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Line planning with multiple modalities: Performance analysis of cost-oriented approaches

Jarkko Jalovaara

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Author Jarkko Jalovaara

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Supervisor Asst. Prof. Philine Schiewe

Advisor Asst. Prof. Philine Schiewe

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Abstract

A functional public transport system is an important part of any contemporary city, providing affordable transport services for citizens, improving environmental sustainability and increasing the overall welfare of the city. The key elements of all public transport systems are the set of lines, i.e., routes for the vehicles and passengers, since they provide the basis for operating the entire transport system and allow passengers to plan their journeys efficiently. The mathematical models for designing the routes and lines optimally in transport systems give strong support for decision-making in transport planning. However, the large size of real-life transport systems and varying properties of different modes of transport require more efficient models that can be used for real-life instances.

In this work, we introduce line planning models and algorithms based on integer optimization for obtaining cost-optimal sets of lines in public transport networks with multiple modes of transport. We formulate the models in terms of external passenger demand and present theoretical analysis based on the properties of public transport networks with multiple modes of transport. Finally, we perform computational experiments using the different models and compare the models using an artificial public transport network and passenger demand.

The results show that the integer optimization methods for public transport networks with multiple transport modes give optimal solutions in terms of line operating costs while also maintaining a reasonable passenger travel time. We also confirm that line planning problems with multiple modes can be reduced to a set of single modality problems at the expense of optimality. Overall, we witness a trade-off between tractability and optimality for the different models. Future work remains on analyzing the behaviour of different models in terms of model parameters and different objectives in public transport line planning.

Keywords Public transport planning, Line planning, Integer optimization, Multimodal network, Public transport network

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Tiivistelmä

Toimiva joukkoliikennejärjestelmä on tärkeä osa nykyaikaista kaupunkisuunnittelua. Joukkoliikenne on asukkaille edullinen matkustusmuoto, joka on ympäristön kannalta kestävää ja kaupungin yleistä asuinmukavuutta parantavaa. Tärkeä osa kaikkia joukkoliikennejärjestelmiä ovat joukkoliikenteen linjat, jotka kuvaavat kulkuneuvojen reittejä liikenteessä. Linjat ovat joukkoliikenteen toiminnan perusta, jonka avulla matkustajat voivat helposti suunnitella matkojensa toteutustavan. Matemaattiset mallit optimaaliseen linjastosuunnitteluun tarjoavat vahvan perustan koko julkisen liikenteen suunnitteluun liittyvälle päätöksenteolle. Kuitenkin liikennejärjestelmien suuri koko ja kulkuneuvotyyppien vaihtelevat ominaisuudet vaativat yhä tehokkaampia malleja, jotka toimivat hyvin myös reaalimaailman sovelluksissa.

Tässä työssä esitellään kokonaislukuoptimointiin perustuvia linjastosuunnittelun malleja ja algoritmeja, joilla määritetään kokonaishinnan kannalta optimaalisia linjastoja usean kulkuneuvotyypin joukkoliikenneverkoille. Käytettävät mallit määritellään ulkoisen matkustajatarpeen mukaan, ja mallien analysointi pohjautuu usean kulkuneuvotyyppien joukkoliikennesysteemien ominaisuuksiin. Työssä myös vertaillaan eri mallien toimintaa laskennalisten testien avulla käyttämällä testaukseen soveltuvaa joukkoliikenneverkkoa ja erillistä matkustajatarvetta.

Työn tulokset osoittavat, että kokonaislukuoptimointiin perustuvat mallit antavat optimaalisia ratkaisuja kokonaishinnan suhteen siten, että linjastoa käyttävien matkustajien matkustusaika säilyy kohtuullisena. Tulokset osoittavat myös, että usean kulkuneuvotyypin linjastosuunnittelun optimointiongelma voidaan jakaa useiksi yhden kulkuneuvotyypin osaongelmiksi ratkaisun optimaalisuuden kustannuksella. Yleishavaintona työssä huomataan, että eri malleilla voidaan optimaalisuutta heikentämällä ratkaista linjastosuunnittelun ongelmia merkittävästi tehokkaamin. Tulevassa tutkimuksessa on syytä tutkia eri linjastosuunnittelun parametrien vaikutuksia optimaalisiin ratkaisuihin sekä muihin linjastosuunnittelun kohdefunktioihin.

Avainsanat Joukkoliikenteen suunnittelu, Linjastosuunnittelu, Kokonaislukuoptimointi, Joukkoliikenneverkko, Multimodaalinen verkko

Contents

1 Introduction

Public transport planning in the context of urban planning is a vital part of designing modern city infrastructure and providing efficient and reliable transportation for citizens. In recent years, the importance of public transport planning has increased considerably due to several sustainability-related factors. These include growing populations in urban areas, environmental sustainability issues and socioeconomic imbalances in cities [16]. A good quality public transport system plays a key role in addressing many of these issues and increasing the overall welfare of cities. Indeed, the recent transport policies that suggest replacing car travel with more sustainable means of transport require more efficient and more economically sustainable transport design to increase the overall attractiveness of public transport [15, 16].

The primary goal of public transport planning is to create good quality transportation services for the population for a reasonable operational cost. The mathematical approach for public transport planning uses various optimization models to create line concepts, timetables and vehicle schedules that optimize different objectives such as passenger travel time and transportation costs. The traditional way to tackle the large size and high complexity of transport planning is to solve the interconnected planning models sequentially, as shown in Figure 1, to obtain the characteristics of public transport systems such as optimal timetables and optimal vehicle routes. With this approach, the optimal results from previous stages are used in formulating the subsequent planning models [10].

This work focuses on the line planning step of mathematical public transport modeling. Line planning is an important aspect of public transport planning since the lines are essential building blocks of a public transport system and important tools for both passengers and transport operators alike. In line planning, the goal is to obtain a set of lines and their corresponding operating frequencies for the vehicles to satisfy external passenger demand. In the context of line planning, a line represents an ordered sequence of stops from the origin to the destination along with the routes between the stops in the network. We assume that a vehicle serves all the stops belonging to a line in order and always drives from the origin to the destination. Line planning is a multi-objective optimization problem with multiple objectives such as the passenger travel time and the line operating costs. In a sequential fashion, the obtained set of lines are used as an input for subsequent public transport planning stages such as timetabling and vehicle scheduling [3].

Traditionally, line planning problems are formulated using only a single transport mode and its corresponding transportation network. Because a majority of the public transport systems in the world use multiple transport modes, it is important to consider the aspects of multimodality in line planning and generally in public transport planning. In addition, different modes of transport have notably different characteristics from the building of the transport infrastructure to the operation of the transport. For example, constructing a metro line involves greater initial investment than a bus line, while generally a metro offers more capacity advantages and reliability than a bus line. Thus, multimodal line planning approaches provide valuable support for public transport decision-making in real-life public transport systems [19].

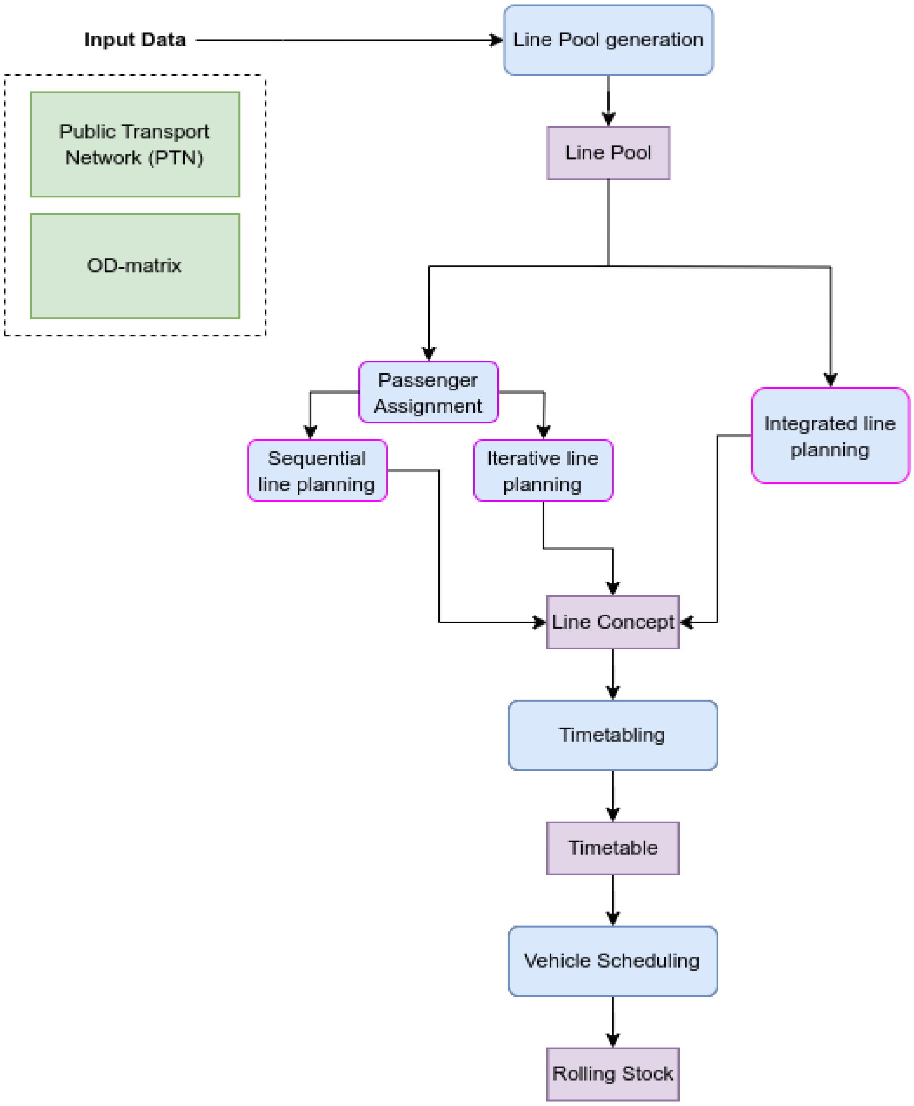


Figure 1: A general overview of different planning steps in mathematical public transport optimization. The order of solving the different planning stages sequentially is presented with the arrows. In the Figure, the blue boxes represent the planning models, the purple boxes represent the output from the planning models and the green boxes represent the input data for the planning models.

In this work, we present mathematical formulations of several line planning models that can be used with multiple different transport modes, each with their corresponding public transport network. We also introduce two general approaches for dealing with multiple transport modes in line planning problems. We present a detailed theoretical analysis of line planning problems with multiple modes and compare the different solution approaches theoretically using various examples. We also introduce new passenger assignment algorithms to obtain line planning problems with varying emphases on objective functions. Finally, we present experimental results and comparison for the line planning models and modified passenger assignment algorithms using passenger demand data and an artificial public transport network. We focus primarily on the line planning models based on cost-minimization, but also evaluate other objective functions to highlight the differences in our proposed line planning models and solution approaches.

1.1 Literature review

The mathematical framework of public transport modeling involves various planning stages that determine the characteristics of a public transportation system. The different planning stages include network design, passenger routing and line planning, as well as timetabling, vehicle scheduling and delay management. Traditionally, the different planning stages have been solved in a sequential fashion, as illustrated in Figure 1. The underlying idea of this sequential approach is to solve each of the planning stages separately and use the optimal solution of the previous stage to formulate the next stage problem. Handling all the planning stages in a single problem usually results in a large and complicated model with no guarantee of tractability even for smaller instances [10].

Due to the greedy-like approach of the sequential planning, recent literature has suggested integrated models that combine some of the different planning stages into a single problem, while still keeping the models relatively simple. For example, [11] suggests combining passenger assignment, line planning and timetabling stages into a single optimization problem instead of solving the problems sequentially. The integrated problems have been shown to result in better solutions than sequential multi-stage problems as presented in [1, 11]. Additionally, the decision-making perspective of integrated models is particularly useful, since they allow making first-stage decisions such that the following second-stage decisions are already optimal. Against their sequential model counterparts, the integrated models always perform at least equally well in comparison. However, the large size of the problem and limited tractability prevents the use of integrated models over sequential models if the real-life public transport networks are too large in size[1, 11].

Combining different planning stages of a sequential model into an integrated model is not always a straightforward process. However, in the literature there exists some general frameworks to convert separate problems into a single optimization problem. As an example, [6] presents an eigenmodel that solves the separate subproblems iteratively to find local optima of the corresponding integrated model. Moreover, [1] introduces a mathematical formulation for general sequential processes along with a partial integration approach for solving only a subset of sequential problems in an integrated fashion.

In this work, we focus on the line planning phase of the sequential public transport modeling. This means our primary interest is to obtain lines for vehicles and their corresponding operating frequencies to fulfill the external passenger demand in the public transport network. In the literature, there exist many different types of line planning models that differ in terms of the objectives and constraints of the line plan [3]. For example, cost-oriented models aim to minimize the costs of operating the set of lines, whereas passenger-oriented models aim to minimize the average travel time of the passengers in the network or maximize the number of direct passengers, that is, the passengers that don't have to transfer between different lines during their journey. Cost-oriented models are analyzed in detail in [8]. A passenger-oriented model with travel time minimization using a modified network with transfer penalties is presented in [5] and a line planning model maximizing the number of direct passengers is introduced in [9].

The public transport systems in real life are usually so large that considering all the possible lines in the line planning phase becomes numerically intractable for the majority of line planning problems. Therefore, choosing a limited set of possible lines called line pool to be considered in line planning models is needed. Methods for obtaining good quality line pools with limited sizes are presented in [2]. Moreover, [7] presents another approach of integrating line pool generation into the actual line planning model using a multi-commodity flow formulation.

Traditionally, line planning models assume that passenger demand is already distributed to the edges of the public transport network. However, this means that line planning is a two-stage problem. In the first stage, passenger assignment, the travel routes are selected for passengers in the public transport network based on the external passenger travel demand data. In the second stage, a sequential line planning optimization model, with constraints obtained from the first stage passenger assignment, is solved. In the literature, the effects of passenger assignment to the optimal solution of the line planning problem are investigated in [4]. A more general perspective on passenger assignment ideas are presented in [10]. In addition, [4] introduces integrated models for line planning phase where the line planning and passenger assignment are combined into a single integrated optimization model.

In this work, we introduce new formulations and algorithms for cost-oriented line planning and passenger assignment that can be used when the underlying public transport network consists of different vehicle modalities, such as metro, tram and bus. The previous literature on line planning in multimodal public transport systems is relatively scarce, although some references can be found. Line planning models with separate stopping patterns on line-level, as presented in [12] and [13], can be considered an elementary version of multimodal line planning, but only in terms of different network structures. In addition, [14] presents a system split procedure so that the original problem can be split into separate problems of single modality and solved separately. In this work, we present two separate approaches to handle multimodal line planning problems. One of the approaches is similar to the system split procedure in a sense that the original problem is also split to separate unimodal problems. In this work, we also provide theoretical background for allowing different types of problem splitting based on the problem instance.

1.2 Thesis outline

The structure of this thesis is presented in the following: Section 2 covers the basic concepts related to the mathematical public transport planning and formally defines the concept of line planning in the context of public transport planning.

In Section 3, we present the two main cost-oriented line planning algorithms, the sequential line planning model and the integrated line planning model, to obtain the optimal sets of lines in line planning problems with a single modality. We also present the general framework of passenger assignment algorithms and illustrate the effects of different passenger assignment algorithms to the optimal sets of lines in line planning problems.

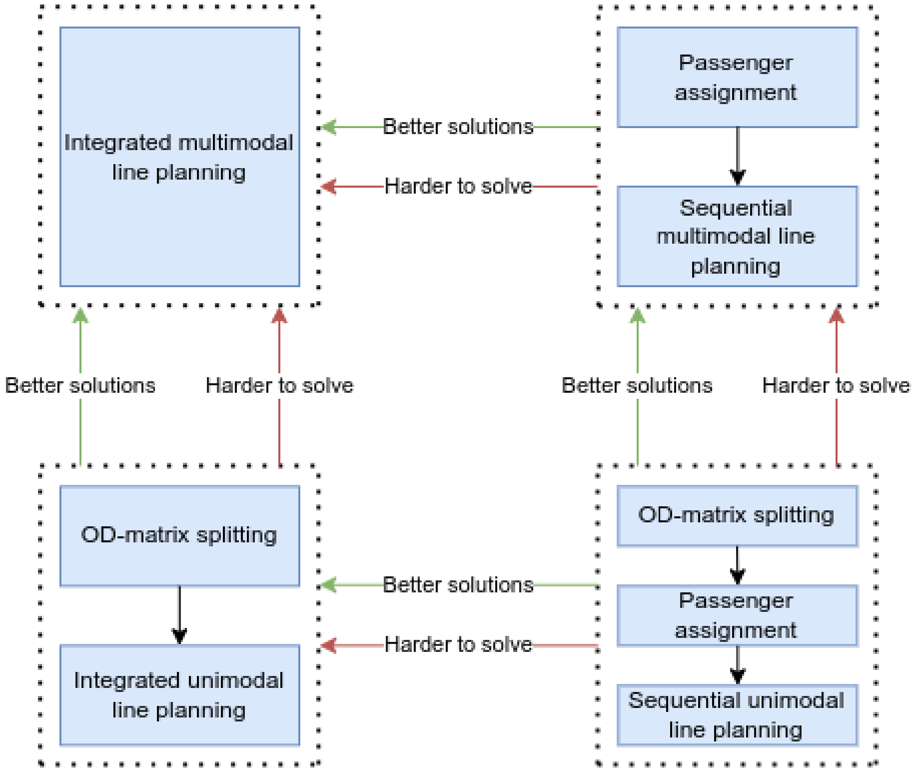


Figure 2: The relationships between the different methods for solving multimodal line planning problems in terms of problem tractability and the optimal solution quality. The black arrows show the order of steps in each method. The colored arrows show the relationship between the different methods.

In Section 4 we introduce extended formulations of the two cost-oriented line planning algorithms that can be used to solve line planning problems with multiple vehicle modalities. We also introduce the concept of OD-matrix splitting, a heuristic approach for dealing with multiple vehicle modalities in line planning that can be used to split the problem to a set of single modality problems. Additionally, we propose an iterative approach for solving line planning problems with multiple modalities. The relationships between the different methods presented in this chapter are illustrated in Figure 2 where the main characteristics of the methods are tractability in large problem instances and the quality of the optimal solution.

Section 5 presents detailed theoretical analysis and comparison of the methods presented in Section 4 and explores the theoretical properties of public transport networks of multiple vehicle modalities with illustrative examples.

Finally, in Section 6 we present experimental results and comparisons of the methods presented in Section 4 using an artificial public transport network and travel demand based on a real-life dataset. We conclude the thesis in Section 7.

2 Modeling public transport planning

2.1 Passenger demand and public transport network

We begin by formally defining the concepts of passenger demand and public transport network that are used to formulate the line planning problems in public transport planning.

