Preference Programming for Spatial Multiattribute Decision Analysis

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Spatial Decision Analysis

• Consequences of alternatives are distributed across a geographical region
• E.g., select the position of a rescue helicopter base, $P^1$ or $P^2$
  • Alternatives imply different response times, i.e., consequences, for each location
  • Locations not equally important? (cf. population density)
• Plenty of other applications
  • Urban, environmental and transportation planning
  • Waste management, hydrology, agriculture, and forestry
  • See, e.g., Malczewski & Rinner 2015, Ferretti & Montibeller 2016
Spatial Value Function

- Value of decision alternative $z$ (Simon, Kirkwood and Keller 2014):
  \[ V(z) = \int_{s \in S} a(s)v(z(s)) ds \]
  
  $a(s)$: spatial weight ("importance") of specific location $s$ in region $S$
  
  $z(s)$: consequence for location $s$ when alternative $z$ is chosen
  
  $v(\cdot)$: consequence value function

- Challenges:
  - Specifying spatial weights $a(s)$ for an infinite number of locations $s$
  - Only a conjecture on the underlying preference assumptions exists
Spatial Value Function

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• Our contribution:
  • Axiomatic basis for preferences that can be represented with the spatial value function
  • Spatial preference programming: Determination of dominances among alternatives based on incomplete specification of weights
Preference Assumptions

- Let $\succeq$ be a binary relation on the set of decision alternatives $Z = \{z: S \to C\}$
  - $S$: set of locations
  - $C$: set of consequences
- Assumptions
  - A1 $\succeq$ is transitive and complete
  - A2 There exist $z^1, z^2 \in Z$ such that $z^1 \not\succeq z^2$
  - A3 “Spatial preference independence”
  - A4 “Consequence consistency”
  - A5 “Spatial consistency”
  - A6 “Divisibility of subregions”
  - A7 “Monotonicity”

A3: Preference between two alternatives does not depend on locations with equal consequence
Additive Spatial Value Function $V(z)$

- **Theorem.** $\succcurlyeq$ satisfies A1-A7 iff there exists a non-atomic measure $\alpha$ on $S$ and a bounded function $v: C \to \mathbb{R}$ such that $z \succcurlyeq z' \iff V(z) \geq V(z')$ where
  \[
  V(z) = \int_S v(z(s))d\alpha(s)
  \]
- Proof based on Savage 1954
- The weighting function $\alpha: 2^S \to \mathbb{R}$
  - Assigns a weight to each subregion $S' \subseteq S$ (cf. relative importance)
  - Connection to Simon’s et al. weighting $a(s): \alpha(S') = \int_S a(s)ds$
- $v$ is a cardinal value function for consequences $c = z(s)$
  - I.e., unique up to positive affine scaling
  - E.g., additive multiattribute $v(z) = \sum_{j=1}^{m} b_j v_j(z_j)$
Incomplete Preference Information

- Small set of feasible weighting functions can be sufficient for ranking alternatives
  - Avoiding the overwhelming task of specifying the exact weighting function $\alpha$
- Stated preferences between pairs of alternatives $\implies$ Constraints on the spatial weighting function $\alpha$ and the vector $b$ of attribute weights
- Multiple preference statements comparing suitable alternatives $\implies$ System of linear constraints on
  - $\alpha(S^1), \ldots, \alpha(S^n)$ where $S^1, \ldots, S^n$ is a partition of $S$
  - $b_1, \ldots, b_m$

$z^1 \succeq z^2 \iff V(z^1) \geq V(z^2) \iff \alpha(S^1) \geq \alpha(S^2)$
"Subregion $S^1$ more important than $S^2$"

Consequences $C$

<table>
<thead>
<tr>
<th>Least preferred</th>
<th>Most preferred</th>
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Dominance

- Constraints from preference statements result in
  - A set of feasible weighting functions $\mathcal{A} \subseteq \{\alpha: 2^S \to \mathbb{R}_+ | \alpha(S) = 1\}$
  - A set of feasible attribute weights $B \subseteq \{b \in \mathbb{R}_+^m | \sum b_j = 1\}$

- Alternative $z^1$ dominates alternative $z^2$ if
  - $V(z^1) \geq V(z^2)$ for all $\alpha \in \mathcal{A}$ and $b \in B$
  - $V(z^1) > V(z^2)$ for some $\alpha \in \mathcal{A}$ and $b \in B$

- Dominance check: bi-level LP problem
  \[
  \inf_{\alpha \in \mathcal{A}, b \in B} V(z^1) - V(z^2) = \min_{b \in B} \min_{\alpha \in \mathcal{A}} \sum_{i=1}^{n} \alpha(S^i) \inf_{s \in S^i} \sum_{j=1}^{m} b_j v_j(z^1_j(s)) - v(z^2_j(s))
  \]
  where $S^1, ..., S^n$ is a partition of $S$

- Solution: Enumerate extreme points of $B$ and solve LP problem in each one
Air Defense Planning: Positioning of Air Bases

- Select positions for 2 main and 3 secondary air bases to maximize air defense capability
  - Main bases: 3 position candidates
  - Secondary bases: 5 position candidates

- Spatial consequences provided by a simulation tool – input parameters:
  - Number of defensive flying units; fuel consumption; weapons consumption; flight speed
  - Positions of air bases; turnaround times; refueling and rearming times; alert, taxi and scramble delays
Attributes of Air Defense Capability

Positions of air bases affect...

• "Engagement frontier" where hostile aircraft can first be intercepted by defensive flying units
  • Attribute #1: Location’s distance to south frontier
  • Attribute #2: Location’s distance to west frontier

• "Force fulfillment”
  • Attribute #3: Average number of defensive flying units available at the location
  • Attribute #4: As attribute #3 with one secondary base destroyed (cf. combat sustainability)

Consequences $C_j$
Preference Statements

• Spatial preference statements ($\alpha$)
  • Major cities > SW coastal area
  • Power plants > SW coastal area
  • SW coastal area > NE coastal area
  • NE coastal area > Other areas

• Attribute preference statements ($\beta$)
  • Engagement frontier attributes > Force fulfillment attributes
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13 non-dominated alternatives
Additional Preference Statements

• Spatial preference statements ($\alpha$)
  • Power plants > Major cities
  • Power plant #1 > Power plant #2
  • City #1 > City #2 > City #3
  • ...

• Attribute preference statements ($b$)
  • West engagement frontier > South engagement frontier
  • Force sustainability > Initial force fulfillment

4 non-dominated alternatives
Conclusion: Spatial Decision Analysis
Benefits from Preference Programming

• The additive spatial value function
  • Axiomatic basis
  • Weighting subregions rather than locations

• Preference programming for spatial decision analysis
  • Incomplete preference information & non-dominated decision alternatives
    • Burden of DM eased considerably by not requiring unique spatial weighting
    • Global sensitivity analysis: Effect of spatial weighting on ranking of alternatives

• Future development
  • Practices and behavioral issues of eliciting the weighting function
  • Spatial decision support systems: Graphical user interface, utilization of GIS data
References

• Ferretti, V. and Montibeller, G., 2016. Key challenges and meta-choices in designing and applying multi-criteria spatial decision support systems. *Decision Support Systems*, 84


