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Portfolio optimization under solvency requirements

An application to a Finnish pension fund

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Abstract

This thesis develops an optimization model that secures financing of future pension obligations of a pension provider in the Finnish regulatory framework. Due to the expected growth of the liabilities of pension funds this asset and liability management problem is increasingly relevant.

We consider and apply relevant risk measures from literature. A formulation based on the Conditional Value at Risk (CVaR) of the solvency capital of the pension fund was found to be the most applicable model. The solution significantly increased the efficiency of the portfolio compared to portfolio returns based CVaR optimization.

We use a scenario-based approach that incorporates the uncertainty of the market assumptions in the optimization. The scenarios are sampled from a multivariate normal distribution.

Because of the complexity of the problem we explore multiple models. All variations are based on how the solvency of the pension provider changes as a function of the portfolio allocation. The aim is to minimize the possibility of bearish scenarios where a fund is not able to meet the required capital. The portfolios are constrained with a solvency requirement to be admissible with the Finnish legislation.

The computational results highlight a trade-off between the complexity and reliability. While the complex statements are likely to reflect reality more accurately the solutions are inconsistent. A reason is that the optimization algorithms may have converged to local optima. Models are evaluated based on their consistency, convergence, objective function value and their practical feasibility.

Keywords Asset and liability management, pension funds, portfolio optimization, risk measures, scenario generation, stochastic optimization

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Tiivistelmä

Tässä diplomityössä kehitetään optimointimalli, joka turvaa työeläkevakuutusyhtiön vakavaraisuuden Suomen lainsäädännön mukaisesti. Työeläkevakuutusyhtiöiden vastuuvelkojen odotetun kasvun vuoksi tämä ongelma on ajankohtainen.

Työssä tarkastellaan ja sovelletaan kirjallisuuden pohjalta kuuluvia riskimittareita. Soveltuvimmaksi malliksi osoittautui vakavaraisuuspääoman ehdolliseen tappioarvoon (CVaR) perustuva formulointi. Ratkaisu lisäsi merkittävästi salkun tehokkuutta verrattuna salkun tuottoon perustuvaan CVaR-optimointiin.

Työssä käytetty skenaarioihin pohjautuva metodologia sisällyttää optimointiin markkinaoletuksiin liittyvän epävarmuuden. Skenaariot poimitaan moniulotteisesta normaalijakaumasta.

Ongelman monimutkaisuuden vuoksi arvioidaan erilaisia malleja, joista kaikki perustuvat siihen, miten työeläkevakuutusyhtiön vakavaraisuus muuttuu salkkupäätöksien mukaan. Pyrimme minimoimaan mahdollisuuden laskuskenaarioihin, joissa työeläkevakuutusyhtiö ei pysty täyttämään vaadittua pääomaa. Salkkuja rajoittaa vakavaraisuusvaatimus, jotta ne täyttävät Suomen lainsäädännön vaatimukset.

Ongelman muotoilun monimutkaisuuden ja tulosten luotettavuuden välillä on käänteinen yhteys. Vaikka monimutkaisilla formuloinneilla saatetaan kuvata todellisuus tarkemmin, niiden ratkaisut ovat epäjohdonmukaisia. Syynä on se, että käytetyt optimointialgoritmit ovat saattaneet konvergoitua paikallisiin optimeihin. Malleja arvioitiin niiden johdonmukaisuuden, konvergenssin, tavoitefunktion arvon ja käytännön toteutettavuuden perusteella.

Avainsanat Portfolio-optimointi, riskimittarit, skenaarioiden luonti, stokastinen optimointi, työeläkevakuutusyhtiö, varojen ja vastuiden hallinta

Contents

Symbols and abbreviations

Symbols

$\boldsymbol{x}$	Portfolio weights
$\boldsymbol{y}$	Return samples
C	Solvency capital
L	Solvency liability
R	Solvency ratio
A	Assets
P	Portfolio returns
r	Solvency requirement
ϱ	Risk class correlation matrix
Σ	Covariance matrix

Abbreviations

FTO	Fund transfer obligation
MVO	Mean-variance optimization
VaR	Value-at-Risk
CVaR	Conditional Value-at-Risk
ALM	Asset-liability management
CMA	Central market assumption
GA	Genetic algorithm

1 Introduction

Asset-liability management (ALM) is central in financial optimization, especially in the pension fund industry. This significance is highlighted by the societal reliance on these funds to fulfill future pension obligations. The essence of ALM lies in formulating a strategic approach to investments that aims to guarantee liability coverage across various time horizons.

This thesis focuses on how a pension fund can optimize its portfolio under the specific rules and conditions the Finnish pension fund system is subjected to. The aim is to find and compare optimization methodologies such that the objective matches the goals of a pension fund; the risk related to the market assumptions is considered in a comprehensive way; and the results are produced in a computationally feasible way.

The main contribution of the thesis is a stochastic programming model. It incorporates the explicit regulations a Finnish pension fund is subject to and minimizes the risk related to the solvency capital. This relies on a scenario-based approach which incorporates uncertainties related to the market assumptions in the optimization. The scenarios are generated by sampling asset class returns from a multivariate probability distribution.

The literature often proposes a multistage stochastic programming (MSP) model for the ALM problem of pension funds because financial results depend on investment returns as well as the decision policy over assets and liabilities. Oliveira et al. (2017) examine a multistage stochastic programming (MSP) model applied to the Brazilian pension fund industry. They apply a probabilistic value at risk constraint to obtain a positive funding ratio with high probability. Duarte et al. (2017) apply a linear stochastic programming model with a concave utility function for the risk aversion of insolvency. The methodology used in this thesis has similarities with these two papers.

The goal of a pension fund is to secure financing of its future pension liabilities. As a single stage optimization model, the objective of a pension fund can be formulated by optimizing the risk and return trade-off of the portfolio while maintaining the legally required solvency metrics. The aim is to minimize the risk related to the deviations within the solvency capital of the fund while maintaining a given level of expected return. Such model yields a set of portfolios that form a constrained efficient frontier. Using this approach the portfolio of a pension fund can be viewed as the allocation within this set that meets the target expected return of the fund. The objective function is linear which makes the model computationally efficient.

A single stage model is limited in that the target expected return of the portfolio needs to be given as an input parameter. For this reason, we explore formulations which minimize the risk of not meeting the future solvency requirements. A key consideration in formulating the problem statement is the reliability of the chosen risk measure and the convergence of the solution. The regulations imply a non-convex problem which may cause the algorithms to converge to local extrema. In navigating this difficulty, we examine multiple alternative problem statements. The aim is to start with a reliable simplification of the problem and then explore more complicated statements. In the more complicated formulations, the objective function is nonlinear, which makes the models computationally more demanding.

The optimization models are solved in three steps. The first step is to generate the asset class return scenarios. The second step is to model the liabilities. These are scenario-dependent, and they are affected by the combined portfolio returns of all private pension providers in Finland. The third step is to run the optimization algorithm to minimize the chosen objective function. The chosen time horizon is between 5 and 10 years.

The results of any portfolio optimization methodology depend on the choice of input values. The assumptions regarding the financial markets can change significantly in relatively short periods of time. Thus, the long term strategic allocation should be robust to small changes in the market assumptions and also the optimization framework should allow the modification of these values in an efficient way. Traditional portfolio optimization methods such as mean-variance have been criticized due to their lack of robustness. However, they provide important benchmarks for comparing the different methodologies.

In the real-world, the investors' goals and the market assumptions are uncertain and thus the choice for the objective function is not obvious. For example, should one prioritize highest returns, lowest risk, diversification, or liquidity. In this thesis, the focus is on the asset-liability metrics which are calculated based on the returns of the generated scenarios. In a practical sense, an optimal portfolio should be one that performs well across a variety of different metrics.

This thesis is structured as follows. Chapter 2 reviews the investment literature on portfolio optimization, risk measures and previous ALM-models. Chapter 3 discusses the Finnish regulatory framework. It also describes how the liabilities are modelled and the how the solvency requirement is accounted for in the optimization. Chapter 4 presents how the asset class returns are sampled from multivariate probability distributions. Chapter 5 discusses the problem statements and the algorithms for solving the problem. Chapter 6 discusses the results, including their sensitivity with regards to the inputs. Chapter 7 concludes.

2 Portfolio optimization theory

2.1 Modern portfolio theory

Modern portfolio theory or mean-variance (MVO) optimization provides guidance to assembling a portfolio of assets such that the expected return is maximized for a given level of risk. It was introduced in a Nobel prize winning article Markowitz (1952). It is a celebrated formalization of diversification in investing. The theory shows that by combining assets in a particular way one can produce a portfolio whose expected return reflects its components, but with considerably lower risk. The key idea is to construct a portfolio whose total return and risk ratio is higher than the individual assets. The risk of the portfolio is measured as the variance of the returns. Investors are assumed to prefer a less risky portfolio for a given expected return.

Let $\boldsymbol{r} \in \mathbb{R}^k$ be the vector describing the rate of return where k is the number of assets. For each asset $i \in k$ the expected return μ is defined as $\mu_i = E(r_i)$. Let $\boldsymbol{x} \in \mathbb{R}^k$ denote the portfolio weights. The rate of return of the portfolio is $\boldsymbol{x}^T\boldsymbol{r}$ with mean $\boldsymbol{\mu}^T\boldsymbol{x}$ and variance $\boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x}$, where $\boldsymbol{\Sigma}$ is the covariance matrix of the rates of returns.

The quadratic form of the mean-variance problem is defined as

$$\min_{x \in \mathbb{R}^k} \; \boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x} \tag{1}$$

$$s.t. \; \boldsymbol{x}^T\boldsymbol{\mu} = R \tag{2}$$

$$\boldsymbol{x}^T\boldsymbol{e} = 1, \tag{3}$$

where $\boldsymbol{e}$ is a vector of ones and R is a fixed level of required portfolio return.

When only equality constraints are applied the problem can be solved analytically. In practice, the mean-variance formulation is often used with inequality constraints such as long only constraint ($x_i \geq 0, \forall i$) and bounds for specific assets. Then the problem can be solved with numerical optimization methods.

The set of optimal portfolios that are obtained when the mean-variance problem is solved as a function of R define the *efficient frontier* which is is typically presented in the mean-volatility space.

Even though the mean-variance optimization is a widely accepted in investment theory, applying it in practice can be problematic, especially for a pension fund. Without the short short selling constraint, it is likely to set large negative weights on some assets which are not available for many institutional investors. On the other hand, with the short selling constraint, the resulting portfolios may be undiversified corner solutions. The lack of diversification stems from the assumption of perfect certainty regarding the input parameters. This assumption is unrealistic since the parameters involve significant uncertainties.

The sensitivity of the mean-variance portfolio has been studied thoroughly in the literature. Michaud and Michaud (2008) and Best and Grauer (1991) empirically show that even small errors in the estimation of an assets expected return might have the effect to disregard it completely from the efficient frontier. The instability of the resulting portfolio over time has also been shown by DeMiguel and Nogales (2009). In practice this makes it challenging to use MVO because the market assumption typically change in short timeframes. Following the MVO portfolio would then likely lead to excessive amount of trading costs. Merton (1980) concludes that the instability is mostly due to the estimation errors regarding the expected return. Chopra and Ziemba (1993) show that the errors within the expected return estimation have more than a ten fold difference compared to the errors within the variances or the covariances.

2.2 Conditional Value-at-Risk optimization

Conditional value-at-risk (CVaR), also known as mean excess loss or tail value-at-risk, is a coherent risk measure (Rockafellar and Uryasev 2000). A function $\varsigma : L^\infty \to \mathbb{R}$ is said to be coherent if it satisfies the following properties for a set of random variables $z \in L^\infty$ representing financial positions:

1. **Normalized:** $\varsigma(0) = 0$
2. **Monotonicity:** Consider two random variables, z_1 and z_2, e.g., the returns from two portfolios. If $z_2 > z_1$ in all feasible future states then $\varsigma(z_2) < \varsigma(z_1)$. This imples that a financial security that always has higher return in all future states has less risk of loss.
3. **Sub-additivity:** $\varsigma(z_1 + z_2) \leq \varsigma(z_1) + \varsigma(z_2)$.
4. **Positive homogeneity:** if $\lambda > 0$, $\varsigma(\lambda z_1) = \lambda\varsigma(z_1)$.
5. **Translation invariance:** For any random variable z the addition of additional outcome with a certain positive return a will reduce the risk by that amount, $\varsigma(z + a) = \varsigma(z) - a$.

CVaR can be viewed as a variant of the popular Value-at-risk (VaR) measure. The β-VaR of a portfolio is the lowest amount α such that, with probability β, the loss will not exceed α. The β-CVaR is the conditional expectation of losses above the amount α. Informally, and non-rigorously this means saying that "what is the average loss in cases so severe they occur only β percent of the time" Based on their definitions a low β-CVaR portfolio must have a low β-VaR as well. The most commonly used β values are 0.9, 0.95 and 0.99.

It has been shown that CVaR has better characteristics than VaR (Artzner et al. 1999). VaR lacks desirable mathematical characteristics such as subadditivity and convexity. It is coherent only when it is based on standard deviations of normal distributions (McKay and Keefer 1996). In the minimization of VaR, there can be multiple local extrema, which makes its use unpredictable. Pflug (2000) proved that CVaR has the following properties: transition-equivariant, positively homogeneous, convex, monotonic w.r.t. stochastic dominance of order 1, and monotonic w.r.t. monotonic dominance of order 2.

Following the notations of Rockafellar and Uryasev (2000), let $f(\boldsymbol{x}, \boldsymbol{y})$ be the loss function associated with decision vector $\boldsymbol{x} \in \mathbb{R}^n$ and random vector $\boldsymbol{y} \in \mathbb{R}^m$. The decision vector represents the portfolio weights that are chosen from the set of available portfolio weights $X \in \mathbb{R}^n$ and the vector $\boldsymbol{y}$ stands for the uncertainties that impact the loss function. For each $\boldsymbol{x}$, the loss $f(\boldsymbol{x}, \boldsymbol{y})$ is a random variable having a distribution in $\mathbb{R}$ induced by $\boldsymbol{y}$. Let $\rho(\boldsymbol{y})$ represent the underlying probability density function of $\boldsymbol{y} \in \mathbb{R}^m$. It is not necessary to have an analytical expression for the density. It it sufficient to generate random samples from $\rho(\boldsymbol{y})$. The procedure for the sampling is described in Chapter 4.

The probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding a threshold α is given by

$$\psi(\boldsymbol{x}, \alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y}) \le \alpha} \rho(\boldsymbol{y})\, d\boldsymbol{y}. \tag{4}$$

For a given $\boldsymbol{x}$, Equation (4) represents the cumulative distribution function of the loss function which is fundamental in defining both Var and CVaR measures. We assume that it is non-decreasing and continuous everywhere with respect to α. In reality, $\psi(\boldsymbol{x}, \alpha)$ is continuous from the right but may not be from the left if there are discontinuous jumps in the distribution.

Let the β-VaR and β -CVaR values for the random variable representing the loss be denoted as $\alpha_\beta(\boldsymbol{x})$ and $\phi_\beta(\boldsymbol{x})$. They are defined as

$$\alpha_\beta(\boldsymbol{x}) = \min \{\alpha \in \mathbb{R} : \psi(\boldsymbol{x}, \alpha) \ge \beta\} \tag{5}$$

and

$$\phi_\beta(\boldsymbol{x}) = \int_{f(\boldsymbol{x},\boldsymbol{y}) \geq \alpha_\beta(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y})\rho(\boldsymbol{y})\, d\boldsymbol{y} \tag{6}$$

Equation (5) for VaR describes the left endpoint of the nonempty interval consisting of the values α such that $\psi(\boldsymbol{x},\alpha) = \beta$. This follows directly from the assumption that $\psi(\boldsymbol{x},\alpha)$ is continuous and non-decreasing with respect to α. In case ψ contains flat spots, the interval may contain more than a one single point. The probability that $f(\boldsymbol{x},\boldsymbol{y}) \geq \alpha_\beta(\boldsymbol{x})$ in Equation (6) is equal to $1-\beta$. This means that $\phi(\boldsymbol{x})$ is the conditional expectation of the loss associated with $\boldsymbol{x}$ relative to that loss being greater or equal to $\alpha_\beta(\boldsymbol{x})$. The key approach for the CVaR is to characterize $\alpha_\beta(\boldsymbol{x})$ and $\phi_\beta(\boldsymbol{x})$ in terms of a function F_β on $X \times \mathbb{R}$. It is defined as

$$F_\beta(\boldsymbol{x},\alpha) = \alpha + (1-\beta)^{-1} \int_{y\in\mathbb{R}^m} [f(\boldsymbol{x},\boldsymbol{y}) - \alpha]^+ \rho(\boldsymbol{y})\, d\boldsymbol{y}, \tag{7}$$

where

$$[t]^+ = \begin{cases} t, & \text{if } t \geq 0\,, \\ 0, & \text{if } t \leq 0\,. \end{cases}$$

The definition of $F_\beta(\boldsymbol{x},\alpha)$ can be approximated by sampling the probability distribution of $\boldsymbol{y}$ based on to its density ρ. Let the collection of vectors that the sampling generates be $\boldsymbol{y_1,...,y_q}$. The approximation to $F_{\boldsymbol{x},\alpha}$ can be stated as

$$\tilde{F}(\boldsymbol{x},\alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \left[f(\boldsymbol{x},\boldsymbol{y}_k) - \alpha\right]^+. \tag{8}$$

This expression for $\tilde{F}(\boldsymbol{x},\alpha)$ is convex and piecewise linear with respect to α. Though it is not differentiable with respect to α, it can be minimized with various optimization algorithms.

Under the above assumptions, Rockafellar and Uryasev (2000) provide the following two theorems.