Definition 1. The public transport network PTN = (V, E) is a directed graph where the nodes V represent the stops for vehicles in the network and edges E represent the direct connections between the stops. In other words, there is a directed edge from stop v_1 to v_2 if the vehicle can drive directly from v_1 to v_2 (e.g. there is a road/railway from v_1 to v_2).

Usually, the PTN is part of the input data of the model. However, there also exists methods for choosing locations of the stops in the PTN as part of the city infrastructure design. An example of such method that formulates and solves a specific stop location problem is described in [18, 3].

Definition 2. An OD-matrix $W \in \mathbb{N}^{|V| \times |V|}$ describes the transport demand in the PTN network such that W_{uv} is the number of passengers willing to travel from stop u to stop v in the network, $u, v \in V$.

The OD-matrix is also part of the input data of the problem and is used to determine the constraints of the line planning problem [3].

2.2 Concept of line planning

Definition 3. A line l is defined as a connected directed path in a PTN where each edge can be used at most once. A line concept of a PTN is then a set of frequencies f_l for all lines available $l \in \mathcal{L}^0$. We refer the line pool $\mathcal{L}^0$ as a collection of potential lines that can be used to build the actual line concept. The cost c_l for each line $l \in \mathcal{L}^0$ can be formulated as

$$c_l = c_{fix} + c_k \sum_{e \in l} d_e$$

where d_e denotes the length of the edge, c_k denotes the unit cost of a line in terms of line length and c_{fix} denotes the fixed cost of the line.

The goal of the line planning problem is then to find a line concept $(\mathcal{L}, f)$ to minimize or maximize some objective function with given constraints. Here $\mathcal{L}$ is a set consisting of lines $l \in \mathcal{L}$ to be operated and f is the vector of corresponding line frequencies of the lines $f_l, l \in \mathcal{L}$. The cost-oriented objective is to minimize the sum of costs for each line in the line concept $\sum_{l \in \mathcal{L}^0} c_l f_l$. Passenger-oriented objective functions include minimizing the number of transfers and minimizing the average travel time of all passengers, when travelling between stops $u, v \in V$. The resulting line concept can then be used to formulate the problems in the following planning stages, such as creating timetables or vehicle schedules in the PTN [3]. In this work, we focus on the cost-oriented objective of minimizing the overall costs in extended multimodal public transport networks. However, we also evaluate the line concepts for the passenger-oriented objectives to see how much the cost reduction affects the passenger-friendliness of the line concept. The formulations of the passenger-oriented objective functions are presented in more detail in Section 6.

In the line planning problem, we generally assume that the set of potential lines, i.e., line pool $\mathcal{L}^0$ is given. According to [2] and [4], the underlying line pool greatly affects the quality of the resulting line concept in the line planning problem, which is certainly intuitive. Preferably, we would like to allow using all possible lines in the problem, because with more lines to choose from, we have more flexibility to adjust different lines in the line concept and therefore increase its quality. In practice, this is generally not possible since the resulting line planning problem would quickly become too large to solve efficiently.

In this work we use the iterative minimum spanning tree approach presented in [2] to construct comparable line pools for all of the problem instances. Technically, we assume line pools along with their corresponding line costs are 'given' in every problem instance to neglect the effects of line pool generation in our experimental and theoretical results. The process of generating the line pools in the experimental analysis is presented in more detail in Section 6.

3 Cost-oriented unimodal line planning

3.1 Sequential line planning problem

We now present the basic sequential approach for cost-oriented line planning. In this approach, the solution to a separate passenger assignment problem is used to formulate the cost minimization problem. In more detail, the approach works as follows.

1. Calculate traffic loads ω_e for every edge $e \in E$ using Passenger Assignment Algorithm 2.
2. Calculate the corresponding lower edge frequencies $f_e^{\min} := \left\lceil \frac{\omega_e}{\text{Cap}} \right\rceil$ for each edge $e \in E$.
3. Solve the resulting line planning problem $\mathbf{LineP}(f^{\min})$ where $\mathbf{LineP}(f^{\min})$ is the basic cost model for line planning [4]:

$$\begin{aligned} \mathbf{LineP}(f^{\min}) : \min & \sum_{l \in \mathcal{L}^0} c_l f_l \\ \text{s.t.} \quad & \sum_{l \in \mathcal{L}^0 : e \in l} f_l \geq f_e^{\min} \; \forall \, e \in E \\ & \sum_{l \in \mathcal{L}^0 : e \in l} f_l \leq f^{\max} \; \forall \, e \in E \\ & f_l \in \mathbb{N} \; \forall \, l \in \mathcal{L}^0 \end{aligned}$$

Here, Cap determines the vehicle capacity in the line concept and $f^{\max}$ is the upper limit for vehicles operating frequency derived from the infrastructural limits of the public transport network. The first constraint of the model guarantees that for each edge in the public transport network, there are enough lines to serve all the passengers assigned to the edges. The second constraint limits the number of lines on the edges due to technical restrictions such as security headways [3]. Naturally, $f^{\max}$ could also be edge dependent, but for simplicity we treat it as constant in this work. Note that with Cap constant and f_l integer, we can rewrite the integer program $\mathbf{LineP}(f^{\min})$ in terms of the traffic loads as an equivalent model

$$\mathbf{LineP}(\omega) : \min \sum_{l \in \mathcal{L}^0} c_l f_l \tag{1}$$

$$\text{s.t. Cap} \sum_{l \in \mathcal{L}^0 : e \in l} f_l \geq \omega_e \; \forall \, e \in E \tag{2}$$

$$\text{Cap} \sum_{l \in \mathcal{L}^0 : e \in l} f_l \leq U \; \forall \, e \in E \tag{3}$$

$$f_l \in \mathbb{N} \; \forall \, l \in \mathcal{L}^0 \tag{4}$$

where $U = \text{Cap} \cdot f^{\max}$ describes the theoretical maximum load for every edge in the network [4]. The basic sequential approach for cost-oriented line planning can now be described with Algorithm 1.

Algorithm 1 Sequential model for cost-oriented line planning

1: Calculate traffic loads ω_e for every edge $e \in E$ using a passenger assignment Algorithm 2
2: Solve the resulting line planning problem $\mathbf{LineP}(\omega)$

The basic cost model for line planning (1)-(4) has been well studied in the literature. Most notably, [8] show that even without upper-frequency constraints, the model is NP-hard by reducing the problem to a set covering problem. This means that for large enough problem instances, the problem becomes computationally intractable. Apart from few special cases, the line planning problems are generally hard to solve efficiently [8].

To obtain the traffic loads ω_e for each $e \in E$, we perform passenger assignment process as described in Algorithm 2. All specific passenger assignment algorithms described later in this work are versions of this general algorithm. For each OD-pair $(u, v) : u, v \in V$, the goal is to distribute passenger demand W_{uv} to different paths $P^1_{uv}, ..., P^{N_{uv}}_{uv}$ in the PTN and set weights $\alpha^1_{uv}, ..., \alpha^{N_{uv}}_{uv} \geq 0$ for each path that describe the proportion of the complete passenger demand for that specific path. After distributing passengers to different paths for each (u, v), we then compute the passenger loads for each edge. The algorithm is shown in detail below:

Algorithm 2 Algorithm for passenger assignment

Input: PTN $= (V, E)$, $W_{uv}\ \forall\, u, v \in V$

for every $u, v \in V$ with $W_{uv} > 0$ **do**
  Calculate a set of paths $P^1_{uv}, ..., P^{N_{uv}}_{uv}$ from u to v in the PTN
  Compute weights for the paths $\alpha^1_{uv}, \dots, \alpha^{N_{uv}}_{uv} \geq 0$ with $\sum_{i=1}^{N_{uv}} \alpha^i = 1$
end for
for every $e \in E$ **do**
  Set $\omega_e := \sum_{u,v \in V} \sum_{i=1,...,N_{uv}: e \in P^i_{uv}} \alpha^i_{uv} W_{uv}$
end for

The passenger assignments obtained with Algorithm 2 are valid in a sense that all passengers are assigned to connected paths from u to v in the network. This way, the passenger assignment always respects the passenger demand from the OD-matrix. Many different approaches exist for determining the weights α^i_{uv} and the set of paths in this algorithm. The most common approach for cost-oriented line planning models is the shortest paths approach, that is $N_{uv} = 1$ for all OD-pairs u, v with $P^1_{uv} = P_{uv}$ a shortest path from u to v [4]. The length of a shortest path is denoted as $SP_{uv} = \sum_{e \in P_{uv}} d_e$. In this work, we propose new approaches for calculating the weights and paths to receive passenger weights resulting in more suitable line planning problems of form (1)-(4). Some of the new approaches are designed to utilize different capacities of vehicles in multimodal public transport networks.

3.2 Passenger assignment problem

We now present three types of passenger assignment algorithms to obtain the traffic loads for each $e \in E$. Each of the algorithms below follow the passenger assignment procedure represented in Algorithm 2, while the process for calculating the set of paths $P^1_{uv}, ..., P^{N_{uv}}_{uv}$ and path weights $a^1_{uv}, ..., a^{N_{uv}}_{uv}$ is different for each algorithm. All of the algorithms are based on shortest paths but the cost-function used for routing is different. Also, the iterative processes for obtaining the weights is slightly different for each algorithm.

First, we formulate the shortest paths passenger routing as follows:

Algorithm 3 Passenger Assignment Algorithm: Shortest Paths

Input: PTN $= (V, E), W_{uv}\ \forall\, u, v \in V$

for every $u, v \in V$ with $W_{uv} > 0$ **do**

  Calculate shortest path P_{uv} from u to v in the PTN according to edge lengths d_e

end for

for every $e \in E$ **do**

  Set $\omega_e := \sum_{u,v \in V : e \in P_{uv}} W_{uv}$

end for

This algorithm computes shortest paths for each OD-pair only once. Using shortest paths in passenger assignment is especially practical from the passengers' point of view, since the goal is to minimize their travel time with this approach. However, routing on shortest paths still doesn't guarantee that passengers will actually take those routes after solving the corresponding line planning problem. For example, the capacity limitations of the vehicles or the upper frequency constraints in line planning may prevent passengers from using the shortest paths when travelling. In essence, Algorithm 3 can be considered the most passenger-friendly cost-oriented heuristic to generate cost-oriented line planning problems [4].

The next two algorithms compute the paths for OD-pairs iteratively until weights converge or the maximum number of iterations has been reached.

The Reduction Algorithm 4 is a cost-oriented iterative approach that aims to concentrate passengers only on a selected number of edges in the network to reduce the costs further, in contrast to mere shortest paths. It does this by iteratively calculating passenger loads and lowering distance-induced costs for the edges that are already used by other passengers after each shortest paths routing. When the passenger loads ω_e converge (or the maximum number of iterations is reached) the edges with zero passenger load are removed from the network. Then, the normal shortest paths passenger routing is performed on the reduced network [17].

The Reward Algorithm 5 is also a cost-oriented iterative algorithm that uses a modified cost function to calculate shortest paths in the network. Similarly to Algorithm 4, the algorithm iteratively calculates passenger loads and lowers costs for already used edges. The algorithm aims to reduce line operating costs by eliminating empty seats in the vehicles. More precisely, costs are reduced more for the edges where there is less space for passengers in terms of vehicle capacity.

Algorithm 4 Passenger Assignment Algorithm: Reduction

Input: PTN $= (V, E), W_{uv}\ \forall\ u, v \in V, \varepsilon, max_{iter}$

$i := 0$

$\omega_e^0 := 0\ \forall\ e \in E$

repeat

  for every $u, v \in V$ with $W_{uv} > 0$ **do**

    Calculate shortest path P_{uv}^i from u to v in the PTN according to

$$\mathrm{cost}_i(e) = d_e + \gamma \cdot \frac{d_e}{\max\{\omega_e^{i-1}, 1\}}$$

  end for

  for every $e \in E$ **do**

    Set $\omega_e^i := \sum_{u,v \in V: e \in P_{uv}^i} W_{uv}$

  end for

  $i = i + 1$

until $\sum_{e \in E} (\omega_e^{i-1} - \omega_e^i)^2 < \varepsilon$ or $i > max_{iter}$

for every $u, v \in V$ with $W_{uv} > 0$ **do**

  Calculate shortest path P_{uv} from u to v in the PTN according to

$$\mathrm{cost}(e) = \begin{cases} d_e, & \omega_e^i > 0 \\ \infty, & \text{otherwise} \end{cases}$$

end for

for every $e \in E$ **do**

  Set $\omega_e := \sum_{u,v \in V: e \in P_{uv}} W_{uv}$

end for

Algorithm 5 Passenger Assignment Algorithm: Reward

Input: PTN $= (V, E), W_{uv}\ \forall\ u, v \in V, \varepsilon, max_{iter}$

$i := 0$
$\omega_e^0 := 0\ \forall\ e \in E$
repeat
  $i = i + 1$
  $\omega_e^i := \omega_e^{i-1}\ \forall\ e \in E$
  for every $u, v \in V$ with $W_{uv} > 0$ **do**
    for $j = 1, \ldots, W_{uv}$ **do**
      Calculate shortest path $P_{uv_j}^i$ from u to v in the PTN according to

$$\text{cost}_i(e) = \max\{d_e \cdot (1 - \gamma \cdot (\omega_e^{i-1} \bmod \text{Cap})/(\text{Cap})), 0\}$$

      for every $e \in P_{uv_j}^{i-1}$ **do**
        Set $\omega_e^i := \omega_e^i - 1$
      end for
      for every $e \in P_{uv_j}^i$ **do**
        Set $\omega_e^i := \omega_e^i + 1$
      end for
    end for
  end for
until $\sum_{e \in E}(\omega_e^{i-1} - \omega_e^i)^2 < \varepsilon$ or $i > max_{iter}$

Unlike with Reduction, separate shortest paths assignment is not performed, and the resulting passenger loads ω_e are obtained directly after the iterative process [4].

Note that Algorithm 5 enables routing passengers within a single OD-pair to different routes if necessary. This is the reason why shortest paths are calculated for each passenger of each OD-pair. This approach enables passengers of same OD-pair to be routed differently each time a single vehicle gets full. This way the chance of having to obtain another vehicle for remaining passengers of single OD-pair will be reduced, possibly decreasing the costs in the process. Ultimately, enabling passenger splitting increases the number of different possible passenger assignments, thus improving the versatility of the algorithm [4].

The following simple example shows how the passenger assignment affects the line planning problem and the resulting optimal solution in the line planning process.

Example 1. Consider the PTN illustrated in Figure 3, with OD-matrix such that $W_{14} = 30$, $W_{13} = 20$ and all other entries zero. Our goal is to find a feasible line concept $(\mathcal{L}, f)$ that minimizes the costs of the network. We first compute passenger assignment ω and then solve **LineP**(ω), according to Algorithm 1. The first passenger assignment is calculated using Shortest Paths Algorithm 3 and we obtain

$$\omega^A = \begin{cases} \omega^A_{(1,2)} = 30 \\ \omega^A_{(2,4)} = 30 \\ \omega^A_{(1,3)} = 20 \\ \omega^A_e = 0, \text{ otherwise} \end{cases}$$

Another passenger assignment is

$$\omega^B = \begin{cases} \omega^B_{(1,2)} = 50 \\ \omega^B_{(2,4)} = 30 \\ \omega^B_{(2,3)} = 20 \\ \omega^B_e = 0, \text{ otherwise} \end{cases}$$

We can now compare the optimal results by solving **LineP**(ω) with the obtained passenger assignments. We assume complete line pool, that is, any possible line can be chosen as part of the line concept. Furthermore, we assume $\text{Cap} = 50$ and $c_l = \sum_{e \in l} d_e$ for each $l \in \mathcal{L}^0$. We obtain

$$\mathbf{LineP}(\omega^A) = 2 + 5 + 4 = 11, \text{ with } f_{12} = 1, f_{24} = 1, f_{13} = 1$$
$$\mathbf{LineP}(\omega^B) = 2 + 5 + 3 = 10, \text{ with } f_{12} = 1, f_{24} = 1, f_{23} = 1$$

We can now see that the choice of passenger assignment affects the optimal cost of the line planning problem, even in simple problem instances. In fact, the choice of passenger assignment affects other objective functions as well: The total travel time of

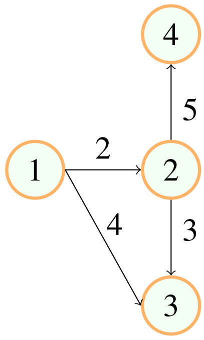


Figure 3: An example of a unimodal public transport network with four nodes and four edges. The numbers on the orange nodes indicate the node index $v \in V$ of each node. The numbers on sides of the edges indicate the length d_e of each edge

problem **LineP**(ω^A) is smaller than **LineP**(ω^B) because the passengers have to take longer route with ω^B to get from node 1 to node 3. Different passenger assignments result in different sets of objective function values so that different objective functions are given more emphasis in the multi-objective optimization task. In this work, we primarily focus on the cost-minimization in line planning but also consider other objective function values in experimental results.

3.3 Integrated line planning problem

Next, we present an integrated model for line planning which can be used to determine the optimal passenger assignment and line concept simultaneously. The objective function, lower frequency constraints and upper frequency constraints remain the same as in problem (1)-(4), but we introduce new decision variables that describe the passenger routes for each OD-pair $(u, v) : u, v \in V$. We also add a constraint to prevent the passenger-weighted average path length from increasing by more than β compared to the shortest path length SP_{uv} for each OD-pair (u, v). The reason for the new constraint is to prevent unreasonably long routes (and hence travel times) for passengers when using non-shortest paths for minimizing cost of the line concept. The integrated line planning problem is formulated in (5)-(11).

$$\textbf{LineA} : \min \sum_{l \in \mathcal{L}^0} c_l f_l \tag{5}$$

$$\text{s.t. Cap} \sum_{l \in \mathcal{L}^0 : e \in l} f_l \geq \sum_{u,v \in V} x_e^{uv} \ \forall\, e \in E \tag{6}$$

$$\text{Cap} \sum_{l \in \mathcal{L}^0 : e \in l} f_l \leq U \ \forall\, e \in E \tag{7}$$

$$\Theta x^{uv} = b^{uv} \ \forall\, u, v \in V \tag{8}$$

$$\sum_{e \in E} d_e x_e^{uv} \leq \beta SP_{uv} W_{uv} \ \forall\, u, v \in V \tag{9}$$

$$f_l \in \mathbb{N} \ \forall\, l \in \mathcal{L}^0 \tag{10}$$

$$x_e^{uv} \in \mathbb{N} \ \forall\, u, v \in V, e \in E \tag{11}$$

where $\Theta \in \mathbb{R}^{|V| \times |E|}$ is the incidence matrix of PTN, that is

$$\Theta(v, e) = \begin{cases} 1 & \text{if } e = (v, u) \text{ for some } u \in V \\ -1 & \text{if } e = (u, v) \text{ for some } u \in V \\ 0 & \text{otherwise} \end{cases}$$

and $b^{uv} \in \mathbb{R}^{|V|}$ where $b_u^{uv} = W_{uv}$, $b_v^{uv} = -W_{uv}$ and the remaining components are 0.

