to the non-dominated solutions when (i) no new employees are hired and (ii) the maximum amount of new employees are hired (corresponding to a 15% increase in total man-hours). The points located between these two frontiers correspond to the non-dominated extreme solutions when the number of extra man-hours is greater than 0% but smaller than 15%. According to the results, the DM can increase the total sales at most by 23.4%, total profit as much as 117.6% or both more evenly, depending on the selected budget level and the DM's preferences.

Figures 4 and 5 present the range of possible input and output levels for individual DMUs across the non-dominated solutions. The ranges are visualized in terms of relative input and output changes $\Delta x^j = x^j/\hat{x}^j$ and $\Delta y^j = y^j/\hat{y}^j$, respectively. As can be seen in Figure 4, the relative changes in floor area (Δx_2^j) remain unchanged for every DMU $j \in \{1, \ldots, 25\}$ in accordance with (23). The relative changes in man-hours (Δx_1^j), on the other hand, vary for most DMUs across the non-dominated solutions. Interestingly, however, the resources (i.e. man-hours) of DMUs $j \in \{3, 4, 10, 11, 23, 25\}$ remain unchanged, resulting in constant sales and profit as well, as can be seen in Figure 5. This can be explained by examining Table 1; accordingly, all the DMUs mentioned above are characterized as efficient (based on their effi-

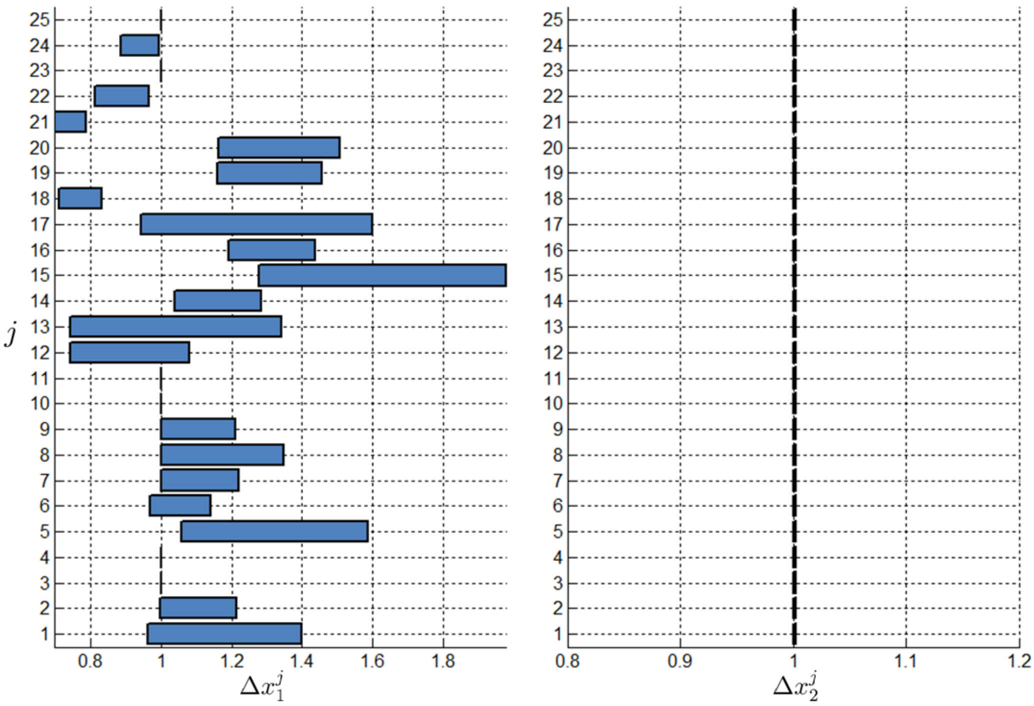

Figure 4: Relative changes in inputs over the non-dominated solutions.

ciency scores), and are thus located on the (weakly) efficient frontier of the PPS. This indicates that these DMUs are performing well with regard to the other DMUs; however, according to the results, their marginal rates of return are not high enough to warrant additional resources. It is important to note that these results are dependent on the assumptions made about the underlying PPS: if some model other than BCC had been used, the results might differ to some extent.

Based on Figure 4, it is evident that resources are always extracted from DMUs $j \in \{18, 21, 22, 24\}$ and re-allocated among the other DMUs. Furthermore, DMUs $j \in \{2, 5, 7, 8, 9, 14, 15, 16, 19, 20\}$ always either receive additional resources or retain their initial amount. Moreover, DMUs $j \in \{1, 6, 12, 13, 17\}$ either receive or distribute some of their resources, depending on the selected allocation.