Theorem 1 *$F_\beta(\boldsymbol{x},\alpha)$ is convex and continuously differentiable as a function of α. The β-CVaR of the loss associated with any $\boldsymbol{x} \in X$ can be determined from*

$$\phi_\beta(\boldsymbol{x}) = \min_{\alpha\in\mathbb{R}} F_\beta(\boldsymbol{x},\alpha). \tag{9}$$

The set consisting of the values of α for which the minimum is attained, given by

$$A_\beta(\boldsymbol{x}) = \underset{\alpha\in\mathbb{R}}{\operatorname{argmin}}\, F_\beta(\boldsymbol{x},\alpha), \tag{10}$$

is a nonempty closed bounded interval. The β-VaR of the loss is given by

$$\alpha_\beta(\boldsymbol{x}) = \text{left endpoint of } A_\beta(\boldsymbol{x}). \tag{11}$$

In particular, it holds that

$$\alpha_\beta(\boldsymbol{x}) \in \underset{\alpha\in\mathbb{R}}{\operatorname{argmin}}\, F_\beta(\boldsymbol{x},\alpha) \tag{12a}$$

$$\phi_\beta(\boldsymbol{x}) = F_\beta(\boldsymbol{x},\alpha_\beta(\boldsymbol{x})). \tag{12b}$$

Theorem 2 *Minimizing the β-CVaR of the loss associated with $\boldsymbol{x}$ over all $\boldsymbol{x} \in X$ is equivalent to minimizing $F_\beta(\boldsymbol{x},\alpha)$ over all $(\boldsymbol{x},\alpha) \in X \times \mathbb{R}$.*

$$\min_{x\in X} \phi_\beta(\boldsymbol{x}) = \min_{(\boldsymbol{x},\alpha)\in X\times\mathbb{R}} F_\beta(\boldsymbol{x},\alpha). \tag{13}$$

in Equation (13) a pair $(\boldsymbol{x}^*,\alpha^*)$ achieves the second minimum if and only if $\boldsymbol{x}^*$ achieves the first minimum and $\alpha^* \in A_\beta(\boldsymbol{x}^*)$. In case where the interval $A_\beta(\boldsymbol{x}^*)$ reduces to a single point, the minimization of $F(\boldsymbol{x},\boldsymbol{\alpha})$ over $(\boldsymbol{x},\alpha) \in X \times \mathbb{R}$ produces a pair $(\boldsymbol{x}^*,\alpha^*)$, which is not necessarily unique, such that $\boldsymbol{x}^*$ minimizes the β-CVaR and α^* gives the corresponding β-VaR. F_β is convex with respect to $(\boldsymbol{x},\alpha)$, and $\phi_\beta(\boldsymbol{x})$ is convex with respect to $\boldsymbol{x}$, when $f(\boldsymbol{x},\boldsymbol{y})$ is convex with respect to $\boldsymbol{x}$, in which case, if the constraints are such that X is a convex set, the joint minimization is an instance of convex programming.

The proofs of Theorems (1) and (2) are in Rockafellar and Uryasev (2000). The usefulness of the formulas in Theorem (1) is that continuously differentiable convex functions can be minimized numerically. Also, the β-CVaR can be calculated without first having to explicitly calculate the β-VaR based on its definition. It can be obtained as the by-product of the methodology, even though the extraction of it requires extra effort in determining the left endpoint of $A_\beta(\boldsymbol{x})$.

Theorem (2) states that for the purpose of determining an $\boldsymbol{x}$ that minimizes β-CVaR it is not needed to work directly with the function $\phi_\beta(\boldsymbol{x})$. This is useful because the value $\alpha_\beta(\boldsymbol{x})$ may have troublesome mathematical properties. The minimization of $F_\beta(\boldsymbol{x},\alpha)$ is in the category of stochastic optimization because of the presence of a expected value in the definition of $F_\beta(\boldsymbol{x}, \alpha)$.

To demonstrate the CVaR optimization and its connection to MVO, let us consider a case where the decision vector $\boldsymbol{x}$ represents portfolio $\boldsymbol{x} = (x_1, ..., x_n)$ where x_j is the position in asset j, expressed as its share of the portfolio. For simplicity, we exclude short positions and assume the assets sum up to one. That is

$$x_j \geq 0, \text{ for } j = 1, ..., n, \text{ with } \sum_{j=1}^{n} x_j = 1. \tag{14}$$

The random vector that represents the return on each asset j is denoted as $\boldsymbol{y} = (y_1, ..., y_n)$. The distribution of $\boldsymbol{y}$ constitutes a joint distribution of the returns for each of the assets and is independent of $\boldsymbol{x}$. It has density $\rho(\boldsymbol{y})$.

The portfolio returns are the sum of the returns of the individual assets in the scaled by the proportions x_j. The loss function is given by

$$f(\boldsymbol{x}, \boldsymbol{y}) = -[x_1 y_1 + \cdots + x_n y_n] = -\boldsymbol{x}^T \boldsymbol{y}. \tag{15}$$

The loss associated with $\boldsymbol{x}$ will be continuous if $\rho(\boldsymbol{y})$ is continuous with respect to $\boldsymbol{y}$ (Kibzun and Kan 1996). In this case, the objective function we focus in connection with the β-VaR and β-CVaR is

$$F_\beta(\boldsymbol{x}, \alpha) = \alpha + (1 - \beta)^{-1} \int\limits_{y \in \mathbb{R}^m} \left[-\boldsymbol{x}^T \boldsymbol{y} - \alpha\right]^+ \rho(\boldsymbol{y})\, d\boldsymbol{y}. \tag{16}$$

Let $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ denote the mean and variance of the return of the portfolio $\boldsymbol{x}$. We can define these in terms of the mean $\boldsymbol{m}$ and covariance V of the vector of asset returns $\boldsymbol{y}$ as

$$\mu(\boldsymbol{x}) = -\boldsymbol{x}^T \boldsymbol{m} \tag{17}$$

$$\sigma^2(\boldsymbol{x}) = \boldsymbol{x}^T V \boldsymbol{x} \tag{18}$$

A constraint that only portfolios that can be expected to return at least a given amount R will be allowed can be added to the optimization. This is

$$\mu(\boldsymbol{x}) \leq -R \tag{19}$$

An approximation of F_β by sampling the probability distribution $\boldsymbol{y}$ yields the function

$$\tilde{F}(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \left[-\boldsymbol{x}^T \boldsymbol{y} - \alpha\right]^+. \tag{20}$$

There is a connection to the minimization the variance

$$\min \sigma^2(\boldsymbol{x}) \text{ over } \boldsymbol{x} \in X. \tag{21}$$

to the minimization β-CVaR and β-VaR. Rockafellar and Uryasev (2000) prove the following proposition

Proposition 1 *Suppose that the loss associated with each $\boldsymbol{x}$ is normally distributed, as holds when $\mathbf{y}$ is normally distributed. If $\beta \geq 0.5$ and the constraint (19) is active in any two of the problems of minimizing variance, VaR, CVaR, then the solutions to those two problems are the same; a common portfolio $\boldsymbol{x}^*$ is optimal by both criteria.*

In an empirical experiment, Rockafellar and Uryasev (2000) reached an accuracy of less than 1% difference between the VaR, CVaR and the minimum variance approaches with using a sample size of 10 000.

2.3 Expected shortage

The expected shortage is a less common risk measure. It is closely related to the CVaR. The expected shortage $ES(\eta, \boldsymbol{x})$ is the conditional expectation of the loss function above a pre-fixed value η. The shortfall probability $\zeta(\eta, \boldsymbol{x})$ is the probability that the loss function is above the given level η. The expected shortage is defined as (see Conejo et al. 2010)

$$ES(\eta, \boldsymbol{x}) = \eta + (1 - \zeta(\eta, \boldsymbol{x}))^{-1} \int\limits_{y \in \mathbb{R}^m} [f(\boldsymbol{x}, \boldsymbol{y}) - \eta]^+ \rho(\boldsymbol{y})\, d\boldsymbol{y}. \tag{22}$$

The definition is similar to the definition of CVaR in Equation (7). Except that instead of having a pre-fixed probability β that determines the amount α, there is a pre-fixed value η that determines the shortfall probability $\zeta(\eta, \boldsymbol{x})$.

Ogryczak and Ruszczyński (1999) consider expected shortage an intuitive multidimensional risk measure, which has the most general dominance relation for all risk-averse preferences. They state that the risk measure has the following properties for a convex loss function: continuous, convex, nonnegative and nondecreasing $\forall \eta \geq \eta_0$. η_0 describes the smallest value of the loss function.

The upside compared to CVaR is that it focuses on a specific benchmark which may not be the tail-end scenarios. It is context dependent which risk measure is most suitable. The use of a fixed target η causes the expected shortage not to satisfy coherence (Conejo et al. 2010).

2.4 Asset and liability management models

Most of previously built ALM models do not consider regulatory requirements explicitly. The regulations are in the center of the decision making process of a pension fund. Thus, it is relevant to include these to the optimization model.

Multistage stochastic optimization models have often been applied to the pension fund industry. Gülpinar and Pachamanova (2013) present an ALM model with time-varying investment opportunities. They numerically compare robust-optimization-based strategies to the classical stochastic programming approach. Yao et al. (2013) investigate a continuous-time mean-variance ALM problem.

Oliveira et al. (2017) present a multistage scenario-based procedure for the ALM problem. They maximize the terminal value of the fund's portfolio and apply a VaR constraint to maintain the required solvency level with high probability. The solution is obtained as the average weights for the first time step. This method is based on the Michaud and Michaud (2008) resampling solution. Because the weights are solved for each scenario separately, they approximate the optimal solution, but thus is not a stochastic program per se.

Ferstl and Weissensteiner (2011) consider a multi-stage setting under time-varying investment opportunities and apply a CVaR optimization to the ALM problem of a pension fund. They use a first-order unrestricted vector autoregressive process to model asset returns and state variables and Nelson/Siegel parameters to account for the change in the yield curve.

Duarte et al. (2017) consider the regulatory framework in the Solvency II project. They offer the closest match to the goal of this thesis which is to find the porfolio that minimizes the risk of insolvency in future timesteps within the regulatory framework of Finland. They define the objective function as

$$\max \sum_{s\in S} p_s \left[\sum_{t\in \tau} \left(N_t(s) - K_t^{min}(s) - \sum_{i\in I} \theta E_t^i(s) \right) d_t(s) \right] \tag{23}$$

which consists of two portions: the assets N above the minimum requirement K^{min} $(N_t(s) - K_t^{min}(s))$ and the penalty for different levels $i \in I$ of insolvency $(\sum_{i\in I} \theta E_t^i(s))$.

The parameter d is the discount factor that is employed to evaluate all stages and scenarios $s \in q$ on the same basis. E_t^i is an auxiliary variable to identify the values of insolvency in each stage and scenario that exceeds the levels $\varphi_1 = 0, \varphi_2 = 0.5, \varphi_3 = 0.7$. It is defined as $\forall i \in I, t \in T \text{and} s \in q$

$$E_t^i(s) = \left[(1 - \varphi)K_t^{min}(s) - W_t(s)\right]^+, \tag{24}$$

where W is the net worth of the fund. Because the objective function in Equation (23) contains both the net assets and the penalty for the insolvency, the solution is a function of the utility parameter θ. This approach leaves the choice for the risk aversion unaddressed. They define values for the penalty parameter as $\theta_1 = 0.1, \theta_2 = 0.1, \theta_3 = 0.2$.

The modelling of the net assets N and the minimum capital requirement K^{min} is based on the Brazilian legislation. The function for the K^{min} is nonlinear which they simplify by first-order Taylor approximation. They simplify the problem to a linear program which makes it computationally efficient.

The solution to the objective function Equation (23) yields a portfolio that maximizes a utility function which accounts for the assets above the minimum requirement and the penalty for insolvency. It is not clear how the penalty parameter should be defined. Furthermore, for the goal of securing the financing of the pension liabilities, it is not obvious why the bonus for the excess assets above the minimum requirement is necessary.

3 Finnish regulatory framework

In Finland the private sector earnings-related pension has been decentralized to pension insurance companies, company pension funds and industry-wide pension funds. The private-sector pension assets totaled 150 billion euros at the end of 2022 of which 95.6% is administered by pension providers. The investment portfolios of individual pension providers differ considerably in size. The investments of all pension providers are distributed among four primary pension providers, of which two managing approximately 35% of the total assets each, the third oversees 25%, and the fourth handles the remaining 5% (*Pension Assets (Private Sector)* 2024).

The earnings-related pension scheme is a defined and partially funded scheme. The partial funding has been realized so that each individual pension is divided into a funded and an unfunded component. The funded components are the responsibility of the pension providers.

Private sector solvency regulations are set in place because the pension providers are liable for the funded pension components with their assets. They also engage in mutual competition with investment profits, although the system includes a joint and several liability. This means that the assets and liabilities of a pension provider that goes bankrupt are shared between the others. The solvency regulations prevent competition with excessive risk-taking since the responsibility for the mitigation of large risks is shared among all parties in the system.

Pension providers invest the funded components of earnings-related pension contributions as securely and efficiently as possible to ensure that there are enough assets when the accrued pensions are to be paid out in the future. The pension providers cover the pension expenditure that is not funded in advance with earnings-related pension contributions accumulated in the year that the pension is paid.

The investment operations of pension providers in the private sector are tightly regulated. Pension providers have accumulated technical reserves for future pensioners which are calculated according to actuarial principles. Private sector pension providers compete not only in efficiency but also in terms of investment profits and services for the policyholders. As a result, there are system-level conditions and limitations regarding the investment operations and related risks, as well as the solvency, of pension providers.

Pension providers have an obligation to transfer assets into funds. This way, the investment return becomes part of the funding of pensions paid under these pension acts and, later, of the actual pensions paid. Pension providers meet their fund transfer obligation with their investment return. If the investment return is not enough to cover the obligation, it must cover the deficit from its solvency buffers. On the other hand, if the investment return exceeds the transfer obligation, the pension provider's solvency buffer increases and the provider can take bolder investment risks, since it is also better equipped to handle investment losses.

The solvency of pension providers is assessed by using various metrics, the most important of which are solvency capital, solvency requirement, solvency position and solvency ratio.

Pension providers can prepare for investment and underwriting risks by using their solvency capital. Solvency capital includes, among other things, equity, valuation differences on investments and buffer funds for risks. In order for a pension provider to be solvent, its solvency capital must exceed the solvency requirement. The solvency requirement is the value calculated based on the structure of the pension provider's investment portfolio, to which the solvency capital is compared. The general rule is that the greater the risks of investments, the higher the solvency requirement required by the regulations. For calculating the solvency requirement, each pension provider must identify the risks in each investment. The regulations on the solvency limit are common for all private sector pension providers. Figure 1 by *Pension Assets (Private Sector)* (2024) illustrates how the solvency framework fits with the total assets of a pension provider.

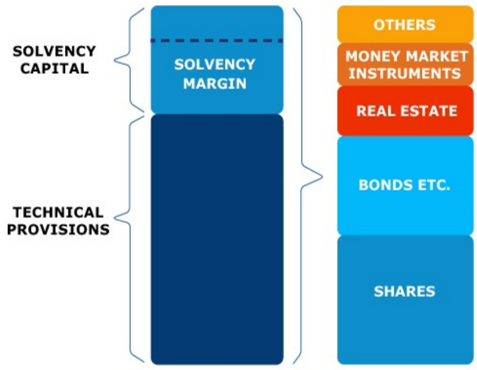

Figure 1: Illustration of the solvency framework of a pension provider

Because the solvency capital must exceed the solvency requirement, each pension provider needs to minimize the risk of violating the constraint. Solvency position describes the ratio between the solvency capital and the solvency requirement. It can be modelled as a bankruptcy if the solvency position of a pension provider falls below one. However, in reality extra restrictions are imposed on the pension provider's activities and bankruptcy occurs after some time has been spent under the limit.

The solvency requirement also depends on the solvency ratio. The current regulation makes it risky for an individual pension provider to deviate much from the system. The liabilities of each provider depend on the average solvency of the system. If the average solvency of the system is significantly higher than the solvency of an individual company the liabilities can grow at a faster pace than the assets. This is due to the limitations on the risk-taking abilities for the company. Furthermore, if the market follows the expected path, this inevitably leads to a downward loop for the solvency of the company, which leads to having a smaller solvency capital than the margin.

3.1 Liability modelling

The liabilities are modelled in two separate ways for the system of private pension providers in Finland on one hand, and the individual company that the optimization is performed on on the other hand. In reality, each company represents a part of the system. However to reduce the complexity and computation time, we assume the solvency of the system and the solvency of the individual pension fund are independent of each other.

The modelling of the system requires assumptions about the weighted average portfolio of the pension providers in Finland. This can be estimated with sufficient accuracy from the annual financial statements of the companies.

The solvency of the system p is defined as

$$p = \sum_f w_f p_f, \tag{25}$$

where

$$p_f = \max\{\frac{C_f}{L_f}, 0.1\}, \tag{26}$$

$$w_f = \frac{\min\{0.15, \frac{L_f}{\sum_f L_f}\}}{\sum_f \min\{0.15, \frac{L_f}{\sum_f L_f}\}} \tag{27}$$

and f represents an individual pension fund. Equation (27) limits the maximum weight that a particular company can have to the system solvency p to 25 %.