The decision variables x_e^{uv} contain the passenger loads for each OD-pair (u, v) and for each edge $e \in E$. Here SP_{uv} stands for the shortest path length between OD-pair (u, v). Note that the sequential model first calculates the passenger loads with a separate algorithm, thus making the passenger assignment a heuristical approach for load generation. The integrated model incorporates the heuristic into the parameter β that allows to reduce the costs at the expense of passenger-oriented objectives such as the average travel time. Note that smaller values of β makes the integrated problem more passenger-friendly, indicating better values for passenger travel times but with higher costs. In turn, increasing β puts more emphasis on cost minimization in the integrated model at the cost of passenger-friendly objectives [4].

Similarly to the basic cost model in the sequential approach, the integrated line planning problem (5)-(11) is also an NP-hard problem. In fact, the model is even harder to solve than sequential model but will often generate better solutions in terms of the different objectives as shown by. Indeed, the motivation behind different solutions methods is the difference in tractability of the model and quality of the solution output [1].

4 Cost-oriented multimodal line planning

In this chapter we extend the unimodal line planning models described in the previous section to accept multimodal public transport network data. The multimodal public transport data usually consists of separate unimodal PTN^m for each modality $m \in M$ and global OD-matrix W for the whole network. We first define the multimodal counterparts of the public transport data.

Denote S as the set of stops in any unimodal PTN^m of any modality $m \in M$. Then we can express $\text{PTN}^m = (S^m, E^m)$ where $S^m \subseteq S$ and $E^m \subseteq \{(s, s') : s, s' \in S^m\}$. Further, denote $T \subseteq S$ as the set of modality transfer stops in the network. We now combine the nodes from separate unimodal PTNs into a single graph and append this set of nodes with a set of transfer nodes and OD-nodes as follows:

$$
\begin{aligned}
V_{node} &:= \bigcup_{m \in M} \{(s, m) : s \in S^m\} \\
V_{trans} &:= \{(s, trans) : s \in T\} \\
V_{OD} &:= \{(s, OD) : s \in S\}
\end{aligned}
$$

We also extend the set of edges in the resulting multimodal PTN to include the possibility to transfer between modes via the transfer nodes and to include edges from each OD-node to their corresponding stops. The new set of edges contains the following:

$$
\begin{aligned}
E_{node} &:= \{((s, m), (s', m)) : (s, s') \in E^m, m \in M\} \\
E_{trans} &:= \{((s, m), (s, trans)) \text{ and } ((s, trans), (s, m')) \text{ and} \\
&\qquad ((s, m'), (s, trans)) \text{ and } ((s, trans), (s, m)) : s \in T, m, m' \in M\} \\
E_{OD} &:= \{((s, OD), (s, m)) \text{ and } ((s, m), (s, OD)) : s \in S^m, m \in M\}
\end{aligned}
$$

Note that the original edges and nodes of each PTN^m are indeed preserved during the creation of the multimodal PTN such that the original edges are all in the set E_{node} and original nodes are all in the set V_{node}.

The transfer nodes V_{trans} and transfer edges E_{trans} allow passengers to transfer from one modality to another at stops $T \subseteq S$ at the cost of mode transfer penalty d_m per transfer. Moreover, the OD-nodes V_{OD} and OD-edges E_{OD} are used in the routing phase with OD-pair $u, v \in S$ such that the start point of the route is $(u, OD) \in V_{OD}$ and the end point is $(v, OD) \in V_{OD}$. The OD-edges connect the OD-nodes to the remaining graph and always have the effective length and penalty of zero when routing.

With the above definitions for different nodes and edges in the multimodal PTN, the complete formal definition can be written as

$$
\text{PTN} = \{(V, E) : V := V_{node} \cup V_{trans} \cup V_{OD}, E := E_{node} \cup E_{trans} \cup E_{OD}\} \tag{12}
$$

4.1 Multimodal sequential line planning problem

We now modify Algorithm 1 to the corresponding multimodal version. This approach uses multimodal PTN = (V, E) created from separate unimodal PTNs as described in (12) with a global OD-matrix W and line pools $\mathcal{L}_m^0$ for each $m \in M$. Utilizing the previously defined set of stops S, the multimodal approach is described in Algorithm 6.

Algorithm 6 Sequential model for multimodal cost-oriented line planning

1: Calculate traffic loads ω_e in the multimodal PTN for every edge $e \in E_{node}$ using a passenger assignment Algorithm 2 given OD-pairs $u, v \in S$

2: Calculate traffic loads $\omega_{(s_1,s_2)}$ for every stop-pair $(s_1, s_2) \in S \times S$ with formula $\omega_{(s_1,s_2)} = \sum_{e \in E_{node}: e=((s_1,\cdot),(s_2,\cdot))} \omega_e$

3: Solve the resulting line planning problem **LinePM**(ω)

This approach shares the same core idea as Algorithm 1. The multimodal cost model for line planning can be formulated as

$$\textbf{LinePM}(\omega): \ \min \sum_{m \in M} \sum_{l \in \mathcal{L}_m^0} c_{ml} f_{ml} \tag{13}$$

$$\text{s.t.} \sum_{l \in L_m^{(s_1,s_2)}} \mathrm{Cap}_m f_{ml} \leq \mathrm{Cap}_m U_m, \ \forall\, m \in M, (s_1, s_2) \in S \times S \tag{14}$$

$$\sum_{m \in M} \sum_{l \in L_m^{(s_1,s_2)}} \mathrm{Cap}_m f_{ml} \geq \omega_{(s_1,s_2)}, \ \forall\, (s_1, s_2) \in S \times S \tag{15}$$

$$f_{ml} \in \mathbb{N}, \ \forall\, m \in M, l \in \mathcal{L}_m^0 \tag{16}$$

Here f_{ml} are the decision variables, that is, the frequencies for lines in their mode-specific line pool $l \in \mathcal{L}_m^0$, c_{ml} are the costs of the corresponding lines and Cap_m denotes the capacity for different modes $m \in M$. U_m denotes the upper frequency for the mode $m \in M$. $\omega_{(s_1,s_2)}$ denotes the load between a stop-pair $(s_1, s_2) \in S \times S$. Note that the loads are not mode specific in this case. Furthermore, we denote $L_m^{(s_1,s_2)}$ as a set of lines of modality m that contain a stop-pair (s_1, s_2) in their line path.

In (13)-(16) the objective function now takes into account the costs of all lines for each modality. With the lower frequency constraint, we are not interested in the modality distribution of lines when fulfilling the lower frequency constraint, so we can simply sum over all modalities' contribution in the constraints. This is precisely the reason why in the load generation process we sum over all modality specific traffic loads for each stop-pair to arrive at global traffic loads $\omega_{(s_1,s_2)}$ to be used in the lower frequency constraint.

The upper frequency constraint limits the traffic load for separate modalities between stop-pairs $(s_1, s_2) \in S \times S$. This is reasonable since, for example, the tram tracks can only be used by trams and metro tracks can only be used by metros, meaning that different modalities have different upper limits on their respective traffic frequencies.

4.2 OD-matrix splitting

We now present another closely related approach for multimodal cost-oriented line planning. This approach presents an alternative idea for line planning with multiple modalities by splitting the global OD-matrix W to modality specific OD-matrices W^m for each $m \in M$. This approach is particularly useful since solving a series of unimodal line planning problems can be computationally much more easier to solve than a single potentially large multimodal problem. Especially in large real-life transport networks, the limits of computation power may raise a notable concern. The cost-oriented approach with OD-matrix splitting is summarized in Algorithm 7.

Algorithm 7 Sequential cost-oriented line planning with OD-matrix splitting

1: Compute a viable set of OD-matrices W^m for each $m \in M$ using an OD-matrix splitting algorithm with global OD-matrix W.
2: For each $m \in M$, solve the unimodal sequential cost-oriented line planning problem with PTN^m and W^m using some passenger assignment algorithm for all modalities and receive f_m and c_m for each $m \in M$
3: Obtain the frequencies of the multimodal line concept $f^* = (f_m)_{m \in M}$ and the compound multimodal cost $c^* = \sum_{m \in M} c_m$

The idea of the approach is greatly similar to the multimodal sequential model with one key difference. Namely, the OD-matrix splitting approach has two separate heuristics involved, whereas Algorithm 6 only has one. The added heuristic is precisely the OD-matrix splitting algorithm in step 1 of Algorithm 7.

To preserve the original passenger demand of the problem, the set of OD-matrices W^m has to be viable. The viability condition ensures that for each OD-pair (u, v), the incoming flow of passengers to the destination and the outgoing flow from the origin matches the actual passenger demand in the OD-matrix W. Moreover, the viability condition ensures that no passengers appear or disappear when routing between the OD-pair (u, v). We now present the formal definition of viability for the set of OD-matrices W^m.

Definition 4. Let $W \in \mathbb{N}^{|S|\times|S|}$. Set of unimodal matrices $\{W^m, m \in M\}$ is viable if the following conditions hold:

- For all $u, v \in S$ with $W^m_{uv} > 0$, u and v are connected in PTN^m
- For each $u, v \in S$, $m \in M$, there exists a set of weights $c^{u,v,m}_{s,t}$, $s, t \in S$ with

$$\sum_{s,t\in S} c^{u,v,m}_{s,t} = W^m_{uv}$$

- For each $s, t \in S$, and for all $v \in V \setminus \{s, t\}$:

$$\sum_{m\in M}\sum_{u\in V} c^{u,v,m}_{s,t} - \sum_{m\in M}\sum_{u\in V} c^{v,u,m}_{s,t} = 0$$

- For each $s, t \in S$,

$$\sum_{m\in M}\sum_{u\in V} c^{s,u,m}_{s,t} - \sum_{m\in M}\sum_{u\in V} c^{u,s,m}_{s,t} = W_{s,t}$$

$$\sum_{m\in M}\sum_{u\in V} c^{u,t,m}_{s,t} - \sum_{m\in M}\sum_{u\in V} c^{t,u,m}_{s,t} = W_{s,t}$$

The following example shows two OD-matrix splits for a single multimodal line planning problem, one that is viable and one that is not.

Example 2. Consider a multimodal line planning problem with the PTN illustrated in Figure 4 with OD-matrix

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An example of viable OD-matrix split is then a set of OD-matrices

$$W^{green} = \begin{bmatrix} 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 0 & 0 & 30 \\ 0 & 0 & 35 & 0 \end{bmatrix}, W^{red} = \begin{bmatrix} 0 & 20 & 0 & 0 \\ 15 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \end{bmatrix}$$

By looking at Figure 4, we can see that for both modalities the OD-pairs with non-zero entries in the respective OD-matrices are indeed connected in their modality-specific network. Next, we can set the following weights

$$
\begin{aligned}
c_{2,3}^{2,1,red} &= W_{21}^{red} \\
c_{2,3}^{1,3,green} &= W_{13}^{green} \\
c_{2,3}^{2,4,red} &= W_{24}^{red} \\
c_{2,3}^{4,3,red} &= W_{43}^{green} \\
c_{3,2}^{3,1,green} &= W_{31}^{green} \\
c_{3,2}^{1,2,red} &= W_{12}^{red} \\
c_{3,2}^{3,4,green} &= W_{34}^{green} \\
c_{3,2}^{4,2,red} &= W_{42}^{red}
\end{aligned}
$$

and set all other weights zero.

The third constraint makes sure that incoming passengers flows to specific node are the same as outgoing passenger flows from that specific node. In our case, the third constraint is indeed satisfied, for example, $c_{2,3}^{2,1,red} = W_{21}^{red} = c_{2,3}^{1,3,green} = W_{13}^{green}$.

The fourth constraint ensures that for global OD-pair $s, t \in S$, the accumulated outgoing passenger flow from u matches the demand of the global OD-matrix W_{uv}, whereas the fifth constraint ensures that accumulated incoming passenger flow to v matches the demand W_{uv}. In our case both constraints are indeed satisfied. For example, $c_{3,2}^{3,1,green} + c_{3,2}^{3,4,green} = c_{3,2}^{4,2,red} + c_{3,2}^{1,2,red} = W_{23}$.

Based on the previous observations, this set of matrices satisfy the viability conditions of Definition 4.

However, the set of OD-matrices in the following is not viable

$$
W^{green} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 \end{bmatrix}, W^{red} = \begin{bmatrix} 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

To see this, let's investigate the weights corresponding to pairs $u, v \in S$ with zeros in the modality-specific OD-matrix. According to the second constraint, we have

$$
\begin{aligned}
&\sum_{s,t \in S} c_{s,t}^{u,v,green} = W_{u,v}^{green} = 0 \\
\implies\ & c_{s,t}^{u,v,green} = 0 \ \forall\ (u, v) \notin (3, 4)
\end{aligned}
$$

and

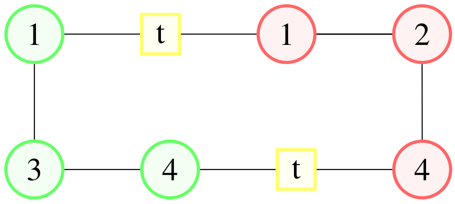


Figure 4: Example of multimodal network with separate modalities marked with colors green and red. The numbers on the nodes refer to stop indices $s \in S$ of the nodes. The nodes with yellow color are transfer nodes

$$\sum_{s,t \in S} c_{s,t}^{u,v,red} = W_{u,v}^{green} = 0$$
$$\implies c_{s,t}^{u,v,red} = 0 \; \forall \; (u,v) \notin (1,4)$$

since by definition all demand and weights have to be positive.
For the stop-pair $(2,3) \in S$, the fourth constraint can be given as

$$\sum_{m \in M} \sum_{u \in V} c_{2,3}^{2,u,m} - \sum_{m \in M} \sum_{u \in V} c_{2,3}^{u,2,m} = W_{2,3}$$
$$\sum_{m \in M} \sum_{u \in V} c_{2,3}^{2,u,m} - \sum_{m \in M} \sum_{u \in V} c_{2,3}^{u,2,m} = 50$$
$$\implies 0 \neq 50$$

which leads to a contradiction. Given the above set of OD-matrices we must attain zero values to some of the weights to satisfy the second constraint. This inevitably leads to situation where the fourth constraint does not hold for some $s, t \in S$. Therefore, the above set of OD-matrices is not viable w.r.t. Definition 4. Recall that we can only use sets of viable OD-matrices as part of the Algorithm 7 for solving multimodal line planning problems. Otherwise, the resulting subproblems do not have the corresponding passenger demand of the original multimodal problem.

From a theoretical perspective, the connection between the OD-matrix splitting and the multimodal cost-oriented line planning is important. It can be shown that for each feasible line concept in (13)-(16) there exists a viable OD-matrix split that will give the same optimal cost when used with Algorithm 7 with some suitable passenger assignment algorithms. However, obtaining the correct OD-matrix split can sometimes be as difficult as solving the sequential model for multimodal line planning.

On the other hand, obtaining merely a viable set of OD-matrices is much more straightforward. In fact, we can formulate a general algorithm that generates viable sets of OD-matrices for any multimodal PTN and the corresponding OD-matrix. Similarly to the path-based approach for passenger assignment, the general algorithm can be formulated as Algorithm 8.

Algorithm 8 Algorithm for generating viable sets of OD-matrices

Input: Multimodal PTN $= (V, E), W_{uv} \ \forall\, u, v \in S$

$W^m = \mathbf{0} \ \forall\, m \in M$

for every $u, v \in S$ with $W_{uv} > 0$ **do**

  Calculate a set of paths $P^1_{uv}, ..., P^{N_{uv}}_{uv}$ from u to v in the PTN

  Compute weights for the paths $\alpha^1_{uv}, ..., \alpha^{N_{uv}}_{uv} \geq 0$ with $\sum_{i=1}^{N_{uv}} \alpha^i = 1$

end for

for every $u, v \in S$ **do**

  for every $P^i_{uv} \in P^1_{uv}, ..., P^{N_{uv}}_{uv}$ **do**

    Set $s^0 = u$

    for every transfer node $(s^j, trans)$ in P^i_{uv} **do**

      Update $W^m_{s^0 s^j} := W^m_{s^0 s^j} + \alpha^i_{uv} W_{uv}$

      Set $s^0 = s^j$

    end for

    Update $W^m_{s^0 v} := W^m_{s^0 v} + \alpha^i_{uv} W_{uv}$

  end for

end for

Algorithm 8 generates paths between all points in the multimodal PTN using some routing algorithm and then distributes the demand to modality-specific OD-pairs that are the start and end points of the corresponding longest modality-specific subpath. As with the general passenger assignment algorithm, the paths can be calculated by any routing algorithm such as the shortest paths algorithm. In the theoretical analysis section of this work, we introduce versions of the above algorithm for obtaining viable sets of OD-matrices for simplified PTNs that result in the same optimal solution obtained from (13)-(16).

4.3 Multimodal passenger assignment problem

4.3.1 Multimodal passenger assignment algorithms

In the multimodal case, the passenger assignment process as described in Algorithm 2 is similar to the unimodal case with only few exceptions. First, we use the multimodal PTN = (V, E) when routing passengers to enable modality transfers in routing. This means that the entries in the OD-matrix now correspond to set of stops S instead of all the nodes in the network as in the unimodal case. In pratice, we choose the origin node as $(u, OD) \in V_{OD}$ and destination node as $(v, OD) \in V_{OD}$ when calculating P_{uv} for each OD-pair $u, v \in S$.

Second, we adjust the definition of cost functions in all of the algorithms such that

$$\text{cost}(\tilde{e}) = \begin{cases} \text{cost}_i(e) & \text{if } \tilde{e} = e \in E_{node} \\ d_t & \text{if } \tilde{e} \in E_{trans}, E_{OD} \end{cases}$$

where d_t is the transfer penalty. Although adjusting transfer penalties for each station could technically be possible, we usually don't consider this amount of detail. The constant value for transfer penalties suffices here for simplicity. With the above changes in mind, every passenger assignment algorithm presented in Section 3.2 can be transformed directly to the multimodal case.

However, in the multimodal case with different capacities Cap_m and transfer possibilities for each $m \in M$, we can employ new heuristics to the passenger assignment algorithms to arrive at more suitable passenger assignments and traffic loads ω_e. More precisely, we employ new cost-functions to be used for both reduction and reward approaches to arrive at more suitable passenger assignments. We first declare the cost functions that can be used for both reward and reduction alike.