As can be seen in Figure 5, the relative changes in sales (Δy_1^j) and profit (Δy_2^j) for each DMU $j \in \{1, \dots, 25\}$ are always greater than or equal to 1 in accordance with (22); this corresponds to our initial assumption that the outputs of each DMUs must not decrease. According to Table 1, some DMUs generate extremely low profits, resulting in substantial relative profit

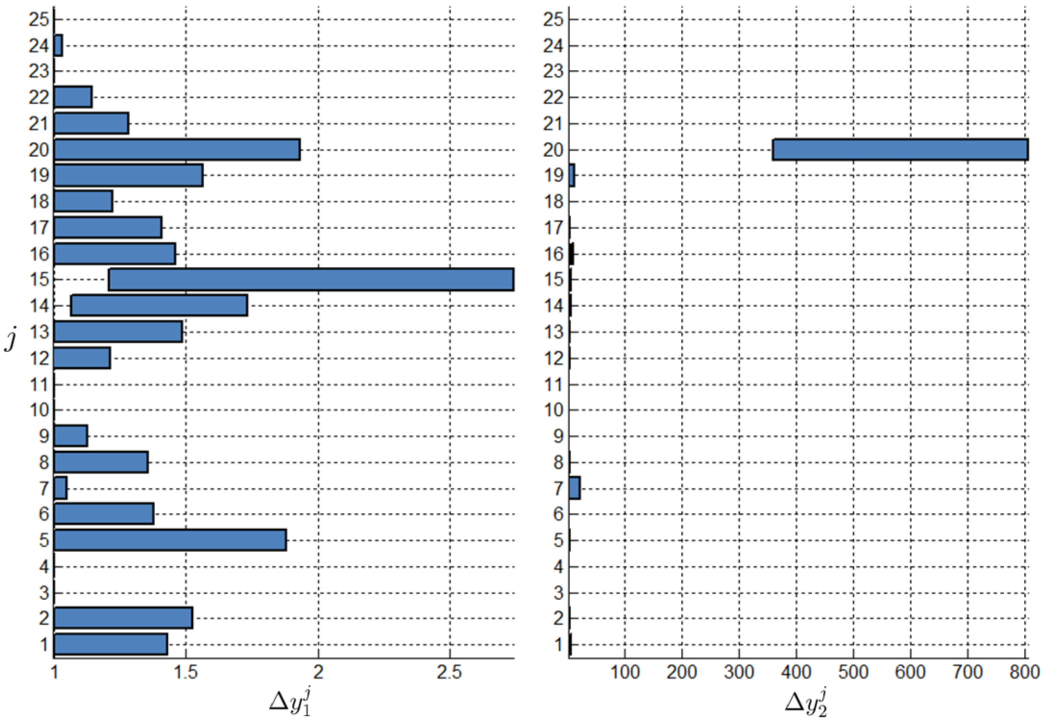

Figure 5: Relative changes in outputs over the non-dominated solutions.

improvements in case they receive additional resources. Unfortunately, this also generates difficulties in visualizing all the improvements in a reasonable manner: for instance, DMU 20 is able to increase its profits by a factor of 360 - 805, depending on the solution, whereas most DMUs are only capable of increasing their profits by a factor of less than 10.

5.1 Incorporating preference information

Suppose that the DM is not fully satisfied with the results and is unable to select the optimal portfolio due to the overwhelming number of non-dominated solutions and corresponding production plans. In order to facilitate this selection, the DM provides preferences on the unit values of the outputs (i.e. sales and profit). Incorporating such preference information into the model and subsequently computing the non-dominated solutions may reduce the set of non-dominated portfolios significantly, thus facilitating the decision-making process.

Suppose that the DM aims to compensate the costs resulting from the hiring of new employees by generating a sufficient amount of profit. To achieve

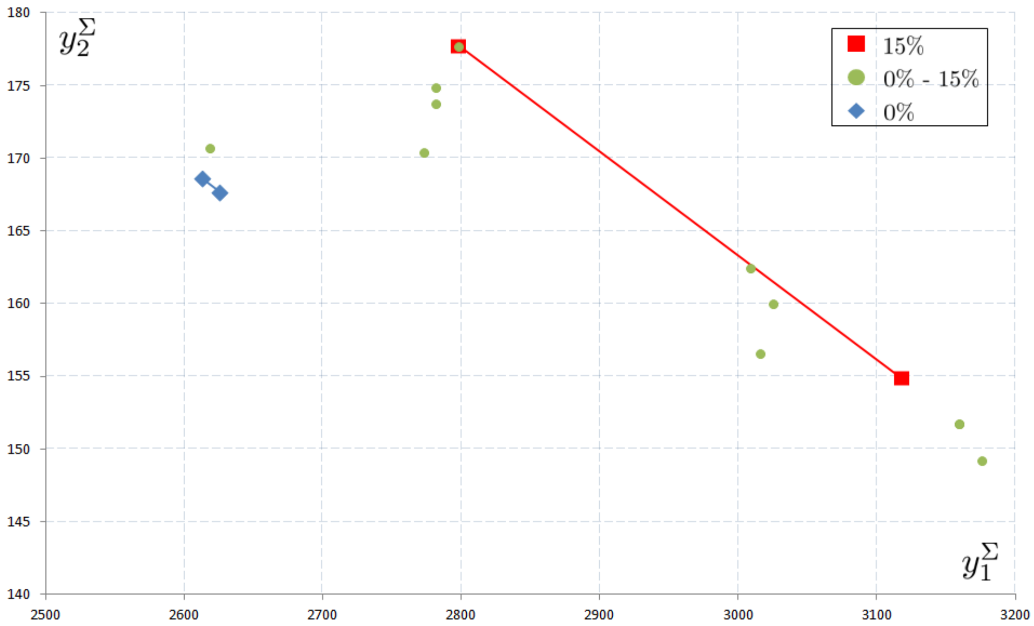

Figure 6: Non-dominated solutions (based on preferences with profit focus) with varying budget constraint corresponding to 0% - 15% increases in total man-hours.