Equity-linked buffer fund (EBF) is used to provide for increased investment risks. The amount of this buffer depends on the pension providers' average returns on listed equities. Let $z_{i,t}$ denote the assumed weight the system has on the asset i at time t and $y_{i,t}$ the sampled returns from $\rho(\boldsymbol{y}_t)$. Then, the equity-linked buffer fund is defined as

$$\text{EBF}_t = (1 + y_{e,t}) - 1.01, \tag{28}$$

where $\boldsymbol{y_e}$ is the equity return of the assumed system portfolio. Because equities are modelled as one asset class, the multiplier for the $\boldsymbol{y_e}$ is one. If equities were examined with a larger universe containing more assets (e.g. US equities as e_1 and European Equities as e_2). The EBF would be

$$\text{EBF}_t = (1 + \frac{\sum_{i=e_1}^{e_n} z_{i,t} y_{i,t}}{\sum_{e_1}^{e_n} z_{i,t}}) - 1.01. \tag{29}$$

The buffer fund for pooled pension components is supplemented annually by the technical rate of interest $b16$. It is defined as

$$b16 = \begin{cases} 0.36(1-\lambda)p - 0.057, & \text{if } p \leq 0.198\,, \\ 0, & \text{if } 0.198 \leq p \leq 0.218\,, \\ 0.15(1-\lambda)p - 0.057, & \text{if } p \leq 0.198\,. \end{cases} \tag{30}$$

The parameter λ is set to be 0.2 by the Finnish law and the discount rate i_0 to 0.03 (Tela 2023). The fund transfer obligation (FTO) is the multiplier at which the liabilities of the pension funds change between consecutive time steps. It is defined as

$$\text{FTO} = b16 + i_0 + \lambda\text{EBF}. \tag{31}$$

We denote the part of the transfer obligation that does not include the technical rate of interest TO. It is defined as

$$\text{TO} = i_0 + \lambda\text{EBF}. \tag{32}$$

The recursive formulas for the solvency metrics for the system are

$$A_t = A_{t-1}P_t, \tag{33}$$

$$P_t = \sum_i z_{i,t}y_{i,t}, \tag{34}$$

$$p_t = \frac{A_{t-1}}{L_{t-1}} - 1, \tag{35}$$

$$\text{EBF}_t = (1 + \boldsymbol{z_{e,t}y}_{e,t}) - 1.01, \tag{36}$$

$$b16_t = \begin{cases} 0.36(1-\lambda)p_t - 0.057, & \text{if } p_t \leq 0.198\,, \\ 0, & \text{if } 0.198 \leq p_t \leq 0.218\,, \\ 0.15(1-\lambda)p_t - 0.057, & \text{if } p_t > 0.218\,, \end{cases} \tag{37}$$

$$\text{FTO}_t = b16_t + i_0 + \lambda\text{EBF}_t, \tag{38}$$

$$\text{TO}_t = i_0 + \lambda\text{EBF}_t, \tag{39}$$

$$L_t = (1 + \text{FTO}_t)l_{t-1}. \tag{40}$$

These solvency metrics can be calculated based on the sampled returns $\boldsymbol{y}$ and the assumed system portfolio $\boldsymbol{z}$. They are input values for the optimization model for the individual company.

3.2 Solvency requirement

For the calculation of the solvency requirement, also known as margin or limit, the Finnish law defines explicit formulas and necessary input values (Finlex 2015). The law ensures that each company has the requisite funds and level of diversification in their portfolio.

The exact calculation of the solvency requirement is computationally intensive and approximations are needed to reduce the computation time. Options are examined through their delta-hedged exposure value instead of market value. This approximates the asset's value change in response to small movements in the underlying asset's price. The computations with regard to options become far less complex and time consuming while still maintaining reasonable level of accuracy. Typically the solvency requirement is calculated based on individual investments. As the goal is to optimize a portfolio, the requirement is calculated based on asset classes instead of individual investments. The law defines eighteen risk classes which serve as the basis for calculating the requirement. Each asset class in the portfolio can be exposed to multiple risk classes. The exposures that each asset class has to a specific risk class can be calculated with a matrix E which is defined such that $E_{i,j}$ is the fraction of weight that asset class i has to the risk class j.

For each risk class j, the law provides the default loss S_j and the expected return m_j which are shown in Table 2. A constant describing the risk of debt τ has been set to 3. Let us denote the correlation matrix between the risk classes as ϱ and the exposure to a given risk class as a. The portion which is debt on a given investment is denoted as l. Table 3 shows the values of ϱ.

To calculate the requirement, one has to solve the stress value vector $\boldsymbol{V} \in \mathbb{R}^j$ and the expected return $\boldsymbol{\mu} \in \mathbb{R}^j$. With a few exceptions these are defined as

$$V_j = \sum_i a_{i,j} \min\{(1 + \tau L_i)S_j, 1\}, \tag{41}$$

$$\mu_j = \sum_i (m_j + L_i(m_j - m_6))a_{i,j}. \tag{42}$$

The risk classes 6–10 and 15–16 are calculated with the modified formulas

$$m_6^* = m_6 d_i^{\gamma}, \tag{43}$$
$$\mu_6 = \sum_i (m_6^* + L_i(m_6^* - m_6))a_{i,j}, \tag{44}$$
$$S_6^* = S_6 d_i - m_6^*, \tag{45}$$
$$V_6 = \sum_i a_{i,j} \min\left[(1+\tau L_i)S_6^*; 1\right], \tag{46}$$
$$V_k = \sum_i a_{i,k} \min\left[(1+\tau L_i)(S_k d_i - (m_k + L_i(m_k - m_6))), 1\right] \tag{47}$$
$$k \in [7, 8, 9, 10], \tag{48}$$
$$V_{15} = l(i_0 + b16 + C - \lambda S_{15}), \tag{49}$$
$$\mu_{15} = -l(i_0 + b16 + D + \lambda m_j), \tag{50}$$
$$V_{16} = \Lambda m_{16}, \tag{51}$$
$$\mu_{16} = \Lambda \min[S_{16}, 1]. \tag{52}$$
$$\tag{53}$$

The constant m_6 describes the expected return of the interest rate risk class. Table 1 lists the input parameters. The value-column describes whether the parameter is a constant or needs to be given as an input value. That is, the values are not determined explicitly by the law and are fund specific.

The solvency requirement r is defined as

$$r = \sqrt{(\boldsymbol{V}+\boldsymbol{\mu})^T \boldsymbol{\varrho}(\boldsymbol{V}+\boldsymbol{\mu}) + \sum_j (\beta^2 B_j^2)} - \sum \mu_j + \sum_k K_k, \tag{54}$$

Notation	Description	Value
d	Modified durations	Input
γ	Curvature of the yield curve	0.134
L	Solvency liability	Input
l	Fraction that is debt on given asset	Input
$b16$	Technical rate of interest	Input
Λ	Old-age and disability pension liabilities up to and including	Input
C	Expected decrease in solvency	0.5
D	Expected increase in the technical rate of interest	0.004

Table 1: Variables in the solvency requirement formulas.

where β is defined as 0.08 for the risk classes 1-4 and 0 for the rest. B_j is the difference between long and short positions. Since options based on their exposure value B is set to 0. K_k is an exception that is added when a company has too much exposure to a single real estate investment. It requires that more than 15% of the assets are invested into the given real estate. In reality, pension funds do not invest into a single real estate within this magnitude and thus it is very unlikely to happen. Therefore K is set to 0.

The formula for approximating the solvency requirement reduces to

$$\tilde{r} = \sqrt{(\boldsymbol{V} + \boldsymbol{\mu})^{\boldsymbol{T}} \varrho (\boldsymbol{V} + \boldsymbol{\mu})} - \sum \mu_j. \tag{55}$$

Furthermore, the minimum requirement is defined as 5% of the total assets of the pension provider. Equation (55) is non-convex. This is due that ϱ is not positive semidefinite. That is, the eigenvalues of ϱ are not strictly positive.

Table A1 and Table A2 in the appendix show the solvency requirement input matrix E and the necessary solvency requirement inputs.

Risk class	S_j	m_j
1	0.320	0.080
2	0.300	0.080
3	0.330	0.080
4	0.350	0.100
5	0.320	0.080
6	0.020	0.033
7	0.000	0.000
8	0.015	0.005
9	0.025	0.010
10	0.050	0.020
11	0.090	0.065
12	0.140	0.075
13	0.145	0.000
14	0.290	0.015
15	0.316	0.080
16	0.008	0.000
17	1.000	0.150
18	*	0.000

Table 2: The stress values and expected return for each risk class.

Table 3: The correlation matrix between the risk classes ϱ.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0.800	0.700	0.700	0.700	-0.200	0	0.600	0.700	0.700	0.200	0.200	0	0	0.890	0.200	0	0.430
0.800	1	0.700	0.700	0.700	-0.200	0	0.600	0.700	0.700	0.200	0.200	0	0	0.890	0.200	0	0.410
0.700	0.700	1	0.700	0.700	-0.200	0	0.600	0.700	0.700	0.200	0.200	0	0	0.860	0.200	0	0.410
0.700	0.700	0.700	1	0.700	-0.200	0	0.600	0.700	0.700	0.200	0.200	0	0	0.860	0.200	0	0.290
0.700	0.700	0.700	0.700	1	0	0	0.600	0.700	0.700	0.200	0.200	0	0	0.780	0.200	0	0.350
-0.200	-0.200	-0.200	-0.200	0	1	0	-0.400	-0.400	-0.400	0	0	0	0	-0.240	0.200	0	-0.210
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-0.180
0.600	0.600	0.600	0.600	0.600	-0.400	0	1	0.900	0.800	0.100	0	0	0	0.690	0.200	0	-0.250
0.700	0.700	0.700	0.700	0.700	-0.400	0	0.900	1	0.900	0.100	0	0	0	0.800	0.200	0	0.150
0.700	0.700	0.700	0.700	0.700	-0.400	0	0.800	0.900	1	0.100	0	0	0	0.790	0.200	0	0.300
0.200	0.200	0.200	0.200	0.200	0	0	0.100	0.100	0.100	1	0.800	0	0	0.120	0.200	0	0.390
0.200	0.200	0.200	0.200	0.200	0	0	0	0	0	0.800	1	0	0	0.120	0.200	0	0.380
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0.050
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0.370
0.890	0.890	0.860	0.860	0.780	-0.240	0	0.690	0.800	0.790	0.120	0.120	0	0	1	0.190	0	0.410
0.200	0.200	0.200	0.200	0.200	0.200	0	0.200	0.200	0.200	0.200	0.200	0	0	0.190	1	0	0.200
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0.430	0.410	0.410	0.290	0.350	-0.210	-0.180	-0.250	0.150	0.300	0.390	0.380	0.050	0.370	0.410	0.200	0	1

4 Scenario generation

The scenario generation lies at the core of portfolio optimization. The presented method aims to quantify the uncertainty into the scenarios. It is important to acknowledge that there is no "optimal" portfolio for all outcomes but the aim is to look beyond traditional portfolio construction methods to prepare for the increasingly uncertain markets (Chan, Meschenmoser, et al. 2021).

Incorporating the uncertainty serves two key purposes. Investors never have the full certainty about the specific value for the expected returns. Accounting for this can be done by adding an estimate of uncertainty with regards to the input variables. The estimates can capture the variation in levels of uncertainty in the cross section of time and asset classes. This makes sense, because the lower ability to estimate returns for a given asset class should warrant higher uncertainty around its expected return. This should be reflected in the results of the portfolio construction.

The expected return and the uncertainty estimates are provided in BlackRock (2024). The methodology of combining the central market assumption (CMA) and the volatility of the asset class is also discussed in Chan, Henderson, et al. (2018).

The scenarios are generated by simulating the mean return expectations as well as the asset returns around those means. The uncertainty estimates and the asset return around the simulated means are assumed to be normally distributed. This approach requires assumptions about how the uncertainty estimates for different asset classes are correlated, which is challenging to quantify. As a baseline, we consider two cases. First, the correlation matrix for the uncertainties is the same as for the returns. Second, the estimates are uncorrelated.

Let us denote the standard deviations of the uncertainty estimates as $\boldsymbol{\sigma_1} \in \mathbb{R}^a$ and the standard deviations around the means as $\boldsymbol{\sigma_2} \in \mathbb{R}^a$, Where a is the number of asset classes. The following procedure illustrates the sampling process:

1. Compose covariance matrix Σ_1 for the uncertainties. That is

$$\Sigma_2 = \text{diag}(\boldsymbol{\sigma_1}) \cdot \text{Corr} \cdot \text{diag}(\boldsymbol{\sigma_1}),$$

   where Corr is the assumed correlation matrix of the uncertainties.

2. Sample n sets of central market assumptions, with mean equal to the expected return and covariance matrix Σ_1.

3. Sample T sets of returns with asset class covariance matrix Σ_2 for each CMA.

Table A3 in the appendix shows the asset class correlation assumptions Corr. The return distribution of this sampling process transforms the standard deviation of the assets based on the covariance matrices Σ_1 and Σ_2. The standard deviation of the sampled returns for each asset follow equation $\sigma_{3i} = \sqrt{\sigma_{1i}^2 + \sigma_{2i}^2 + 2\,\mathrm{Cov}(u_i, g_i)}, \forall i \in a$, where u is the random variable for the CMA and g is the random variable for the assumptions around the CMA. Let us denote the covariance matrix of the sampled returns as Σ_3. This is one way of robustifying the scenario generation with regard to the uncertainty within the market assumptions. Also other ways of robustification such as shrinkage have been proposed (Ledoit and Wolf 2003).

The market assumptions were updated in September 2023. The covariance matrix is calculated based on historical returns of proxy indices. The number of assets was limited to 13 to reduce the computation time of the optimizations. Table 4 shows the asset classes, the corresponding Blackrock references and the Bloomberg tickers for the proxy indices. The sampled returns $\boldsymbol{y}_{s,t}$ are denoted such that there is a vector of asset returns for each time $t \in T$ and scenario $s \in q$.

Blackrock does not differentiate between Investment grade bonds and Corporate bonds. These are assumed to have the same expected returns and uncertainties. The market assumptions are provided separately in terms of Euros and US dollars. Currency is modelled as the difference between the Euro equity assumptions and the US dollar equity assumptions.

Figure 2 shows the densities of the cumulative returns of Equity, Government bonds, Currency and Real estate. The x-axis indicates the total return. These are

Asset class	**Abbreviation**	**Blackrock reference**	**Ticker**
Cash	CASH	EMU cash	EGB0
Government bonds	GOV	EMU treasury bonds	BERPG3
Emergin market debt	EMD	EM debt	JGENVUEG
Investment grade	IG	EMU corporate bonds	QW5A
High yield	HY	Global high yield bonds	IBOXXMJA
Equity	EQ	Europe large cap equities	MSDEE15N
Private equity	PE	Global private equity	LPX50TR
Private debt	PD	Global aggregate bonds	CDLI
Hedge funds	HF	Hedge funds	HFRIAWC
Real estate	RE	Global core real estate	NPNCRE
Corporate finance	CF	EMU corporate bonds	QW5A
Infrastructure	INFRA	Global infrastructure equity	DJBEIET
USDEUR	USDEUR	*	*

Table 4: Asset classes, Blackrock references and Bloomberg tickers.

created with 20000 samples. The densities illustrate the risk related to each asset. The log-normal distribution of the cumulative returns of government bonds take the smallest deviations and equities have the largest. The upside potential in equity returns is significantly higher than that of other asset classes.

The chosen time step for the arithmetic returns is 1 year. The reason behind this is two-fold. First, the focus of the optimization horizon is between 5–10 years and the number of time steps needs to be limited to reduce computation time. Second, this gives an intuitive way to handle the annualized input data.

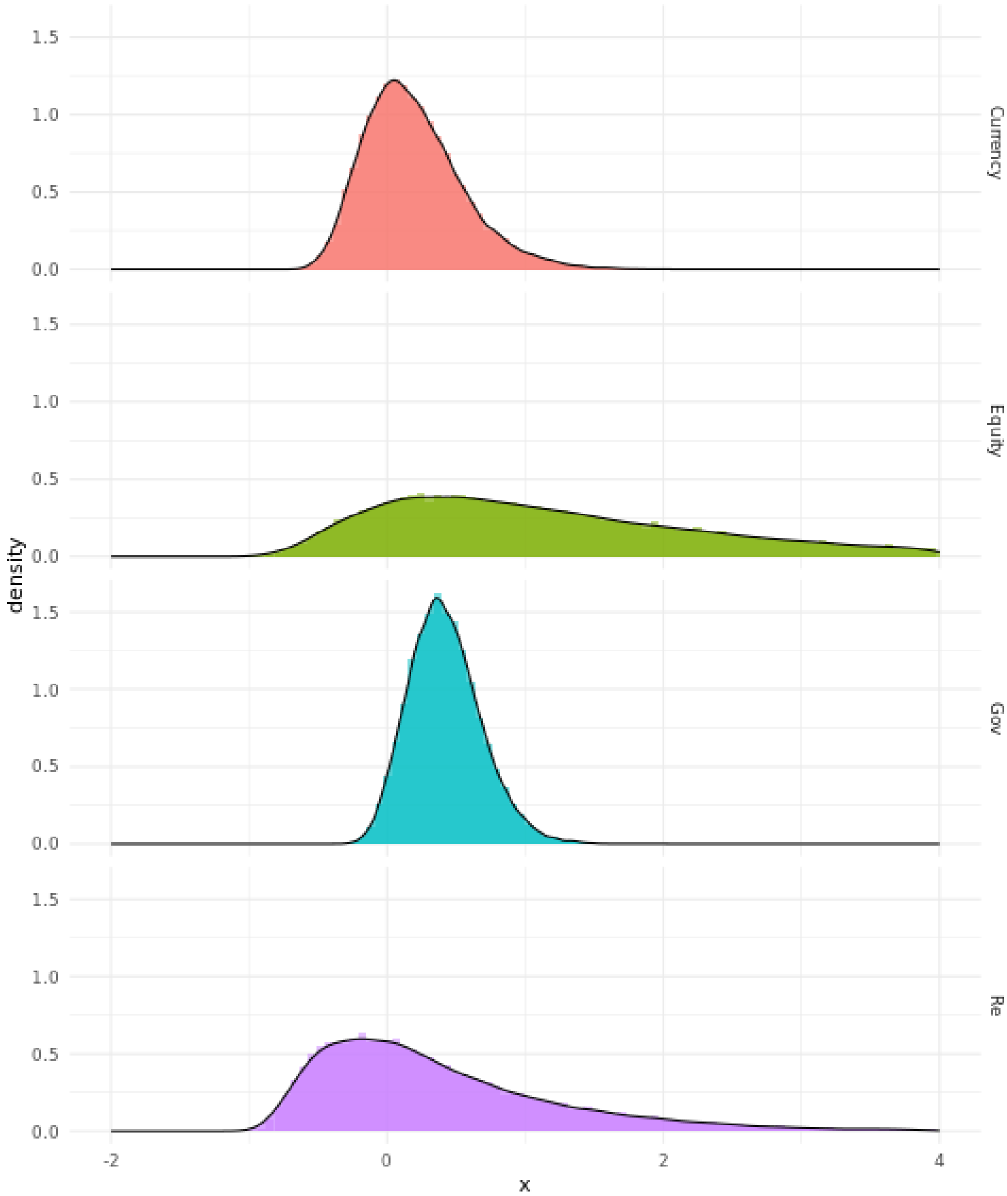

Figure 2: Densities of the 10-year cumulative returns of the sampled distributions.

