Equilibrium Paths in Discounted Supergames

Kimmo Berg and Mitri Kitti
Aalto University

July 13, 2010

Research Questions

- Setup: infinitely repeated game with discounting
  - perfect monitoring
  - pure strategies
  - stage game with finitely many actions

Research Questions

- Setup: infinitely repeated game with discounting
  - perfect monitoring
  - pure strategies
  - stage game with finitely many actions
- Research questions:

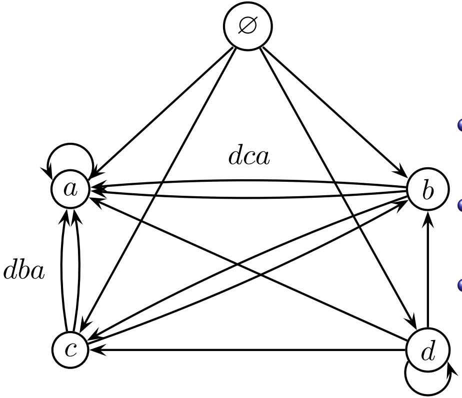
Research Questions

- Setup: infinitely repeated game with discounting
  - perfect monitoring
  - pure strategies
  - stage game with finitely many actions
- Research questions:
  - What are the subgame perfect equilibrium (SPE) paths?
  - What about the payoff set?

Research Questions

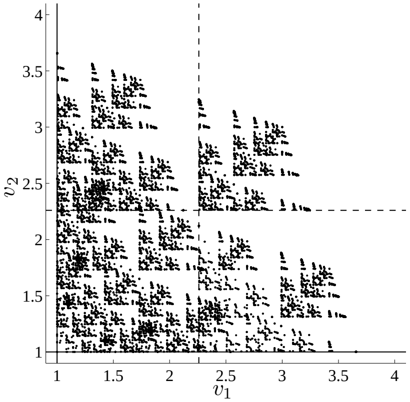
- Setup: infinitely repeated game with discounting
  - perfect monitoring
  - pure strategies
  - stage game with finitely many actions
- Research questions:
  - What are the subgame perfect equilibrium (SPE) paths?
  - What about the payoff set?
  - What happens when the discount factors change?

Main Results: Analyze and Compute SPE Paths



- Complex equilibrium behavior collapses into elementary subpaths
- SPE paths can be represented with directed multigraph
- Analyze complexity of SPE paths

Main Results: Analyze and Compute Payoff Set



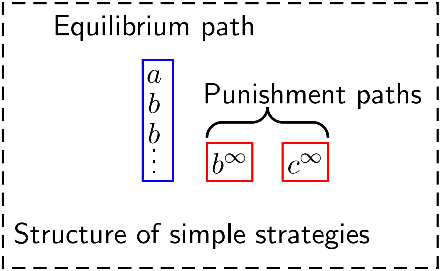
- Payoff set is a particular fractal
- Graph directed self-affine set
- Estimate Hausdorff dimension

Characterization of SPE strategies

- All SPE paths are attained by simple strategies: Abreu (1988)

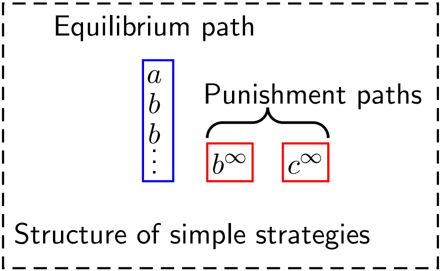
Characterization of SPE strategies

- All SPE paths are attained by simple strategies: Abreu (1988)



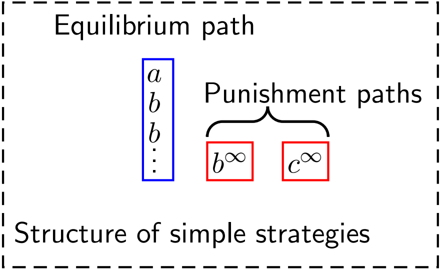
Characterization of SPE strategies

- All SPE paths are attained by simple strategies: Abreu (1988)
  - Equilibrium path that the players follow
  - History-independent punishment paths for each player



Characterization of SPE strategies

- All SPE paths are attained by simple strategies: Abreu (1988)
  - Equilibrium path that the players follow
  - History-independent punishment paths for each player
  - Punishment paths are played if the players deviate from the current path
  - These are equilibrium paths that give the minimum payoffs $v_i^- = \min\{v_i : v \in V^*\}$.



