Travel-time Optimal Line Plans on Trees

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Abstract

Due to the urban transformation of past decades, the need for efficient and effective public transportation systems continues to increase. Consequently, many methods have been developed to optimize different components of public transportation systems using various criteria. One often desired result for these methods is to create a line plan where the travel time of the passengers is minimal. Schobel and Scholl developed a formulation for the line optimization problem that achieves this result and manages to consider the total travel time for all passengers. However, due to the large size and the high complexity of the formulation, it cannot be solved in any straightforward manner.

This thesis aims to expand on the work of Schöbel and Scholl by developing a formulation for star-shaped station networks that acts as a simpler alternative to their binary frequency formulation. The alternative formulation is developed by leveraging the special features of star-shaped networks and tree networks in general: There exists only one simple path between any two nodes of a tree. Thus, we show that the driving time of any passenger is constant for their trip, and the travel time only depends on the passenger's transfers. Consequently, the developed formulation tracks only the transfers, not the entire route of the passengers, and optimizes the total travel time by minimizing the total number of said transfers.

As part of this thesis, we prove the developed formulation to be equivalent to the binary frequency formulation of Schöbel and Scholl in star-shaped trees. It is also found that the developed formulation is significantly smaller than the formulation of Schöbel and Scholl, especially for larger star networks. This implies that the developed formulation is simpler to solve, as desired. The successfully developed formulation thus proves that the formulations of the line planning problems can be simplified by limiting the underlying networks to trees. Additionally, the developed formulation has limited practical applications and benefits.

Keywords Line planning, travel-time optimization, star-shaped trees

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Tiivistelmä

Viime vuosikymmenten kaupungistumisen takia tarve tehokkaille ja toimiville julkisille liikennejärjestelmille kasvaa jatkuvasti. Täten, liikennejärjestelmien eri osien optimointiin on myös kehitetty useita metodeja, useille eri optimaalisuuskriteereille. Yksi yleinen tavoite tällaiselle metodille on muodostaa järjestelmälle linjastosuunnitelma, jossa matkustajien matkustusaika on mahdollisimman pieni. Schöbel and Scholl ovat kehittäneet tähän optimointiongelmaan mallin, joka saavuttaa tämän tavoitteen minimoiden onnistuneesti kaikkien matkustajien yhteenlasketun matkustusajan. Kyseistä mallia ei kuitenkaan voida ratkaista ilman uudelleenmuotoilua sen suuren koon ja kompleksisuuden takia.

Tämän työn tavoitteena täydentää Schöbelin ja Schollin työtä kehittämällä optimointiongelmaan vaihtoehtoisen mallin tähdenmuotoisissa pysäkkiverkostoissa, joka on Schöbelin ja Schollin mallia yksinkertaisempi ratkaista. Vaihtoehtoinen malli kehitettiin tähdenmuotoisten verkkojen ja yleisesti puu-verkkojen erikoisominaisuuksien avulla: Puun kahden solmun välille voidaan luoda täsmälleen yksi yksinkertainen polku. Tämän ominaisuuden pohjalta työssä osoitetaan, että aika, jonka matkustaja käyttää ajamiseen on aina tietylle matkalle vakio. Matkustajan matkustusaika siis riippuu vain hänen tekemistään vaihdoista. Täten, kehitetty malli pitää kirjaa vain matkustajien tekemistä vaihdoista, ei matkustusreitistä, ja optimoi yhteenlasketun matkustusajan minimoimalla tehtyjen vaihtojen summan.

Todistamme tässä työssä, että kehitetty malli sekä Schöbelin ja Schollin malli ratkaisevat saman ongelman. Lisäksi huomataan, että kehitetty malli on kooltaan huomattavasti Schöbelin ja Schollin mallia pienempi, erityisesti suurien pysäkkiverkostojen tapauksissa. Tämä vihjaa kehitetyn mallin olevan myös helpompi ratkaista tavoitteiden mukaisesti. Tämä onnistuneesti kehitetty malli osoittaa siis, että linjastosuunnittelun ongelmiin voidaan muodostaa yksinkertaisempi malli, kun infrastuktuuria kuvaava pysäkkiverkko rajoitetaan puuverkkoihin. Kehitettyä mallia voidaan myös soveltaa ja hyödyntää linjastosuunnittelusta rajoitetusti.

Avainsanat Linjastosuunnittelu, matkustusajan optimointi, tähdenmuotoiset puut

Contents

1 Introduction

Due to the urban transformation of the past decades and the expanding need for more ecological alternatives to private car ownership, efficient and effective public transportation has become increasingly important. Consequently, as Schöbel [7] points out, great efforts have been committed to researching the optimization of public transportation systems.