this, the DM strives to emphasize increases in profits by stating that a unit increase in profits is more important than a 12 unit increase in sales. This corresponds to the linear constraint $v_2 \geq 12 \cdot v_1$ and results in an information set S_v with extreme points $v^1 = (0, 1)$ and $v^2 = (1, 12)/13$. These preferences are incorporated in the model by defining the matrix $V_{ext} = (v^1, v^2)^T$ and replacing the objectives (20) by

$$\underset{x^j, y^j, \lambda^j}{\text{v-max}} \quad \left[V_{ext} \Big(\sum_{j=1}^{25} y^j \Big), -\sum_{j=1}^{25} x_1^j \right]. \tag{25}$$

Solving the problem (21) - (24) with the objectives (25) produces 15 non-dominated extreme solutions that can be achieved with 40 different production plans. Figure 6 presents the overall sales $y_1^\Sigma = \sum_{j=1}^{n} y_1^j$ and profit $y_2^\Sigma = \sum_{j=1}^{n} y_2^j$ that correspond to these solutions with varying budget levels.

Comparing Figures 3 and 6, it is evident that the set of non-dominated extreme solutions based on the given preferences is indeed a subset of the initial set of non-dominated extreme solutions that was computed in the absence of any preference information. Since the number of non-dominated extreme solutions is significantly lower, the process of selecting the optimal

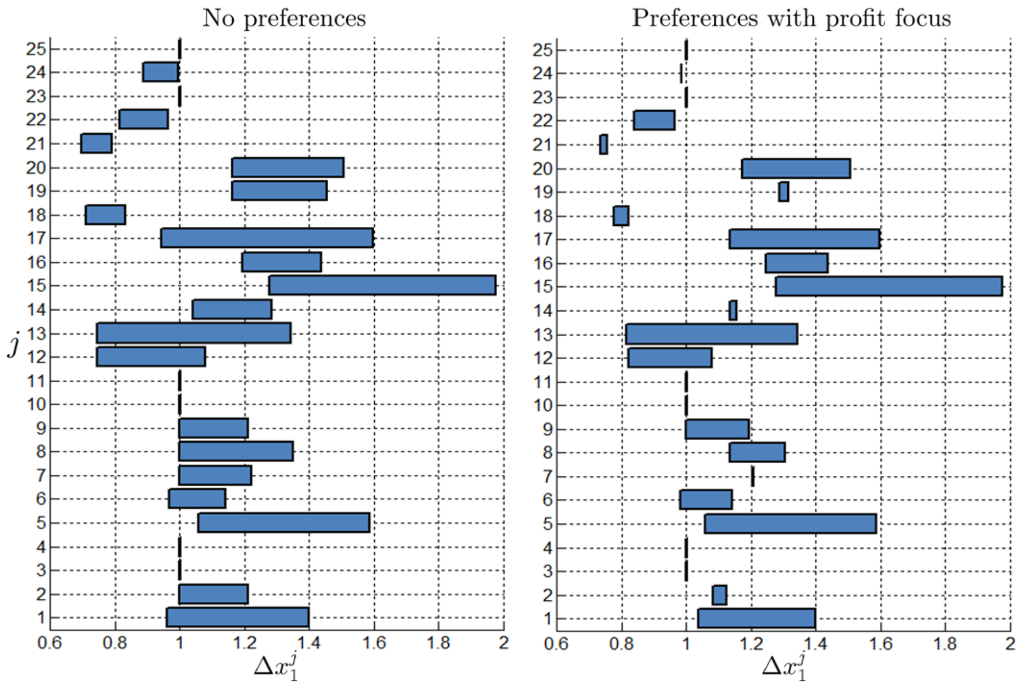

Figure 7: Relative changes in man-hours over the non-dominated solutions.

portfolio becomes substantially easier.

Figure 7 presents the relative changes in the DMUs' man-hours over both the initial set of non-dominated solutions and the refined set of non-dominated solutions computed with regard to the DM's preferences. It can be seen that the value intervals shrink with the introduction of additional preference information, as implied by Corollary 1. Comparing the two situations, we can observe that, for instance, DMUs $j \in \{1, 2, 7, 8, 17\}$ henceforth always receive additional man-hours, as opposed to the initial case with no preferences. The DM can exploit this information, for example, by preparing to allocate at least the minimum amount of resources to the corresponding DMUs, regardless of the selected final solution.

Similarly, Figure 8 presents the relative changes in the DMUs' sales over the initial and the refined set of non-dominated solutions. We can observe that DMUs 7 and 24 no longer generate additional sales, and most DMUs generally produce less sales when the preferences of the DM are included in the model. This makes sense since these preferences emphasize increases in profits rather than sales, and these two outputs are somewhat connected such that higher profits can be achieved with lower sales.

