Cracking the Code of Repeated Games

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    - perfect monitoring
    - pure strategies
    - stage game with finitely many actions

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- Research questions:
    - What are the subgame perfect equilibrium (SPE) paths?
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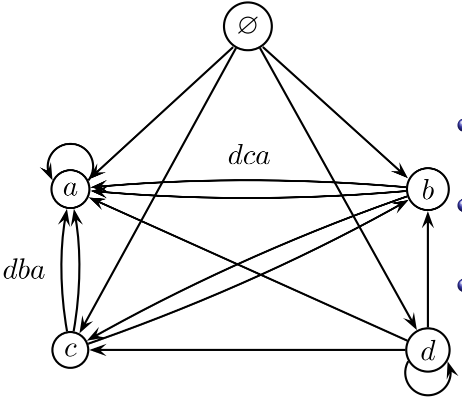
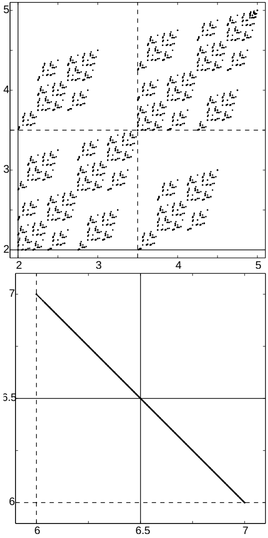
    - What if the stage game and the discount factors change?
    - Can we measure the complexity of equilibria?
    - What affects the complexity?

Main results: methods to compute and analyze equilibria



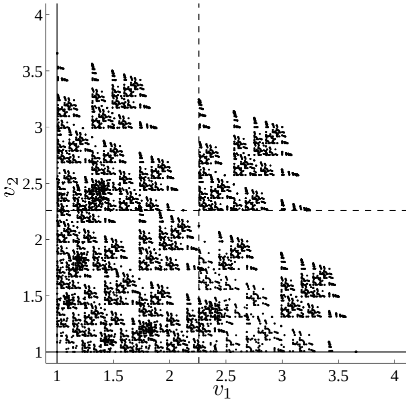
- Complex equilibrium behavior collapses into elementary subpaths
- SPE paths can be represented with directed multigraph
- Analyze complexity of SPE paths

Main results: classification of 2x2 supergames



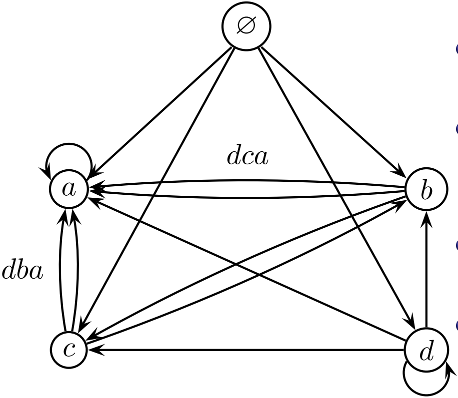
- 12 symmetric ordinal 2x2 games can be classified into 3 groups
- Stag Hunt is more "interesting" than Battle of the Sexes
- SPE paths in BoS: repetition of stage game's NE $(b^{\mathbf{N}} c^{\mathbf{N}})^{\infty}$
- Stag Hunt: suitable combinations of all actions a,b,c,d

Main Results: Measuring Complexity



- Payoff set is a graph-directed self-affine set
- Estimate its Hausdorff dimension
- We can also analyze the paths: their dimension, cardinality and entropy

Main results: what affects the complexity?



- Properties of the multigraph: the cycles and the contractions
- Change in discount factors create continuous change in path dimension
- Change in cycles create discontinuous change
- Related to the eigenvalues of the adjacency matrix

Characterization of equilibria

- Stage game:

	L	R
T	$3, 3$	$0, 4$
B	$4, 0$	$1, 1$

a	b
c	d

- Path d^∞ is SPE but there are others
- SPE strategies consists of SPE and punishment paths
- There are no one-shot deviations from SPE paths
- Here, path d^∞ is the punishment path

The building block of SPE paths

- A path is first-action feasible (FAF) if the first action is incentive compatible when any SPE path follows the path
- $bdca$ is FAF if there are no profitable one-shot deviations from b and the path continues incentive compatible