According to several authors, this process of optimization consists of several phases responsible for planning one component of the public transportation system, such as the location of the stops or the timetable of the vehicles transporting passengers. One of these phases is called line planning, and the goal of this phase is to ascertain which paths in the network of stations referred to as lines should be serviced in the public transportation system. [2, 6, 7] This set of lines is referred to as the line concept of the system.

The line planning phase is executed by solving an optimization problem called a line planning problem which determines the optimal set of lines according to the desired metric. Since the criteria for optimality can be freely determined, there are multiple approaches to constructing the line planning problems. As Schöbel [6, 7] mentions, these approaches can, however, be divided into two categories: cost-oriented models, which minimize the cost while adequate service is ensured, and passenger-oriented models, which maximize passenger comfort within the constraints of the given budget.

The methodology used to construct line planning problems varies especially in the case of passenger-oriented problems. This remains true even when we are only focusing on passenger-oriented problems, in which we aim to minimize the travel times of the passengers. For example, some formulations aim to maximize the number of direct passengers, some minimize the total time spent traveling in a vehicle, and a model developed by Schöbel and Scholl minimized the total travel time including the transfers [7].

Even though all of these formulations produce viable line concepts, they do have their issues, as indicated by Schöbel and Scholl: The direct passenger method easily disregards the comfort of numerous passengers, and minimizing the time spent in vehicles may result in passenger routes with unnecessarily long or numerous transfers. The formulation by Schöbel and Scholl resolves both of these issues by tracking all possible driving connections and transfers. However, due to this, the size of the formulation prevents it from being solvable in any straightforward manner. [8]

The goal of this thesis is to expand upon the formulation developed by Schöbel and Scholl [8] by providing a formulation that acts as a simpler alternative in the case of star-shaped trees: The specific features of trees should reduce the number of variables required to track passenger behavior. Thus, we aim to leverage these special features to develop a line-planning problem formulation which is both equivalent to the formulation in [8] and easier to solve.

2 Background

As indicated by several authors in the research papers [1, 2, 3, 8], the line planning problem, referred to later as LPP, has been extensively studied since late 20th century. Since there are multiple ways to define optimal in the case of public transportation, this research has led to the development of numerous LPP models. Schöbel and Scholl [7, 8] distinguish that the developed models can be separated into two categories, "the cost-oriented approach" and "the customer-oriented approach".

In the cost-oriented approach, the model aims to find a line concept with minimal costs while servicing all customers using the transportation network. These costs can contain the initial, daily operation or management costs, or any combination of these. In the customer-oriented approach, the aim is to find a line concept that maximizes customer comfort, while still fulfilling the constraints imposed by the underlying infrastructure and limited budget. [6, 7, 8] Some models, such as the ones given by Borndörfer, Grötschel, and Pfetsch [1, 2, 3], belong to both of these categories since they combine the two competing approaches using a weighted objective function. We shall focus on the literature relating to models implementing the customer-oriented approach since our new model is a customer-oriented model.

As explained by Schöbel [7] and Borndörfer et al.[1, 2], one of the earlier developed methods for LPPs using the customer-oriented approach is the direct passenger method. In this method, the aim is to maximize the number of passengers making zero transfers while traveling, possibly within the limitations of additional budget and vehicle capacity constraints. [7] One model utilizing this method was explored by Bussieck et al.: They created a mixed integer linear programming formulation of a LPP using integer variables to model the number of direct travelers on a line traveling from a specific destination to another, and to model the frequency of vehicles on some lines. In the formulation, the sum of the direct passengers is maximized while three capacity and coverage constraints ensure that all solutions are realistic and have sufficient coverage. [4]

Models utilizing the direct passenger method can be solved utilizing conventional algorithms for integer programming: In their paper, Brussieck et al. demonstrate that a mixed integer formulation maximizing direct travelers is possible to be solved utilizing LP-relaxations and solvers utilizing the branch-and-bound method. However, Bussieck et al. did have to reduce the formulation by replacing the numbers of passengers with their upper and lower bounds before this was possible since the original formulation was too large for the solvers. [4] On the other hand, Borndörfer et al. [1, 2, 3] mention that the direct passenger method does not consider the waiting times associated with transfers. Additionally, the method does not consider the driving times of the lines or the paths of non-direct passengers, as indicated by Schöbel and Scholl [8]. Consequently, as Schöbel and Scholl [8] mention, the method allows non-direct passengers to make arbitrarily many transfers and makes direct passengers travel using only one line even if another path including transfers might be quicker. Therefore, the models using the direct passenger method may assign unnecessarily complicated paths to passengers.