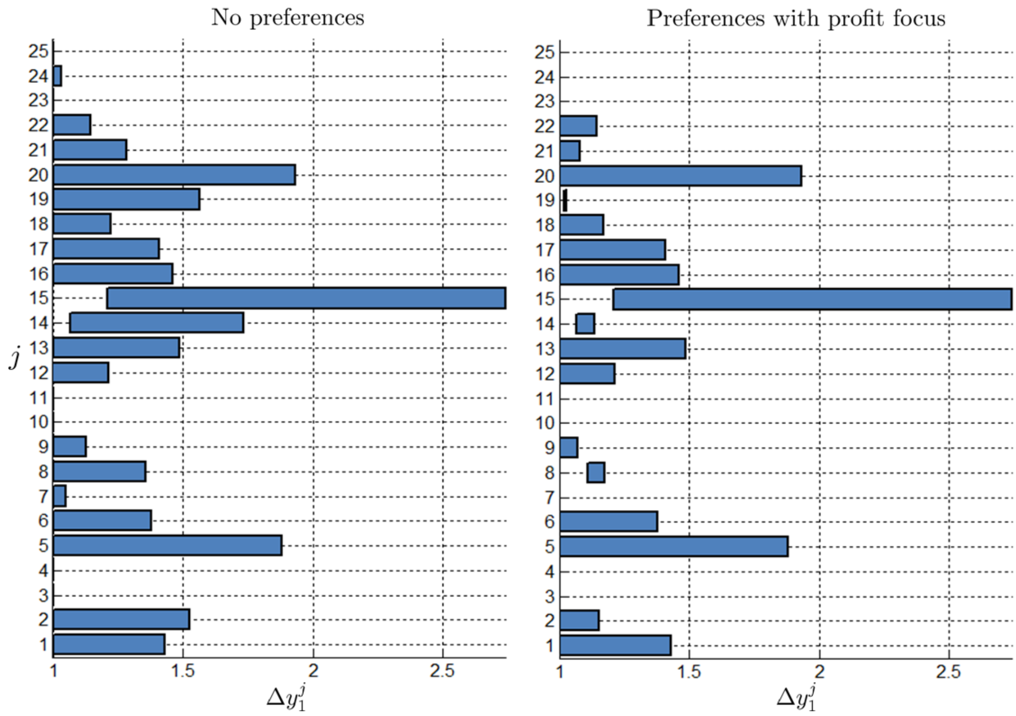

Figure 8: Relative changes in sales over the non-dominated solutions.

Figure 9 presents the absolute changes (rather than relative ones) in the DMUs' profits due to the difficulties of visualizing the relative changes. It can be seen that the minimum (guaranteed) amount of profit produced by the DMUs increases with the introduction of preferences with profit focus, as opposed to the initial case with no preference information.

The benefits of incorporating the preferences to the resource allocation problem are obvious: the number of different non-dominated extreme solutions and production plans is significantly smaller; moreover, the DM may now estimate the performances of the individual DMUs more accurately, since the ranges of possible input and output values have been reduced.

5.1.1 Forcing an allocation

According to Table 1, DMU 13 is the most inefficient unit; additionally, it is one of the largest DMUs in terms of input consumption. Nevertheless, based on the results in Figure 7, it (DMU 13) may possibly receive additional man-hours, depending on the selected production plan.

Suppose that the DM consequently decides that DMU 13 should not receive

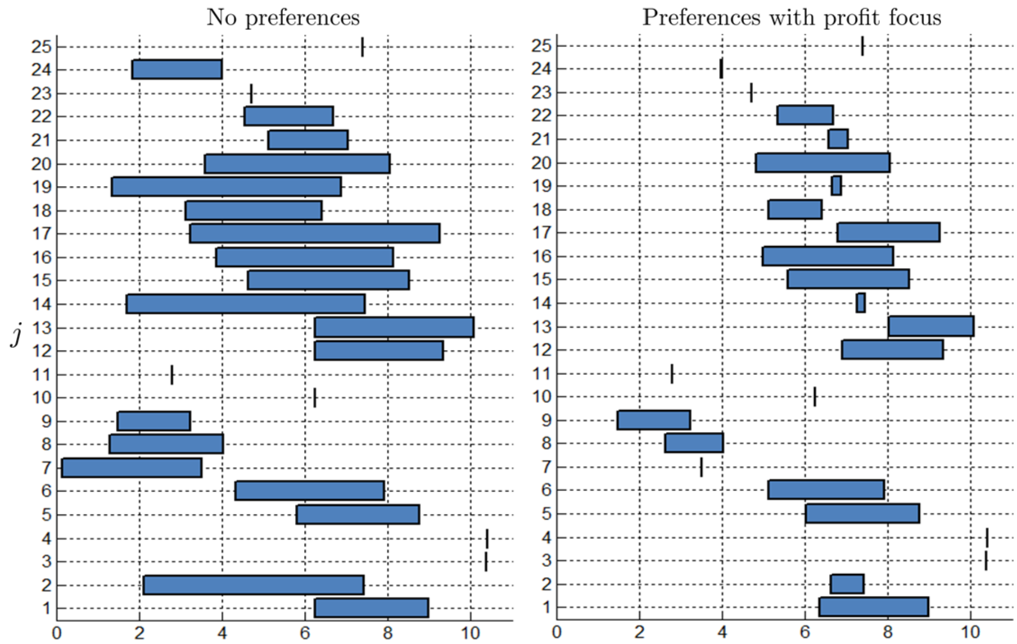

Figure 9: Absolute changes in profits over the non-dominated solutions.

additional resources (man-hours) and wishes to identify those non-dominated extreme solutions that satisfy the additional constraint ($x_1^{13} \leq 0$). The refined set of non-dominated extreme solutions can be computed directly from the current set by eliminating those solutions where DMU 13 receives a positive amount of man-hours. Figure 10 presents these non-dominated extreme solutions.