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- $bbaa$ is a FAF path

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$$a\,b\,b\,a\,a\,\ldots$$

- Thus, ABBA can be played infinitely

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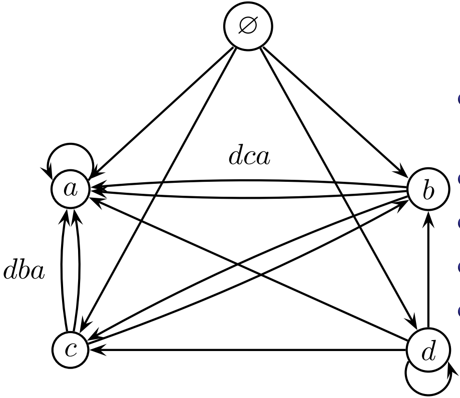

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  - Hausdorff dimension of the payoff set
  - graph directed construction: Mauldin and Williams (1988)
  - arcs correspond to contractions
  - if $p = abc$ is played on an arc, then contraction mapping on the arc is $r_p = \delta^{|p|} = \delta^3$

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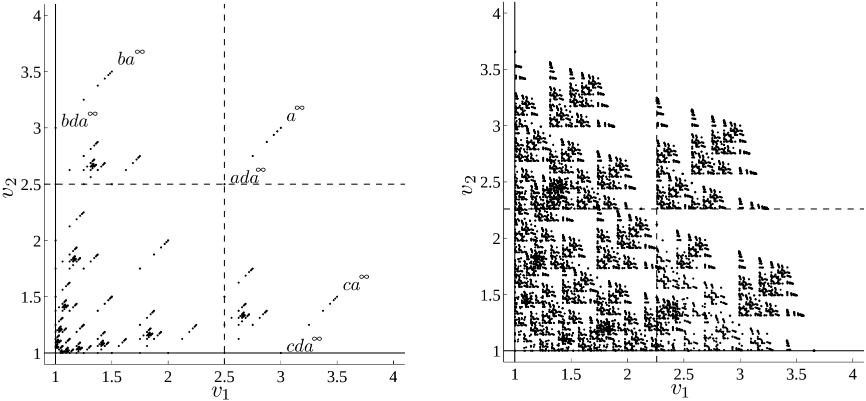
  - if $p = abc$ is played on an arc, then contraction mapping on the arc is $r_p = \delta^{|p|} = \delta^3$
  - exact dimension when open set condition is satisfied ($\delta < 0.5$)
  - otherwise, lower and upper bound estimates: Edgar and Golds (1999)

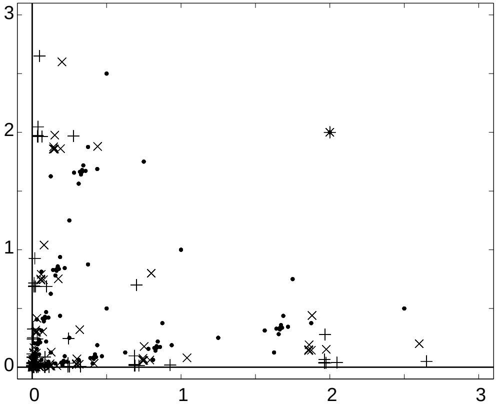
Effects of discounting: SPE paths increase

$\delta = 0.5$, $\dim_H = 0$ (limit)

$\delta = 0.58$, $\dim_H \approx 1.4$

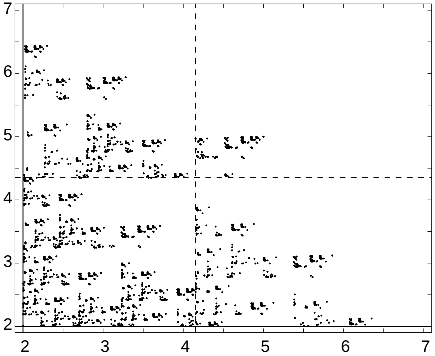


Effects of discounting: payoff set not monotone



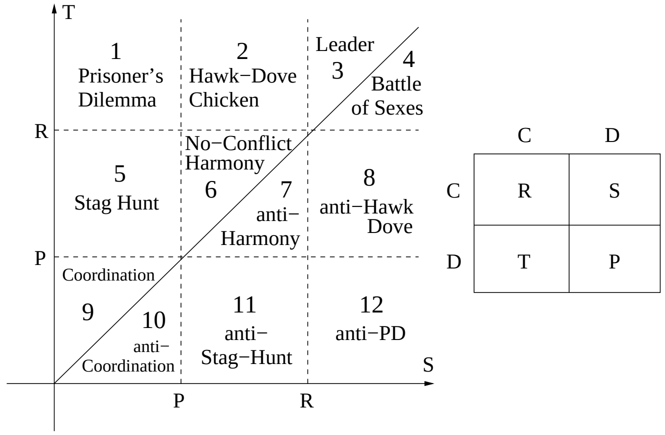
- PD with $\delta = 0.35$ (+), $\delta = 0.4$ (x), $\delta = 0.5$ $(\cdot)$
- maximum payoff around 2.5 decreases, path ca^{∞}
- Mailath, Obara and Sekiguchi (2002)