Schöbel mentions in her literature review [7] that another frequent method for formulating LPPs of the customer-oriented approach is to minimize the total travel time of all passengers while simultaneously routing them. The routing is done while respecting the given budgetary constraints. According to Schöbel [7], the first integer programming formulations utilizing this total travel time minimization method were introduced by Scholl and herself, and the formulations were later developed by the researchers together in [8] and separately in other papers. The formulation of Schöbel and Scholl utilizes the change&go notation where the underlying undirected infrastructure network of stations and connecting routes (PTN) is converted to directed change&go-graph consisting of nodes of station and line pairs and edges representing either getting on or off a line, traveling using a line or transferring between lines. Consequently, to route the passengers, the formulation uses variables that indicate whether an edge is used by passengers traveling from a specific origin to their desired destination. [8]

According to Schöbel [7], Borndörfer et al. developed formulations that also use the same total travel time minimization method, independently from the work of Schöbel and Scholl: Formulations in [1, 2, 3] aim to minimize both the total riding time while ignoring the time taken by transfers, and the establishment and operating costs of the lines. To achieve this, the formulations in [1, 2] model passenger behavior using variables that denote the amount of passenger flow in different paths and, unlike the formulation by Schöbel and Scholl in [8], construct the lines dynamically based on the flow variables instead of using a set of predetermined possible lines. In the formulation in [3], a variable is assigned for each possible path and possible line, which are used to solve the formulation using the branch-and-price algorithm. The possible lines are also generated dynamically in this formulation using a column-generation algorithm instead of using a predetermined set of lines.

Since the total travel time minimization used by Schöbel and Scholl in [8] and Borndörfer et al. in [1, 2, 3] automatically investigates the behavior of all passengers, it generates reasonable paths for both direct and transferring passengers, contrary to the direct passenger method. Additionally, Schöbel's and Scholl's formulation [8] finds the true quickest path of a passenger since the model compares the routes using the travel times instead of directness as the comparison criteria. This does not completely apply to the formulations of Borndörfer et al. [1, 2, 3] since the model ignores the time taken by transfers when comparing the travel times. Therefore, the formulation [1, 2, 3] may incorrectly prefer transferring routes in cases where the true travel times between a transferring and a direct route are very close.

Despite not being plagued by some of the insufficiencies of the direct passenger method, the solving of formulations using total travel time minimization methods is also hampered by the size and complexity of these methods: Both Schöbel and Scholl [8] and Borndörfer et al. [1, 2, 3] ascertain that their formulations or at least their LP-relaxations are NP-hard problems even in simple cases. This is especially true for formulation developed by Schöbel and Scholl since they mention that due to the size of the problem, its LP-relaxation cannot be solved in any straight-forward manner [8]. Even though Schöbel and Scholl as well as Borndörfer et al. present solution methods to their formulation in [8] and [1, 2, 3], this issue with solvability and size inspired us to attempt to solve the problem of these formulations by applying the ideas of

Schöbel and Scholl to star-shaped trees and consequently create a formulation that is easier to solve.

Later research that utilizes or improves upon these two customer-oriented approach methods is scarce, however, research using other approaches can be found. For example, Goerick and Schmidt [5] developed a model for minimizing travel time in a more randomized setting, and Zhou et al. [11] created a multi-objective optimization model that considers several limitations and effects of the underlying infrastructure. Goerick's and Schmidt's model is a bi-level optimization model where the upper level is the line plan optimization, and the lower level is the passenger route optimization minimizing the individual travel time. This way the model does not assume that the passenger path is either independent of the line plan or assigned by the staff, but rather decided independently by each passenger on their own. Goerick and Schmidt also show the viability of this model's integer formulation for problems containing up to 10 stations and provide a generic solution algorithm for problems containing up to 250 stations. [5]. On the other hand, the mixed integer non-linear model of Zhou et al. optimizes cost and travel times while considering capacity restrictions of the vehicles, travel and transfer times, and the effects of the same tracks being used by multiple lines. Zhou et al. also verify that the model is feasible and efficient to solve using an outer approximation of the objective function by studying it in the simplified Hong Kong transportation network. [11]

Much of the recent research related to customer-oriented LPPs aims to combine solving the LPP responsible for the line concept, the optimization of the timetables, and other later phases of the line planning. Some examples are the paper by Yan and Goverde [10] as well as the paper by Schöbel [9]. Yan and Goverde create a method for combining the line concept with the timetabling by developing a multi-frequency line planning model and a multi-period timetabling optimization model and creating an iterative framework that can be utilized for simultaneous solving: the iterative framework consists of feedback loops between the two models so that the regularity constraints of the timetabling model are gotten from the solutions of the line planning model, and the values of the solutions for the timetabling model are inputted to the frequency constraints of the line planning model. [10] Schöbel, contrarily, integrates line planning, timetabling, and vehicle scheduling by proposing an eigenmodel that iterates between the three phases as well as several heuristic approaches to solve the integrated problem. [9] Additionally, Schöbel argues that models that integrate several phases into one are always better optimizers than using several models to optimize the phases individually [9]. However, public transportation plans still are executed sequentially, which means that models meant to create the line concept while not considering the later phases still retain utility.