Characterization of SPE paths

- SPE paths are characterized by one-shot deviation principle

Characterization of SPE paths

- SPE paths are characterized by one-shot deviation principle
- A path p that strategy σ induces is a SPE path if and only if it satisfies the incentive compatibility (IC) constraints:

Characterization of SPE paths

- SPE paths are characterized by one-shot deviation principle
- A path p that strategy σ induces is a SPE path if and only if it satisfies the incentive compatibility (IC) constraints:

$$(1-\delta_i)u_i(a^k(\sigma)) + \delta_i v_i^k \geq \max_{a_i \in A_i} \left[(1-\delta_i)u_i(a_i, a_{-i}^k(\sigma)) + \delta_i v_i^- \right],$$

$\forall i \in N$, $k \geq 0$, and where the continuation payoff after $a^k(\sigma)$ is $v_i^k = (1-\delta_i)\sum_{j=0}^{\infty} \delta_i^j u_i(a^{k+1+j}(\sigma))$.

New Concept

Definition

A finite path $p \in A^k(a)$ is a first action feasible (FAF) path if the first action profile a is incentive compatible when any SPE path follows the finite path:

New Concept

Definition

A finite path $p \in A^k(a)$ is a first action feasible (FAF) path if the first action profile a is incentive compatible when any SPE path follows the finite path:

$$(1-\delta_i)\sum_{k=0}^{|p|-1} u_i(i(p_k)) + \delta_i^{|p|} v_i^- \geq \max_{a_i \in A_i} (1-\delta_i) u(a_i, a_{-i}) + \delta_i v_i^-,$$

$\forall i \in N$.

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^{\infty}$ a SPE path?

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^{\infty}$ a SPE path?

$$a\,b\,b\,a\,a\cdots$$

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^{\infty}$ a SPE path?

$$\boxed{a}\, b\, b\, a\, a \cdots$$

- a is a FAF path

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^\infty$ a SPE path?

$$a\,\boxed{b\,b\,a\,a}\cdots$$

- $bbaa$ is a FAF path

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^\infty$ a SPE path?

$$a\,b\,\boxed{b\,a}\,a\cdots$$

- ba is a FAF path

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^\infty$ a SPE path?

$$a\,b\,b\,\boxed{a}\,a\cdots$$

- a is a FAF path

Illustrative Example

- We can check that a path is IC with the FAF paths
- FAF paths are a, ba, and $bbaa$
- Is a path $p = (abba)^{\infty}$ a SPE path?

$$a\,b\,b\,a\,a\cdots$$

- Thus, $p = (abba)^{\infty}$ is a SPE path

Recursive Definition of FAF Paths

Definition

A vector $\mathsf{con}(a)$ gives the least payoffs that make action a IC

$$(1-\delta_i)u_i(a)+\delta_i \mathsf{con}_i(a)=\max_{a_i\in A_i}\left[(1-\delta_i)u_i(a_i,a_{-i})+\delta_i v_i^{-}\right],$$

$\forall i \in N$.

Recursive Definition of FAF Paths

Definition

A vector con(a) gives the least payoffs that make action a IC
$(1-\delta_i)u_i(a)+\delta_i\text{con}_i(a)=\max_{a_i\in A_i}\left[(1-\delta_i)u_i(a_i,a_{-i})+\delta_i v_i^-\right]$,
$\forall i\in N$.

Definition

For any $p\in A^k(a)$, $k\geq 2$, and $p=p^{k-1}a$,
$\text{con}_i(p)=\delta_i^{-1}\left[\text{con}_i(p^{k-1})-(1-\delta_i)u(a)\right]$.

Recursive Definition of FAF Paths

Definition

A vector con(a) gives the least payoffs that make action a IC

$$(1-\delta_i)u_i(a)+\delta_i\text{con}_i(a)=\max_{a_i\in A_i}\left[(1-\delta_i)u_i(a_i,a_{-i})+\delta_i v_i^-\right],$$

$\forall i\in N$.