Based on the results, it is no longer beneficial to consume the maximum available budget (15%) with the introduction of the constraint that prevents DMU 13 from increasing its resources.

5.2 Evaluating the approximate Benson's algorithm

In this section, we evaluate the accuracy and performance of the approximate Benson's algorithm (Shao and Ehrgott, 2008) by solving resource allocation problems with up to four objective functions and several hundred decision variables. We use the supermarket data in Table 1 and formulate a problem similar to that presented in (Korhonen and Syrjänen, 2004).

Suppose that the DM is only interested in maximizing the aggregate output production by allocating available resources (inputs) among the DMUs. The

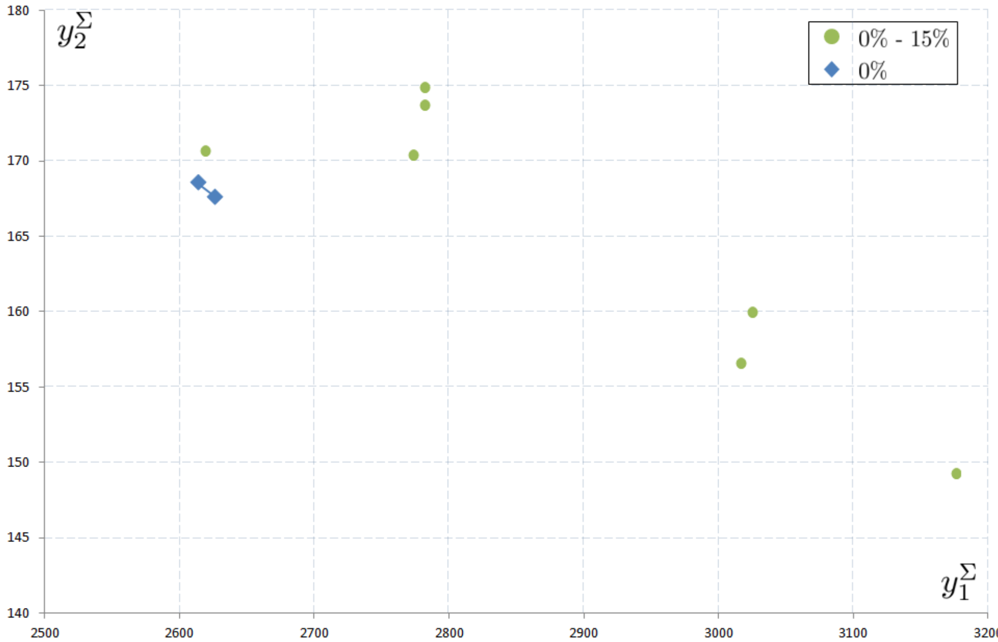

Figure 10: Non-dominated solutions (profit focus) with varying budget when resources of DMU 13 are not allowed to increase.

resources of each DMU are allowed to increase at most 30% and decrease at most 10%, and the total amount of resources is allowed to increase only 1% from the current amount. Furthermore, the BCC-O efficiency scores of the DMUs must remain unchanged. The PPS for each DMU is estimated from observed data based on the BCC model, i.e. $\Lambda^f = \Lambda^{BCC} = \{\lambda \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i = 1\}$ in (13). These assumptions lead to the following MOLP-problem.

$$\begin{aligned}
\underset{x^j, y^j, \lambda^j}{\text{v-max}} \quad & \sum_{j=1}^{25} y^j && (26)\\
\text{s.t.} \quad & (x^j, \sigma^j y^j)^T \in P^j = \big\{(x^j, y^j)^T \in \mathbb{R}^{2+2}_+ \mid x^j \geq \hat{X}\lambda^j, y^j \leq \hat{Y}\lambda^j, \\
& \qquad\qquad \lambda^j \in \Lambda^{BCC}\big\}, \quad j = 1, \ldots, 25 && (27)\\
& 0.9\hat{x}^j \leq x^j \leq 1.3\hat{x}^j, \quad j = 1, \ldots, 25 && (28)\\
& \sum_{j=1}^{25} x^j \leq 1.01 \sum_{j=1}^{25} \hat{x}^j. && (29)
\end{aligned}$$

Problem (26) - (29) has two objective functions and 725 decision variables. As such, it is easily solvable with Benson's algorithm (in comparison, the problem is practically unsolvable with the multi-objective Simplex due to the large number of decision variables). We use Löhne's (2012) Matlab implementation of Benson's algorithm in solving the set of non-dominated solutions. The implementation also allows adjusting the approximation parameter ϵ.

Solving the problem (26) - (29) with the exact Benson's algorithm takes approximately 30 seconds using a laptop with 2.1 GHz dual-core Intel processor and 2 GB RAM. The algorithm returns 68 non-dominated extreme solutions that can be achieved with 115 different allocation plans (i.e. input/output combinations of the DMUs).

In comparison, the approximate algorithm solves the problem in less than 5 seconds with an approximation error of $\epsilon = 0.1$. However, the approximate algorithm returns only 11 non-dominated extreme solutions, corresponding to 17 different allocation plans. Figure 1 presents the aggregate outputs $y_1^{\Sigma} = \sum_{j=1}^{n} y_1^j$ and $y_2^{\Sigma} = \sum_{j=1}^{n} y_2^j$ that correspond to the non-dominated extreme solutions for both algorithms.