SELECTIVE ASSORTATIVITY

$$\mathrm{cost}_i(e) = d_e + \gamma \cdot \frac{d_e}{\max\{\omega_e^{i-1}, 1\}} + \alpha \cdot \frac{d_e}{\max\{\sum_{e' \in E_{pred}(e)} \beta + \sum_{e' \in E_{succ}(e)} \beta, 1\}} \tag{17}$$

WEIGHTED ASSORTATIVITY

$$\mathrm{cost}_i(e) = d_e + \gamma \cdot \frac{d_e}{\max\{\omega_e^{i-1}, 1\}} + \alpha \cdot \frac{d_e}{\max\{\sum_{e' \in E_{pred}(e)} \omega_e^{i-1} + \sum_{e' \in E_{succ}(e)} \omega_e^{i-1}, 1\}} \tag{18}$$

SELECTIVE DIRECT PASSENGERS

$$\mathrm{cost}_i(e) = d_e + \gamma \cdot \frac{d_e}{\max\{\omega_e^{i-1}, 1\}} + \alpha \cdot \frac{d_e}{\max\{\sum_{e' \in E_{pred}(e) \setminus E_{trans}} \beta + \sum_{e' \in E_{succ}(e) \setminus E_{trans}} \beta, 1\}} \tag{19}$$

WEIGHTED DIRECT PASSENGERS

$$\mathrm{cost}_i(e) = d_e + \gamma \cdot \frac{d_e}{\max\{\omega_e^{i-1}, 1\}} + \alpha \cdot \frac{d_e}{\max\{\sum_{e' \in E_{pred}(e) \setminus E_{trans}} \omega_e^{i-1} + \sum_{e' \in E_{succ}(e) \setminus E_{trans}} \omega_e^{i-1}, 1\}} \tag{20}$$

Here $E_{pred}(e)$ is the set of edges incoming to the start node of e and $E_{succ}(e)$ is the set of edges outgoing from the end node of of e and E_{trans} is the set of transfer edges in the multimodal PTN. The parameter β should be chosen to represent the usual scale of weights appearing in the weighted approach. A particularly simple approach is to set β as the average of passengers entries in the OD-matrix.

All cost functions (17)-(20) share the same first two terms appearing in the cost function of Algorithm 4. Indeed, the core idea is to concentrate passenger loads to already used edges to reduce costs in the line concept. The magnitude of this effect is controlled by parameter γ.

In the Selective and Weighted Assortativity approaches, we focus on those edges with high assortativity index, i.e., number of edges incident to the start and end nodes of the edges. In the Selective and Weighted Direct Passengers approaches, we give separate bonuses for edges that have high assortativity in terms of the edges that share the same modality. The magnitude of both effects is controlled by parameter α. We argue that preferring routes with edges of high assortativity leads to lower overall cost of the line plan due to increased flexibility in transfers. In particular, the passengers enjoying significant benefit for transfers can be routed to transport hubs and other locations where more possibilities for transfers occur. In turn, this means that the actual line concept creation has more possibilities to choose the set of lines that would allow cost reductions for only small decrease in passenger friendliness.

Last, we present a cost function that aims to benefit from different capacities of different modalities to reduce line plan costs. The following cost-function can be interpreted as the multimodal version of the Algorithm 5. The idea is to reduce the number of empty seats in all of the operating vehicles by pre-calculating suitable passenger amounts that eliminate the empty seats for a specific set of capacities. More formally, the cost functions is defined as

MULTIMODAL REWARD

$$\mathrm{cost}_i(e) = \max\{d_e \cdot (1 - \gamma \cdot h(\omega_e^{i-1}), 0\} \tag{21}$$

where

$$h(\omega_e) = \begin{cases} 0 & \text{if } \omega_e \in K \\ \frac{\omega_e}{\min\{k \in K : k > \omega_e\} - \max\{k \in K : k < \omega_e\}} & \text{otherwise} \end{cases}$$

with $K = \{Cap^T x : x \in \mathbb{N}^{|M|}\}$ and Cap is the vector of capacities for each $m \in M$. The set K contains all passenger amounts that result in zero empty seats within an edge if line frequencies are chosen accordingly. In practice, K doesn't contain all possible passenger amounts since the infinite sequence is impossible to calculate explicitly. Instead, we use bound $Cap^T x \leq \sum_{u,v \in S} W_{uv}$, that is, the largest computed passenger amount can be at most the sum of all passenger demand in the OD-matrix. If all demand is traversed through a single edge in the PTN, this is the theoretical maximum load.

Similarly to the unimodal reward, this approach doesn't guarantee that in the resulting line planning optimization problem, the seats are filled to eliminate empty seats. However, we argue that (21) is still a good heuristic to eliminate empty seats in multimodal scenario. Particularly in situations where modality specific edges promote more passenger load for filling seats, the multimodal reward takes into account other modalities that can be used to fit the complete stop-pair demand.

4.3.2 Passenger assignment with Change&Go-network

It should be noted that all of the passenger assignment methods described above distribute passengers on the paths P_{uv} on the multimodal PTN without any information on the resulting line concept. Should the information about the line concept be available, the passengers could be assigned more efficiently on the edges of the PTN. However, with the traditional line planning framework, we need full passenger assignment information in order to build the line concept.

To overcome this phenomenon, several approaches have been implemented to include information about the potential line concept already in the passenger assignment step. The information about possible line configurations allows to consider transfer options between lines already in the passenger assignment phase, possibly resulting in better solutions in the line planning phase as shown by [5]. We now present the central concept of these approaches, namely a Change&Go-network, that will be used in the following section when formulating the multimodal iterative line planning method.

A multimodal Change&Go-network is a directed graph where nodes of the graph represent either the station-line pairs for each modality or the global start locations and end locations of the passengers. Formally, we have

$$\begin{aligned}\mathcal{V}_{CG} &:= \{(s,l,m) \in S \times \mathcal{L}^0_m \times M : l \in \mathcal{L}^0_m(s), m \in M, s \in V^m\}\\ \mathcal{V}_{OD} &:= \{(s,0) : s \in S\}\end{aligned}$$

where $\mathcal{V}_{CG}$ are the stop-line-modality triples and $\mathcal{V}_{OD}$ are the origin-destination nodes.

Further, we define different types of edges in the Change&Go-network to allow transfers between lines of the same modality, transfers between lines of different modalities, regular line edges and origin-destination edges as follows:

$$\begin{aligned}\mathcal{E}_{LineChange} &:= \{((s,l_1,m),(s,l_2,m)) \in \mathcal{V}_{CG} \times \mathcal{V}_{CG} :\\ &\qquad m \in M, s \in V^m, l_1, l_2 \in \mathcal{L}^0_m(s)\}\\ \mathcal{E}_{ModalChange} &:= \{((s,l_1,m_1),(s,l_2,m_2)) \in \mathcal{V}_{CG} \times \mathcal{V}_{CG} :\\ &\qquad m_1, m_2 \in M, s \in V^m, l_1 \in \mathcal{L}^0_{m_1}(s), l_2 \in \mathcal{L}^0_{m_2}(s)\}\\ \mathcal{E}_{l_m} &:= \{((s,l,m),(s',l,m)) \in \mathcal{V}_{CG} \times \mathcal{V}_{CG} : (s,s') \in E(l) \times E(l)\}\\ \mathcal{E}_{go} &:= \{\bigcup_{m \in M} \bigcup_{l \in \mathcal{L}^0_m} \mathcal{E}_{l_m}\}\\ \mathcal{E}_{OD} &:= \{((s,0),(s,l,m)) \in \mathcal{V}_{OD} \times \mathcal{V}_{CG}, ((t,l,m),(t,0)) \in \mathcal{V}_{CG} \times \mathcal{V}_{OD} :\\ &\qquad s,t \in S, m \in M, l \in \mathcal{L}^0_m(s), l \in \mathcal{L}^0_m(t)\}\end{aligned}$$

Here $E(l)$ is a set of edges in line l.

With these definitions, the multimodal Change&Go-network is formally defined as:

$$G_{CG} = \{(\mathcal{V},\mathcal{E}) : \mathcal{V} := \mathcal{V}_{CG} \cup \mathcal{V}_{OD}, \mathcal{E} := \mathcal{E}_{LineChange} \cup \mathcal{E}_{ModalChange} \cup \mathcal{E}_{go} \cup \mathcal{E}_{OD}\} \tag{22}$$

The intuition of extending the network is similar to that of extending a unimodal PTN to a multimodal PTN with a few key differences. First, different types of transfer edges $\mathcal{E}_{LineChange}$ and $\mathcal{E}_{ModalChange}$ can have different constant transfer penalties. Second, instead of distributing passengers to network edges as in the case of multimodal PTN, we now distribute passengers to the lines of the network. Moreover, we now use only different types of transfer edges and omit the transfer nodes presented in the multimodal PTN formulation to align with the original formulation of [5].

With a Change&Go-network $G_{CG} = (\mathcal{V},\mathcal{E})$ constructed, the passenger assignment process for all previous algorithms with different cost functions can be performed with few modifications listed below:

1. Declare **Input** : $G_{CG} = (\mathcal{V}, \mathcal{E}), W_{uv}\ \forall\, u, v \in S$

2. Declare

$$\text{cost}(\tilde{e}) = \begin{cases} \text{cost}_i(e) & \text{if } \tilde{e} = e \in \mathcal{E}_{go} \\ d_t & \text{if } \tilde{e} \in \mathcal{E}_{LineChange} \\ d_m & \text{if } \tilde{e} \in \mathcal{E}_{ModalChange} \\ \max\{d_m, d_t\} & \text{if } \tilde{e} \in \mathcal{E}_{OD} \end{cases}$$

where d_t is the penalty for transfer between same modality lines and d_m is is the penalty for transfer between different modality lines. In practice, it may be reasonable to assume that $d_t \leq d_m$ for any transfer option in the graph. Here $\max\{d_m, d_t\}$ is a sufficiently high penalty that prevents infeasible transfers via OD-edges.

As we can see from the definition of nodes $\mathcal{V}_{CG}$, the lines used to construct the Change&Go-network (22) are indeed all the lines in each modality specific line pool $\mathcal{L}_m^0$ for each $m \in M$. If the line pool is reasonably good, we can already arrive at better cost in the line planning problem when using the Change&Go-network for passenger assignment [4]. However, the line pool only gives information about lines that are possible to create and not about the lines that actually will be created [5]. In the following section, we describe the iterative line planning model that solves multiple line planning problems iteratively and uses the line concepts from the previous optimal solution to create the Change&Go-network to be used for next passenger assignment iteration.

4.4 Multimodal iterative line planning problem

Using the multimodal version of the sequential line planning problem along with the multimodal passenger assignment problem, we now formulate the iterative approach for solving the optimal line plan for multimodal PTN = (V, E) with associated OD-matrix W. In Section 6, we investigate whether solving the line planning optimization problem multiple times iteratively results in better cost or even better cost-travel ratio for a multimodal PTN. The idea is to fix a set of lines obtained from solving (13)-(16) and modify the optimization problem to relax the problem constraints using these fixed lines. We also reroute passengers after each iteration using the resulting line concept from the previous iteration. In more detail, the framework of the iterative approach is summarized in Algorithm 9.

Algorithm 9 Iterative model for multimodal cost-oriented line planning

1: Solve the line planning problem **LinePM**(ω) with weights $\omega_{(s_1,s_2)}$ obtained from the passenger assignment.
2: Define a set of fixed lines as L_m^* for each modality $m \in M$
3: Perform passenger assignment using the C&G-graph induced by the previous solution of **LinePM**(ω) or **LinePMI**(ω) and obtain new weights $\omega^i_{(s_1,s_2)}$
4: Solve the resulting line planning problem **LinePMI**(ω^i)
5: Repeat steps 2,3,4 until all of the lines are fixed and obtain f_{ml} for each $m \in M$ and $l \in \mathcal{L}_m^0$

The line planning problem described in Step 4 of Algorithm 9 is similar to **LinePM**(ω) but the effect of the fixed line decision variables on the constraints for each edge is accounted. Formally, the modified problem is the following:

$$\textbf{LinePMI}(\omega^i) : \min \sum_{m\in M} \sum_{l\in\mathcal{L}_m^0} c_{ml} f_{ml} \tag{23}$$

$$\text{s.t.} \sum_{l\in L_m^{(s_1,s_2)}} \text{Cap}_m f_{ml} \leq \text{Cap}_m U_m - \sum_{l^*\in L_m^{(s_1,s_2)}\cap L_m^*} \text{Cap}_m f_{ml^*},\ \forall\, m \in M, (s_1,s_2) \in S\times S \tag{24}$$

$$\sum_{m\in M} \sum_{l\in L_m^{(s_1,s_2)}} \text{Cap}_m f_{ml} \geq \omega^i_{(s_1,s_2)} - \sum_{m\in M} \sum_{l^*\in L_m^{(s_1,s_2)}\cap L_m^*} \text{Cap}_m f_{ml^*},\ \forall\, (s_1,s_2) \in S\times S \tag{25}$$

$$f_{ml} \in \mathbb{N},\ \forall\, m \in M, l \in \mathcal{L}_m^0 \tag{26}$$

Here, everything else remains the same as in **LinePM**(ω) but the lines are now considered in two groups, namely fixed lines $l_m^* \in L_m^*$ and free lines $l_m \in \mathcal{L}_m^0 \setminus L_m^*$. In essence, the decision variables of the new problem (23)-(26) are all the lines in the pool but the fixed lines $l_m^* \in L_m^*$ (and their frequencies) affect the right-hand sides of the constraints. As in the basic problem, $\mathcal{L}_m^0$ denotes the given set of lines in a modality specific line pool.

The passenger assignment in Step 3 of Algorithm 9 follows the procedure described in Section 4.3.2. We simply build the corresponding Change&Go-network from the solution of the previous iteration. Note that the choice of fixed lines L_m^* does not affect the resulting Change&Go-network because we take all lines with nonzero frequency into account when building the Change&Go-network. We will also stick to the same passenger assignment algorithm used in Step 1 of Algorithm 9 when routing in Change&Go-network to maintain consistency in different assignment iterations.

In Section 6, we experimentally evaluate whether the iterative approach can give better solutions than the traditional sequential approach. According to [5], the use of Change&Go-network in routing can indeed improve the optimal cost under certain circumstances. For this reason, we want to verify whether iterating the Change&Go-network for each step can result in even better cost or improved passenger travel time than regular routing approaches with Change&Go-network.

Choosing the set of fixed lines is the key heuristic in the iterative approach since it determines the optimization problem to be solved in the next iteration. In the experimental analysis we fix all the lines of a single modality in the problem for a single iteration and alter the order of different modalities to be set fixed. Selecting smaller sets of lines to be fixed can be difficult since evaluating the importance of different lines is difficult in general. By fixing all lines within a single modality, we simplify the analysis and make the process of selecting fixed lines more straightforward.

4.5 Multimodal integrated line planning problem

To conclude this section, we present the multimodal counterpart of the integrated line planning problem (5)-(11). Interestingly, the multimodal integrated line planning problem, does not require any additional parameters in the model formulation. Thus, comparing unimodal results from the integrated approach to the multimodal integrated results gives valuable insight on the effects of multimodality in the line planning phase. The integrated model is formulated as follows:

$$\textbf{LineAM} : \min \sum_{m\in M} \sum_{l\in\mathcal{L}_m^0} c_{ml} f_{ml} \tag{27}$$

$$\text{s.t.} \sum_{l\in L_m^{(s_1,s_2)}} \text{Cap}_m f_{ml} \geq \sum_{u,v\in S} x_e^{uv} \; \forall\, e \in \{\{(s_1,m),(s_2,m)\} : (s_1,s_2) \in E^m, m \in M\} \tag{28}$$

$$\sum_{l\in L_m^{(s_1,s_2)}} \text{Cap}_m f_{ml} \leq \text{Cap}_m U_m \; \forall\, m \in M, (s_1,s_2) \in S\times S \tag{29}$$

$$\Theta x^{uv} = b^{uv} \; \forall\, u,v \in S \tag{30}$$

$$\sum_{e\in E} d_e x_e^{uv} \leq \beta SP_{uv} W_{uv} \; \forall\, u,v \in S \tag{31}$$

$$f_{ml} \in \mathbb{N} \; \forall\, m \in M, l \in \mathcal{L}_m^0 \tag{32}$$

$$x_e^{uv} \in \mathbb{N} \; \forall\, u,v \in S, e \in E \tag{33}$$

Here $\Theta \in \mathbb{R}^{|V|\times|E|}$ is the incidence matrix of the multimodal PTN, that is

$$\Theta(v,e) = \begin{cases} 1 & \text{if } e = (v,u) \text{ for some } u \in V \\ -1 & \text{if } e = (u,v) \text{ for some } u \in V \\ 0 & \text{otherwise} \end{cases}$$

and $b^{uv} \in \mathbb{R}^{|V|}$ where $b_u^{uv} = W_{uv}$, $b_v^{uv} = -W_{uv}$ and the remaining components are 0.

Here, $L_m^{(s_1,s_2)}$ is the set of lines in the mode-specific line pool that contain an edge with s_1 its start point and s_2 its end point. Here, SP_{uv} stands for the shortest path length between OD-pair (u,v). Recall from the definition of the multimodal PTN that in the lower frequency constraint we are iterating over all edges $e \in E_{node}$. Note also that $L_m^{(s_1,s_2)} \subseteq \mathcal{L}_m^0$ for each $m \in M$. Otherwise, the multimodal integrated model (27)-(33) has the same properties as its unimodal counterpart.

As a concluding remark, we point out that multimodal integrated line planning problems can also be split into unimodal integrated problems using OD-matrix splitting. The process follows Algorithm 7 where we solve integrated models with same parameter value β instead of sequential models with same passenger assignments. Recall that OD-matrix splitting is used to create a set of unimodal problems that can be solved more easily than the large multimodal problem. This approach can be beneficial for large integrated multimodal problems since integrated models are generally much harder to solve than sequential models [1]. In Section 6, we compare the OD-matrix splitting approach runtimes and solutions for both integrated models and sequential models to validate the effects of OD-matrix splitting.

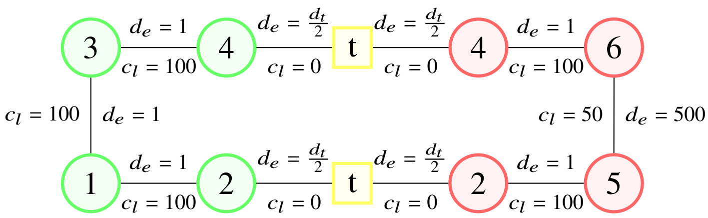


Figure 5: Example of a multimodal public transport network where line costs c_l and edge lengths d_e are not necessarily correlated. The numbers on the nodes refer to stop indices $s \in S$ of the nodes. The nodes with yellow color are the transfer node.