Unequal discount factors



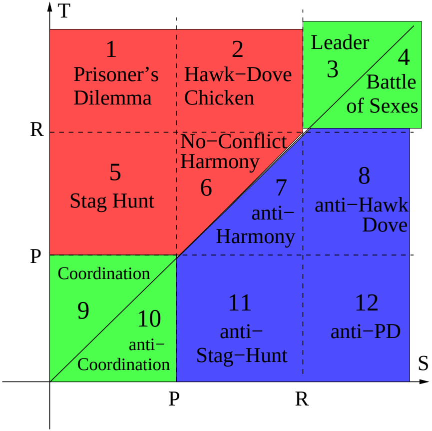
- PD with $\delta_1 = 0.57$ and $\delta_2 = 0.53$
- payoff set tilted to one side, more sparse on southern side
- some actions to player 2 are not possible as he is less patient
- Lehrer and Pauzner (1999)

Twelve symmetric strictly ordinal 2x2 games



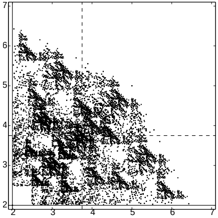
Robinson and Goforth (2005)

Classification into three groups

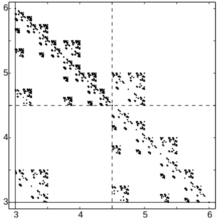


red: high complexity, green: low complexity, blue: only one SPE

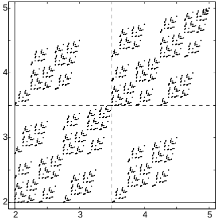
Payoff sets with high complexity



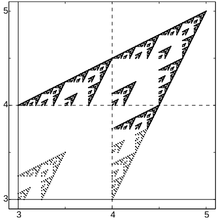
Prisoner's Dilemma, $\delta = 0.65$



Chicken, $\delta = 0.5$

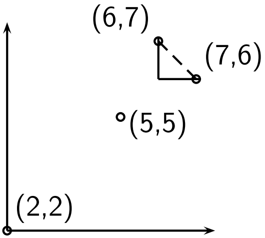


Stag Hunt, $\delta = 0.5$

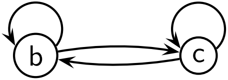


No Conflict, $\delta = 0.5$

Payoff sets with low complexity



	C	D
C	$5,5$	$6,7$
D	$7,6$	$2,2$



- Payoff sets similar in Leader, Battle of the Sexes, Coordination and anti-Coordination games
- repetition of two equilibria
- $\dim_H = 1$ when δ from $1/2$ to $0.6 \dots 0.8$
- when $\delta < 1/2$, isolated points between b and c

Path dimensions

game/δ	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0.69	1.23*	3.37*	5.91*	12.88*
2	0.58	0.81	1.24	2.03*	3.33*	5.80*	12.75*
5	0.73	1.10	1.49	2.26*	3.46*	5.85*	12.76*
6	0	0	1.39	2.12*	3.33*	5.71*	12.44*
Sierpinski	0.91	1.20	1.59	2.15	3.08	4.92	10.43
Upper bound	1.15	1.51	2	2.71	3.89	6.21	13.16
3	0.58	0.76	1	1.36	1.94	3.11	5.52*
4	0.58	0.76	1	1.36	2.12**	3.83**	6.40*
9	0.58	0.76	1	1.46**	2.51**	4.47*	10.57*
10	0.58	0.76	1	1.36	2.25**	4.09*	10.07*

FAF path length restricted to 8 (*) and 12 (**)

Summary

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- SPE paths are characterized by finite subpaths
- Useful multigraph presentation
- Hausdorff dimensions for paths and payoffs

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- Useful multigraph presentation
- Hausdorff dimensions for paths and payoffs
- Classification of 2x2 games
- Equilibria for wide range of discount factors

Thank you!

Any questions?