As can be seen in figure 11, the end points of the true non-dominated set are accounted for in the approximation. Additionally, the approximate algorithm provides a decent representation of the true non-dominated set. This is also the case in several case studies (see, e.g. Shao and Ehrgott, 2008; Hamel et al., 2013). Thus, since the number of non-dominated extreme points is typically overwhelming, it is most likely not necessary to calculate all of them to obtain reliable results. In this case, accommodating the value of the approximation parameter ϵ becomes a priority.

Changing the value of ϵ can have a significant impact on both the computation time and the number of non-dominated extreme points produced by the algorithm. To illustrate, suppose that in addition to maximizing the aggregate outputs, the DM wishes to minimize the total resource consumption. This results in the problem (26) - (29) with the following additional objectives

$$\underset{x^j, y^j, \lambda^j}{\text{v-min}} \quad \sum_{j=1}^{25} x^j. \tag{30}$$

With the addition of (30), the problem has four objective functions instead of two, thus increasing the execution time of the algorithm dramatically. Table 2 presents computation times (s), number of non-dominated extreme points

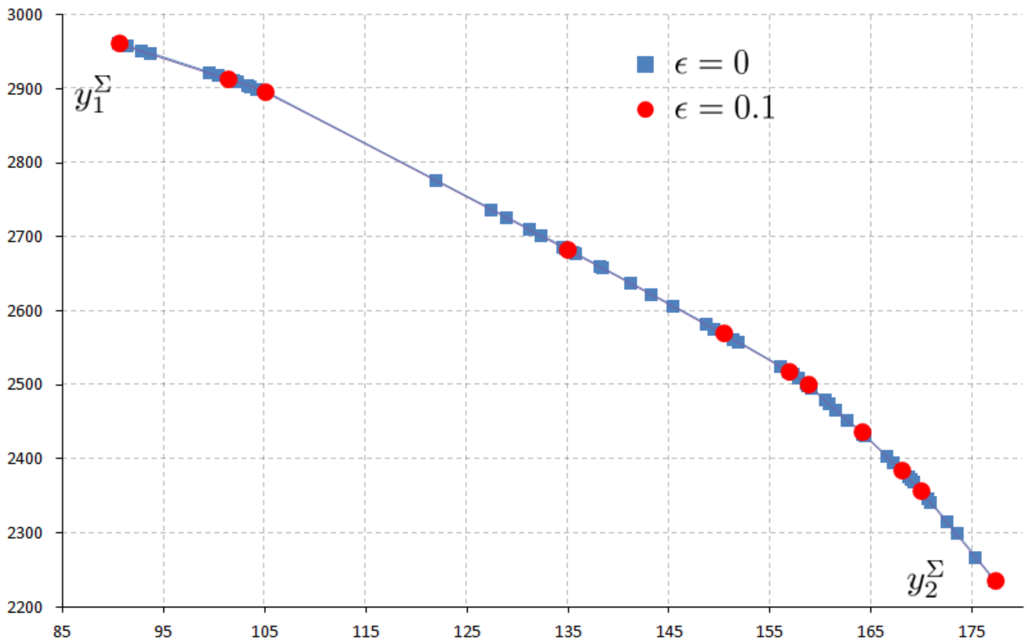

Figure 11: True and approximate non-dominated set of the problem (26) - (29).

Table 2: Computation times for the problem (26) - (30) with different values of ϵ.

ϵ	time	#ND	#LP
$\epsilon = 0.8$	255 s	343	2366
$\epsilon = 0.6$	602 s	501	4887
$\epsilon = 0.4$	1442 s	859	9506
$\epsilon = 0.2$	7263 s	1862	24639

(#ND) and number of LPs solved (#LP) with different values of ϵ for the problem (26) - (30). The effect of increasing the value of ϵ is evident: a change from $\epsilon = 0.2$ to $\epsilon = 0.8$ decreases the computation time nearly by a factor of 30.

6 Conclusions

In this paper, we have presented a resource allocation model for organizations with a centralized decision-making environment, in which the decision maker (DM) strives to allocate available resources among several decision making units (DMUs) to improve the organization as a whole. The developed model is based on a multiple objective linear programming (MOLP) formulation, extending the previous work of Korhonen and Syrjänen (2004). We applied principles of Portfolio Decision Analysis (PDA; Salo et al., 2011) to build a framework that supports the DM in selecting the optimal solution from a large set of non-dominated allocation portfolios.

The presented model was applied to a hypothetical resource allocation problem (based on a real-life data set), in which the DM of a supermarket chain is planning to hire additional employees and allocate them among the individual DMUs (supermarkets) so that the aggregate outputs (i.e. sales and profit) are maximized. A benefit-cost analysis was performed in order to help the DM in evaluating the amount of overall sales and profit that could be achieved with varying budget constraint. The set of non-dominated portfolios was first computed without preference information; subsequently, the DM's preferences were incorporated in the model and the refined set of non-dominated portfolios was finally calculated in view of those preferences.

Initially, several non-dominated extreme solutions were found; however, with the introduction of the DM's preferences, the number of such solutions decreased significantly, thus facilitating the process of selecting the optimal portfolio. Additionally, an example was presented, demonstrating how to further narrow down the set of non-dominated solutions by imposing an upper bound on the resource consumption of a single DMU. Furthermore, DMU-level information across the non-dominated solutions was presented, enabling the DM to assess the performances of the individual DMUs while focusing on the main goal of maximizing the aggregate sales and profit.