5 Theoretical analysis of multimodal networks

In this section, we analyze the relationship between the multimodal sequential cost-oriented line planning, described in Algorithm 6 and sequential cost-oriented line planning with OD-matrix splitting, described in Algorithm 7. We know that Algorithm 7 can be considered a heuristic version of multimodal line planning since the OD-matrix splitting algorithm is an auxiliary heuristic that doesn't appear in the regular multimodal or unimodal cost-oriented line planning algorithms. On the other hand, Algorithm 7 solves a series of problems with unimodal PTNs, whereas in sequential multimodal line planning, we use multimodal PTN (12) as the underlying PTN.

A central question related to the two approaches is their comparable performance: What are the optimal costs of the two approaches and how does the chosen OD-matrix splitting algorithm affect the solution? The following example shows that even when using a simple shortest paths passenger assignment, the choice of OD-matrix splitting can drastically alter the optimal cost in the problem. In fact, it is possible to obtain worse or better optimal costs than in the multimodal problem based on the graph structure and the choice of the OD-matrix split.

Example 3. Consider a sequential multimodal line planning problem using shortest paths passenger assignment (Algorithm 3) in the PTN described in Figure 5 with an OD-matrix where $W_{16} = 10$ and all other entries are zero.

Next, consider two unimodal line planning problems using shortest paths passenger assignment for PTN^{green} and PTN^{red} described in Figure 5 where the corresponding modality specific OD-matrices W^{green}, W^{red} are obtained using an OD-matrix splitting algorithm. Assume $W^{green}_{12} = 10$ and all other entries zero and $W^{red}_{26} = 10$ and all other entries zero. This set of OD-matrices is viable since all passengers are on a single connected path from 1 to 6 in the whole graph. Finally, assume equal capacities for both modalities $\text{Cap}_{green} = \text{Cap}_{red} = 10$. We assume complete line pool for both modalities.

First, we compute the optimal cost for the multimodal problem. Using shortest paths passenger assignment, we obtain a set of loads

$$\omega_e = 10, \ \forall e \in \{(1,3),(3,4),(4,6)\}$$
$$\omega_e = 0 \text{ otherwise}$$

Now since the capacity of a single line matches each nonzero load in ω, we can simply choose to use the corresponding lines from the complete line pool. The optimal cost obtained is therefore,

$$c^* = \textbf{LinePM}(\omega) = 100 \cdot 3 = 300$$

Then we compute the optimal costs for both unimodal problems in a similar fashion. We obtain the following sets of loads for the unimodal problems

$$\begin{aligned} \omega_e^{green} &= 10, \ \forall e \in \{(1,2)\} \\ \omega_e^{green} &= 0 \text{ otherwise} \\ &\text{and} \\ \omega_e^{red} &= 10, \ \forall e \in \{(2,5),(5,6)\} \\ \omega_e^{red} &= 0 \text{ otherwise} \end{aligned}$$

The optimal costs for the unimodal problems are therefore

$$\begin{aligned} c^*_{green} &= \textbf{LineP}(\omega^{green}) = 100 \\ c^*_{red} &= \textbf{LineP}(\omega^{red}) = 100 + 50 = 150 \\ \sum_{m \in M} c^*_m &= c^*_{green} + c^*_{red} = 250 \end{aligned}$$

Comparing the compound optimal cost from line planning with OD-matrix splitting to the multimodal line planning optimal cost, we can indeed see that line planning with OD-matrix splitting gives better cost than the multimodal line planning algorithm, when the passenger assignment algorithm and the underlying PTN are comparable for both approaches.

The reason for this behavior is that OD-matrix splitting greatly affects the resulting passenger loads for each line planning problem, even when the actual passenger assignment algorithms are identical. This way, the OD-matrix splitting can be considered an additional heuristic in the process that produces constraints for the actual passenger assignment algorithm. For example, it is easy to see that if we chose OD-matrices $W_{14}^{green} = 10$ and all other entries zero and $W_{46}^{red} = 10$ and all other entries zero, we would have obtained the same optimal costs for both problems since the OD-matrix split now allows the same passenger assignment as obtained with shortest paths in the multimodal PTN. On the other hand, with identical passenger assignment as in the example, but $c_{\{(5,6)\}} = 150$ instead of $c_{\{(5,6)\}} = 50$, we would have obtained worse cost for line planning with OD-matrix splitting, namely $\sum_{m \in M} c^*_m = 100 + 100 + 150 = 350$. This indicates line planning with OD-matrix splitting is also sensitive to the underlying graph and line properties.

The previous observations raise a notable concern related to the OD-matrix splitting approach: How can we make sure that the obtained unimodal line planning problems from OD-matrix splitting are comparable to the actual multimodal line planning problem? It turns out that with the correct viable OD-matrix split, we can always arrive at the same multimodal sequential optimal solution using line planning with OD-matrix splitting.

Theorem 5.1. *Let $c^*(\mathcal{L}^A)$ be the optimal cost of the multimodal sequential line planning problem for multimodal PTN induced from PTN^m, $m \in M$, with OD-matrix W when using passenger assignment $\omega_e, e \in E$. Then there always exists a set of OD-matrices $\{W^m, m \in M\}$ that are viable w.r.t. Definition 4 s.t. the optimal costs $c^*_m(\mathcal{L}^{B^m})$ of the unimodal sequential line planning problems for PTN^m with OD-matrix W^m when using passenger assignment $\omega^m_e, e \in E$ satisfy*

$$c^*(\mathcal{L}^A) = \sum_{m \in M} c^*_m(\mathcal{L}^{B^m})$$

Proof. Denote A the multimodal sequential line planning problem with optimal cost of $c^*(\mathcal{L}^A)$ and B^m the unimodal sequential line planning problem with optimal cost of $c^*_m(\mathcal{L}^{B^m})$

Let $(\mathcal{L}^A)$ be the optimal line concept of A such that $c^*(\mathcal{L}^A)$ is the optimum and denote the corresponding passenger assignment of the multimodal PTN as $\omega_e, e \in E$. Let $(\mathcal{L}^{A,m})$ be the line concept of modality m of solution $\mathcal{L}^A$ with passenger assignment $\omega^m_e, e \in E$ such that ω^m_e contains the passenger load of edges $e \in E^m$ for each $m \in M$. For each passenger assignment $\omega^m_e, e \in E, m \in M$, the corresponding OD-matrix is defined as

$$W^m = \begin{cases} W^m_{st} = \omega^m_{(s,t)} \ \forall \ (s,t) \in E^m \\ W^m_{st} = 0 \ \forall \ (s,t) \notin E^m \end{cases}$$

This set of OD-matrices $\{W^m, m \in M\}$ for each modality satisfies Definition 4 because all the OD-matrices are derived from the connected paths of a feasible line concepts.

Now suppose that $(\mathcal{L}^{A,m})$ is not the optimal solution to the problem B^m with passenger assignment ω^m_e, that is, there exists a feasible solution $(\mathcal{L}^{B^m})$ for B^m with the same passenger assignment ω^m_e for some modality m so that

$$c^*_m(\mathcal{L}^{B^m}) < c(\mathcal{L}^{A,m}) .$$

Then the following line concept for multimodal PTN

$$\mathcal{L}^{A,m'} = \begin{cases} \mathcal{L}^{A,m} & \text{if } m' \neq m \\ \mathcal{L}^{B^m} & \text{otherwise} \end{cases}$$

is a feasible solution of problem A because $\mathcal{L}^{A,m}$ and $\mathcal{L}^{B^m}$ are both obtained from the same passenger assignment ω_e^m.

According to our initial assumption

$$c(\mathcal{L}^{A,m'}) = \sum_{m \in M \setminus m'} c(\mathcal{L}^{A,m}) + c_m^*(\mathcal{L}^{B^m}) < \sum_{m \in M} c(\mathcal{L}^{A,m}) = c^*(\mathcal{L}^A)$$

This is a contradiction since $c^*(\mathcal{L}^A)$ is the optimal solution of A. Therefore, the line concept $(\mathcal{L}^{A,m})$ is the optimal solution to problem B^m for each $m \in M$. Moreover, each optimal solution $(\mathcal{L}^{A,m})$ is obtained from the set of feasible OD-matrices $\{W^m, m \in M\}$. Based on the optimality of $(\mathcal{L}^{A,m})$, we have $c_m^*(\mathcal{L}^{A,m}) = c_m^*(\mathcal{L}^{B^m})$ and therefore

$$c^*(\mathcal{L}^A) = \sum_{m \in M} c_m^*(\mathcal{L}^{A,m}) = \sum_{m \in M} c_m^*(\mathcal{L}^{B^m})$$

□

The above theorem clarifies the relationship between the multimodal problem and corresponding set of unimodal problems obtained by OD-splitting. By solving the sequential multimodal problem, we always arrive at the actual optimum, but with correct choice of OD-matrix split, we can also do so with OD-matrix splitting. However, solving a set of unimodal problems from OD-splitting is generally much more tractable than solving a single, potentially larger multimodal problem instance. Therefore, being able to find the correct viable OD-matrix split is of great interest.

Note that Theorem 5.1 also generalizes to the multimodal integrated line planning problems. In the integrated case, we can simply construct the corresponding passenger loads ω_e^m directly from the decision variables $x_e^{u,v}$ after which the proof is similar to the sequential cases.

In the following we analyze the properties of simple multimodal PTNs and whether the line planning problems can be easily split to the corresponding unimodal problems using OD-matrix splitting. We also investigate the effect of modality transfer points in the PTN for both unimodal and multimodal approaches.

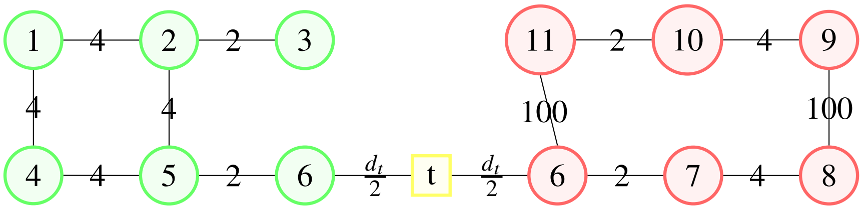


Figure 6: Example of a network with separate PTNs connected by a single transfer point. The numbers on the edges refer to the lengths of the edges d_e and numbers on the nodes refer to stop indices $s \in S$ of the nodes. The node with yellow color is a transfer node.

5.1 Separate PTNs with a single common transfer point

We begin the analysis of multimodal networks by investigating the case of only one single transfer point in multimodal PTN. In this case, the multimodal PTN consists of n unimodal PTNs, each of which share a common stop $s^* \in S$. In our case, s^* is also the only common stop index for all PTN$^m, m \in M$. All of the multimodal PTNs of this form can then be expressed in the following format:

$$\text{PTN} = \{(V', E') : V' := V'_{node} \cup V'_{trans} \cup V'_{OD}, E' := E'_{node} \cup E'_{trans} \cup E'_{OD}\} \quad (34)$$

where

$$V'_{node} := \bigcup_{m \in M} \{(s, m) : s \in S^m\}$$
$$V'_{trans} := \{(s^*, trans)\}$$
$$V'_{OD} := \{(s, OD) : s \in S\}$$

and

$$E'_{node} := \{((s, m), (s', m)) : (s, s') \in E^m, m \in M\}$$
$$E'_{trans} := \{((s^*, m), (s^*, m')) \text{ and } ((s^*, m'), (s^*, m)) : \ s^*, \ \forall m, m' \in M, m \neq m'\}$$
$$E'_{OD} := \{((s, OD), (s, m)) \text{ and } ((s, m), (s, OD)) : s \in S^m, m \in M\}$$

Note that the only transfer node in (34) is s^* and the only transfer edges are the incoming and outgoing edges of $(s^*, trans)$ for each PTN$^m, m \in M$. Figure 6 illustrates an example of this type of network.

For this type of multimodal PTN, we can formulate an especially strong result related to cost-oriented line planning. Namely, the line planning problem for (34) with OD-matrix W can be solved by splitting it into modified separate unimodal line planning problems with unimodal PTNs for each modality using a specified version of Algorithm 8. The OD-matrix splitting algorithm works as follows:

Algorithm 10 OD-matrix splitting for separate PTNs with one single transfer point

1: Declare $\tilde{W}^m = \mathbf{0}$ and $\bar{W}^m = \mathbf{0}$, $\tilde{W}^m, \bar{W}^m \in \mathbb{N}^{|V^m| \times |V^m|}$ for each $m \in M$
2: For each W_{uv} where $u \in V^m, v \in V^{m'} : m \neq m'$, update $\tilde{W}^m_{us^t} := \tilde{W}^m_{us^t} + W_{uv}$ and $\tilde{W}^{m'}_{s^t v} := \tilde{W}^{m'}_{s^t v} + W_{uv}$
3: For each W_{uv} where $u \in V^m, v \in V^m$, update $\bar{W}^m := \bar{W}^m + W_{uv}$

Here s^t is the closest transfer point to u, although in the case with single s^*, there is only one transfer point available for a graph so $s^t = s^*$. The set of OD-matrices $\tilde{W}^m, m \in M$ is viable, since it satisfies Definition 4. Moreover, note that Algorithm 10 is a special case of Algorithm 8 that always produces a set of viable OD-matrices.

After using the OD-matrix splitting algorithm to obtain the mode-specific matrices $\tilde{W}^m$ and W^m for each $m \in M$, we obtain the following result

Theorem 5.2. *Let c^* be the optimal cost of multimodal sequential line planning problem for (34) with OD-matrix W when using shortest-paths passenger assignment. Let c^*_m be the optimal cost of unimodal sequential line planning problem for PTN^m with OD-matrix $W^m = \tilde{W}^m + \bar{W}^m$ when using shortest-paths passenger assignment. Then*

$$c^* = c^*_m + c^*_{m'}$$

Proof. For each W_{uv} where $u \in V^m, v \in V^{m'} : m \neq m'$, all passenger shortest paths P_{uv} have to satisfy $P_{uv} = P_{us^*} \cup P_{s^*v}$, where s^* is the transfer point. We therefore have

$$c^* = \mathbf{LinePM}(\omega(\bar{W}^m, \mathrm{PTN}^m) + \omega(\bar{W}^{m'}, \mathrm{PTN}^{m'}) + \omega(\tilde{W}^m, \mathrm{PTN}^m) + \omega(\tilde{W}^{m'}, \mathrm{PTN}^{m'}))$$

On the other hand the summed costs from unimodal problems can be expressed as

$$\begin{aligned} c^*_m + c^*_{m'} = &\,\mathbf{LineP}(\omega(\bar{W}^m, \mathrm{PTN}^m) + \omega(\tilde{W}^m, \mathrm{PTN}^m)) \\ &+ \mathbf{LineP}(\omega(\bar{W}^{m'}, \mathrm{PTN}^{m'}) + \omega(\tilde{W}^{m'}, \mathrm{PTN}^{m'})) \end{aligned}$$

However, since PTN^m and $\mathrm{PTN}^{m'}$ are both different unimodal PTNs, and there are no overlapping edges in any of the separate PTNs, we can factor

$$\begin{aligned} c^* = &\,\mathbf{LinePM}(\omega(\bar{W}^m, \mathrm{PTN}^m) + \omega(\tilde{W}^m, \mathrm{PTN}^m)) \\ &+ \mathbf{LinePM}(\omega(\bar{W}^{m'}, \mathrm{PTN}^{m'}) + \omega(\tilde{W}^{m'}, \mathrm{PTN}^{m'})) \end{aligned}$$

The solution to multimodal sequential line planning problem is equal to the unimodal sequential line planning problem if the underlying PTN for both problems is unimodal and passenger assignments are identical. Thus,

$$
\begin{aligned}
c^* &= \mathbf{LineP}(\omega(\bar{W}^m, \mathrm{PTN}^m) + \omega(\tilde{W}^m, \mathrm{PTN}^m)) \\
&\quad + \mathbf{LineP}(\omega(\bar{W}^{m'}, \mathrm{PTN}^{m'}) + \omega(\tilde{W}^{m'}, \mathrm{PTN}^{m'})) \\
c^* &= c^*_m + c^*_{m'}
\end{aligned}
$$

□

We also witness that the average travel time can also be compared against the transfer penalty d_t. In fact, we see that

Theorem 5.3. *Let T^* be the total travel time of multimodal sequential line planning problem for (34) with OD-matrix W when using shortest-paths passenger assignment. Let T^*_m be the optimal total travel time of unimodal sequential line planning problem for PTN^m with OD-matrix $W^m = \tilde{W}^m + \bar{W}^m$ when using shortest-paths passenger assignment. Further, let q be the sum of all entries in W where $u \in V^m, v \in V^{m'}$. Assume line pool $\mathcal{L}^0$ is the same for all modalities. Then*

$$T^* = \sum_{m \in M} T^*_m + q d_t$$

Proof. For each W_{uv} where $u \in V^m, v \in V^{m'} : m \neq m'$, all passenger shortest paths P_{uv} have to satisfy $P_{uv} = P_{us^*} \cup P_{s^*v}$, where s^* is the transfer point. Therefore, for each such entry, the corresponding travel time of multimodal problem is $SP_{uv} = SP_{us^*} + SP_{s^*v} + \frac{1}{2} \cdot 2d_t$, where SP indicates the length of the shortest path (we assume edge length is equal to the travel time). No other modality transfers occur in any of the problems, so with a set of unimodal problems with same passenger assignment for each, we confirm that

$$T^* = \sum_{m \in M} T^*_m + q d_t$$

□

The result above confirms that under certain conditions, OD-matrix splitting can be performed using a relatively simple algorithm such that the resulting multimodal cost and sum of unimodal costs are always equal. In this relatively simple case, the travel times are also equal if we simply account for the compulsory modality transfer penalties that are not present in any of the unimodal problems.

5.2 Separate PTNs with more than one common transfer point

Next, we analyze the case where separate PTNs are connected with $n > 1$ common transfer points. Given the following example network in Figure 7 with two transfer points, we shall see if the OD-matrix splitting from the previous section also gives the same cost as the multimodal counterpart.

$$\text{PTN} = \{(V', E') : V' := V'_{node} \cup V'_{trans} \cup V'_{OD}, E' := E'_{node} \cup E'_{trans} \cup E'_{OD}\} \quad (35)$$

where

$$V'_{node} := \bigcup_{m \in M} \{(s, m) : s \in S^m\}$$
$$V'_{trans} := \{(s, trans) : s \in T\}$$

and

$$E'_{node} := \{((s, m), (s', m)) : (s, s') \in E^m, m \in M\}$$
$$E'_{trans} := \{((s^*, m), (s^*, m')) \text{ and } ((s^*, m'), (s^*, m)) : \ s^* \in \{s^*_1, s^*_2, ..., s^*_n\}, m, m' \in M\}$$
$$E_{OD} := \{((s, OD), (s, m)) \text{ and } ((s, m), (s, OD)) : s \in S^m, m \in M\}$$

Example 4. Consider a sequential multimodal line planning problem for the graph presented in Figure 7 with OD-matrix W such that $W_{18} = 50$ and all other entries are zero. Assume we are using shortest path passenger assignment.