Definition

For any $p\in A^k(a)$, $k\geq 2$, and $p=p^{k-1}a$,

$$\text{con}_i(p)=\delta_i^{-1}\left[\text{con}_i(p^{k-1})-(1-\delta_i)u(a)\right].$$

Definition

A finite path $p\in A^k(a)$, $k\geq 2$, is a FAF path if and only if con$(p)\leq$ con(a).

New Concept

Definition

A finite path $p \in A^k(a)$ is a first action infeasible (FAI) path if the first action profile a is not incentive compatible no matter what SPE path follows:

New Concept

Definition

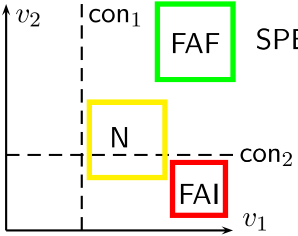
A finite path $p \in A^k(a)$ is a first action infeasible (FAI) path if the first action profile a is not incentive compatible no matter what SPE path follows:

$$\text{con}_i(p) > \bar{v}_i, \quad \text{for some } i \in N,$$

where $\bar{v}_i = \max\{v_i : v \in V^*\}$, $i \in N$.

Interpretation of FAF and FAI paths

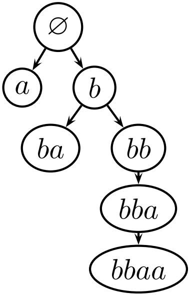
- We can classify all finite paths by using con(a)
- Future payoffs weigh less due to discounting



SPE continuation payoffs

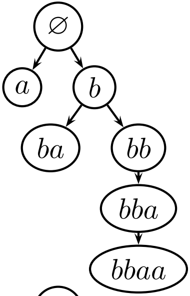
Construction of SPE paths

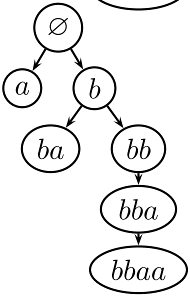
1. Compute FAF paths and represent as tree



Construction of SPE paths

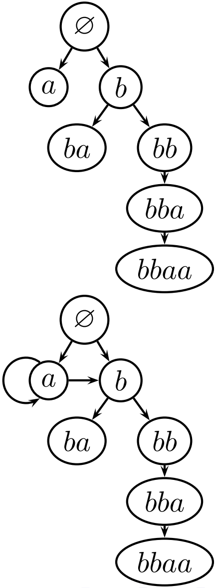
1. Compute FAF paths and represent as tree
2. Form graph: Nodes are from the tree
3. Form arcs:
   - inner nodes: arcs from the tree





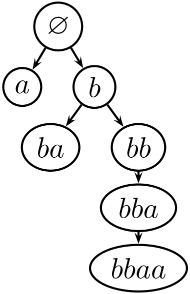
Construction of SPE paths

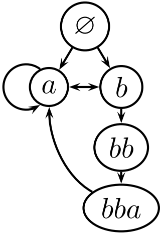
1. Compute FAF paths and represent as tree
2. Form graph: Nodes are from the tree
3. Form arcs:
   - inner nodes: arcs from the tree
   - leaf nodes connected to root: arcs to root node's children



Construction of SPE paths

1. Compute FAF paths and represent as tree
2. Form graph: Nodes are from the tree
3. Form arcs:
   - inner nodes: arcs from the tree
   - leaf nodes connected to root: arcs to root node's children
   - other leaf nodes: find p_k in the tree.
     - If p_k found in tree, arc from p_1 to p_k.
     - If longest common path with p an inner node in tree, p is infeasible.
     - Else set $k = k + 1$.

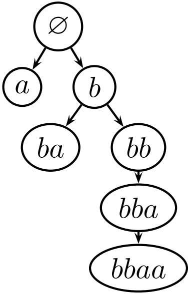


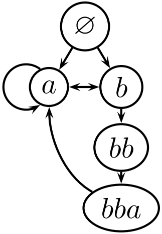


Construction of SPE paths

1. Compute FAF paths and represent as tree
2. Form graph: Nodes are from the tree
3. Form arcs:
    - inner nodes: arcs from the tree
    - leaf nodes connected to root: arcs to root node's children
    - other leaf nodes: find p_k in the tree.
        - If p_k found in tree, arc from p_1 to p_k.
        - If longest common path with p an inner node in tree, p is infeasible.
        - Else set $k = k + 1$.