We used Löhne's (2012) Matlab implementation of Benson's algorithm to solve the resource allocation problems presented in this paper. Computationally, the algorithm had little trouble solving the studied problem with three objective functions and 725 decision variables using a laptop with 2.1 GHz dual-core processor and 2 GB RAM. In addition, we evaluated the performance and accuracy of the approximate Benson's algorithm using the previous supermarket data. Based on the results, the approximate algorithm is capable of solving similar resource allocation problems with (at least) up to four objective functions and several hundred decision variables in a reasonable time; additionally, the algorithm produced a decent approximation of the true set of non-dominated solutions in the studied example.

This paper paves way towards building an interactive decision support system that facilitates the selection of an optimal portfolio by allowing the DM to (i) incorporate preference statements on the unit values of different inputs and outputs, (ii) define aspiration levels for the individual DMUs' input consumption and output production and (iii) perform a benefit-cost analysis to identify all the non-dominated portfolios with varying budget constraints. Future avenues for improving the model could include, for instance, incorporating interdependencies between DMUs, modelling various uncertainties and developing robust decision rules to support the DM.

References

Banker, R. D., Charnes, A., and Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science*, 30(9):1078–1092.

Benson, H. (1998). An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. *Journal of Global Optimization*, 13:1–24.

Charnes, A. (1994). *Data Envelopment Analysis: Theory, Methodology, and Application*. Kluwer Academic Pub.

Charnes, A., Cooper, W. W., and Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2(6):429–444.

Ehrgott, M. (2005). *Multicriteria Optimization*. Springer Berlin, second edition.

Golany, B. and Tamir, E. (1995). Evaluating efficiency-effectiveness-equality trade-offs: a data envelopment analysis approach. *Management Science*, 41(7):1172–1184.

Hamel, A. H., Löhne, A., and Rudloff, B. (2013). Benson type algorithms for linear vector optimization and applications. *ArXiv e-prints*. http://arxiv.org/pdf/1302.2415v3.pdf. Accessed: 01/09/2013.

Korhonen, P. and Syrjänen, M. (2004). Resource allocation based on efficiency analysis. *Management Science*, 50(8):1134–1144.

Löhne, A. (2012). bensolve-1.2. Matlab implementation of Benson's algorithm to solve linear vector optimization problems. http://ito.mathematik.uni-halle.de/~loehne/page_en/downloads.php. Accessed: 03/01/2013.

Liesiö, J., Mild, P., and Salo, A. (2007). Preference programming for robust portfolio modeling and project selection. *European Journal of Operational Research*, 181(3):1488–1505.

Liesiö, J., Mild, P., and Salo, A. (2008). Robust portfolio modeling with incomplete cost information and project interdependencies. *European Journal of Operational Research*, 190(3):679–695.

Salo, A., Keisler, J., and Morton, A. (2011). *Portfolio Decision Analysis:*

Improved Methods for Resource Allocation. New York, NY: Springer New York.

Shao, L. and Ehrgott, M. (2008). Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning. *Mathematical Methods of Operations Research*, 68(2):257–276.

Toppila, A., Liesiö, J., and Salo, A. (2011). A resource allocation model for r&d investments: A case study in telecommunication standardization. In *Portfolio Decision Analysis*, pages 241–258. Springer.

A Appendix

Proof of Theorem 1 (i). Suppose contrary to the statement that there exists a portfolio $p' = (x', y') \in P_N(\tilde{S})$ such that $p' \notin P_N(S)$. Consequently, there exists a portfolio $p = (x, y) \in P_N(S)$ such that $p \succ_S p'$. According to definition 4, this means that the following hold:

$$I_u(x) \leq I_u(x') \quad \text{for all } u \in S_u, \tag{31}$$
$$O_v(y) \geq O_v(y') \quad \text{for all } v \in S_v \tag{32}$$

and one of the inequalities is strict for some $u^* \in S_u$ or $v^* \in S_v$. Since $\tilde{S} \subset S$, we must also have

$$I_u(x) \leq I_u(x') \quad \text{for all } u \in \tilde{S}_u, \tag{33}$$
$$O_v(y) \geq O_v(y') \quad \text{for all } v \in \tilde{S}_v. \tag{34}$$

Let $(\tilde{u}, \tilde{v}) \in \tilde{S}_u \times \tilde{S}_v$ and $(u^0, v^0) \in \text{int}(\tilde{S}_u) \times \text{int}(\tilde{S}_v)$ such that $\tilde{u} = \alpha u^0 + (1-\alpha)u^*$ and $\tilde{v} = \alpha v^0 + (1-\alpha)v^*$ for some $\alpha \in (0, 1)$. According to (31) and (32), either $I_{u^*}(x) < I_{u^*}(x')$ or $O_{v^*}(y) > O_{v^*}(y')$ must hold. Suppose that $I_{u^*}(x) < I_{u^*}(x')$; we get

$$\begin{aligned}
I_{\tilde{u}}(x) - I_{\tilde{u}}(x') &= \sum_{i=1}^{m} \tilde{u}_i x_i - \sum_{i=1}^{m} \tilde{u}_i x'_i \\
&= \sum_{i=1}^{m} (\alpha u_i^0 + (1-\alpha)u_i^*)x_i - \sum_{i=1}^{m} (\alpha u_i^0 + (1-\alpha)u_i^*)x'_i \\
&= \sum_{i=1}^{m} (\alpha u_i^0 + (1-\alpha)u_i^*)(x_i - x'_i) \\
&= \alpha \sum_{i=1}^{m} u_i^0 (x_i - x'_i) + (1-\alpha) \sum_{i=1}^{m} u_i^* (x_i - x'_i) \\
&= \alpha (I_{u^0}(x) - I_{u^0}(x')) + (1-\alpha)(I_{u^*}(x) - I_{u^*}(x')) \\
&< 0.
\end{aligned}$$