Using Algorithm 3 to calculate loads ω in the multimodal case, gives us the following set of loads:

$$\omega_e = 50, \ \forall e \in P_{18}$$
$$\omega_e = 0 \text{ otherwise}$$

where

$$P_{18} = \{(1, 4), (4, 5), (5, 6), (6, t), (t, 6), (6, 7), (6, 8)\}$$

This will be used as part of the lower frequency constraint in the line planning problem **LinePM**(ω).

We then compare the results generated in the multimodal case to the unimodal counterpart with OD-matrix splitting. Using the OD-matrix splitting Algorithm 10 presented in the previous chapter, we obtain two unimodal OD-matrices, with colors representing the modalities. Here $\bar{W}^{green}_{13} = 50$, otherwise zero and $\bar{W}^{red}_{38} = 50$, otherwise zero. This OD-matrix split is also viable, since the path $P_{18} = (P_{13}, P_{38})$ is connected.

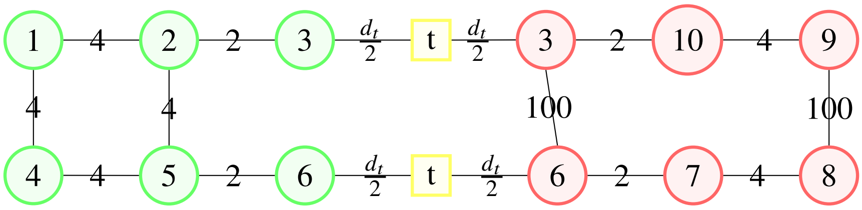


Figure 7: Example of a network with separate PTNs connected by two transfer points. The numbers on the edges refer to the lengths of the edges d_e and numbers on the nodes refer to stop indices $s \in S$ of the nodes. The node with yellow color is a transfer node.

Using Algorithm 3 to calculate loads ω for both the unimodal cases, gives us the following set of loads, first for GREEN problem:

$$\begin{aligned} \omega_e^{green} &= 50, \ \forall e \in \{(1,2),(2,3)\} \\ \omega_e^{green} &= 0 \text{ otherwise} \end{aligned}$$

and then for RED problem

$$\begin{aligned} \omega_e^{red} &= 50, \ \forall e \in \{(3,6),(6,7),(7,8)\} \\ \omega_e^{red} &= 0 \text{ otherwise} \end{aligned}$$

Now let us calculate the optimal values for all the example problems presented above. We assume a complete line pool for all problems without upper-frequency constraints and $\text{Cap}_m = 1$ for each $m \in M$ and $c_l = \sum_{e \in l} d_e$ for all lines. We have

$$c^* = \textbf{LinePM}(\omega) = 16 \cdot 50 = 800$$

and

$$\begin{aligned} c^*_{green} &= \textbf{LineP}(\omega^{green}) = 6 \cdot 50 = 300 \\ c^*_{red} &= \textbf{LineP}(\omega^{red}) = 106 \cdot 50 = 5300 \\ \sum_{m \in M} c^*_m &= c^*_{green} + c^*_{red} = 5600 \end{aligned}$$

In this case with $n > 1$ common transfer points, we have $c^* < \sum_{m \in M} c^*_m$.

The previous example shows that using the OD-matrix splitting does not work generally with $n > 1$ common transfer points, although for case $n = 1$ we always obtain the multimodal optimal value. The naive approach for OD-matrix splitting with

OD-pair $(u, v), u \in V^m, v \in V^{m'}, m \neq m'$ is to find the shortest path from u to any of the transfer points and then find shortest path from that transfer point to v. It turns out this approach is not sufficient, as can be seen from the example by looking at nodes 1 and 8. In essence, a shortest path to any transfer point is not necessarily a subpath of actual shortest path P_{uv} when the number of transfer points is $n > 1$.

From the previous observations we conclude that with shortest paths passenger assignment, in order to obtain the multimodal sequential optimal value for graph type (35) using OD-matrix splitting, we have to calculate shortest paths P_{uv} for each (u, v) in the multimodal PTN and perform the OD-matrix splitting based on those shortest paths. In practice, we use Algorithm 8 so that paths in the algorithm are now shortest paths. With this approach, we can consistently solve large instances of line planning problems for (35) potentially faster than with Algorithm 6 by splitting the large problem into smaller modality-specific parts. Moreover, this OD-matrix splitting results in multimodal optimal line plans given the shortest paths passenger assignments for all problems. However, as we can see in the following, overlapping edges on multimodal networks may prevent obtaining the correct set of OD-matrices easily.

5.3 Identical PTNs with modality transfers

We now turn to multimodal networks where identical unimodal PTNs, all with the same set of stops S and set of edges E, which are connected by a set of transfer points. In this kind of network it is possible to transfer from every modality to any other modality if there is a transfer point available. Such PTNs can always be expressed as:

$$\text{PTN} = \{(V', E') : V' := V'_{node} \cup V'_{trans} \cup V'_{OD}, E' := E'_{node} \cup E'_{trans} \cup E'_{OD}\} \quad (36)$$

where

$$\begin{aligned} V'_{node} &:= \bigcup_{m \in M} \{(s, m) : s \in S\} \\ V'_{trans} &:= \{(s, trans) : s \in T\} \\ V_{OD} &:= \{(s, OD) : s \in S\} \end{aligned}$$

and

$$\begin{aligned} E'_{node} &:= \{((s, m), (s', m)) : (s, s') \in E, m \in M\} \\ E'_{trans} &:= \{((s^*, m), (s^*, m')) \ : \ s^* \in \{s^*_1, s^*_2, ..., s^*_n\}, \ \forall m, m' \in M\} \\ E'_{OD} &:= \{((s, OD), (s, m)) \text{ and } ((s, m), (s, OD)) : s \in S^m, m \in M\} \end{aligned}$$

Here, we assume that edge lengths d_e are equal for every corresponding edge of each modality. We first realize that in such a network with shortest paths approaches for passenger assignment, there is no reason to use modality transfers at all, since all OD-pairs can be reached even without transfers that accumulate more length to every shortest path. When line pools $\mathcal{L}_m^0$ are the same for each modality, the line costs c_{lm} and capacities Cap_m still affect the outcome in the line planning process.

For example, it is easy to see that if capacities for each line $l \in \mathcal{L}_m^0$ are equal for all modalities, we always choose the modality with lowest cost per line. On the other hand, if costs for each available line $l \in \mathcal{L}_m^0$ are equal for all modalities, then choosing the modality with higher capacity is always the better option when minimizing the overall cost of the line concept.

In reality, it is not practical to assume that different modality lines would all have the same costs. This is mainly due to differences in infrastructure related costs, such as costs related to different types of rails, tunnels and so on.

For this type of PTN, the intuitive approach of OD-matrix splitting is to find a set of unimodal OD-matrices such that $W = \sum_{m \in M} W^m$. This is reasonable because with identical PTNs, the dimensions of OD-matrices are also identical. In detail, the unimodal OD-matrices can be represented as $W^m = a_m W, m \in M$ such that $\sum_{m \in M} a_m = 1$. Next, we show that with PTN of type (36), each such set of OD-matrices is viable and can be used to split the multimodal line planning problem into unimodal parts.

Theorem 5.4. *Let $W \in \mathbb{N}^{|S| \times |S|}$ and let the underlying PTN be of type (36). A set of OD-matrices $\{W^m = a_m W, m \in M\}$ with $\sum_{m \in M} a_m = 1$, $a_m \geq 0$ is viable w.r.t. Definition 4.*

Proof. Since the underlying PTNs are identical, that is $\mathrm{PTN} = \mathrm{PTN}^m$, $\forall m \in M$, then $\forall u, v \in S$, u and v are connected and the first constraint holds.■

By setting the weights

$$c_{u,v}^{u,v,m} = W_{uv}^m = a_m W_{uv}, \; \forall\, u, v \in S, m \in M$$

and all other weights zero, we arrive at

$$\sum_{s,t \in S} c_{s,t}^{u,v,m} = c_{u,v}^{u,v,m} = W_{uv}^m, \; \forall\, u, v \in S, m \in M$$

so the second constraint also holds.■

With the weights given, the third constraint holds for each $s, t \in S$, $v \in V \setminus \{s, t\}$ because $c_{s,t}^{s,t,m}$, $s, t \in S$, are the only nonzero entries in our set of weights and $v \in V \setminus \{s, t\}$, so all the entries appearing in this constraint are zero.■

The fourth and fifth constraint in our case can be expressed as

$$\sum_{m\in M}\sum_{u\in V} c_{s,t}^{s,u,m} - \sum_{m\in M}\sum_{u\in V} c_{s,t}^{u,s,m} = \sum_{m\in M} c_{s,t}^{s,t,m} - 0$$
$$= \sum_{m\in M} W_{s,t}^{m}$$
$$= \sum_{m\in M} a_m W_{s,t} = W_{s,t}\ \forall\ s,t \in S$$

and

$$\sum_{m\in M}\sum_{u\in V} c_{s,t}^{u,t,m} - \sum_{m\in M}\sum_{u\in V} c_{s,t}^{t,u,m} = \sum_{m\in M} c_{s,t}^{s,t,m} - 0$$
$$= \sum_{m\in M} W_{s,t}^{m}$$
$$= \sum_{m\in M} a_m W_{s,t} = W_{s,t}\ \forall\ s,t \in S,$$

Therefore, we have found a set of weights that satisfy all five constraints, so the proof is complete. □

We now present an interesting example that promotes the use of different capacities to lower the costs in the multimodal line planning problem. The example also shows that overlapping edges with different capacities in the multimodal PTN can make it difficult to obtain an OD-matrix split corresponding to the optimal value of **LinePM**(ω). From Theorem 5.1, we know that we can always obtain the same optimal cost than in multimodal line planning with some OD-matrix split that satisfies Definition 4, independent of the number of transfer points. However, in some cases $c^* < \sum_{m\in M} c_m^*$ with any $W = \sum_{m\in M} W^m$, and that modality transfers are required to obtain the multimodal optimum. That is, there does not exist an OD-matrix split of the form $W = \sum_{m\in M} W^m$ that results in the same optimal objective value for the multimodal problem and the sum of unimodal problems.

Example 5. Consider a sequential multimodal line planning problem for the graph presented in Figure 8. This simple graph is an example of (36), with transfer possibilities for each node in the network. Assume the multimodal OD-matrix $W_{16} = 15, W_{26} = 15, W_{36} = 15, W_{46} = 15$ and all other entries 0. Due to simplicity of the graph, the corresponding passenger loads are independent of the routing scheme. We have $\omega_{16} = 15, \omega_{26} = 15, \omega_{36} = 15, \omega_{46} = 15$ and therefore $\omega_{56} = 15 \cdot 4 = 60$

Now let's calculate the optimal values for the multimodal problem. We assume complete line pool without upper frequency constraints for all problems. We further assume two modalities, red and green with $\text{Cap}_{red} = 15, \text{Cap}_{green} = 20$ and $c_{red_l} = 8, c_{green_l} = 10,\ \forall\ l \in \mathcal{L}_m^0$. Then we have

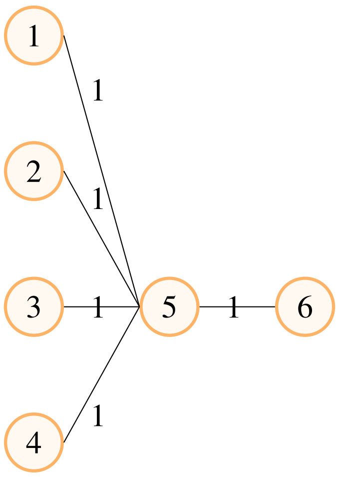


Figure 8: Example of a network with two identical PTNs with transfer points between the modalities for each node. The numbers on the edges refer to the lengths of the edges d_e and numbers on the nodes refer to stop indices $s \in S$ of the nodes. The transfer nodes are omitted from the picture for clarity.

$$c^* = \mathbf{LinePM}(\omega) = 4 \cdot 8 + 3 \cdot 10 = 62$$

It turns out that in this specific example, there does not exist a pair of unimodal matrices W^{red} and W^{green} such that $W^{red} + W^{green} = W$ for which $\sum_{m \in M} c^*_m = c^*$. To achieve the optimal cost, we must allow modality transfers from red to green at point 5, indicating that the correct OD-matrix split contains a nonzero entry W^{green}_{56}, while $W_{56} = 0$ in the original OD-matrix.

So in this case $c^* < \sum_{m \in M} c^*_m$ independent of choosing OD-matrices W^{red} and W^{green} such that $W^{red} + W^{green} = W$.

The result is remarkable, because it shows that OD-matrix split that the results in the multimodal line planning problem optimum is not always of the form $W = \sum_{m \in M} W^m$. Note that the overall travel time increases due to the transfer penalties. The result shows that using other passenger routes than shortest paths can be used to decrease costs even in simple multimodal networks due to overlapping edges with different capacities. The result indicates that solving the multimodal problem should be preferred, since obtaining the correct OD-matrix split when using custom passenger assignments even with identical PTNs can be difficult.

5.4 PTNs with limited transfer points

Thus far, we have investigated only the case of identical PTNs where the number of modality transfer points does not affect the problem at all. This happens when routing with shortest paths in (36) as PTN, because shortest paths routing algorithms never use transfers due to non-zero transfer cost and corresponding edge lengths being equal for all modalities. When routing with the Change&Go-graph, modality transfers can be utilized since the graph does not always contain all edges or nodes of the original multimodal PTN. Moreover, changing lines within a single modality can overall be more costly than changing to different modality lines. Nevertheless, the choice of passenger assignment is still the primary factor affecting the optimal cost of the multimodal line planning problem.

However, in the multimodal integrated line planning problem (27)-(33) the passenger assignment is incorporated into the parameter β that controls the relationship between the cost of the line concept and passenger travel time. Recall that increasing β indicates more emphasis on lowering the cost of the resulting line concept. If we set β high enough, the costs of the line concept are minimized close to the global minimum, while the passenger travel time is not taken into account. In such a case, adding more transfer points to the existing set of transfer points gives more possibilities to lower the costs even further at the expense of passenger-friendliness.

To conclude the theoretical analysis of multimodal networks, we show that adding a single transfer point $s^* \in S$ in a multimodal PTN can only decrease the cost of the line concept with multimodal integrated line planning problem when β is large enough. In the following theorem, we show that this observation holds independent of the network structure.

Theorem 5.5. *Let c_1^* be the optimal cost of the multimodal integrated line planning problem for a multimodal PTN with OD-matrix W with a set of transfer edges E'_{trans} in the underlying PTN. Let $s^* \in S$ be a single stop in the same multimodal PTN. Let c_2^* be the optimal cost of the multimodal integrated line planning problem for the same multimodal PTN with OD-matrix W with a set of transfer edges $\tilde{E}'_{trans} = E'_{trans} \cup \{((s^*, m), (s^*, m')) \notin E'_{trans}, m, m' \in M\}$ in the underlying PTN. Assume line pools are the same for all modalities and assume β sufficiently high. Then*

$$c_2^* \leq c_1^*.$$

Proof. Comparing the two problems $\mathbf{LineAM}(E'_{trans}) = c_1^*$ and $\mathbf{LineAM}(\tilde{E}'_{trans}) = c_2^*$, we see that the objective function is the same for both problems. Comparing the constraints, we see that with β sufficiently high, the constraint

$$\sum_{e \in E} d_e x_e^{uv} \leq \beta SP_{uv} W_{uv} \ \forall\, u, v \in S$$

becomes redundant for both problems since the right hand size becomes sufficiently high. The upper-frequency constraint

$$\sum_{l \in L_m^{(s_1, s_2)}} \mathrm{Cap}_m f_{ml} \leq \mathrm{Cap}_m U_m \ \forall\, m \in M, (s_1, s_2) \in S \times S$$

also stays the same since transfer edges don't appear in the constraint.
However, the flow constraint

$$\Theta x^{uv} = b^{uv} \ \forall\, u, v \in S$$

is affected. Indeed, the set of entries in x^{uv} are appended with the added entries corresponding to the added transfer edges. This means there now exists new feasible combinations of values of $x_{e'}^{uv}, e' \in \{\{(s_1, m), (s_2, m)\} : (s_1, s_2) \in E^m, m \in M\}$ that satisfy the flow constraint. In turn, this means that the right-hand size of the constraint

$$\sum_{l \in L_m^{(s_1, s_2)}} \mathrm{Cap}_m f_{ml} \geq \sum_{u, v \in S} x_e^{uv} \ \forall\, e \in \{\{(s_1, m), (s_2, m)\} : (s_1, s_2) \in E^m, m \in M\}$$

has now more feasible values. Consequently, the decision variables f_{ml} on the left-hand side that directly affect the objective function value have now more feasible combinations. On the other hand, by setting $x_{e'}^{uv} = 0$, we always obtain

$$\mathbf{LineAM}(E'_{trans}) = \mathbf{LineAM}(\tilde{E}'_{trans})$$

In conclusion,

$$c_2^* \leq c_1^*$$

□

Note that the goal of setting β sufficiently high is to make sure that the route constraint (31) is completely relaxed from the integrated optimization problem and only the cost of the line concept is accounted. It turns out we can always compute a fixed value for β so that this constraint does not affect the problem at all. By setting $\beta = \max\{d_e, e \in E\} \cdot \max\{W_{uv}, u, v \in S\} \cdot |E|$, the left-hand side of the constraint can never be higher than the right-hand side, essentially neglecting the effect of the route constraint.

The above result confirms that adding more transfer points to the existing set of transfer points in any multimodal PTN can only make the optimal cost better when performing multimodal integrated line planning with only the cost of the line concept accounted (that is, with β sufficiently high). In essence, the result shows that with the most cost-oriented passenger assignment possible, adding more transfer options can only decrease the cost of the resulting line concept. It should be noted that simply increasing the number of transfer points does not guarantee improvement. Indeed, even with same number of transfer points for two multimodal problems where transfer points are different, we can arrive at different optimal values for the cost of the line concept.