Figure 3 shows the density distribution of the arithmetic return samples for equities and government bonds. The normal distribution based on the volatility of the asset is added as the dashed line. The sampled arithmetic returns are normally distributed with a standard deviation σ_3. Table 5 shows the standard deviations of the input and output values of the sampling process.

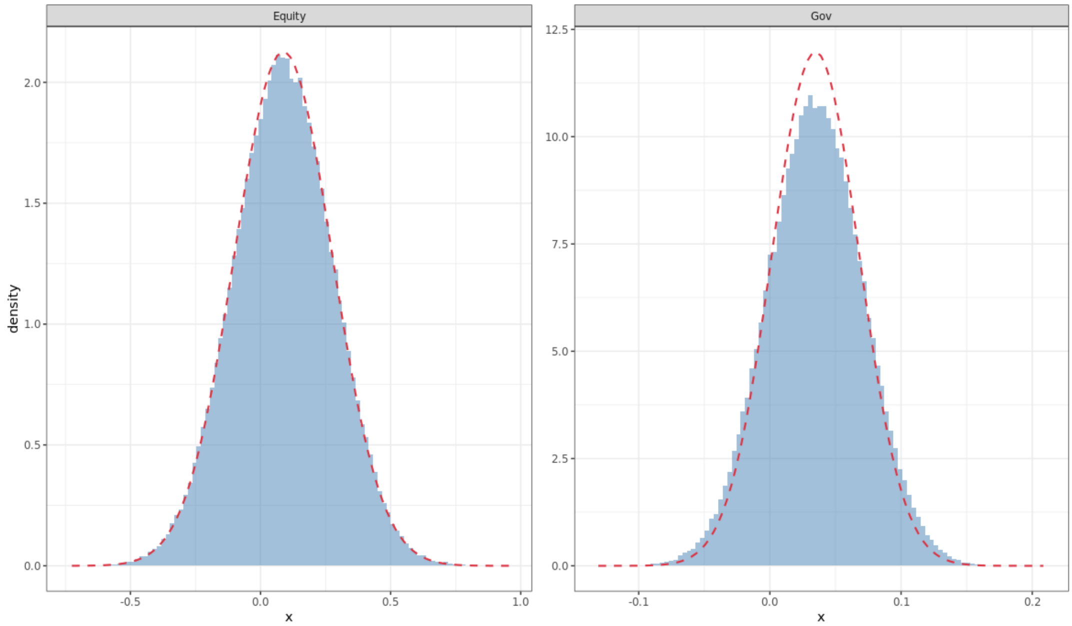

Figure 3: The densities of arithmetic returns.

Table 5: Standard deviations (%) of sampling inputs and outputs.

Asset class	σ_1	σ_2	σ_3
CASH	0.0	1.4	1.4
GOV	1.6	3.3	3.7
EMD	4.0	8.7	9.5
IG	1.1	2.7	2.9
HY	2.2	4.7	5.2
EQ	3.3	18.8	19.0
PE	13.1	26.0	29.1
PD	0.6	9.1	9.1
HF	9.4	3.4	10.0
RE	5.3	12.7	13.8
CF	1.1	2.4	2.6
INFRA	20.1	16.2	25.8
USDEUR	0.6	9.2	9.2

The incorporation of the uncertainty with regards to the CMA becomes increasingly important when the time horizon is long. Figure 4 illustrates the development of the interquartile range of the cumulative returns, with different time horizons. The difference between the adjusted and the CMA interquartile range expands exponentially with time. The summary statistics of the arithmetic returns are shown in Table 6.

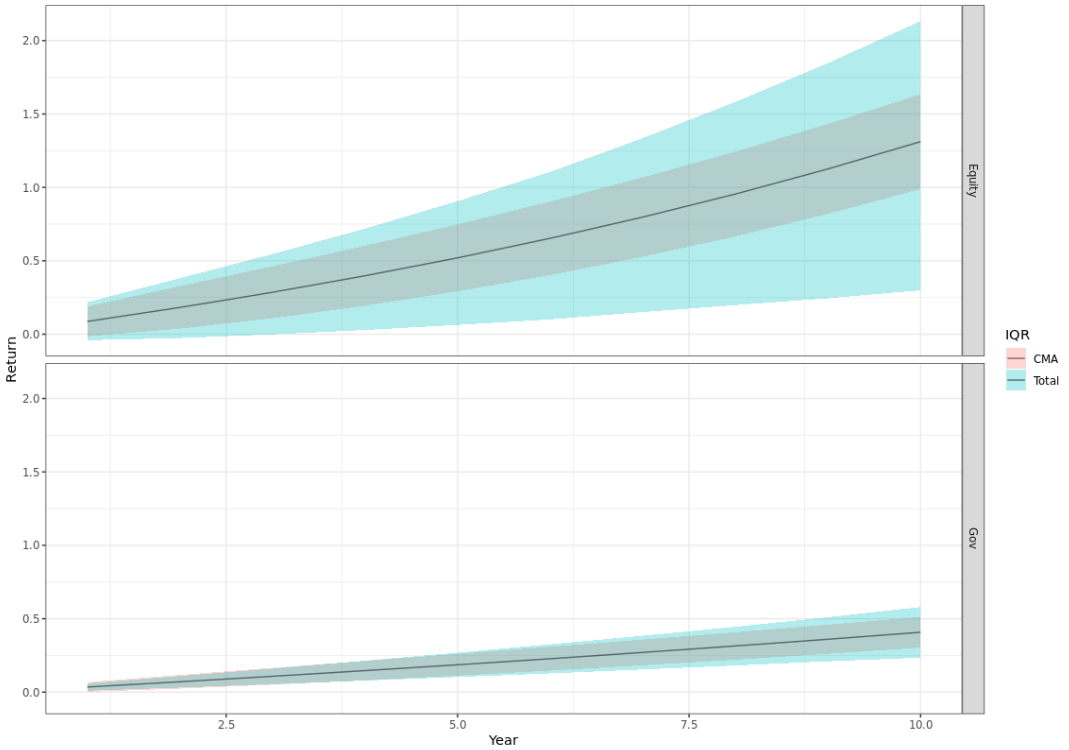

Figure 4: Interquartile ranges for equities and government bonds.

The correlation assumption of the uncertainty estimates does not influence the means or the standard deviations of the individual assets. It only changes the correlations between the assets in the case that the uncertainty estimates are assumed to have zero correlation.

Table 6: Summary statistics of the arithmetic returns.

	Min.	1st Qu.	Mean	3rd Qu.	Max.	Sd	Skewness	Kurtosis
CASH	-0.039	0.019	0.029	0.039	0.087	0.014	-0.009	3.000
GOV	-0.140	0.010	0.035	0.059	0.193	0.037	0.003	2.998
EMD	-0.378	-0.012	0.052	0.116	0.454	0.095	-0.007	2.990
IG	-0.094	0.020	0.040	0.059	0.163	0.029	-0.002	2.984
HY	-0.179	0.019	0.053	0.088	0.288	0.052	-0.007	2.995
EQ	-0.725	-0.040	0.089	0.218	0.954	0.190	-0.002	2.987
PE	-1.266	-0.124	0.072	0.267	1.381	0.291	0.001	3.009
PD	-0.412	-0.025	0.037	0.098	0.438	0.091	-0.003	3.002
HF	-0.399	-0.003	0.065	0.132	0.552	0.100	0.013	3.003
RE	-0.630	-0.070	0.023	0.116	0.622	0.138	-0.001	2.991
CF	-0.088	0.022	0.040	0.057	0.171	0.026	0.00003	3.003
INFRA	-1.136	-0.100	0.073	0.247	1.488	0.258	-0.007	3.005
CURRENCY	-0.417	-0.045	0.017	0.079	0.429	0.092	-0.001	3.009

5 Models

The alternative problem statements are described in the following subsections. Multiple problem statements are explored because there is trade-off between complexity and reliability of the result due to the difficulty of obtaining the extremum. The target is to define a statement that matches the goal of a pension fund and is numerically solvable. The solution should secure the financing of the future pension liabilities.

We formulate the statement stochastically. It means that we optimize across the scenarios where the portfolio used in each scenario is consistent. If we solved the problem for each scenario separately, the program would lose its stochastic nature and we would have to lean on the resampling solution (Michaud and Michaud 2008).

We start with a model which is based on the fundamental investment theory for risk and return. We use CVaR as it is a tested risk measure which can be specifically directed for the tail risk of the solvency capital of a pension fund, instead of portfolio returns. This can be formulated within a linear objective function, making it computationally efficient and reliable in finding the extremum. To achieve this, the parameter β and the target return have to be specified. This is a disadvantage of the simplification, as it is not known which the parameter values should be. We have to use common sense values which can be assumed based on intuition and industry standard. As the simplification is founded on the standard framework, it enables comparison with the traditional CVaR optimization. The difference between the two solutions gives insight into how the risk modification influences the results. The mere difference between the two can be informative.

We then continue by aiming to formulate the statement, without having to specify the parameter β and the target return. This is done by modifying the loss function of the problem statement to include the solvency requirement. This makes the objective function nonlinear which makes it computationally inefficient. Furthermore, it is not clear whether the objective function should or can be based on minimizing the probability of bankruptcy or the monetary amount in case of falling short of the solvency requirement. This is important for solving the problem statement in terms of finding the extremum. The pros and cons are discussed for each model.

5.1 Minimization of CVaR

In this section we formulate a problem statement using the CVaR risk measure. Let us define the loss function for the problem to be the cumulative return on the capital Pc

$$f_1(\boldsymbol{x}, \boldsymbol{y}) = -Pc. \tag{56}$$

The return on the solvency capital Pc depends on the cumulative portfolio returns P_T, the initial solvency capital C_{t_0} and the cumulative fund transfer obligation FTO_T, i.e,

$$Pc = (1 + C_{t_0})P_T - \mathrm{FTO}_T. \tag{57}$$

The cumulative return P_T is calculated by compounding the annual return

$$P_{s,T} = \prod_{t=1}^{T}(1 + \boldsymbol{x}^\intercal \boldsymbol{y}_{s,t}) - 1, \forall s \in q, \tag{58}$$

where the vector $\boldsymbol{x}$ denotes the asset weights and $\boldsymbol{y}_{s,t}$ the sampled asset return vector at time $t \in T$ and scenario $s \in q$.

FTO_T is calculated by compounding the annual fund transfer obligation

$$\mathrm{FTO}_{s,T} = \prod_{t=1}^{\intercal}(1 + \mathrm{FTO}_t) - 1, \forall s \in q. \tag{59}$$

The annual fund transfer obligation FTO_t is calculated using the formulas (33–40). Let us define a parameter pt for a solvency position target. This parameter defines the amount of buffer with which the resulting portfolio is above the legal limit. The law requires that the parameter needs to be larger than one. As the baseline we use a position target of 1.2 but this is an adjustable parameter. If the parameter value is high, the portfolio has a smaller risk for breaching the legal limit in the near future. However, as a consequence, it constrains the feasible portfolio to be less efficient. This means that in the long term using a less efficient portfolio increases the risk for insolvency. In theory, we want to keep the parameter as small as possible but big enough to accommodate the portfolio adjustments in market stresses. This is to always maintain the solvency position above one.

The problem statement is

$$\min_{\boldsymbol{x}} \ \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \left[-Pc_{s,T} - \alpha_{s,T}\right]^{+} \tag{60}$$

$$\text{s.t.} \tag{61}$$

$$P_{s,T} = \prod_{t=1}^{T} (1 + \boldsymbol{x}^{\top} \boldsymbol{y}_{s,t}) - 1, \forall t \in T, \forall s \in q, \tag{62}$$

$$Pc_{s,T} = (1 + C_{t_0}) P_T - \mathrm{FTO}_T, \tag{63}$$

$$\sum_{i}^{k-1} x_i = 1, \tag{64}$$

$$\mathrm{pt} r_{t_0} \leq C_{t_0}, \tag{65}$$

$$tr \leq \frac{1}{q} \sum_{s=1}^{q} P_{s,T}, \tag{66}$$

$$x_i \geq 0, \forall i \in k, \tag{67}$$

$$x_i \leq 1, \forall i \in k, \tag{68}$$

where the specified target return denoted as tr. Here the solvency requirement constraint $\mathrm{pt}r_{t_0} \leq C_{t_0}$ ensures that the portfolio is feasible for any given starting solvency capital and the solvency position is above target level pt. Currency is modelled as an overlay. All non-overlay assets must sum to 1 and the currency weights can take values from the range 0 to 1. Short positions are not allowed.

Since the $\mathrm{FTO}_{s,T} \forall s \in q$ is determined as a function of the portfolio of the system of pension providers, the suggested program captures the risk of deviating from the systems portfolio. The β probability is given as an input and controls the amount of the tail risk to be included in the optimization. We use the standard 0.9 as the default, but test the sensitivity of the model by also using alternative values. The fact that a constant β needs to be specified for each solvency level is a drawback of this formulation. Intuitively, the lower the solvency capital, the bigger the tail that should be included in the optimization because the risk of breaching the solvency limit is higher. The time parameter T defines for how long the return on the capital is accumulated.

5.2 Minimization of expected shortage

Instead of investing in capital efficient portfolios, the objective of a pension fund can also be defined in terms of meeting the future requirements. In theory such formulation is a more accurate effort. This is because the CVaR formulation does not address the optimal risk level that the fund should take and also requires an assumption for the β-probability parameter.

We define the loss function $f_2(\boldsymbol{x}, \boldsymbol{y})$ as the difference between the requirement and the solvency capital. When the parameter η is set to zero the expected shortage equates to minimizing the expected amount that the fund is insolvent

$$f_2(\boldsymbol{x}, \boldsymbol{y}) = r - C. \tag{69}$$

The loss function is calculated as a percentage of the liabilities in the beginning but it describes the difference between the capital and the requirement. For each timestep we have the distribution of the loss function. Having the loss function in terms of the actual amounts makes it possible to examine across all timesteps with a discount factor d.

The portfolio weights vector $\boldsymbol{x}$ is defined to be constant across all timesteps $t \in T$ and scenarios $s \in q$. The fact that the weights stay the same over all scenarios makes the program stochastic. The restriction that at each timestep the weights remain the same simplifies the problem. In this case, the problem statement answers to the following question. Which constant portfolio weights minimize the expected amount by which the pension provider violates the regulations on a given time horizon and initial state? The problem statement is defined as follows

$$\min_{\boldsymbol{x}} \frac{1}{qT(1-\zeta)} \sum_{k=1}^{q} \sum_{t=1}^{T} \left[r_{s,t} - C_{s,t} \right]^{+} d_t \tag{70}$$

$$\text{s.t.} \tag{71}$$

$$P_{s,t} = \sum_{i=1}^{k} x_i y_{s,t,i}, \tag{72}$$

$$A_{s,t} = A_{s,t-1} P_{s,t}, \tag{73}$$

$$C_{s,t} = A_{s,t} - L_{s,t}, \tag{74}$$

$$L_{t_0} = 1, \tag{75}$$

$$\sum_{i}^{k-1} x_i = 1, \tag{76}$$

$$r_{s,t_0} \leq C_{s,t_0}, \tag{77}$$

$$x_i \geq 0, \forall i, \tag{78}$$

$$x_i \leq 1, \forall i. \tag{79}$$

The parameter d_t is the discount factor which is set to 3%. The discount factor treats all timesteps on the same basis. The total number of points where the loss function is evaluated is the number of scenarios q times the number of timesteps within those scenarios T. The assumption about the discount rate impacts the risk level of the resulting portfolio. This is because the higher the discount rate, the more importance is given to breaching the requirement in the near future. If the capital level is sufficient to cover the increases in the liabilities for a few years, this gives a lower risk level portfolio.

The requirement at each time step and scenario $r_{s,t}$ is calculated with Equation (55). The solvency liabilities and the necessary input parameters for each timestep and scenario can be pre-calculated with the recursive formulas (33)–(40) such that the only decision variables are the portfolio weights $\boldsymbol{x}$. The necessary input values are the $b_{16 s,t}$, the FTOs, t and the TO$_{s,t}$ which are calculated based on the performance of the system of private pension providers.

The portfolio weights need to satisfy the requirement at the initial state t_0. The initial state is the same for each scenario and thus does not include stochasticity. Without the requirement constraint for the initial state the solutions might not be applicable. That is, the risk measure might be minimized even if the requirement is not satisfied in the beginning. The solution may then include a loss for the beginning state in case it decreases the loss for future timesteps. Whether the constraint is active reflects the a balance between the discount factor and the behavior of the loss function between timesteps.

Using the expected shortage risk measure instead of CVaR has the benefit that it is not known which β level most effectively limits the violation of the solvency requirement. Furthermore, a change in the initial solvency ratio changes the position at which we want to optimize the density distribution of the loss funtion. Consider the case where the initial solvency capital is small and the company is near bankruptcy. The probability of ending up below the requirement is roughly 50% after the first year. Minimizing a a typical β level of 0.9 is then far too conservative. Fixing the η-parameter to zero in Equation (22) forces the program to minimize the part of the distribution that violates the requirement regardless of the initial solvency. Figure 5 illustrates this. The x-axis describes the values of the loss function in Equation (69)

The reason why the expected shortage is preferred over to the shortfall probability is two-fold. In theory, minimizing the shortfall probability is aligned with the goals of a pension fund but it has limitations in practice. When the probabilities are calculated based on scenarios the values of the objective function are discrete. The algorithms used to solve such problems require a stopping criterion to monitor the change in the objective function wrt. a change in the decision variables. A discrete objective function values can cause the algorithms to converge prematurely, often at the first

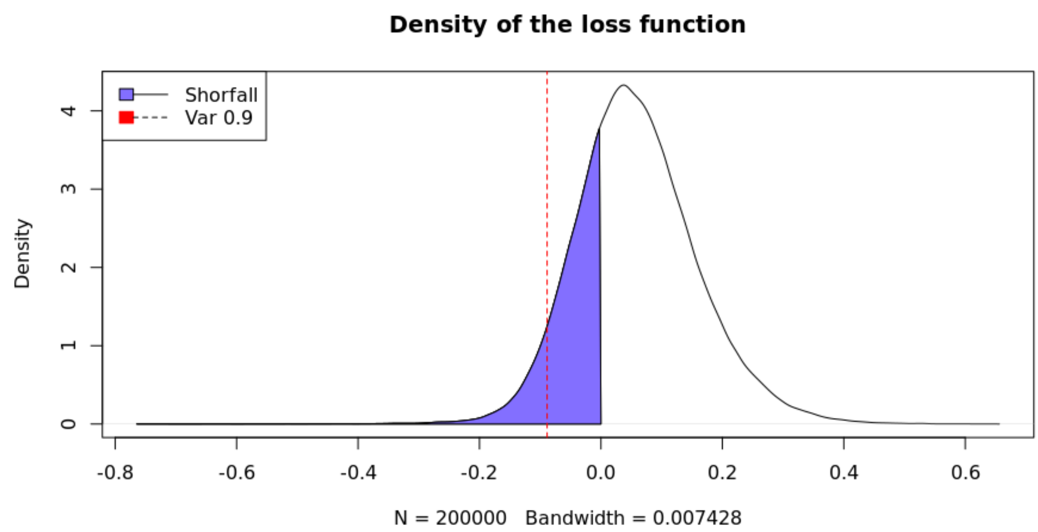

Figure 5: The density of the loss function

iteration. How effectively the solution space is searched for depends on the choice of the algorithm and the number of samples but the problem of premature convergence remains.