Otherwise, if $O_{v^*}(y) > O_{v^*}(y')$, we get

$$\begin{aligned}
O_{\tilde{v}}(y) - O_{\tilde{v}}(y') &= \sum_{i=1}^{s} \tilde{v}_i y_i - \sum_{i=1}^{s} \tilde{v}_i y_i' \\
&= \sum_{i=1}^{s} (\alpha v_i^0 + (1-\alpha) v_i^*) y_i - \sum_{i=1}^{s} (\alpha v_i^0 + (1-\alpha) v_i^*) y_i' \\
&= \sum_{i=1}^{s} (\alpha v_i^0 + (1-\alpha) v_i^*)(y_i - y_i') \\
&= \alpha \sum_{i=1}^{s} v_i^0 (y_i - y_i') + (1-\alpha) \sum_{i=1}^{s} v_i^* (y_i - y_i') \\
&= \alpha (O_{v^0}(y) - O_{v^0}(y')) + (1-\alpha)(O_{v^*}(y) - O_{v^*}(y')) \\
&> 0.
\end{aligned}$$

Thus, either $I_{\tilde{u}}(x) < I_{\tilde{u}}(x')$ or $O_{\tilde{v}}(y) > O_{\tilde{v}}(y')$, which together with (33) and (34) imply that $p \succ_{\tilde{S}} p'$ in accordance with Definition 4. This is a contradiction to $p' \in P_N(\tilde{S})$; thus, it must be that $p' \in P_N(S)$. □

Proof of Theorem 1 (ii). The set of efficient portfolios P_E is calculated from problem (1) - (3) and the set of non-dominated portfolios $P_N(S)$ with regard to information set $S = S_u \times S_v$ is calculated from (10) - (12). Thus, it is sufficient to show that (1) is identical to (10) when there is no preference information, i.e. $S_u = \Delta^m$ and $S_v = \Delta^s$, since the problems (1) - (3) and (10) - (12) are then identical.

The extreme points of the simplices Δ^m and Δ^s correspond to the unit vectors; thus, the extreme point matrices U_{ext} and V_{ext} are identity matrices. Consequently, (10) is identical to (1) and the proof is complete.

□

Proof of Lemma 1. From definition 4, we have

$$
\begin{aligned}
p \succ_S p' &\Leftrightarrow \begin{cases} I_u(x) \leq I_u(x') & \forall u \in S_u \\ O_v(y) \geq O_v(y') & \forall v \in S_v \end{cases} \\
&\Leftrightarrow \begin{cases} \max_{u \in S_u} \sum_{i=1}^{s} u_i(x_i - x_i') \leq 0 \\ \min_{v \in S_v} \sum_{j=1}^{m} v_j(y_j - y_j') \geq 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \sum_{i=1}^{s} u_i(x_i - x_i') \leq 0 & \forall u \in \mathrm{ext}(S_u) \\ \sum_{j=1}^{m} v_j(y_j - y_j') \geq 0 & \forall v \in \mathrm{ext}(S_v) \end{cases} \\
&\Leftrightarrow \begin{cases} U_{ext}x \leq U_{ext}x' \\ V_{ext}y \geq V_{ext}y' \end{cases}
\end{aligned}
$$

since the minimum and maximum of LP-problems are always found in the extreme points of the feasible sets.

Furthermore, according to Definition 4, there exists some $u \in S_u$ such that $I_u(x) < I_u(x')$ or some $v \in S_v$ such that $O_v(y) > O_v(y')$. This is equal to

$$
\begin{aligned}
&\begin{cases} \min_{u \in S_u} \sum_{i=1}^{s} u_i(x_i - x_i') < 0 \text{ or} \\ \max_{v \in S_v} \sum_{j=1}^{m} v_j(y_j - y_j') > 0 \end{cases} \\
\Leftrightarrow &\begin{cases} \sum_{i=1}^{s} u_i(x_i - x_i') < 0 \text{ for some } u \in \mathrm{ext}(S_u) \text{ or} \\ \sum_{j=1}^{m} v_j(y_j - y_j') > 0 \text{ for some } v \in \mathrm{ext}(S_v) \end{cases} \\
\Leftrightarrow &\begin{cases} U_{ext}x < U_{ext}x' \text{ or} \\ V_{ext}y > V_{ext}y' \end{cases}
\end{aligned}
$$

Thus, we also have $(U_{ext}x, -V_{ext}y) \neq (U_{ext}x', -V_{ext}y')$ and the proof is complete. □

Proof of Corollary 1. According to Theorem 1, $P_N(\tilde{S}) \subseteq P_N(S)$. Thus, the ranges of the individual DMUs' possible input-output mixes across $P_N(\tilde{S})$ must belong to $P_N(S)$ as well. The claims then follow.

□