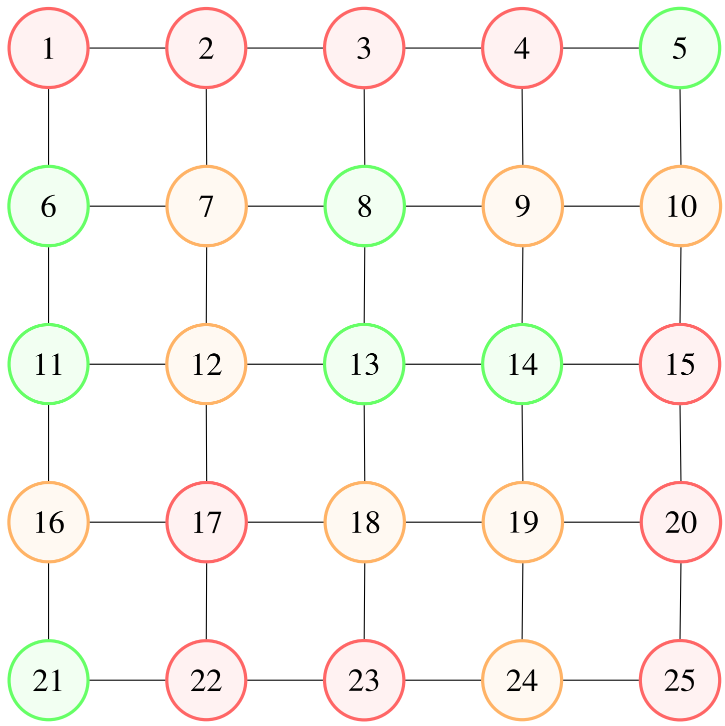


Figure 9: Illustration of the multimodal PTN used in the experiments. In the picture, red nodes represent bus stops, green nodes represent tram stops and orange nodes represent stops with both modalities with transfers allowed to other modalities. For simplicity, the transfer nodes and edges are omitted from the picture. Note also that all edge lengths in the network are equal.

6 Experiments and Results

In this section, we present the computational results of the multimodal line planning algorithms and analyze their performance in terms of runtime, parameters and optimal objective function values. For each of the experiments, we use the same grid-like multimodal PTN shown in Figure 9. We also use the same set of problem parameters presented in Appendix A and the same OD-matrix for each experiment to make the results comparable. All computations and evaluation for the experiments are performed using the LinTim algorithm environment for mathematical public transport optimization that contains algorithms for line planning in a single software [20].

The line pools for all of the problems are generated separately for both modalities. We first split the OD-matrix using the shortest paths OD-matrix splitting to obtain modality specific OD-matrices. That is, we use Algorithm 8 with $N_{uv} = 1$ where all paths P^1_{uv} are shortest paths. Then we generate loads for both modalities using the shortest path passenger assignment with Algorithm 3. Finally, we obtain the line pools for both modalities using the iterative minimum spanning tree approach presented in [2]. In all of the following experiments, we use the same line pools for each line planning problem. Thus, we assume that the line pools are given to neglect the effects of line pool generation in our experiments.

In the experiments, the optimal cost c^* is the optimal objective function value of the corresponding cost-oriented line planning problem. The average travel time is harder to calculate accurately, since passengers usually choose shortest paths in the public transportation system, although the designed passenger assignment for the system could be different. Moreover, the passenger behavior in the transport system depends on other factors as well that can be difficult to model accurately. For all the sequential models, the average travel time is calculated as

$$T_1 = \frac{\sum_{u,v\in S} \tilde{SP}_{uv} \cdot W_{uv}}{\sum_{u,v\in S} W_{uv}} \quad .$$

Here $\tilde{SP}_{uv}$ is the shortest path in a multimodal Change&Go-graph induced by the optimal line concept, including penalties for mode transfers and line transfers within a single modality. Note that the above formula is an approximation of the actual travel time due to the limited capacities of the vehicles. The actual travel time can be higher since all the passengers choosing the shortest route might not fit into the vehicles with limited capacities.

For comparisons of the integrated models, we use the average travel time calculated as

$$T_2 = \frac{\sum_{u,v\in S} \sum_{e\in E} d_e x_e^{uv} \cdot W_{uv}}{\sum_{u,v\in S} W_{uv}} \quad .$$

Note that the above formula is also an approximation since it does not always take into account the line transfer penalties within a single modality. The actual travel time can be higher when the integrated model is used with a PTN and not with a Change&Go-graph. We use this travel time approximation for the integrated model because the passenger routes in the integrated model are incorporated into the decision variables. We obtain far more realistic comparison of the travel times over different parameter values, since we use actual passenger routes and the proposed capacities are never exceeded. This way, the comparison of different instances of the integrated models is far more accurate. However, we also recognize that realistic travel time evaluation in cost-oriented line planning is difficult since the effect of passenger assignment to the actual travel time cannot always be determined reliably.

6.1 Comparison of the line planning algorithms

We begin the experimental analysis by comparing the different line planning algorithms and their effects on the objective function values in line planning. For sequential approaches, we use the different passenger assignment algorithms to create multimodal line planning problems of type (13)-(16) and present their corresponding optimal line costs, average passenger travel times and number of direct passengers. For integrated approaches, we generate problems of type (27)-(33) and present the same results. For iterative approaches, we perform two iterations where we fix all lines belonging to a single modality in the second iteration. More precisely, we first route passengers in a multimodal-line-pool-induced Change&Go-network using shortest paths passenger assignment. Then we fix all the lines belonging to a single modality, route passengers on shortest-paths in the line-concept-induced Change&Go-network and solve the resulting problem (23)-(26), resulting in two sets of fixed lines that form the optimal line concept.

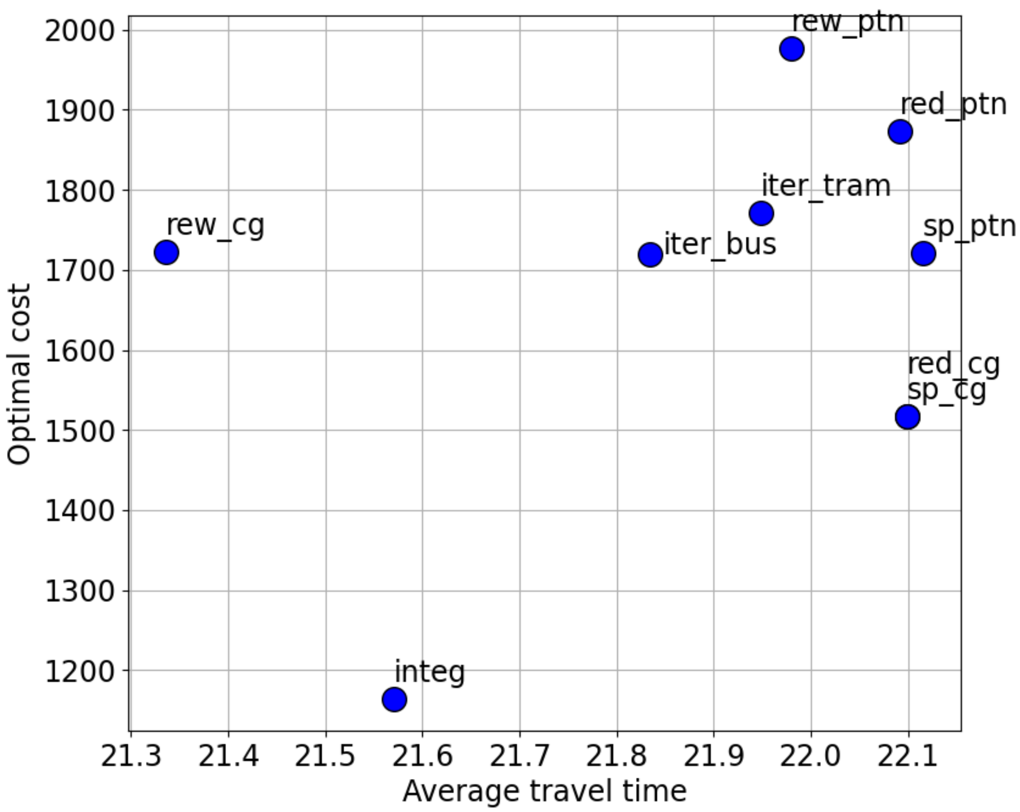


Figure 10: Objective function values of the different multimodal line planning algorithms

From Figure 10 and Table 1 we first see that approaches using the Change&Go-graph give notably lower costs than their PTN-specific counterparts. This is expected since Change&Go-graphs allow considering line transfer penalties already in the passenger assignment phase. However, the runtime for Change&Go-graph approaches is clearly larger as can be seen from Table 1. This is also expected since the size of the graph increases from the regular PTN when the graph contains all lines from the line pool.

On the other hand, we note that Algorithm 4 and Algorithm 5 respectively for both PTN and Change&Go-graph give generally better average travel times and higher costs than the corresponding shortest path approaches. This is in strong contrast to the

Table 1: Experimental results of the different multimodal line planning algorithms. For sequential models, the runtime is the compound runtime of running the passenger assignment and solving the corresponding line planning problem

Algorithm	Optimal cost	Average travel time (T_1)	Runtime (s)	Direct passengers	Parameters
Shortest paths in PTN	1721.60	22.12	2.89	1358	-
Reduction in PTN	1873.90	22.09	10.17	1357	$\gamma = 50$
Reward in PTN	1976.70	21.98	31.71	1373	$\gamma = 10$
Shortest paths in Change&Go-graph	1517.55	22.10	10.78	1369	-
Reduction in Change&Go-graph	1517.55	22.10	141.55	1369	$\gamma = 50$
Reward in Change&Go-graph	1723.10	21.34	2213.33	1484	$\gamma = 10$
Integrated model in PTN	1164.70	21.57	204.29	1464	$\beta = 1$
Iterative model (first fixed tram lines)	1772.05	21.95	12.99	1364	-
Iterative model (first fixed bus lines)	1719.65	21.83	12.73	1391	-

results presented in [4] and our initial assumption that stated Reward and Reduction could be used to decrease the costs of the line concept. Based on these results, it seems that the graph structure and the line planning problem parameters such as capacities and upper frequency constraints can severely affect the heuristic approaches and their desired effect on the results of the line planning process. These effects should be evaluated further in the future research.

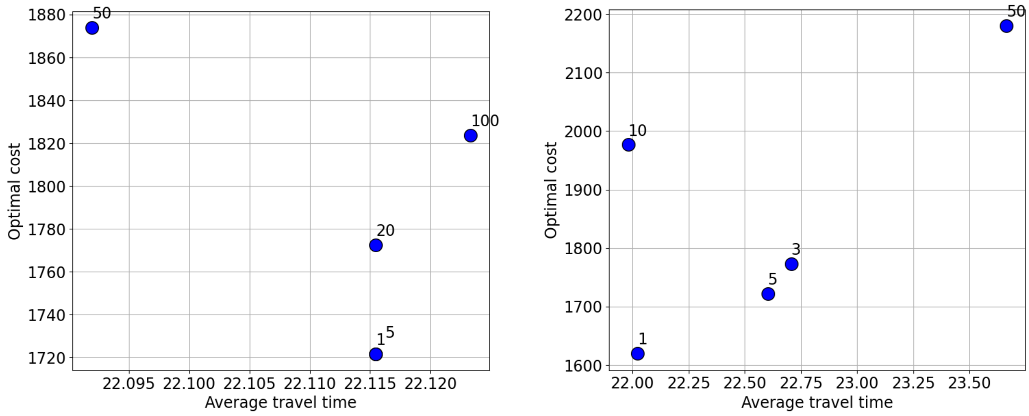
As can be expected, the integrated model outperforms the other approaches in terms of the cost and mostly outperforms the approaches also in terms of travel time. However, this comes with huge increase in runtime, even when using PTN as the underlying graph. In the experiments, we omit the use of Change&Go-graph with Integrated model, since the runtimes for calculating the optima were large in contrast to any other models.

Finally by looking at Figure 10 and Table 1, we note that neither of the iterative approaches attains lower cost than the regular shortest-paths approach in the Change&Go-graph. However, the iterative approaches attain better travel time than most other methods in comparison. The runtimes for iterative models are slightly larger than regular Change&Go shortest-paths, but the differences are relatively small. Based on the results, the iterative models could be useful to achieve middle ground for cost minimization and travel time minimization. However, the actual effects of fixing the set of lines in the iterative process must first be analyzed in the future research.

6.2 Performance analysis of the multimodal passenger assignments

We now investigate how the parameter values for different passenger assignments affect the sequential line planning problems and their corresponding objective function values. We present the results of different parameter values for Reduction (Algorithm 4) and Reward (Algorithm 5).

We also present the results of the new multimodal passenger assignment cost functions such as Weighted-Assortativity and Multimodal Reward with parameter combinations that gave the most distinct results compared to the basic Reduction and Reward.



(a) Results for Reduction

(b) Results for Reward

Figure 11: Objective function values of Reduction and Reward with different values of γ

Table 2: Experimental results of Reduction and Reward with different values of γ

Algorithm	Optimal cost	Average travel time (T_1)	Direct passengers	γ
Reduction in PTN	1721.60	22.115	1358	1
	1721.60	22.115	1358	5
	1772.35	22.115	1358	20
	1873.85	22.092	1357	50
	1823.55	22.123	1353	100
Reward in PTN	1619.80	22.022	1428	1
	1773.70	22.706	1313	3
	1722.35	22.602	1334	5
	1976.70	21.980	1373	10
	2180.15	23.665	1307	50

In Figure 11 and Table 2, we see that both Reward and Reduction approaches generally generate higher costs when increasing the model parameter that gives more emphasis to the model heuristic in contrast to the regular shortest paths. The result shows that the heuristics aimed to minimize the line costs to concentrate passengers on similar lines does not work as intended.

We present a simple explanation for this unexpected and unwanted behaviour. We noted that the lines in both of the line pools for our experiments were relatively short with the number of edges being at most 5. This suggests that regular shortest-path routes may be particularly attractive also in terms of cost minimization since line transfers even within a single modality are often mandatory. This is further supported by the fact that the number of direct passengers in each of the results is only roughly a half of the total passengers in the network (see Appendix A). The line pool generation may also support shortest paths approaches more since the loads for both modalities were created using shortest-paths passenger assignment with shortest paths OD-matrix

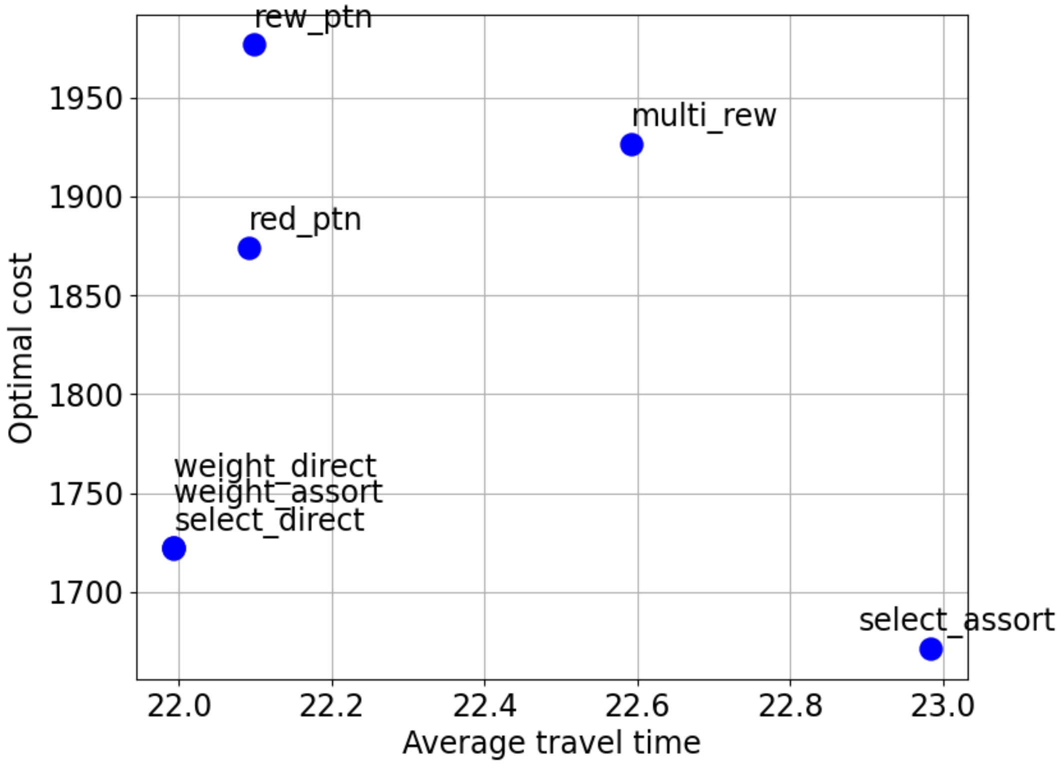


Figure 12: Objective function values of the new passenger assignment cost functions compared to basic Reduction and Reward

Table 3: Experimental results of the new passenger assignment cost functions. Multimodal Reward is used with Algorithm 5 and the rest of the cost functions are used with Algorithm 4

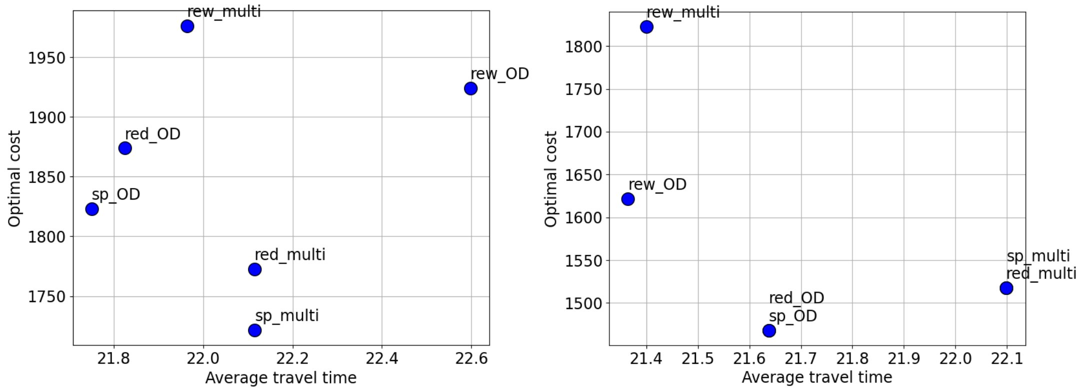
Algorithm	Optimal cost	Average travel time (T_1)	Runtime (s)	Direct passengers	Parameters
Weighted Assortativity	1722.50	21.99	247.85	1378	$\gamma = 10$, $\alpha = 500$
Selective Assortativity	1671.30	22.98	1329	1357	$\gamma = 10$, $\alpha = 500$, $\beta = 4.49$
Weighted Direct	1722.50	21.99	306.25	1378	$\gamma = 10$, $\alpha = 500$
Selective Direct	1722.50	21.99	130.55	1378	$\gamma = 10$, $\alpha = 500$, $\beta = 4.49$
Multimodal Reward	1926.40	22.59	2008.30	1325	$\gamma = 10$

splitting.

From Table 3 and Figure 12 we see that the new passenger assignment algorithms with selected parameters give different results in contrast to regular Reward and Reduction. Apart from Multimodal Reward, which underperforms in runtime and both objective function values, all methods give reasonable results using the selected parameters. We argue that most of the algorithms giving the same result is due to the network structure in our experiments. Since most of the new algorithms are based on the assortativity of the network, the grid-like network in our experiments with similar assortativity over all edges makes the different algorithms perform similarly in this case. It remains subject for future research whether these new algorithms are beneficial also in other settings, such as with larger graphs where the assortativity of the edges varies in different parts of the graph.

