Characterization of Equilibrium Paths in Discounted Stochastic Games

Kimmo Berg
Systems Analysis Laboratory
Aalto University, Finland

July 19, 2012

Outline of the presentation

- Motivation with repeated games
  - What can be studied with new methodology?
  - Compute and analyze equilibrium paths and payoffs
  - Visualize equilibrium payoffs
- Generalization to stochastic games
  - Extend the notion of elementary subpaths
  - Modifications to algorithms, graphs and measures

The setup

- Infinitely repeated stochastic game
- Perfect monitoring
- Pure strategies (no mixed, no correlation devices)
- Discounting (can be unequal discount factors)
- Finite number of states
- Time-independent transition probabilities
- Stage games with finitely many actions

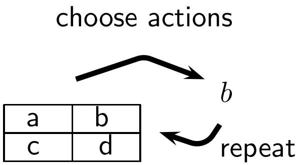
How do repeated games work?

	L	R
T	$0,0$	$2,1$
B	$1,2$	$0,0$

a	b
c	d

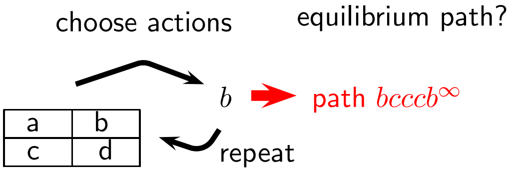
How do repeated games work?

	L	R
T	$0,0$	$2,1$
B	$1,2$	$0,0$



How do repeated games work?

	L	R
T	$0, 0$	$2, 1$
B	$1, 2$	$0, 0$



How do repeated games work?

choose actions

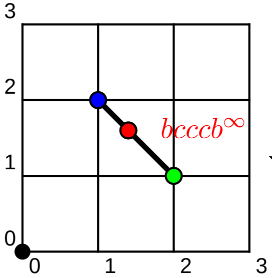
equilibrium path?

	L	R
T	$0,0$	$2,1$
B	$1,2$	$0,0$

a	b
c	d

b → path $bcccb^{\infty}$

repeat



are there one-shot deviations?

utility $\geq$ deviation $+$ punishment

need the smallest payoffs!

c^{∞} for 1, b^{∞} for 2

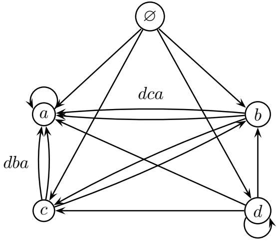
there are other eq. paths

The building blocks of SPE paths

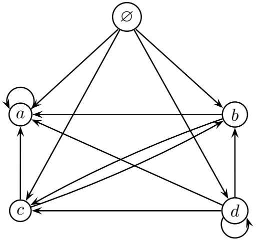
- Elementary subpaths generate recursively all equilibrium paths
- These paths are incentive compatible when followed by SPE paths that satisfy payoff requirements for the following actions

$$a\,b\,b\,a\,c\,d\,a\,a\,\ldots$$

- Equilibrium paths can be compactly represented by graph



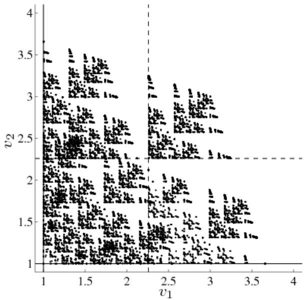
Analyzing equilibrium paths: number of paths



$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

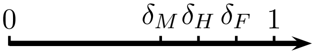
- One-length paths: a, b, c, d
- Two-length paths: aa, ba, bc, ca, cb, da, db, dc, dd
- The number of k-length paths is simply from D^k
- The principal eigenvalue $\rho(D)$ is the asymptotic growth rate
- It tells the size of the equilibrium set

Analyzing equilibrium payoffs: density

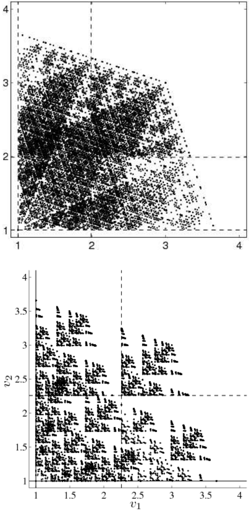


- Hausdorff dimension $\dim_H$ measures how set covers the space
- Difficult to estimate exactly due to overlaps
- Affinity dimension $\dim_A = -\log \rho(D) / \log \delta$ with the graph
- Use topological pressure when unequal discount factors