Second, there is a need for robustness in representing reality. Bankruptcy does not necessarily occur exactly at $\eta = 0$ of the loss function. That is when extra restriction are posed to the pension providers activities and the actual bankruptcy occurs in case the investment profits are not sufficient for a quick recovery. This means that the penalty is proportional to the amount of violating the requirement. In practice, optimizing the expected shortage versus the shortfall probability can be a hedge against a model risk. That is, the liability modelling and the calculation of the requirement both contain some approximation. This can lead to some errors. The solutions between the two risk measures should resemble one another similarly to the difference between CVaR and VaR.

By not including the negative part of the loss function means that the solvency position is not directly optimized. This is desirable as high solvency position serves no benefit other than cover for short term fluctuations. On the other hand the positive part of the loss function leads to bankruptcy which ought to be minimized.

5.2.1 Dynamic strategy optimization

The restriction that the portfolio weights remain constant is a major simplification. In reality pension funds can adjust their portfolio at any given time. Pension funds may want to meet the changes in the solvency metrics during the scenarios with a change in their portfolio weights. In bearish scenarios where the the capital hits the requirement pension funds need to modify the portfolio such that requirement is satisfied up to the point where the requirement is 5% of the assets.

In this section, we allow for the modification of weights during the scenarios while maintaining the stochastic nature of the program. Consider solving the minimization of expected shortage with varying the initial solvency ratio. This yields a strategy that serves as a approximation for the problem formulation.

In order to find how the weights should be modified, we examine a rule based allocation policy based on the solvency metrics. The solvency ratio $R_{s,t}$ remains a consistent measure of the risk taking abilities of the pension fund at each timestep and scenario. Consider the $k \times m$ matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{km} \end{bmatrix}$$

where each column represents a different solvency ratio and each row represents the portfolio weights. At any timestep and scenario the portfolio of the pension fund is a column $X_{\cdot j}$, where the index j is selected as a function of the solvency ratio.

The range on different solvency ratios R is chosen to be [0.05,0.5] and the number of columns m is set to 10. The index j can then be selected with the following expression

$$j(R) = \begin{cases} 1, & \text{if } R \leq 0.05\,, \\ \left\lfloor \frac{\text{R}\times 100}{5} \right\rfloor & \text{if } 0.05 \leq R \leq 0.5\,, \\ 10, & \text{if } R \geq 0.5\,. \end{cases} \tag{80}$$

where $\lfloor \cdot \rfloor$ represents the floor function which rounds down to nearest integer. Let us denote the vector of the ending points of the sections ($[0.05, 0.1, \ldots, 0.5]$) as $\boldsymbol{v}$.

The range of different solvency ratios is limited to $[0.05, 0.5]$ which represents the range at which all the pension providers operate. In practical terms, a solvency ratio beyond 0.5 means that the company is solvent and the risk of violating the requirement is small. The minimum requirement is 5% of the total assets and thus a solvency ratio of 0.05 is the bound of the minimum acceptable solvency level. The number of columns m determines how closely the strategy is matched to the solvency ratios. There are two key considerations to take into account. First, m impacts on the number of samples needed to accurately find a feasible solution for each column. Consider an example where $m = 1000$. Then the probability that the solvency ratio at any timestep or scenario is within a specific section is small. It is possible that the objective function is evaluated so that a given column does not impact the solution. In this case, the number of samples would also have to be increased. Second, in this program following a strategy does not include trading costs. Dividing the strategy into larger sections mitigates this drawback because a shift in the weights between consecutive timesteps occurs only when the solvency ratio changes sections ($j_t \neq j_{t+1}$).

The problem statement is formulated as follows

$$\min_X \frac{1}{qT(1-\zeta)} \sum_{s=1}^{q} \sum_{t=1}^{T} \left[r_{s,t} - C_{s,t}\right]^+ d_t \tag{81}$$

$$\text{s.t.} \tag{82}$$

$$P_{s,t} = \sum_{i=1}^{k} x_{i,j_{t-1}} y_{s,t,i}, \tag{83}$$

$$A_{s,t} = A_{s,t-1} P_{s,t}, \tag{84}$$

$$C_{s,t} = A_{s,t} - L_{s,t}, \tag{85}$$

$$L_{t_0} = 1, \tag{86}$$

$$R_{s,t} = \frac{C_{s,t}}{L_{s,t}}, \tag{87}$$

$$j_t = \begin{cases} 1, & \text{if } R_{s,t} \leq 0.05\,, \\ \left\lfloor \frac{R_{s,t} \times 100}{5} \right\rfloor, & \text{if } 0.05 \leq R_{s,t} \leq 0.5\,, \forall s \in q, t \in T, \\ 10, & \text{if } R_{s,t} \geq 0.5\,, \end{cases} \tag{88}$$

$$\sum_{i}^{k-1} x_{i,j} = 1, \forall j \in m, \tag{89}$$

$$x_{i,j} \geq 0, \forall i \in k, \forall j \in m, \tag{90}$$

$$x_{i,j} \leq 1, \forall i \in k, \forall j \in m, \tag{91}$$

$$r(X_{.,j}) \leq v_j, \forall j \in m. \tag{92}$$

Each column of X is subject to similar constraints as the constant allocation optimization. The columns are constrained to sum to one without the currency overlay and no short positions are allowed. The requirement constraint is added for each column separately. This is to make the strategy feasible for every given initial solvency ratio.

The strategy optimization need to maintain the stochastic nature even though the portfolio weights follow the solvency ratio. This is because the allocation policy is fixed across all scenarios.

The impact of the initial solvency for the strategy. If the the number of time steps would be large the initial solvency should not have significant impact on the solution. However, because the number of timesteps is 10 it may affect them.

The index j could be selected in multiple ways based on the solvency ratio. The floor function is chosen because it guarantees that the initial solvency requirement constraint is satisfied. This choice also impacts the convergence of the optimization.

Testing implies that for example linear interpolation between the columns severely increases the number of iterations.

5.3 Minimization of insolvency probability

An intuitive objective function would be to minimize the number of scenarios where a violation of the solvency requirement happens. We introduce a binary variable $B_{s,t}$ to indicate whether the company is violating the law at a scenario s and timestep t. That is

$$B_{s,t} = \begin{cases} 0, & \text{if } r_{s,t} - C_{s,t} < 0\,, \\ 1, & \text{if } r_{s,t} - C_{s,t} \geq 0\,. \end{cases} \tag{93}$$

The following problem statement scan be defined as

$$\min_X \frac{1}{q} \sum_{s=1}^{q} \prod_{t=1}^{T} B_{s,t} \tag{94}$$

$$\text{s.t.} \tag{95}$$

$$(83) - (92), \tag{96}$$

The cumulative product of the binary variable B in Equation (94) captures whether a violation has happened during a scenario. The optimal value for this problem statement minimizes the probability of breaching the limit with a fixed allocation policy. This provides an obvious benefit and matches closely to the goals of a pension fund. This statement is the same as maximizing the number of scenarios where a violation does not happen.

To the author's knowledge, such an objective function has not been studied in the literature. It contains the non-convexity of the requirement within the objective function and gets discrete values based on the number of samples. Finding a global optimum to this statement is difficult and potentially unrealistic. However, using the algorithms presented in the next chapter, we seek solutions that may be useful and could be compared to the previous problem statements.

5.4 Algorithms and convergence

This section describes how the results are obtained for each problem statement. Because of the non-convexity of the solvency requirement, the choice of the algorithm and initial guess have a key role in obtaining the results. The used algorithms are:

- SLSQP: Sequential quadratic programming
- Cobyla: Constrained Optimization by Linear Approximations
- GA: Genetic algorithm

These algorithms are chosen because they can handle nonlinear constraints, are derivative free and return a solution within reasonable runtime.

SLSQP algorithm, based on the implementation Kraft (1994) is used to solve the CVaR formulation. This algorithm is fast. Furthermore, we assume it finds the extremum with a feasible initial guess. With feasbile initial guess SLSQP algorithm and Cobyla returned the same solution. However, without one the SLSQP found a better solution. It numerically approximates the Jacobian which typically makes it more accurate compared to other mentioned algorithms. The algorithm may encounter difficulties when faced with discrete objective function values, potentially hindering its progression. Testing showed that in the probability based optimization this returned the second iteration. SLSQP is able to handle nonlinear constraints. The algorithm optimizes successive second-order (quadratic/least-squares) approximations of the objective function via BFGS updates, with first-order approximations of the constraints.

The Cobyla algorithm is used to solve the models with a nonlinear loss function. It has the following characteristics: it leverages linear approximations, operates without derivatives, maintains continuous iteration even with discrete objective function values, and is fast. The methodology involves constructing successive linear approximations of the objective function and constraints using a simplex of $n+1$ points (in n dimensions), and optimizing these approximations within a trust region at each step. This algorithm was introduced by Powell (1998).

The GA can be used to test the convergence of the solution. This is done by giving the solution of the Cobyla algorithm as an initial guess. If the GA manages to improve the value of the objective function it proves that the initial guess was not a global optimum. Without such an initial guess, the GA is too slow in improving the objective value. The algorithm continues iterating until a specified number of iterations have passed without improvement. The implementation is based on Scrucca (2013). The GA can be modified and parallelized marginally for faster performance.

The approach to solve the problem statements is a local convergence to the global minimum. The main idea is that if the start is close enough to the global optimum, the algorithm converges to it with high probability. The strongest results of an optimization problem would be convergence no matter how the algorithm is initialized, as would be the case in convex optimization. We are not able to achieve this in the models with a nonlinear loss function. We assume this is due to the non-convexity of the requirement. We first solve the minimization of expected shortage. This result is used as an initial guess in the dynamic strategy optimization.

Nonlinear loss functions can be very difficult to minimize. Work on this thesis made it evident that the runtime of these models increases exponentially. Moreover, the initial guess employed for optimization can drastically alter the final result, introducing unpredictability and potential inaccuracies. Recognizing these challenges, the effort was to reformulate the problem to render it convex. One approach involved approximating ϱ, by the nearest positive semi-definite matrix. While this adjustment aims to enhance tractability, it introduces considerable changes in the requirement values. For this reason this approach was not used.

6 Results

This section presents the solutions for all the models, carries out sensitivity analyses and compares the solutions with each other. The computations were performed in an Azure Kubernetes Service (AKS) pod configured with 1 vCPU and 6 GB of RAM.

6.1 CVaR

Figure 6 shows the capital efficient frontier of the problem statement (60)–(68). We obtain the frontier by varying the target return between the return of the minimum risk portfolio and the maximum of the expected return of the individual assets.

The X-axis indicates the risk related to the 0.9-CVaR of the capital returns $Pc_{s,1} \forall s \in [1, 20000]$. The Y-axis reports the specified target return. The figure contains the frontier also for the traditional CVaR (red dots) where the risk related to the capital is calculated separately.

The optimizations are run for a given initial solvency ratio. The portfolios (in the figures) were produced with an initial solvency ratio of 0.2. Some other solvency ratio could have been chosen as well.

Both portfolios are constrained to be above the position target 1.2. In Figure 6 the uncertainty estimates are assumed to be uncorrelated. The highlighted green area describes the difference in efficiency between the traditional CVaR and the presented model. The difference of efficiency is the largest with the lowest risk portfolios and the difference shrinks as the expected return increases.

The CVaR formulations are solved with SLSQP algorithm. The initial guess for each portfolio on the frontier is the previous point. Before optimizing we do not know what the maximum target return is that still satisfies the position target constraint. Finding the feasible range requires to monitor, if the optimization produces portfolios that satisfy both the target return and the position target. Because this point is not known beforehand, solving the full frontier gives values for which there is no feasible solution. The unfeasible portfolios can be filtered based on whether or not both constraints are satisfied. Solving the frontier of 50 portfolios with 20000 scenarios takes roughly 2 minutes.

Table 7 shows weights for selected portfolios of the capital efficient frontier. The weights on the frontier are in Figure 7. Table 7 includes the range for the filtered weights, based on if both the position and the return target are satisfied. The weights plot includes the unfiltered weights. We divided the range into 50 points. This resulted in filtering 9 of the last portfolios.

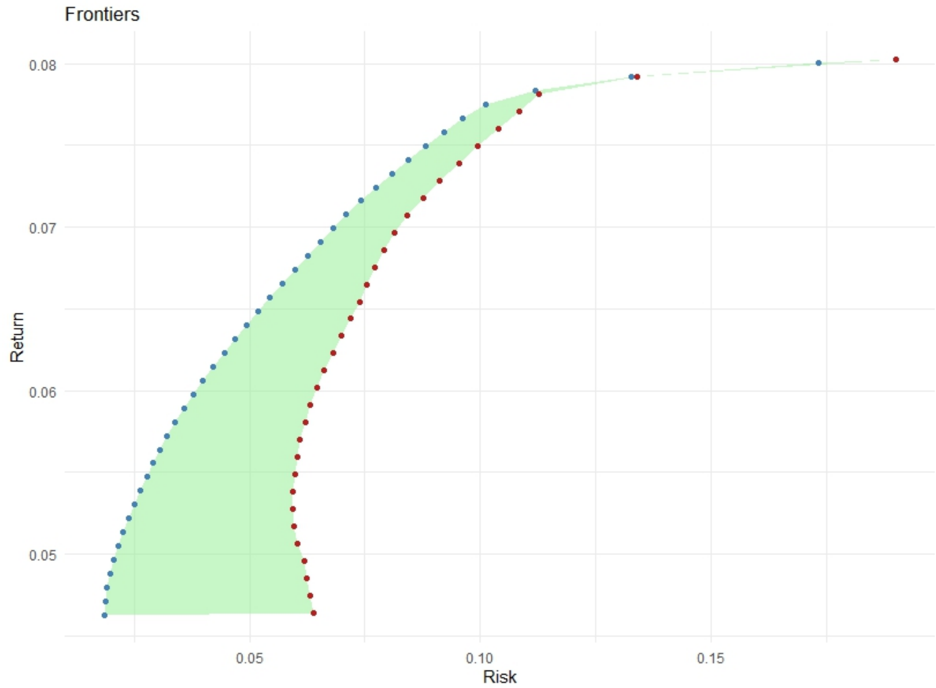

Figure 6: Capital efficient frontier for uncorrelated uncertainties.

Table 7: Capital efficient portfolios for uncorrelated uncertainties.

Expected return	4.6	5.0	5.4	6.2	7.1	8.0
CASH	43.528	29.914	12.735	0	0	0
GOV	0	0	0	0	0	0
EMD	0	0	0	0	0	0
IG	13.479	15.190	18.010	8.351	0	0
HY	5.836	10.226	15.769	33.148	46.850	13.489
EQ	18.899	19.614	20.495	23.404	28.409	26.127
PE	0	0	0	0	0	0
PD	0	0	0	0	0	0
HF	3.255	5.529	8.293	15.904	24.741	31.030
RE	1.087	0.490	0	0	0	0
CF	13.725	18.823	24.442	19.023	0	0
INFRA	0.190	0.214	0.257	0.169	0	29.354
USDEUR	3.384	5.373	8.074	17.204	29.843	53.384

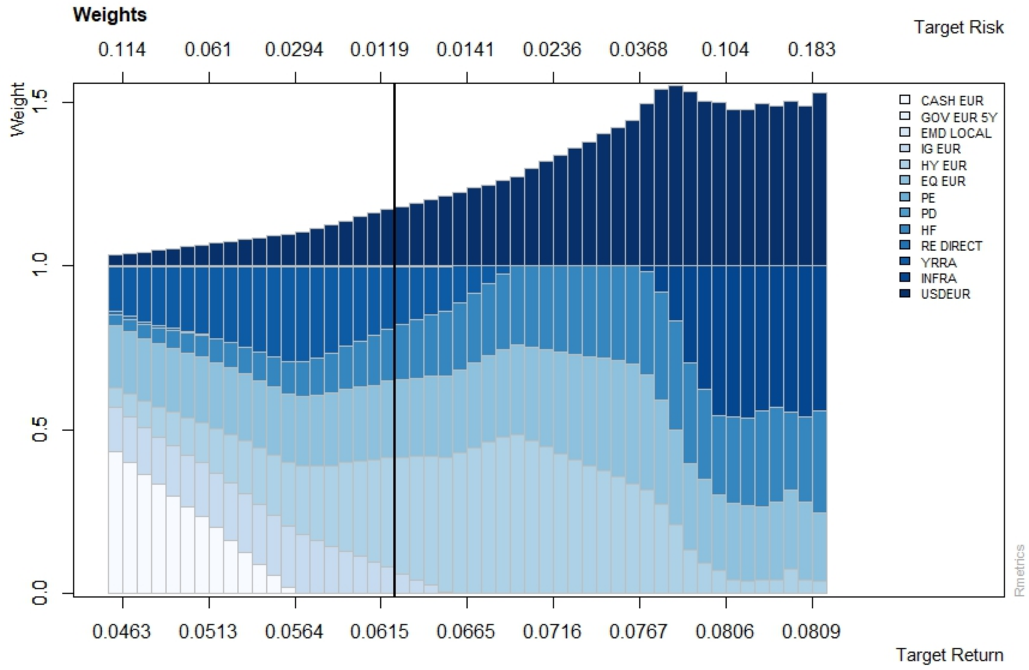

Figure 7: Capital efficient frontier weights for uncorrelated uncertainties.