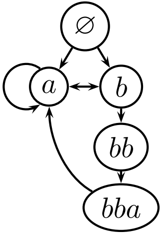
- Note that FAF paths may have infeasible parts.

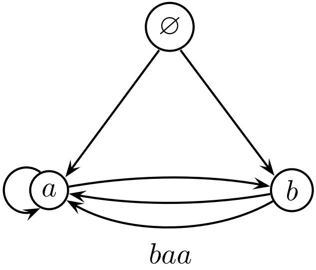




Multigraph Representation

- When FAF paths with infeasible parts are removed, we get the elementary subpaths of the game
- Graph can be simplified by removing the states with only one destination





Analysis with the Multigraph

- Examine complexity of SPE paths
  - cycles in multigraph related to dimension
  - number and length of elementary subpaths
  - entropy of action profiles

Analysis with the Multigraph

- Examine complexity of SPE paths
  - cycles in multigraph related to dimension
  - number and length of elementary subpaths
  - entropy of action profiles
- Examine complexity of payoff set
  - where are the SPE payoffs and how dense are they?

Analysis with the Multigraph

- Examine complexity of SPE paths
  - cycles in multigraph related to dimension
  - number and length of elementary subpaths
  - entropy of action profiles
- Examine complexity of payoff set
  - where are the SPE payoffs and how dense are they?
  - Hausdorff dimension of the payoff set
  - graph directed construction: Mauldin and Williams (1988)
  - arcs correspond to contractions
  - if $p = abc$ is played on an arc, then contraction mapping on the arc is $r_p = \delta^{|p|} = \delta^3$

Analysis with the Multigraph

- Examine complexity of SPE paths
  - cycles in multigraph related to dimension
  - number and length of elementary subpaths
  - entropy of action profiles
- Examine complexity of payoff set
  - where are the SPE payoffs and how dense are they?
  - Hausdorff dimension of the payoff set
  - graph directed construction: Mauldin and Williams (1988)
  - arcs correspond to contractions
  - if $p = abc$ is played on an arc, then contraction mapping on the arc is $r_p = \delta^{|p|} = \delta^3$
  - exact dimension when open set condition is satisfied ($\delta < 0.5$)
  - otherwise, lower and upper bound estimates: Edgar and Golds (1999)

Example of Prisoners' Dilemma

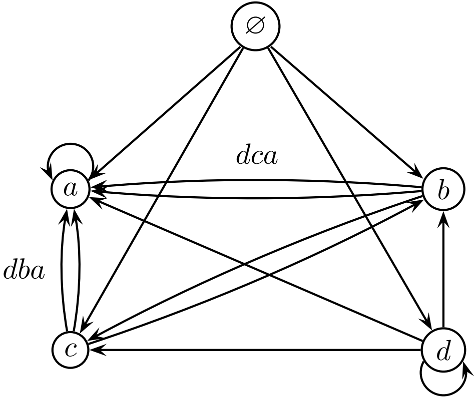
- Stage game:

	L	R
T	$3, 3$	$0, 4$
B	$4, 0$	$1, 1$

- $A = \{a, b, c, d\}$ from left to right, top to bottom
- For $\delta_1 = \delta_2 = 1/2$ the finite elementary sets

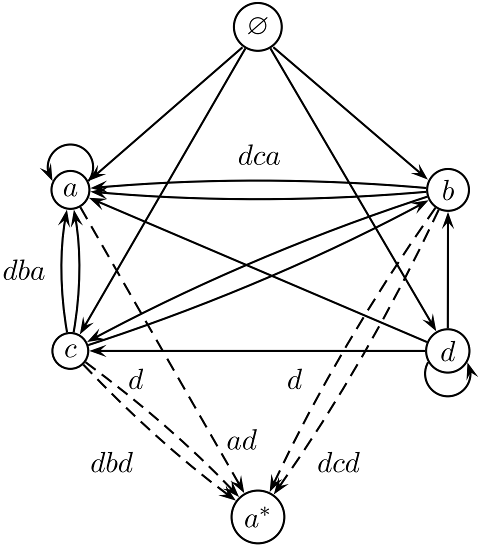
	a	b	c	d
P^1	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$
P^2	$\{aa\}$	$\{ba, bc\}$	$\{ca, cb\}$	$\emptyset$
P^4	$\emptyset$	$\{bdca\}$	$\{cdba\}$	$\emptyset$