6.3 Performance analysis of OD-matrix splitting

Next, we compare the OD-matrix splitting approach (Algorithm 7) to the sequential multimodal line planning (Algorithm 6) and integrated line planning (27) - (33) experimentally. We use shortest paths OD-matrix splitting for each experiment. That is, we use the same set of modality-specific OD-matrices for each line planning algorithm such that in Algorithm 8, $N_{uv} = 1$ where all paths P^1_{uv} are shortest paths. We measure the gap between the optimal cost of the multimodal approach and the OD-matrix splitting approach for the shortest-paths passenger assignment and the integrated model with $\beta = 1$. We also show comparisons for other line planning algorithms to show the effect of OD-matrix splitting on the optimal cost and travel time.



(a) Results in PTN

(b) Results in Change&Go-network

Figure 13: Objective function values of multimodal approaches and OD-matrix splitting approaches in sequential line planning models

Table 4: Experimental results of the line planning models with OD-matrix splitting.

Algorithm	Optimal cost	Average travel time (T_1)	Runtime (s)	Direct passengers	Parameters
Shortest paths in PTN (sp_ptn)	1823.10	21.75	0.96	1406	-
Reduction in PTN (red_ptn)	1873.85	21.83	2.93	1394	$\gamma = 20$
Reward in PTN (rew_ptn)	1924.30	22.60	142.97	1356	$\gamma = 20$
Shortest paths in Change&Go-graph (sp_cg)	1468.30	21.64	5.62	1445	-
Reduction in Change&Go-graph (sp_cg)	1468.30	21.64	67.84	1445	$\gamma = 20$
Reward in Change&Go-graph (sp_cg)	1621.45	21.36	1600.22	1486	$\gamma = 20$
Integrated model 1 in PTN (integ_1)	1316.50	21.44	7.82	1482	$\beta = 1$
Integrated model 2 in PTN (integ_2)	1166.05	21.89	7.39	1438	$\beta = 2$

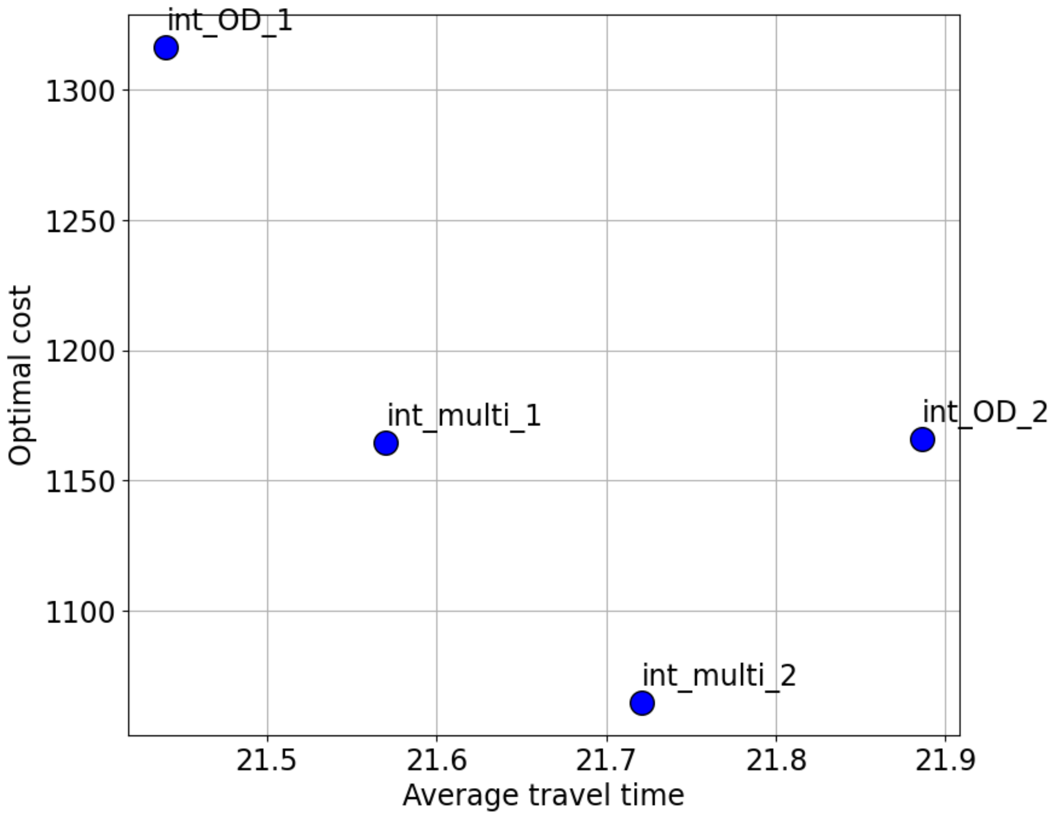


Figure 14: Objective function values of multimodal approaches and OD-matrix splitting approaches in integrated line planning models. The numbers 1 and 2 on the labels indicate the value of the β parameter in the integrated model.

From Figure 13 and Table 4 we see that the OD-matrix split with shortest paths in PTN gives roughly 100 units higher cost than the corresponding multimodal shortest-paths. We also see that both OD-matrix split integrated models give slightly higher costs than multimodal ones, as can be seen from Figure 14. From Table 4 and Table 1, we confirm that all OD-matrix split algorithms, except Reward in PTN, perform significantly faster in terms of runtime than the multimodal ones.

This result supports our theoretical results in the previous section. Given an OD-matrix split that allows the same passenger routes than in the multimodal algorithm with shortest paths, we can arrive at equal or higher cost using OD-matrix splitting whether the unimodal shortest paths are precisely the subpaths of the multimodal shortest paths. This results shows an important trade-off between the tractability (runtime) and the quality of the solution in terms of cost minimization. Overall, OD-matrix splitting can now be confirmed a great tool to obtain close to optimal solutions in multimodal line planning with significantly lowered runtime and increased tractability.

As can be seen from Table 1 and Table 4, the OD-matrix split approaches with shortest paths in Change&Go-network gives better performance in terms of both objective functions and runtime than the multimodal one. This result shows that splitting the multimodal OD-matrix using shortest paths in PTN environment makes the actual shortest paths assignment infeasible in the corresponding Change&Go-network. In other words, the shortest paths in PTN are not necessarily the same shortest paths in Change&Go-network and therefore the OD-matrix splitting approach does not allow the use of actual shortest paths in the Change&Go-network. This limitation can change the objective function values in the OD-matrix splitting approach and in our case, we indeed obtain better cost compared to the multimodal shortest paths. However, we recognize that the cost improvement in this case probably comes with an increase of actual travel time in the line concept, although the evaluated travel time seems to be decreasing. Since the travel times of the passenger assignments are higher in all of the OD-matrix split approaches, we argue that the evaluated travel times in Figure 13 and Table 4 are not sustainable with the capacity limitations of the actual line concepts.

6.4 Performance analysis of the integrated multimodal line plans

We proceed with the performance analysis of the multimodal integrated model (27)-(33). We investigate the effect of parameter β to the objective function values for the integrated case. We use T_2 for the average travel time evaluation to capture the effects of β on the travel time more realistically with the integrated model.

Figure 15 illustrates that increasing the β parameter in the multimodal integrated model decreases the cost of the line concept by increasing the average travel time for the passengers. This is clearly intended behavior as increasing the β parameter relaxes the required travel time constraint in the integrated model. Interestingly, the decrease in the optimal cost is most notable with only a subtle increase of β. Since travel time is not part of the objective, even small increase in β can prevent obtaining

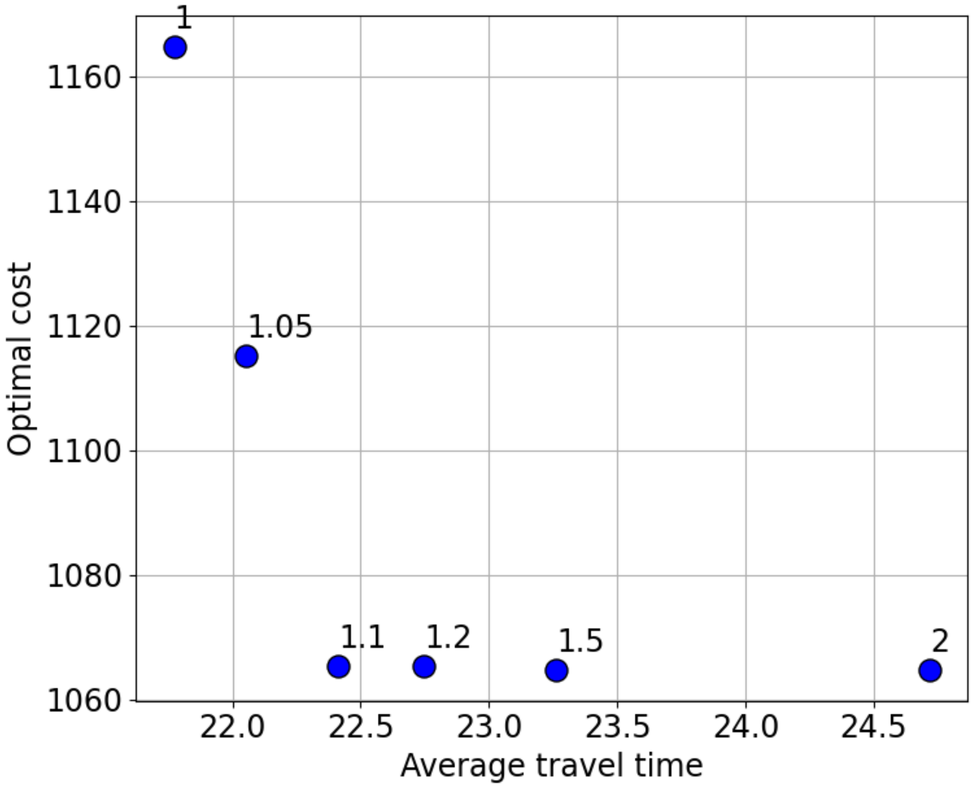
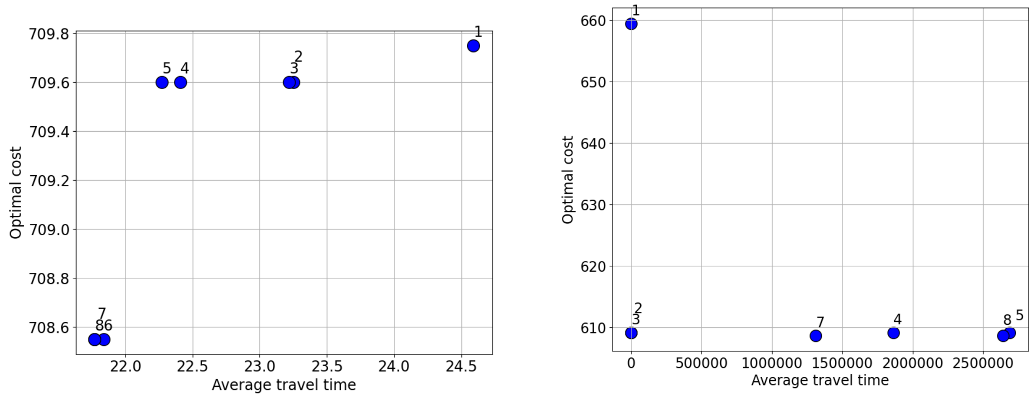


Figure 15: Objective function values of the multimodal integrated model with different parameter values

Table 5: Experimental results of the multimodal integrated model with different parameter values

Algorithm	Optimal cost	Average travel time (T_2)	Direct passengers	β
	1164.70	21.77	1464	1
	1115.15	22.05	1470	1.05
Integrated model in PTN	1065.45	22.41	1456	1.1
	1065.45	22.75	1446	1.2
	1064.85	23.26	1467	1.5
	1064.85	24.72	1451	2

optimal solutions with lower travel time. From Table 5, we see that after the point where $\beta = 1.1$, the decrease in the cost is really small, while the average travel still increases considerably. There is still room for future research to measure the effects of altering β for different problem parameters and different underlying PTN structures.



(a) Results with $\beta = 1$

(b) Results with $\beta = 100000$

Figure 16: Objective function values of multimodal integrated model for different number of transferable stops. Note that we always add stops to the previous set of transferable stops in a fixed order when increasing the number of transfer options. With high β, the entry 6 with infeasibly high travel time is removed from the plot for clarity.

6.5 Cost efficiency of integrated line planning

The final part of the experiments in this work is to examine the cost efficiency of integrated line planning with different numbers of transferable stops in the network. As Theorem 5.5 shows, the optimal cost of the integrated line planning problem (27) - (33) can only decrease when adding more transfer points to the existing set of transfer points, assuming β is sufficiently high. Using our example dataset, we test how much decrease there is for each added transfer point and investigate whether some stops provide more efficient solutions than other added stops.

We also test the effect of adding more transfer possibilities to the network when β is set lower, and show how the Objective function values change when adding more transfer stops. Again, we investigate whether adding certain stops give notable improvement to all of the objectives and see how different objective values vary with different sets of transfer stops. As an average travel time evaluation, we use T_2 to capture the changes in the travel time more realistically with the integrated model.

In this section, we use different problem parameters (Appendix A) than in the previous experiments to limit the number of reasonable line concepts in the problem. Namely, we use different capacities and different upper frequencies for the problem, but still use the same OD-matrix and PTN structure as in the previous section. The reason for this is to reduce the runtime for calculating the solutions, since the original parameters resulted in huge runtimes for obtaining the results. Moreover, we add transfer stops in a fixed order in each of the experiments to make sure that the results from different experiments are comparable.

Based on Figure 16 and Table 6, we confirm that increasing the number of stops can only improve the optimal cost for sufficiently high β, which is according to Theorem

Table 6: Experimental results of multimodal integrated model for different number of transferable stops

Algorithm	Optimal cost	Average travel time (T_2)	Direct passengers	Transfer stops
	709.75	24.59	1452	1
	709.60	23.25	1442	2
	709.60	23.22	1442	3
Integrated model in PTN (β = 1)	709.60	22.41	1442	4
	709.60	22.27	1406	5
	708.55	21.83	1442	6
	708.55	21.77	1442	7
	708.55	21.77	1442	8
	659.45	26.88	1421	1
	609.15	27.69	1288	2
	609.15	27.81	1288	3
Integrated model in PTN (β = 100000)	609.15	1861773.70	1288	4
	609.15	2686595.80	1272	5
	608.7	12246686.94	1271	6
	608.7	1311888.48	1310	7
	608.7	2639461.11	1310	8

5.5. On the other hand, we witness that adding more transfer options improves both objective functions when $\beta = 1$. This observation clearly shows that in our example network, the line concepts with low travel times also imply low costs for the line concept. Again, this implies dominance of passenger assignments with low travel times even for cost optimization, which could be due to shorter line lengths in the line pool.

From Figure 16 and Table 6, we also see that with high β the costs for all sets of transfer points are significantly lower than with any of the $\beta = 1$ transfer options. This is expected since higher β gives more emphasis on the cost minimization. Moreover, after 3 transfer points, we receive line concepts with impractically long travel times that could not be used in real life scenarios. This is known behavior as explained in the theoretical examples of [4]. With β sufficiently high, only the costs of the line concept are accounted in the model and the overall practicality of the solution is completely neglected.

The final key observation in this experiment is that for different values of β, the number of transfer options with the most significant cost increase is also different. Here we recognize that finding the most significant cost improvements using different model parameters and different transfer options depends mostly on the properties of the underlying PTN. Therefore, further research remains to investigate whether efficient transfer options for best possible solutions can be obtained analytically. For example, comparing the effects of model parameters against different transfer options for large number of different PTNs could reveal a connection between good parameter choices and properties of the transfer options. Moreover, separate heuristics for obtaining efficient transfer options could be investigated in the future.

7 Summary and Conclusions

In this work, we introduced a formal definition for multimodal public transport networks and an approach for constructing multimodal public transport networks from regular public transport networks with a single vehicle modality. We developed several cost-oriented line planning models and algorithms that support multimodal public transport networks in line planning and presented mathematical formulations for these models. In addition, we have introduced an OD-matrix splitting approach that can be used to split multimodal line planning problems into a set of unimodal problems to improve the computational performance in multimodal line planning.

The results show that the sequential line planning model with shortest paths passenger assignment and the integrated line planning model attain optimal line operating costs with a reasonable passenger travel time for the given problem instance. However, the alternate passenger assignments designed to lower the line operating costs produce generally higher or equal optimal costs than the regular shortest path approaches with altering passenger travel times. We argue that these results arise from the network properties and the properties of the line pools in our multimodal problem instance. We see that by altering model parameters, these models produce different distinct results in terms of different objectives in line planning. Therefore, the next steps should be taken to analyze the effects of multimodal network properties affecting the performance of different line planning models in terms of both the model parameters and problem parameters.

The results also show that by splitting the multimodal problems to sets of unimodal problems with OD-matrix splitting, we can significantly decrease the total runtime of the line planning procedure in multimodal line planning. We see that the significant increase in computational performance comes with a trade-off. The compound solutions of split problems generally give worse cost optima than their multimodal line planning counterparts. However, we also witness cases where the OD-matrix split limits the possible passenger assignments such that the split problems attain lower compound cost than their multimodal line planning counterpart with effects on the optimal travel time.

Our theoretical analysis shows that different OD-matrix splits have significant effects on the properties of split problems and that OD-matrix splitting can be considered an additional heuristic in the sequential line planning framework. The theoretical analysis also suggests that for each sequential multimodal line planning problem, there exists an OD-matrix split that gives compound solutions identical to the multimodal line planning problem. Future studies should be aimed to further investigate the effects of OD-matrix splits to the line planning problem, especially whether different types of splits can be used to obtain cost-optima with different emphasis on passenger objectives. Moreover, general approaches for obtaining suitable OD-matrix splits for multimodal problems to generate desired compound solutions should also be investigated.

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A Parameters for computational experiments

- Vehicle capacities

  $\mathrm{Cap}_{green} = 50$

  $\mathrm{Cap}_{red} = 60$

- Upper frequencies

  $U_{green} = 40$

  $U_{red} = 50$

- Max iterations for iterative passenger assignments

  $max_{iter} = 15$

- Modality transfer penalty

  $d_m = 10$

- Line transfer penalty

  $d_t = 6$

- Edge length

  $d_e = 6$

- Number of lines in the line pool

  $|\mathcal{L}^0_{green}| = 20$

  $|\mathcal{L}^0_{red}| = 15$

- Total number of passengers in the OD-matrix

  $\sum_{u,v \in S} W_{uv} = 2546$

Modified parameters for cost efficiency experiments in Section 6.5

- Vehicle capacities

  $\mathrm{Cap}_{green} = 100$

  $\mathrm{Cap}_{red} = 120$

- Upper frequencies

  $U_{green} = 20$

  $U_{red} = 25$