Table 8 shows the weights for selected number of portfolios of the traditional CVaR frontier. The weights of this frontier are visualized in Figure 8.

Table 8: Portfolio returns efficient portfolios for uncorrelated uncertainties.

Expected return	4.6	5.0	5.4	6.2	7.1	8.0
CASH	13.351	0.239	0	0	0	0
GOV	0	0	0	0	0	0
EMD	0	0	0	0	0	0
IG	18.366	19.633	12.190	0	0	0
HY	20.753	25.141	35.413	54.215	48.341	10.817
EQ	2.591	3.277	5.038	8.941	18.136	24.584
PE	0	0	0	0	0	0
PD	0	0	0	0	0	0
HF	10.047	12.037	16.544	25.645	33.523	30.596
RE	0	0	0	0	0	0
CF	34.625	39.387	30.709	11.200	0	0
INFRA	0.268	0.286	0.106	0	0	34.003
USDEUR	8.502	10.619	16.259	27.563	44.848	52.966

Figure 9 shows the capital efficient frontier for the returns with correlated uncertainty

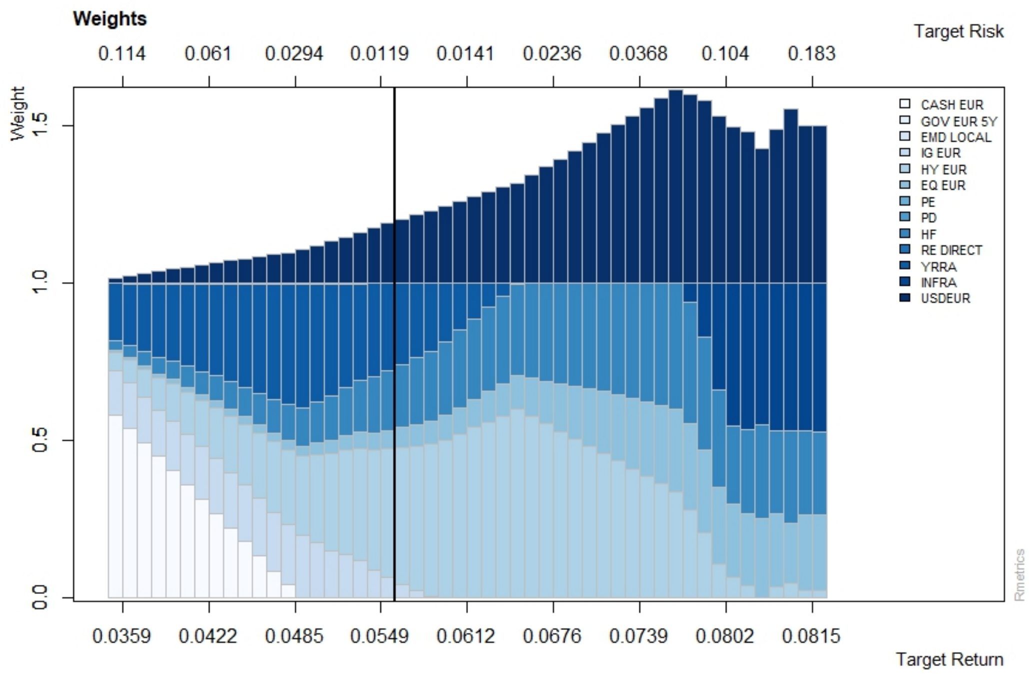

Figure 8: Portfolio returns efficient frontier weights for uncorrelated uncertainties

assumption. The efficiency improvement is not as significant as with the uncorrelated case. The amount of improvement the model achieves compared to the traditional CVaR optimization dependens on the underlying market assumptions. Figure 10 shows the relative difference in Sharpe ratios between the capital based and the portfolio returns based optimizations across the frontiers. Under both uncertainty correlation assumptions the difference is the greatest with the lowest expected return and decreasing with higher expected returns.

Table 9: Portfolio returns efficient frontier for correlated uncertainties.

Expected return	3.5	3.9	4.5	5.6	6.7	8.2
CASH	62.272	44.154	21.791	0	0	0
GOV	0	0	0	0	0	0
EMD	0	0	0	0	0	0
IG	14.537	15.452	16.220	3.807	0	0
HY	3.531	9.537	16.817	38.412	60.646	10.117
EQ	0.916	1.710	2.774	5.992	11.598	23.804
PE	0	0	0	0	0	0
PD	0	0	0	0	0	0
HF	2.240	4.914	8.443	16.772	27.005	31.969
RE	0.231	0	0	0	0	0
CF	16.273	24.233	33.955	35.017	0.750	0
INFRA	0	0	0	0	0	34.110
USDEUR	1.457	4.484	8.399	19.689	34.521	53.474

6.1.1 Sensitivity

Table 11 shows the resulting portfolios with varying the values of the β parameter. This has a very small effect on the composition of the portfolios. These were calculated with a fixed target return of 6%.

Table 12 shows the resulting portfolios with varying the initial solvency ratio and with fixed target return of 6%. The requirement constraint depends on the solvency ratio. The smaller the initial solvency ratio, the more the set of allowed portfolios is reduced. This results, for example, in higher weight for government bonds and smaller equity weight. In the higher initial solvency ratios, the requirement constraint is not active with 6% target return. The resulting portfolios are different because the solvency ratio determines with which ratio the portfolio returns and the change in the liabilities are weighted.

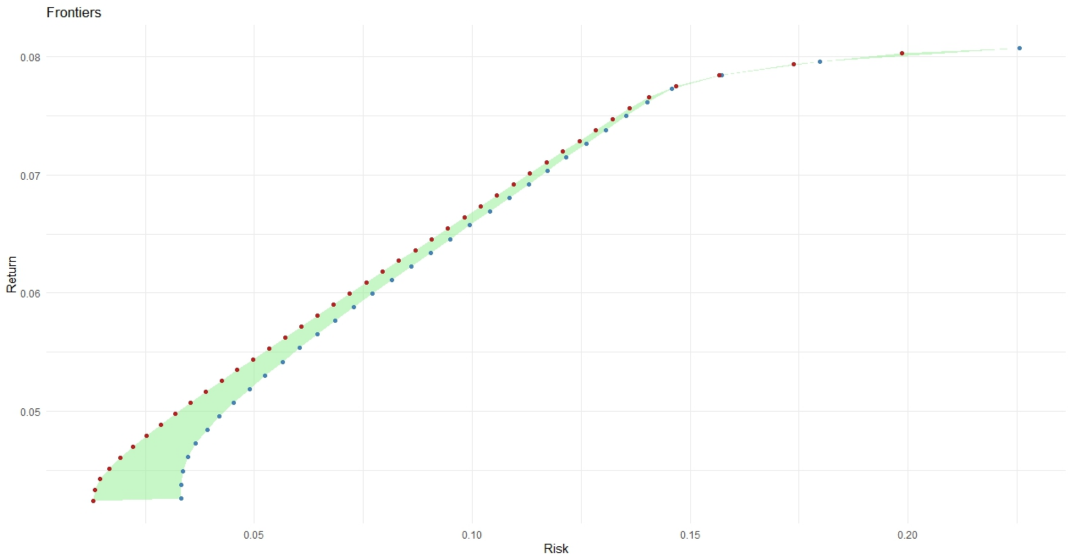

Figure 9: Capital efficient frontier for correlated uncertainties.

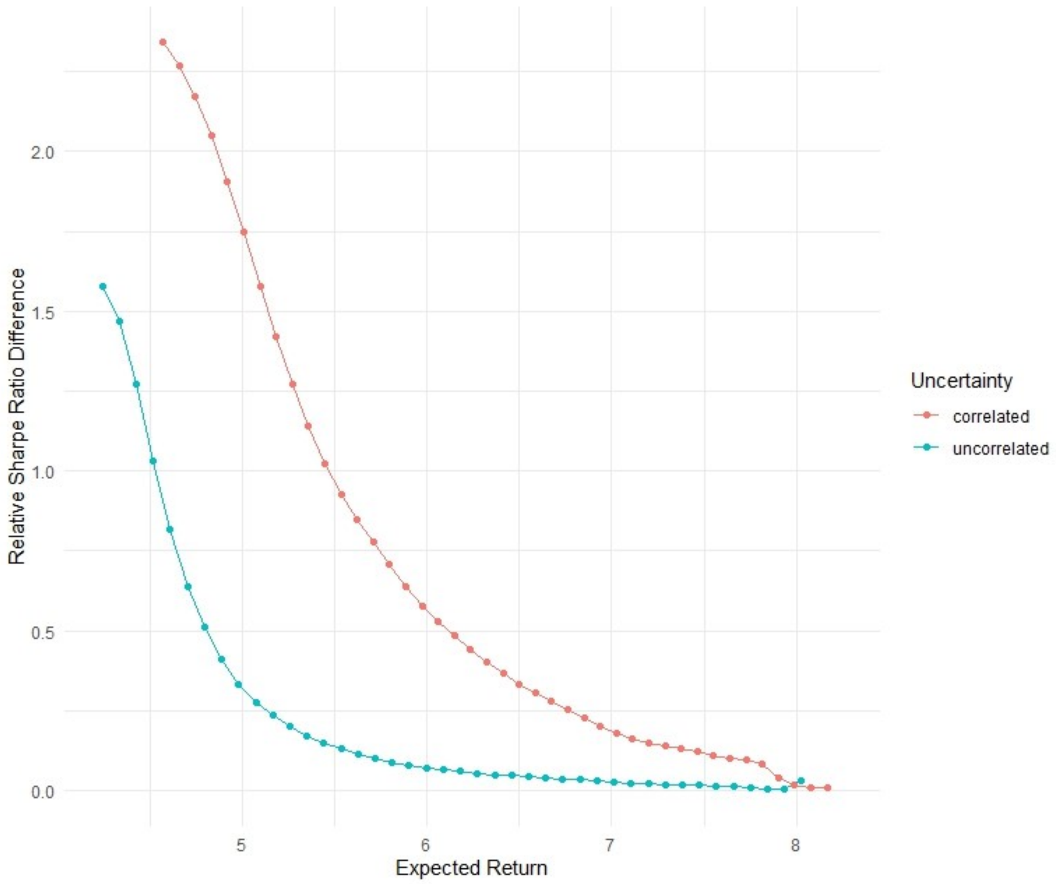

Figure 10: Relative difference in Sharpe ratios across the frontier under both uncertainty correlation assumptions.

Table 10: Capital efficient frontier for correlated uncertainties

Expected return	4.6	4.9	5.4	6.2	7.1	8.1
CASH	46.318	33.351	15.645	0	0	0
GOV	0	0	0	0	0	0
EMD	0	0	0	0	0	0
IG	14.494	16.030	16.899	5.961	0	0
HY	4.037	8.102	14.405	31.598	49.131	20.558
EQ	19.186	20.015	20.804	23.410	27.933	28.182
PE	0	0	0	0	0	0
PD	0	0	0	0	0	0
HF	2.965	5.194	7.942	14.892	22.936	33.639
RE	1.264	0.248	0	0	0	0
CF	11.736	17.060	24.305	24.139	0	0
INFRA	0	0	0	0	0	17.621
USDEUR	3.107	5.520	8.317	17.690	29.516	55.636

Table 11: Sensitivity of asset class weights with regards to the β-parameter.

β	0.9	0.8	0.7	0.6	0.5
CASH	0	0	0	0	0
GOV	0	0	0	0	0
EMD	0	0	0	0	0
IG	12.289	11.553	11.507	12.024	11.169
HY	27.692	28.183	28.028	28.484	28.432
EQ	22.295	22.326	22.206	22.016	21.982
PE	0	0	0	0	0
PD	0	0	0	0	0
HF	13.508	13.510	13.835	13.914	13.975
RE	0	0	0	0	0
CF	23.932	24.321	24.376	23.442	24.325
INFRA	0.284	0.107	0.049	0.122	0.116
USDEUR	14.212	14.061	14.162	14.101	14.146

Table 12: Asset class weights with varying intital solvency ratio.

r_{t_0}	0.05	0.1	0.15	0.2	0.25	0.3
CASH	0	0	0	0	0	0
GOV	0.256	0.042	0	0	0	0
EMD	0	0	0	0	0	0
IG	0	0.009	0.131	0.123	0.119	0.113
HY	0.138	0.283	0.268	0.277	0.284	0.292
EQ	0.167	0.229	0.230	0.223	0.218	0.212
PE	0	0	0	0	0	0
PD	0	0	0	0	0	0
HF	0.158	0.128	0.130	0.135	0.139	0.143
RE	0	0	0	0	0	0
CF	0.099	0.304	0.239	0.239	0.237	0.237
INFRA	0.182	0.006	0.003	0.003	0.002	0.002
USDEUR	0.120	0.137	0.136	0.142	0.146	0.150

6.2 Minimization of expected shortage

Figures 11 and 12 show the results for the minimization of expected shortage. The are performed with both the uncorrelated uncertainty returns and the correlated uncertainty returns.

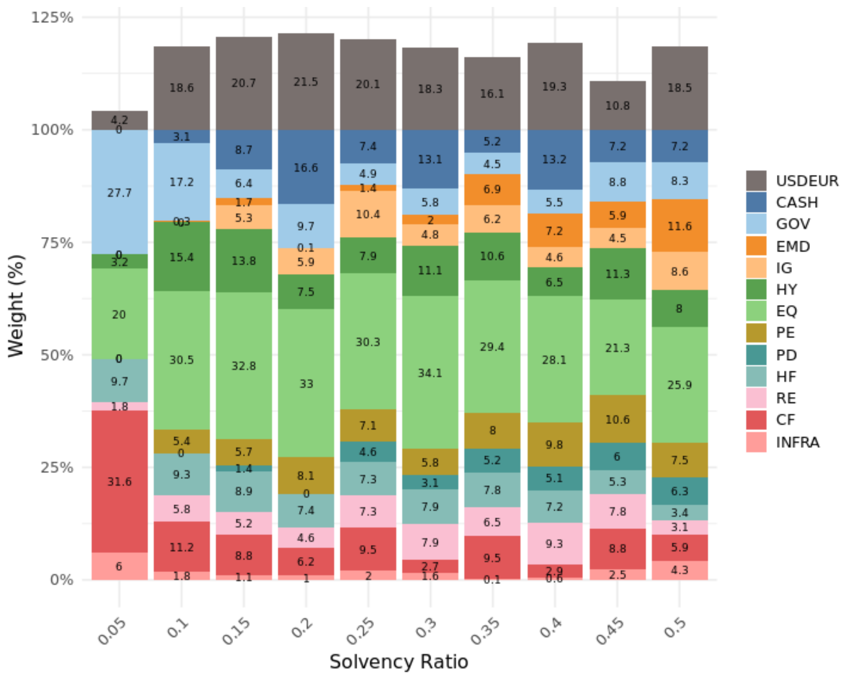

Figure 11: Minimization of expected shortage with varying solvency ratios (correlated uncertainties).

6.2.1 Dynamic strategy optimization

Figures 13 and 14 show the portfolios for the dynamic strategy optimization for each solvency ratio. Figure 15 shows the evolution of the objective function for the GA. The shape of the progression indicates that the objective could be improved even further with more iterations. The fact that the objective value is improved and the cobyla algorithm's stopping criterion was satisfied shows that the cobyla algorithm had converged to a local extrema.

6.3 Minimization of insolvency probability

Figure 16 and Figure 17 show the portfolios for the minimization of insolvency probability for each solvency ratio.

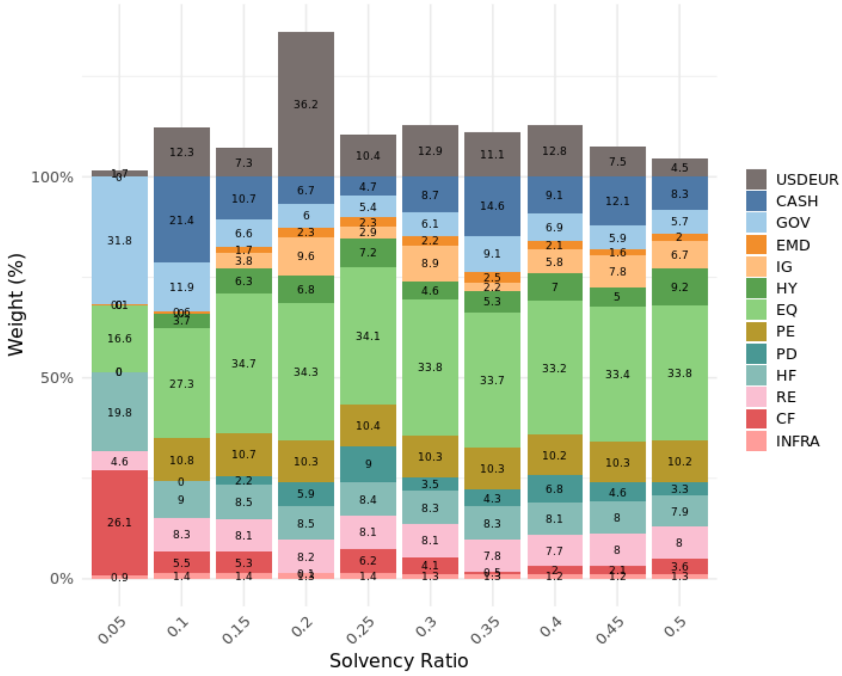

Figure 12: Minimization of expected shortage with varying solvency ratios (uncorrelated uncertainties).