Multigraph of Prisoners' Dilemma



- Finite elementary subpaths
- Note: arc labels contain the information for creating SPEPs
  - no label = same as the node pointed at

Multigraph of Prisoners' Dilemma

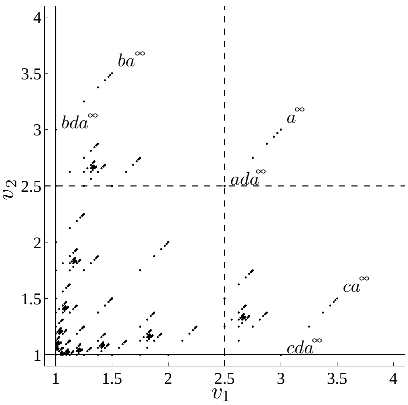


- Finite elementary subpaths
- Note: arc labels contain the information for creating SPEPs
  - no label = same as the node pointed at
- Infinite elementary subpaths

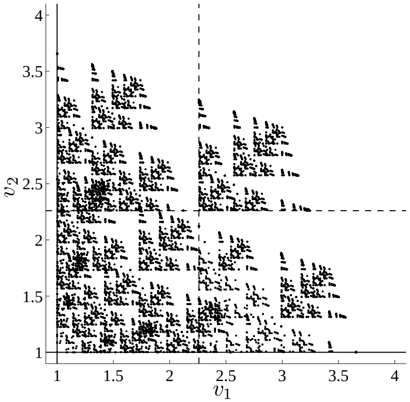
$P^{\infty}(a) = \{ada^{\infty}\}$,
$P^{\infty}(b) = \{bda^{\infty}, bdcda^{\infty}\}$,
$P^{\infty}(b) = \{cda^{\infty}, cdbda^{\infty}\}$

Payoffs in Prisoner's Dilemma

$\delta = 0.5$, $\dim_H = 0$ (limit)



$\delta = 0.58$, $\dim_H \approx 1.4$



Results

Proposition

A path $p \in A^\infty(a)$ is a SPEP if and only if for all $j \in \mathbb{N}$ either $p_j^k \in P^k(i(p_j^k))$ for some k or $p_j \in P^\infty(i(p_j))$.

Results

Proposition

A path $p \in A^{\infty}(a)$ *is a SPEP if and only if for all* $j \in \mathbb{N}$ *either* $p_j^k \in P^k(i(p_j^k))$ *for some* k *or* $p_j \in P^{\infty}(i(p_j))$.

Proposition

For any $\varepsilon > 0$ *there is* k *such that* $p \in A^{\infty}(a)$, $a \in A$, $v(p_1) \geq \text{con}(a) + \varepsilon$, *imply that* $p_j^l \in P^l(i(p_j))$ *for some* $l \leq k$.

Results

Proposition

A path $p \in A^\infty(a)$ is a SPEP if and only if for all $j \in \mathbb{N}$ either $p_j^k \in P^k(i(p_j^k))$ for some k or $p_j \in P^\infty(i(p_j))$.

Proposition

For any $\varepsilon > 0$ there is k such that $p \in A^\infty(a)$, $a \in A$, $v(p_1) \geq \mathsf{con}(a) + \varepsilon$, imply that $p_j^l \in P^l(i(p_j))$ for some $l \leq k$.

Proposition

When syntax $S(u, T)$ contains finitely many paths, then all SPEPs are represented by a multigraph.

Infinite Elementary Subpaths

- Payoffs are on the boundary, i.e., $v_i(p) = con_i(a)$ for some i
- We can either try to find the infinite subpaths or construct a subset of SPE paths
- We know roughly what paths are missing and what payoffs they give

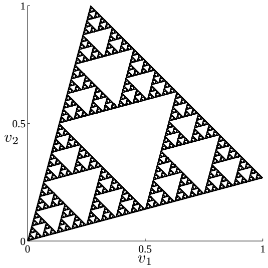
Infinite Elementary Subpaths

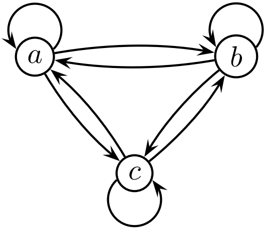
- Payoffs are on the boundary, i.e., $v_i(p) = con_i(a)$ for some i
- We can either try to find the infinite subpaths or construct a subset of SPE paths
- We know roughly what paths are missing and what payoffs they give
- For high discount δ, we have to restrict anyways the length of FAF paths in computation