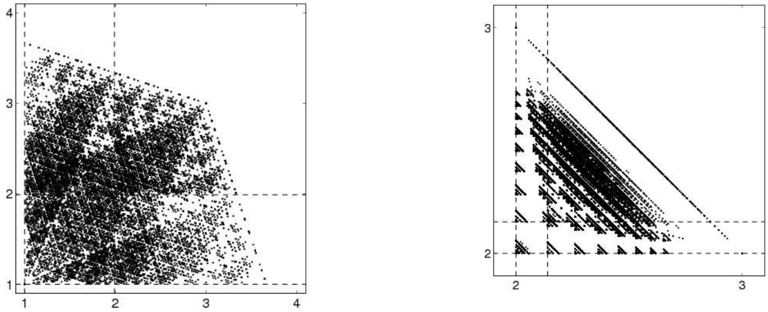
Three critical values for discount factors



- Folk theorem point δ_F
- Hausdorff point δ_H
- Minmax point δ_M

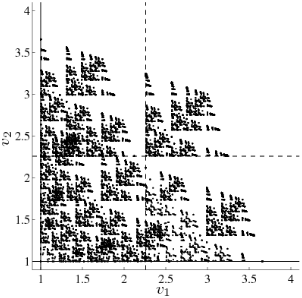


Folk theorem point



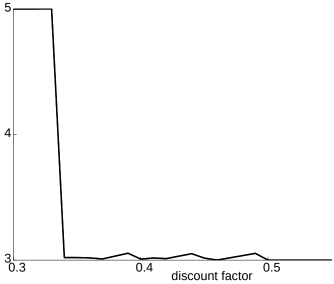
- Payoffs fill the feasible and individual rational points
- Depends on the game

Hausdorff point

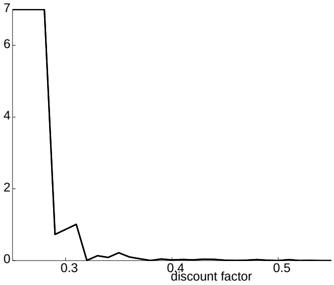


- Payoff set becomes full dimensional somewhere
- When $\dim_H = 2$ for two-player game
- When $\delta_i < 0.5$, $\dim_H = \dim_A$
- If random disturbances in payoffs, $\dim_H = \min(n, \dim_A)$ for all discount factors (Jordal et al. 2007)

Minmax point



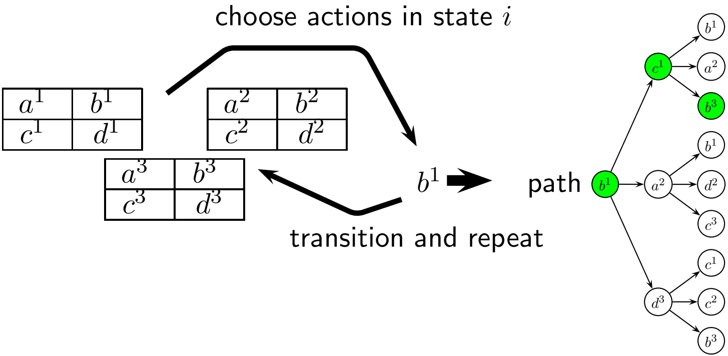
	L	R
T	$5, 5$	$3, 4$
B	$4, 3$	$2, 2$



	L	M	H
L	$10, 10$	$3, 15$	$0, 7$
M	$15, 3$	$7, 7$	$-4, 5$
H	$7, 0$	$5, -4$	$-15, -15$

- When (effective) minmax is reached
- Smallest payoffs are important for generating payoffs

How do stochastic games work?



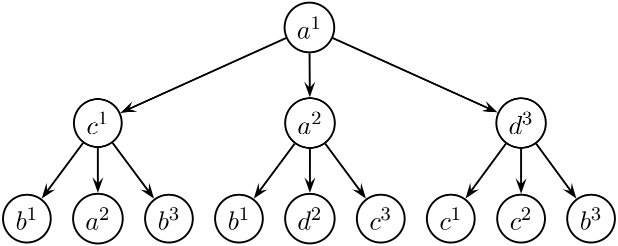
construct graph
analyze paths and payoffs
visualize payoffs

are there one-shot deviations?
need smallest payoffs for all states
and for all players

Equilibrium conditions

- IC conditions are similar: no one-shot deviations
- Punishment paths are state dependent
- Paths that have the smallest payoffs in the state

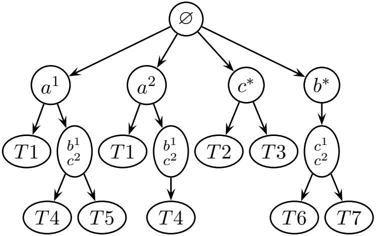
Elementary subpaths in stochastic games

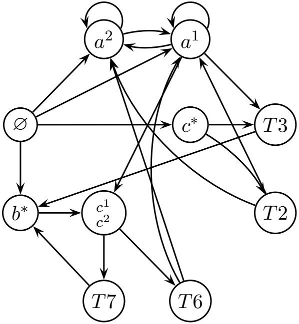


- Elementary subpaths are IC when followed by SPE paths that satisfy payoff requirements for the following actions
- Direct extension from repeated games

Graph of equilibrium paths

- Can be constructed from elementary subpaths
- Algorithm makes a tree and converts it to a graph





Characterization results

Proposition

Path $p \in A^{\infty}$ is SPE path if and only if for all $i \in \mathbb{N}$, $1 \leq j \leq m^{i-1}$, either the k-length start of $sub(p^{i,j})$ is k-length elementary tree or $sub(p^{i,j})$ is infinitely long elementary tree.