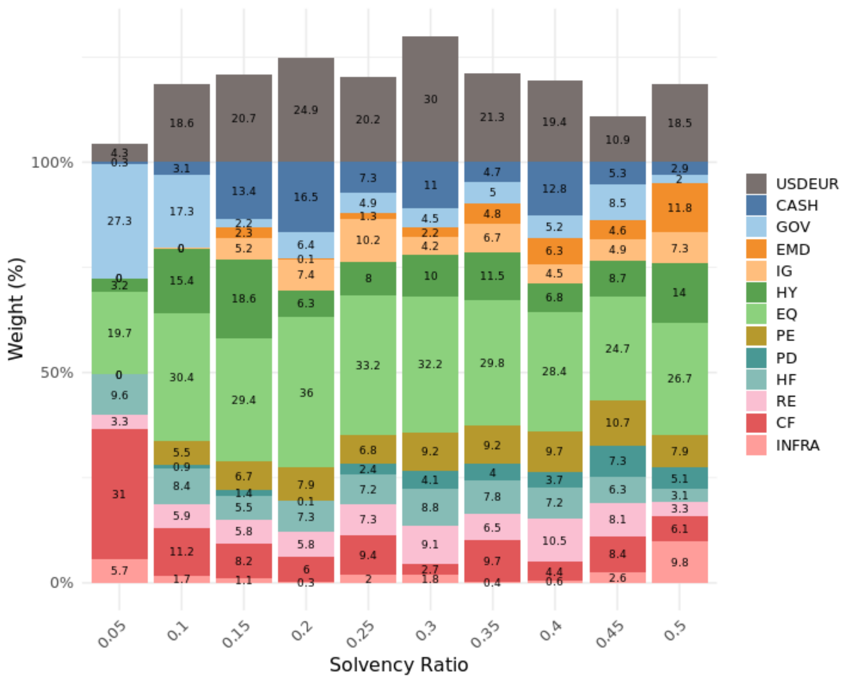

Figure 13: Dynamic strategy optimization with varying solvency ratios (correlated uncertainties).

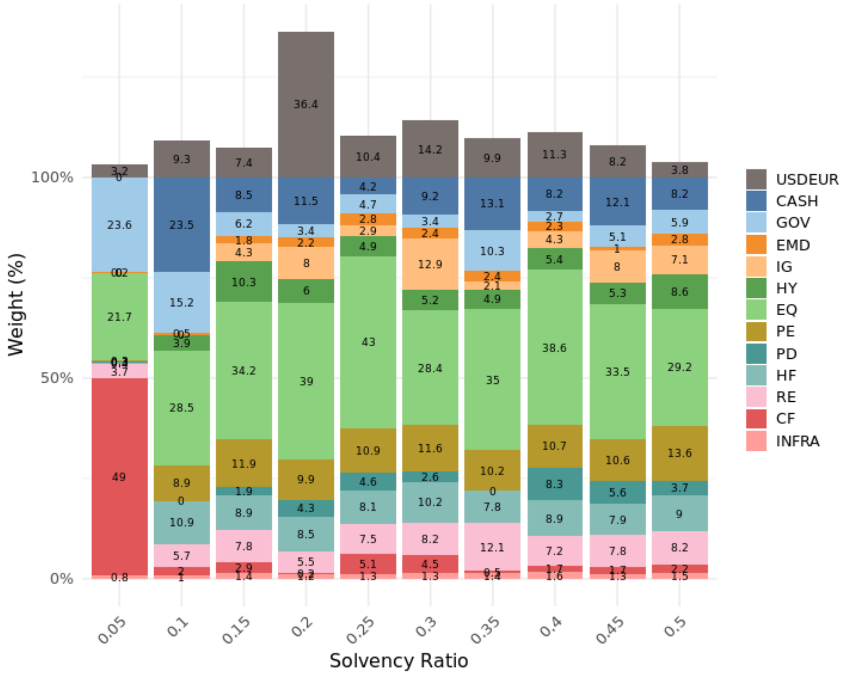

Figure 14: Dynamic strategy optimization with varying solvency ratios (uncorrelated uncertainties).

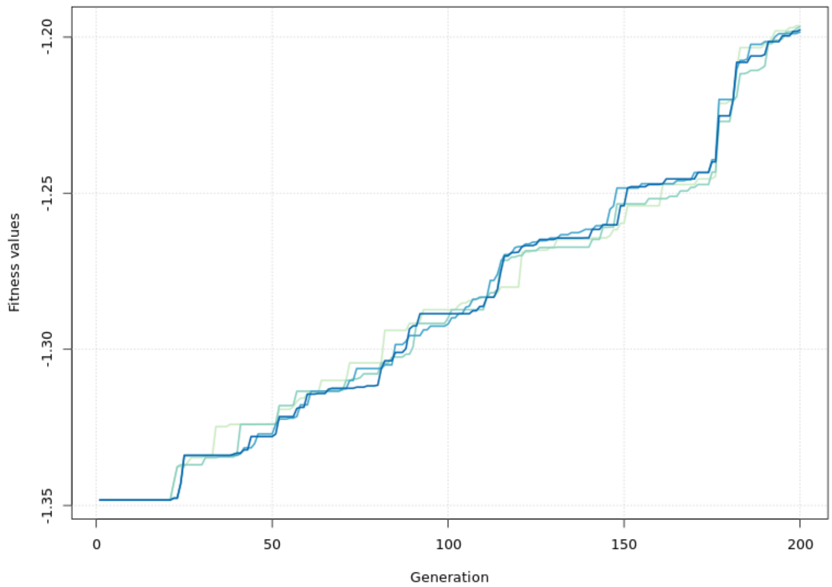

Figure 15: Evolution of the GA.

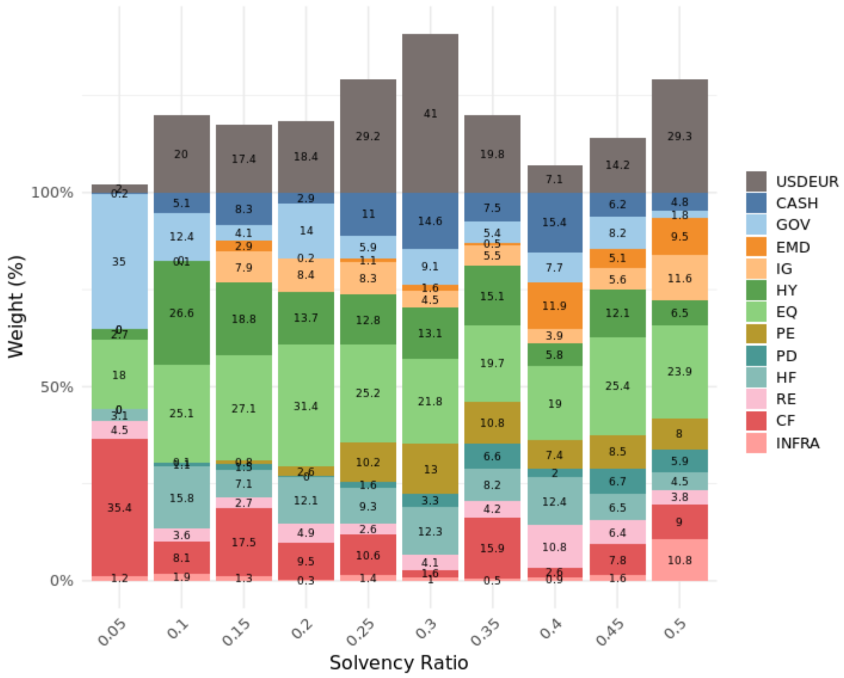

Figure 16: Minimization of insolvency probability with varying solvency ratios (correlated uncertainties).

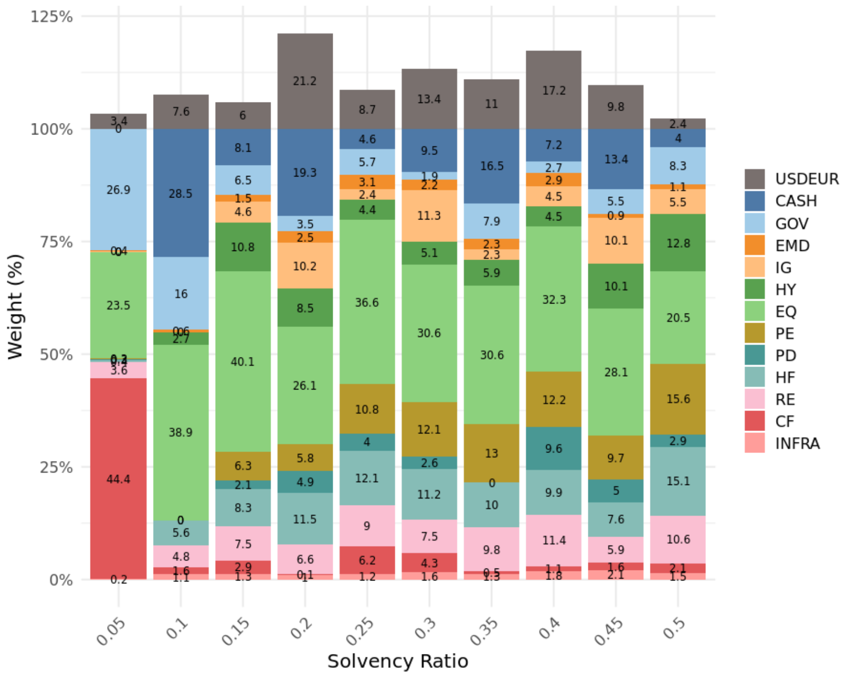

Figure 17: Minimization of insolvency probability with varying solvency ratios (uncorrelated uncertainties).

6.4 Comparative analysis

Table 13 and Table 14 show metrics for the different optimization results. The minimization of expected shortage is calculated as a constant portfolio with different initial solvency ratios but the metrics are calculated based on the strategy it yields.

When the uncertainty correlations are assumed to be uncorrelated, the shift from the constant to the dynamic formulation reduces the expected shortage by 59.4% reducing the number of bankruptcy scenarios by 230 out of 10000 scenarios. The minimization of insolvency probability finds a strategy with the lowest probability of bankruptcy (49.3%). This increases the expected shortage of the strategy significantly. The runtimes of the strategy based algorithms are between 8 and 31 hours which makes them impractical in terms of the sensitivity analysis on the parameters and assumptions.

With the correlated uncertainty assumption the minimization of insolvency probability achieved an objective value 24.5%.

Figure 18 shows the mean Fund transfer obligation for each timestep and the long term average. It has an upward trend where the value for the first year is 5.3% and for the 10-th year 7.3%. The 10-year annual average is 6.01%. The slope of the curve seems to flatten between the 3-10 year period. The initial decline is likely due to the b_{16} formula. The initial value is given as an input value while the rest are calculated. The upward trend suggests that the optimizations are likely to yield higher risk portfolios the longer the time period is selected.

Figure 19 and Figure 20 show the optimization results within the mean-volatility space. The 10-year annual average fund transfer obligation is marked as the dashed line. The portfolios are numbered within the order of the solvency ratios (1=0.05, ..., 10=0.5).

These results are not associated with clear frontiers, which highlights the difficulty of drawing definitive conclusions based on these optimizations. With both uncertainty correlation assumptions, the constant minimization of expected shortage exhibits smaller dispersion within the portfolios compared to the dynamic approach. This is expected as the dynamic strategy optimization includes the freedom to modify the risk level based on the solvency ratio. The minimization of insolvency probability has the largest dispersion within the portfolios. There is an increased concentration of portfolios around expected return of 6.25%.

The correlation assumption for the uncertainties affects significantly the volatility of the portfolio returns. When they are assumed to be correlated the volatilities range between 4.3–10.5% and when uncorrelated between 10.9–28.3%.

Table 13: Cross comparison of the objective values (uncorrelated).

	Expected shortage	Probability of bankruptcy	Runtime (s)
ES (constant)	2.950	0.575	≈ 30000
ES (strategy)	1.196	0.552	≈ 80806
Probability	2.422	0.493	≈ 59700

Table 14: Cross comparison of the objective values (correlated).

	Expected shortage	Probability of bankruptcy	Runtime (s)
ES (constant)	0.459	0.413	≈ 33600
ES (strategy)	0.414	0.433	≈ 108585
Probability	0.990	0.246	≈ 33000

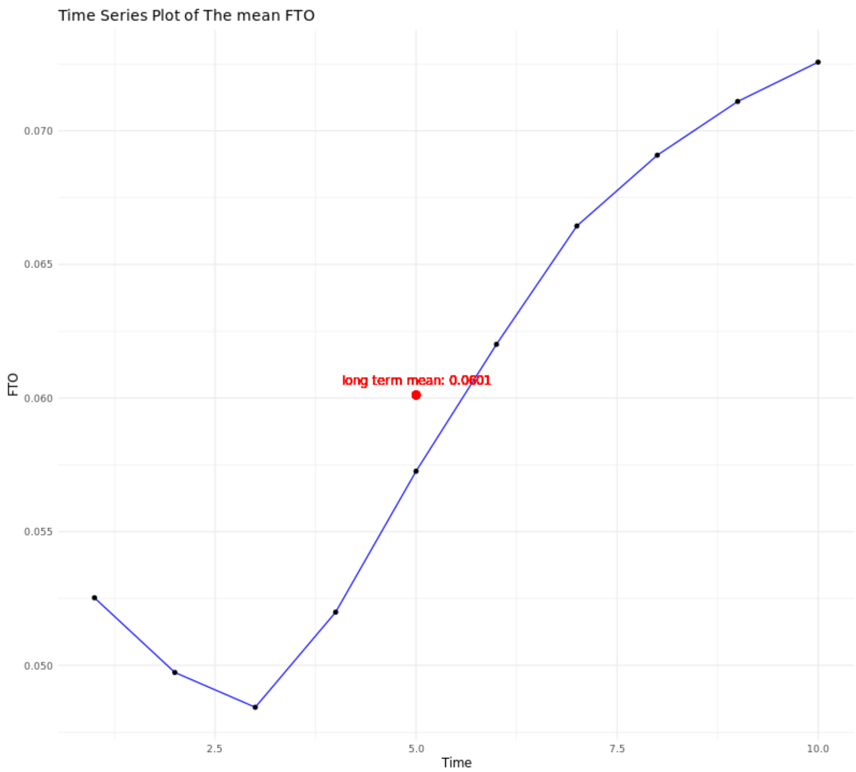

Figure 18: Time series of the mean FTO.

The lowest expected return is observed with the lowest solvency ratios. These portfolios are below the long term mean FTO due to the solvency requirement constraint. There are also portfolios below the long term mean FTO where the solvency requirement constraint is not active. This is unexpected and potentially brings forth shortcomings of the models with the selected parameters. With a portfolio where the expected return is below this line, the solvency ratio of the pension fund is decreasing on average. It may be that with a time period of ten years, the program is not able to capture this, the sample size of 10000 is not enough, or the solutions have converged to local extrema.

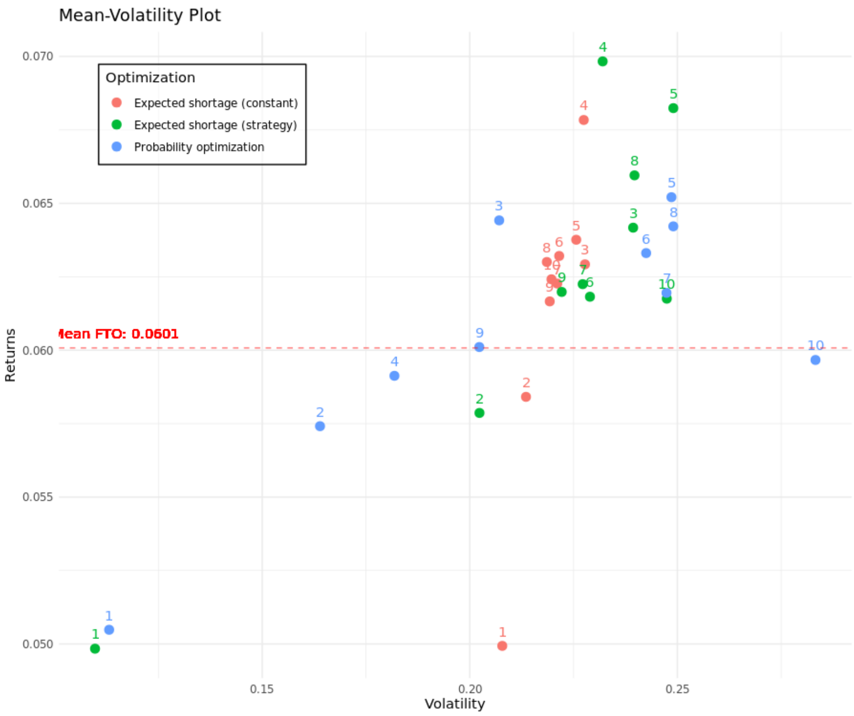

Figure 19: Optimization results in the mean-volatility space (uncorrelated uncertainties).

Figure 20 and Figure 19 could also be presented with the solvency capital CVaR risk measure. However, the solvency ratio affects the value of the risk, and thus it is clearer to compare portfolios with different solvency ratios with a risk measure that is not affected by it.

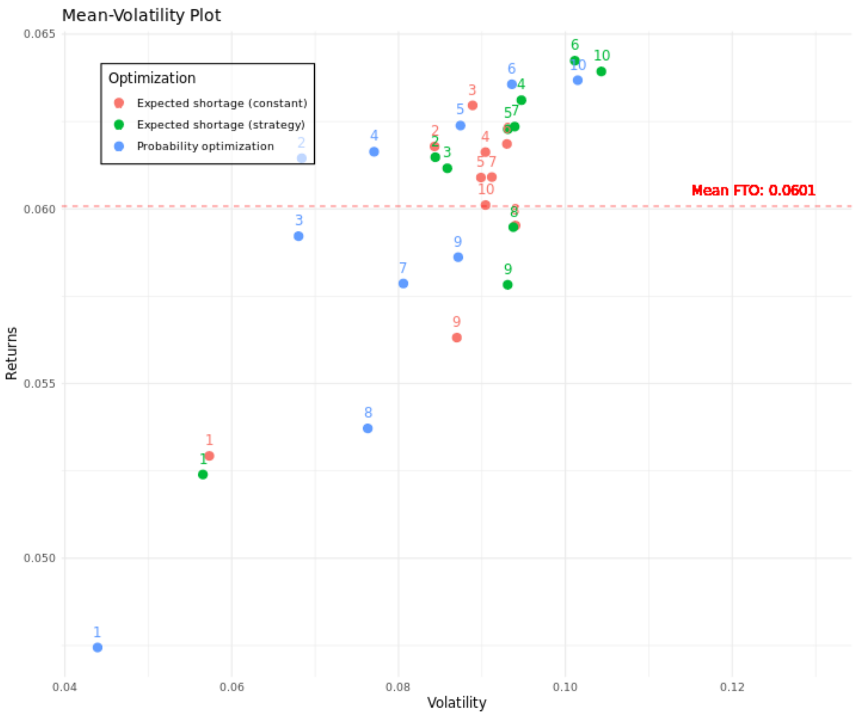

Figure 20: Optimization results in the mean-volatility space (correlated uncertainties).