Sierpinski Game

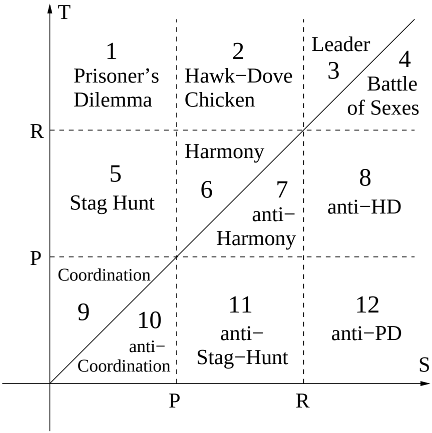
	L	M	R
T	$2-\sqrt{3}, 1$	$-1,-1$	$-1,-1$
C	$-1,-1$	$1, 2-\sqrt{3}$	$-1,-1$
B	$-1,-1$	$-1,-1$	$0, 0$

- $A = \{a, b, c\}$ on the diagonal, $\dim_H = \ln 3 / \ln 2 \approx 1.585$
- For $\delta_1 = \delta_2 = 1/2$, the finite elementary subpaths: a, b, c





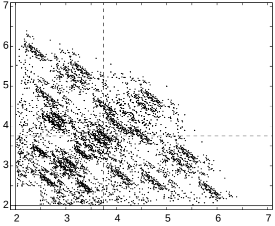
Twelve Symmetric Ordinal 2x2 Games



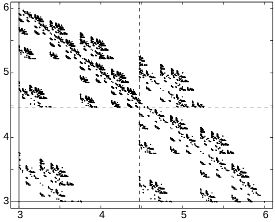
	C	D
C	R	S
D	T	P

Robinson and Goforth (2005)

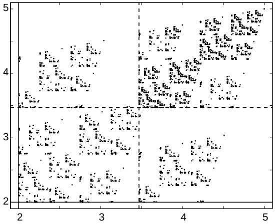
Payoff sets with high complexity



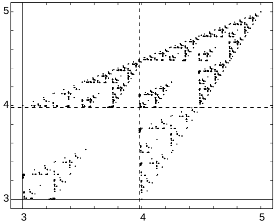
Prisoner's Dilemma, $\delta = 0.65$



Chicken, $\delta = 0.5$

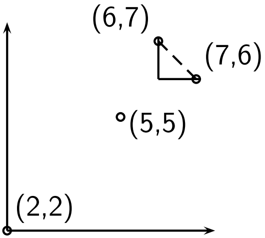


Stag Hunt, $\delta = 0.5$

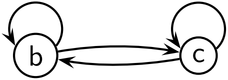


No Conflict, $\delta = 0.5$

Payoff sets with low complexity



	C	D
C	$5, 5$	$6, 7$
D	$7, 6$	$2, 2$



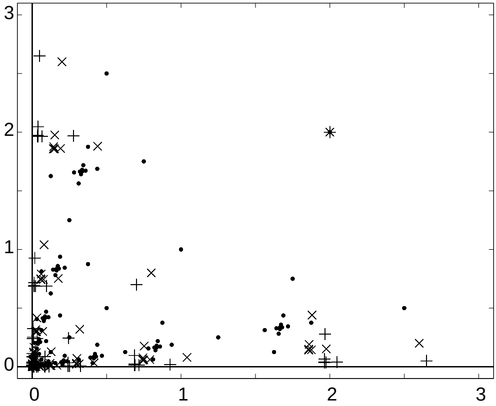
- Payoff sets similar in Leader, Battle of the Sexes, Coordination and anti-Coordination games
- repetition of two equilibria
- $\dim_H = 1$ when δ from $1/2$ to $0.6 \ldots 0.8$
- when $\delta < 1/2$, isolated points between b and c