Proposition

If there are finitely many elementary trees, finitely or infinitely long, then all SPE paths can be represented with a graph

Analysis of equilibria

- Equilibria can be easily analyzed with the graph: asymptotic growth rate and fractal dimensions
- It is possible to incorporate probabilities to the measures
- Many different ways to measure the complexity

Example of two prisoner's dilemmas

state 1

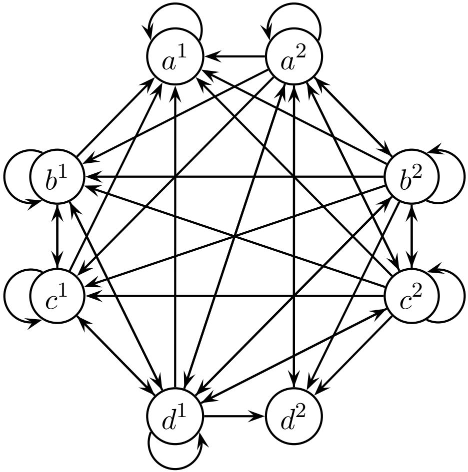
	L	R
T	4,4 #	0,5 #
B	5,0 #	1,1

state 2

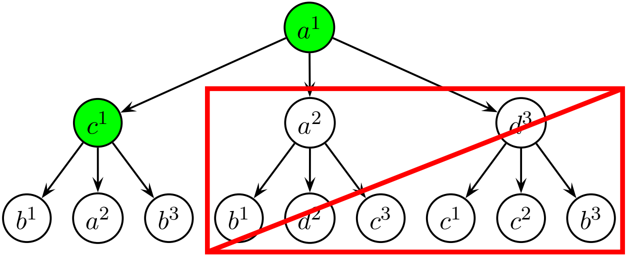
	L	R
T	0,0	-4,1
B	1,-4	-3,-3 #

- $\delta = 0.45$, $q(1|1, a-c) = q(2|2, d) = 1$ and $q(1|1, d) = q(2|2, a-c) = 0.5$
- $\# =$ state stays the same, otherwise randomize
- Some elementary subpaths: b^*, c^*, state 1: a^a_*, d^a_*, d^d_*, where $*$ denotes any action, state 2: a^a_*, a^*_a and d^*_a

Example of two prisoner's dilemmas 2

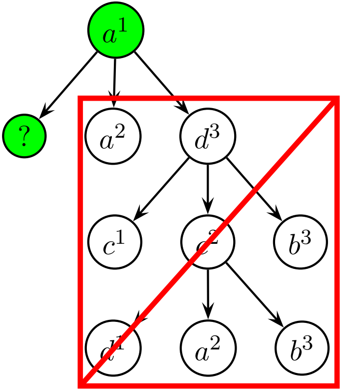


Regeneration effect



- When state 1 is realized, players need not worry about commitments in states 2 and 3
- All unrealized commitments can be forgotten

Regeneration effect 2



- Any elementary subpath in state 1 is possible if it is realized

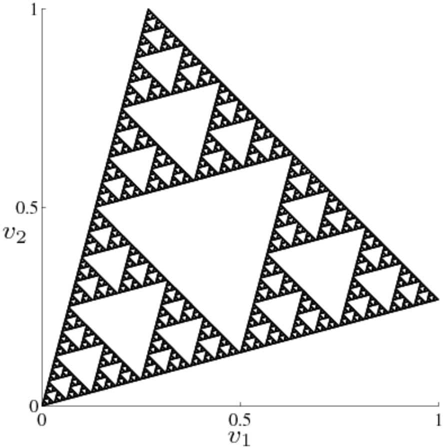
Conclusion

- New methods to compute and analyze equilibria
- SPE paths are characterized by elementary trees
- Useful graph presentation and measures for paths and payoffs
- Regeneration effect for commitments

References

- Berg, Kitti. Equilibrium paths in discounted supergames. Manuscript
- Berg, Kitti. Computing equilibria in discounted 2×2 supergames. Computational Economics, to appear
- Berg, Kitti. Fractal geometry of equilibrium payoffs in discounted supergames. Manuscript
- Jordan, Pollicott, Simon. Hausdorff dimension for randomly perturbed self affine attractors. Communications in mathematical physics 270: 519-544
- Kitti. Subgame perfect equilibria in discounted stochastic games. Manuscript

That's all folks...



Thank you!