7 Summary

This thesis has explored multiple problem statements for optimizing the portfolio of a pension fund within the Finnish regulatory framework. All the optimizations were based on how the liabilities change as a function of the portfolio decisions. We succeeded in formulating the problem in a way which captures the solvency requirement by programming the equations defined in the law, making the necessary assumptions and gathering the needed input values.

We were not able to explicitly formulate a definite and solvable minimization problem. It is why we explored multiple models. The key difference between the models is the difference between the loss functions applied and their feasibility in practical use.

The capital based CVaR formulation uses a linear loss function which is easier to solve but requires adjustment of extra parameters. These are the β-probability, the target return, the position target, and time. This formulation shows consistent results. The portfolios did not exhibit large deviations in asset weights with the adjustment of the input parameters. Furthermore, it can be compared easily to the traditional portfolio returns based CVaR formulation. This makes it possible to examine how the weights of the asset classes change when the risk is directed to the solvency capital of the pension provider. The portfolio returns based formulation is used as a benchmark. The comparison showed significant differences in the sharpe ratios when the risk was defined in terms of the 0.9-CVaR of the solvency capital. It can be concluded that the minimization of CVaR is the most useful method for practice.

In the models with a nonlinear the loss function, the expected return is solved indirectly and there is no need to define the parameter β nor the target return. The difference between the dynamic strategy optimization and the minimization of expected shortage is the assumption whether the portfolio policy is constant or dynamic. Both models are included for comparison. The solution for the minimization of expected shortage is also used to produce the initial guesses for the dynamic strategy optimization. They minimize the expected amount that the fund is insolvent. Minimization of insolvency probability is included for its theoretical accuracy in minimizing the bankruptcy probabilities.

All models with a nonlinear loss function exhibit large deviations in asset weights across the strategies. Each model returned the lowest objective value for their respective objective function. The shift from the constant to the dynamic strategy created more dispersion within the portfolios as expected. The dynamic strategy improved the objective values when the model explicitly minimized not meeting the future solvency requirements. This highlights the significance of the program having the freedom to modify the weights during the scenarios in such formulations.

The genetic algorithm showed that we were not able to find global extremum with the models that had nonlinear loss function. Without a reliable extremum it is hard to identify what causes the inconsistency within the results. The runtimes of the algorithms prohibited adequate sensitivity analysis. Increasing the sample size could lead to more consistent results. The discount factor of the expected shortage risk measure is likely to affect the resulting portfolios. An increase within selected time period of the program may also yield higher risk portfolios. The dynamic strategy optimization consists of defining a $k \times m$ matrix, where each column represents a different solvency ratio and each row represents the portfolio weights. The number of portfolios m in a given strategy is likely to effect the number of required samples for consistent results.

Due to the unreliability of the solutions the most useful way to use the algorithms with a nonlinear loss function would be to start with a predetermined strategy as an initial guess and monitor the change within the objective value and the asset weights.

The choice of the algorithm could be researched further. It is likely that there are suitable algorithms for such problems in addition to those presented in previous sections. Almost certainly the GA could be tailored further for this task.

Two fixed correlation assumptions of the uncertainties were examined in the sampling process, assuming that they are either correlated or fully uncorrelated. This was practical decision to simplify the process. In reality, the correlations of the uncertainties are likely to be somewhere in between these two extreme cases. An idea for further research is to link the correlation assumption of the uncertainties to a parameter. This would allow for a continuous analysis between the two extremes. While having the two sets of returns makes it challenging to single out a specific portfolio as the best one, it does not prohibit the comparison of the models.

A Appendix

Table A1: The exposures each asset class has to a specific risk class E (Ch. 3.2).

Asset class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
CASH	0	0	0	0	0	0.200	0	0.100	0.100	0	0	0	0	0	0	0	0	0
GOV	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
EMD	0	0	0	0	0	1	0.070	0.050	0.440	0.440	0	0	0	0	0	0	0	0
IG	0	0	0	0	0	1	0.050	0.070	0.870	0.010	0	0	0	0	0	0	0	0
HY	0	0	0	0	0	1	0	0.025	0.025	0.950	0	0	0	0	0	0	0	0
EQ	0.480	0.320	0.150	0.050	0	0	0	0	0	0	0	0	0	0	0	0	0	0
PE	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
PD	0	0	0	0	0.160	0.840	0	0	0	0.840	0	0	0	0	0	0	0	0
HF	0.008	0.093	0	0.232	0	0.077	0	0.213	0	0.024	0	0	0	0.038	0	0	0.099	0
RE	0	0	0	0	0.065	0.030	0	0	0	0.035	0.300	0.600	0	0	0	0	0	0
CF	0	0	0	0	0	1	0	0.300	0.680	0.020	0	0	0	0	0	0	0	0
INFRA	0	0	0	0	0.150	0	0	0	0	0	0	0	0	0	0	0	0	0.850
USDEUR	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0

Table A2: Inputs for solvency requirement calculation.

	Asset class	Modified duration	Spread duration	Lever
1	CASH	0.250	0.250	0
2	GOV	3.200	3.200	0
3	EMD	5	5	0
4	IG	6	6	0
5	HY	3	3	0
6	EQ	0	0	0
7	PE	0	0	0
8	PD	0.600	2.600	0.010
9	HF	4.500	4.500	0
10	RE	2	2	0.100
11	CF	0.580	3.500	0
12	INFRA	0	0	0.012
13	USDEUR	0	0	0

Table A3: Asset class correlation assumptions.

	CASH	GOV	EMD	IG	HY	EQ	PE	PD	HF	RE	CF	INFRA	USDEUR
CASH	1	0.420	-0.100	0.140	-0.140	-0.440	-0.460	-0.440	-0.470	-0.500	0.140	-0.390	-0.180
GOV	0.420	1	0.290	0.770	0.320	0.110	0.066	-0.073	-0.096	-0.410	0.770	0.200	-0.011
EMD	-0.100	0.290	1	0.800	0.950	0.720	0.700	0.460	0.680	-0.370	0.800	0.520	0.420
IG	0.140	0.770	0.800	1	0.810	0.530	0.500	0.270	0.380	-0.450	1	0.480	0.240
HY	-0.140	0.320	0.950	0.810	1	0.780	0.750	0.500	0.720	-0.330	0.810	0.620	0.360
EQ	-0.440	0.110	0.720	0.530	0.780	1	0.940	0.680	0.860	0.023	0.530	0.820	0.230
PE	-0.460	0.066	0.700	0.500	0.750	0.940	1	0.710	0.850	0.042	0.500	0.750	0.210
PD	-0.440	-0.073	0.460	0.270	0.500	0.680	0.710	1	0.800	0.460	0.270	0.700	0.150
HF	-0.470	-0.096	0.680	0.380	0.720	0.860	0.850	0.800	1	0.097	0.380	0.710	0.340
RE	-0.500	-0.410	-0.370	-0.450	-0.330	0.023	0.042	0.460	0.097	1	-0.450	0.280	-0.160
CF	0.140	0.770	0.800	1	0.810	0.530	0.500	0.270	0.380	-0.450	1	0.480	0.240
INFRA	-0.390	0.200	0.520	0.480	0.620	0.820	0.750	0.700	0.710	0.280	0.480	1	0.010
USDEUR	-0.180	-0.011	0.420	0.240	0.360	0.230	0.210	0.150	0.340	-0.160	0.240	0.010	1

Tables A4–A15 show the results of the optimizations with nonlinear loss function.

Table A4: Expected shortage portfolios with varying solvency ratio (0.05-0.25) for uncorrelated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0	21.411	10.655	6.684	4.729
GOV	31.824	11.922	6.617	5.972	5.353
EMD	0.112	0.644	1.652	2.277	2.300
IG	0	0.001	3.819	9.580	2.929
HY	0.001	3.703	6.336	6.753	7.170
EQ	16.611	27.261	34.657	34.333	34.069
PE	0.000	10.836	10.709	10.336	10.423
PD	0.002	0.004	2.229	5.886	8.954
HF	19.770	9.043	8.548	8.507	8.410
RE	4.641	8.289	8.110	8.199	8.128
CF	26.130	5.527	5.289	0.143	6.172
INFRA	0.910	1.359	1.379	1.330	1.362
USDEUR	1.662	12.311	7.277	36.168	10.419

Table A5: Expected shortage portfolios with varying solvency ratio (0.3-0.5) for uncorrelated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	8.690	14.618	9.105	12.131	8.311
GOV	6.098	9.106	6.897	5.878	5.704
EMD	2.241	2.491	2.149	1.578	1.952
IG	8.915	2.161	5.811	7.803	6.710
HY	4.569	5.321	6.951	4.964	9.194
EQ	33.818	33.692	33.166	33.411	33.779
PE	10.310	10.318	10.201	10.339	10.193
PD	3.483	4.338	6.783	4.601	3.328
HF	8.319	8.287	8.050	8.034	7.926
RE	8.132	7.850	7.694	7.965	7.999
CF	4.142	0.547	1.995	2.093	3.610
INFRA	1.282	1.270	1.197	1.203	1.294
USDEUR	12.919	11.147	12.756	7.460	4.498

Table A6: Expected shortage portfolios with varying solvency ratio (0.05-0.25) for correlated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0.00001	3.095	8.699	16.575	7.359
GOV	27.741	17.217	6.405	9.666	4.864
EMD	0.00001	0.260	1.745	0.065	1.362
IG	0.00003	0.0001	5.326	5.913	10.390
HY	3.164	15.404	13.836	7.547	7.871
EQ	20.030	30.514	32.761	33.043	30.332
PE	0.0001	5.408	5.692	8.052	7.072
PD	0.001	0	1.449	0.040	4.594
HF	9.690	9.329	8.909	7.385	7.330
RE	1.764	5.805	5.229	4.571	7.306
CF	31.633	11.207	8.832	6.172	9.529
INFRA	5.977	1.761	1.117	0.970	1.989
USDEUR	4.200	18.599	20.714	21.516	20.079

Table A7: Expected shortage portfolios with varying solvency ratio (0.3-0.5) for correlated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	13.145	5.220	13.240	7.218	7.184
GOV	5.796	4.515	5.484	8.791	8.337
EMD	2.011	6.918	7.249	5.856	11.554
IG	4.766	6.207	4.600	4.457	8.612
HY	11.102	10.586	6.459	11.340	8.018
EQ	34.148	29.410	28.123	21.320	25.865
PE	5.783	8.018	9.779	10.608	7.544
PD	3.079	5.174	5.078	6.022	6.285
HF	7.862	7.784	7.160	5.315	3.412
RE	7.917	6.510	9.281	7.810	3.061
CF	2.745	9.511	2.916	8.797	5.858
INFRA	1.646	0.147	0.629	2.467	4.269
USDEUR	18.280	16.148	19.345	10.790	18.488

Table A8: Dynamic strategy optimization portfolios with varying solvency ratio (0.05-0.25) for uncorrelated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0.005	23.492	8.510	11.533	4.187
GOV	23.567	15.240	6.195	3.444	4.738
EMD	0.177	0.502	1.785	2.232	2.846
IG	0.035	0.046	4.260	8.032	2.915
HY	0.018	3.929	10.312	6.037	4.882
EQ	21.656	28.499	34.204	39.049	42.989
PE	0.324	8.854	11.870	9.948	10.918
PD	0.288	0.005	1.946	4.336	4.627
HF	0.446	10.855	8.873	8.483	8.097
RE	3.653	5.662	7.772	5.496	7.462
CF	49.044	1.956	2.885	0.159	5.079
INFRA	0.787	0.961	1.388	1.248	1.261
USDEUR	3.219	9.267	7.408	36.411	10.383

Table A9: Dynamic strategy optimization portfolios with varying solvency ratio (0.3-0.5) for uncorrelated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	9.231	13.085	8.243	12.057	8.153
GOV	3.404	10.251	2.715	5.102	5.899
EMD	2.428	2.432	2.334	0.984	2.845
IG	12.862	2.115	4.318	8.033	7.148
HY	5.219	4.936	5.434	5.340	8.645
EQ	28.443	35.039	38.563	33.548	29.226
PE	11.575	10.195	10.669	10.640	13.581
PD	2.612	0.038	8.253	5.565	3.687
HF	10.156	7.837	8.931	7.900	8.960
RE	8.248	12.150	7.213	7.819	8.184
CF	4.509	0.531	1.728	1.723	2.177
INFRA	1.313	1.392	1.601	1.290	1.496
USDEUR	14.187	9.877	11.316	8.167	3.842

Table A10: Dynamic strategy optimization portfolios with varying solvency ratio (0.05-0.25) for correlated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0.298	3.131	13.436	16.477	7.276
GOV	27.337	17.258	2.165	6.397	4.905
EMD	0.020	0.045	2.329	0.137	1.262
IG	0.011	0.047	5.250	7.414	10.232
HY	3.150	15.409	18.604	6.272	7.992
EQ	19.655	30.438	29.435	35.953	33.229
PE	0.013	5.508	6.740	7.878	6.822
PD	0.002	0.936	1.386	0.058	2.404
HF	9.575	8.401	5.536	7.317	7.190
RE	3.255	5.851	5.804	5.849	7.275
CF	30.988	11.237	8.241	5.978	9.401
INFRA	5.696	1.738	1.075	0.270	2.011
USDEUR	4.327	18.648	20.728	24.922	20.180

Table A11: Dynamic strategy optimization portfolios with varying solvency ratio (0.3-0.5) for correlated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	11.016	4.735	12.759	5.343	2.936
GOV	4.535	4.985	5.154	8.532	2.032
EMD	2.236	4.848	6.323	4.555	11.777
IG	4.237	6.732	4.530	4.857	7.331
HY	9.968	11.472	6.800	8.652	13.991
EQ	32.184	29.766	28.377	24.686	26.652
PE	9.244	9.201	9.747	10.702	7.909
PD	4.117	3.972	3.713	7.344	5.077
HF	8.846	7.786	7.185	6.264	3.078
RE	9.092	6.464	10.465	8.063	3.272
CF	2.710	9.681	4.389	8.449	6.138
INFRA	1.815	0.358	0.558	2.554	9.806
USDEUR	30.015	21.268	19.392	10.850	18.523

Table A12: Minimization of insolvency probability portfolios with varying solvency ratio (0.05-0.25) for correlated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0.219	5.136	8.331	2.907	11.003
GOV	34.953	12.411	4.073	13.961	5.893
EMD	0.022	0.047	2.869	0.154	1.098
IG	0.017	0.087	7.928	8.443	8.315
HY	2.681	26.641	18.777	13.714	12.789
EQ	17.967	25.113	27.083	31.372	25.196
PE	0.013	0.074	0.807	2.565	10.245
PD	0.002	1.106	1.487	0.039	1.559
HF	3.083	15.816	7.124	12.092	9.304
RE	4.453	3.575	2.725	4.923	2.590
CF	35.363	8.069	17.547	9.535	10.629
INFRA	1.227	1.927	1.251	0.295	1.379
USDEUR	2.003	19.987	17.417	18.446	29.247

Table A13: Minimization of insolvency probability portfolios with varying solvency ratio (0.3-0.5) for correlated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	14.593	7.502	15.444	6.179	4.756
GOV	9.061	5.442	7.698	8.238	1.832
EMD	1.606	0.528	11.866	5.060	9.505
IG	4.461	5.503	3.889	5.583	11.616
HY	13.123	15.124	5.841	12.077	6.479
EQ	21.812	19.695	19.043	25.360	23.896
PE	13.030	10.839	7.442	8.515	8.019
PD	3.337	6.576	2.044	6.712	5.882
HF	12.303	8.150	12.448	6.532	4.524
RE	4.063	4.249	10.810	6.358	3.755
CF	1.602	15.927	2.600	7.782	8.953
INFRA	1.008	0.465	0.874	1.606	10.783
USDEUR	40.957	19.840	7.118	14.181	29.293

Table A14: Minimization of insolvency probability portfolios with varying solvency ratio (0.05-0.25) for uncorrelated uncertainties.

	0.05	0.1	0.15	0.2	0.25
CASH	0.008	28.474	8.144	19.305	4.565
GOV	26.864	16.034	6.535	3.501	5.662
EMD	0.387	0.616	1.470	2.533	3.128
IG	0.028	0.037	4.565	10.177	2.393
HY	0.020	2.722	10.802	8.460	4.365
EQ	23.544	38.927	40.135	26.074	36.600
PE	0.186	0.004	6.258	5.837	10.821
PD	0.313	0.006	2.071	4.854	3.963
HF	0.444	5.645	8.302	11.511	12.076
RE	3.586	4.847	7.497	6.587	8.973
CF	44.438	1.560	2.910	0.126	6.235
INFRA	0.183	1.129	1.312	1.034	1.218
USDEUR	3.352	7.594	5.953	21.236	8.706

Table A15: Minimization of insolvency probability portfolios with varying solvency ratio (0.3-0.5) for uncorrelated uncertainties.

	0.3	0.35	0.4	0.45	0.5
CASH	9.510	16.469	7.201	13.368	4.002
GOV	1.861	7.910	2.652	5.500	8.311
EMD	2.241	2.313	2.851	0.868	1.084
IG	11.327	2.289	4.546	10.116	5.495
HY	5.097	5.851	4.454	10.073	12.793
EQ	30.620	30.633	32.255	28.147	20.533
PE	12.120	12.997	12.185	9.708	15.562
PD	2.646	0.030	9.582	5.002	2.850
HF	11.173	9.975	9.884	7.642	15.138
RE	7.515	9.768	11.421	5.853	10.575
CF	4.330	0.473	1.129	1.649	2.133
INFRA	1.560	1.293	1.839	2.073	1.524
USDEUR	13.384	10.956	17.231	9.805	2.358

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