Path Dimensions

game/δ	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0	0	0.69	1.23*	3.37*	5.91*	12.88*
2	0.58	0.81	1.24	2.03*	3.33*	5.80*	12.75*
5	0.73	1.10	1.49	2.26*	3.46*	5.85*	12.76*
6	0	0	1.39	2.12*	3.33*	5.71*	12.44*
Sierpinski	0.91	1.20	1.59	2.15	3.08	4.92	10.43
Upper bound	1.15	1.51	2	2.71	3.89	6.21	13.16
3	0.58	0.76	1	1.36	1.94	3.11	5.52*
4	0.58	0.76	1	1.36	2.12**	3.83**	6.40*
9	0.58	0.76	1	1.46**	2.51**	4.47*	10.57*
10	0.58	0.76	1	1.36	2.25**	4.09*	10.07*

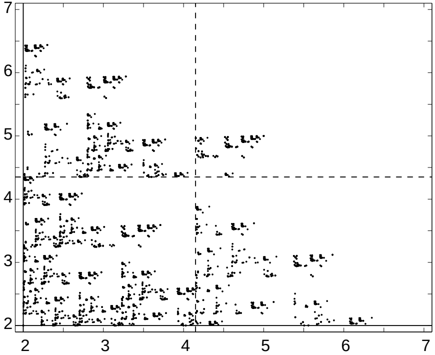
FAF path length restricted to 8 (*) and 12 (**)

Changing Discount Factor



- PD with $\delta = 0.35$ (+), $\delta = 0.4$ (x), $\delta = 0.5$ ($\cdot$)
- maximum payoff around 2.5 decreases, path ca^{∞}
- Mailath, Obara and Sekiguchi (2002)

Unequal Discount Factors



- PD with $\delta_1 = 0.57$ and $\delta_2 = 0.53$
- payoff set tilted to one side, more sparse on southern side
- some actions to player 2 are not possible as he is less patient
- Lehrer and Pauzner (1999)

Summary

- SPEPs are characterized by elementary subpaths
  - all SPEPs are obtained by combining elementary subpaths
  - finite elementary subpaths can be rather easily computed
  - one implication: equilibrium behavior is “easily predicted”

Summary

- SPEPs are characterized by elementary subpaths
  - all SPEPs are obtained by combining elementary subpaths
  - finite elementary subpaths can be rather easily computed
  - one implication: equilibrium behavior is “easily predicted”
- The set of SPE payoffs is a self-affine set
  - finite number of elementary subpaths $\Rightarrow$ graph-directed self-affine set
  - dimension estimates for the payoff set
  - insight into folk theorem: payoff set becomes richer due to having more SPEPs and due to less contractive mappings

Methodological Framework

- The set of SPE payoffs is characterized by a fixed-point equation
  - imperfect monitoring: Abreu, Pearce, and Stacchetti (1986,1988), hereafter APS
  - perfect monitoring: Cronshaw and Luenberger (1994)
  - computation: Cronshaw (1997), Judd, Yeltekin, and Conklin (2003)
  - application to prisoners' dilemma game: Mailath, Obara, and Sekiguchi (2002)

Methodological Framework

- The set of SPE payoffs is characterized by a fixed-point equation
  - imperfect monitoring: Abreu, Pearce, and Stacchetti (1986,1988), hereafter APS
  - perfect monitoring: Cronshaw and Luenberger (1994)
  - computation: Cronshaw (1997), Judd, Yeltekin, and Conklin (2003)
  - application to prisoners' dilemma game: Mailath, Obara, and Sekiguchi (2002)
- Analogy with dynamic programming

Dynamic Programming	Repeated Games
Bellman Equation	APS
Euler Equation	This work!

The APS Theorem

Proposition

The set of SPE payoffs V^* is the (unique) largest (in set inclusion) compact set that satisfies

$$V^* = \bigcup_{a \in F(V^*)} C_a(V^*) = \bigcup_{a \in F(V^*)} B_a(C_a(V^*)).$$

The APS Theorem

Proposition

The set of SPE payoffs V^* is the (unique) largest (in set inclusion) compact set that satisfies

$$V^* = \bigcup_{a \in F(V^*)} C_a(V^*) = \bigcup_{a \in F(V^*)} B_a(C_a(V^*)).$$

- $B_a(v) = (I - T)u(a) + Tv$, i.e., discounted average of $u(a)$ and v, here T is the matrix with $\delta_1, \ldots, \delta_n$ on diagonal
- V^* is a fixed-point of a particular iterated function system
    - V^* is a subset of a self-affine set W for which $W = \cup_{a \in F(V^*)} B_a(W)$