3 Mathematical Models and Notation

In this section, we shall review the methodology and notation used to model and study LPP formulations in a star-shaped transportation network. The section is structured as follows: In subsection 1, we outline and define the concepts and notation used throughout this section and thesis. In subsection 2, the formulation for the LPP with minimal perceived travel time using the existing methodology of Scholl and Schöbel [8] is introduced. In subsection 3, we introduce a new compact formulation for the LPP with minimal travel time.

3.1 Basic Methodology and Definitions

As previously indicated, this thesis studies the LPP with minimal perceived travel time specifically in a star-shaped transportation network. Here, a star-shaped transportation network is defined to be a transportation network with one central station and many leaf stations. In addition, the central station has a direct connection to all leaf stations, however, leaf stations only have a connection to the central station. More formally, a star-shaped transportation network refers to a finite undirected tree-graph $PTN = (S, E)$, where the node set S represents stops, the edge set E represents possible direct connections between two stations, and only one node in S has a degree larger than 1. Additionally, to unify the notation of lines later, we add a dummy leaf station which will not be the destination or origin of any passengers. Therefore, in this thesis, we define the node set S to be $S = \{-1, 0, 1, ..., n\}$ where node 0 is the central node, -1 is the dummy station node, and n is the number of tangible leaf station nodes. Hence, the degree of node 0 is $n + 1$, and the degree for node $i \in \{-1\} \cup \{1, ...n\}$ is 1. Additionally, we define the edge set E to be of the form $E = \{\{0, j\} \mid j \in S \setminus \{0\}\}$. Figure 1 shows an example of a PTN as this paragraph outlines, when the number of tangible leaf nodes n is 3.

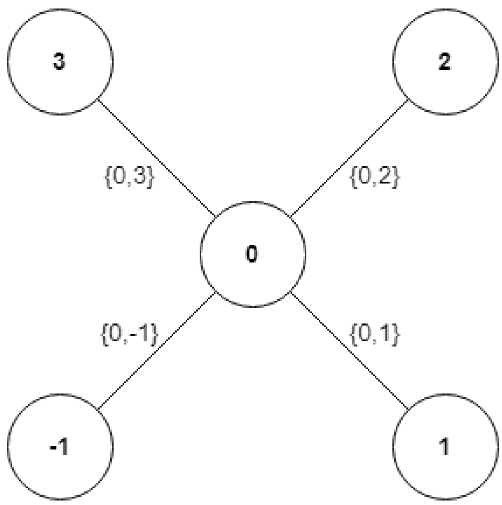

Figure 1: A picture of the PTN graph when the number of tangible leaf nodes $n = 3$.

Let us assume that the PTN used in LPP is as described above and fixed. Now let us define the line pool $\mathcal{L}$, meaning the set of possible lines in the PTN. We assume that no line travels the same edge more than once to ensure that all created lines are cost-efficient, meaning that $\mathcal{L}$ is the set of possible simple paths in the PTN. Since the described PTN is a tree, between two nodes of the PTN, there exists only one simple path, meaning that each line can be specified using the origin station and the terminus of the line, which we assume to be two different stations. All of the lines can also be specified by a sequence of stops that it travels through. Since the line is a simple path, if the line starts or ends at the central node, this sequence includes only two stations. Otherwise, it includes three. To unify the notation, we add the dummy station as the third station to the two stop lines. Hence, we can make the following definition

Definition 3.1. Line Pool

The line pool for the LPP is defined as follows

$$\mathcal{L} := \{[i, j] \mid i, j \in S \ \setminus \ \{0\}, \ i \neq j\}$$

where each of the lines corresponds to a sequence of stops as follows $[i, j] := (i, \ 0, \ j)$. Here, line $[-1, a] := (-1, \ 0, \ a)$, for example, would correspond to the two-stop line originating from the center node and ending at the leaf node a.

Additionally to the line pool, we define an origin-destination set $\mathcal{R} \subset S \setminus \{0\} \times S \setminus \{0\}$, which tracks all of the trips, meaning the origins and the corresponding desired destination, the passengers wish to take in the PTN using origin-destination pairs, later referred to as the OD-pairs. We assume that the origin and the destination have to be different nodes. Additionally, we assume that if a leaf node in the PTN is set as an origin in the $\mathcal{R}$, there is at least one passenger that wishes to travel from there to another leaf node. Similarly, we assume that if a leaf node is a destination in the $\mathcal{R}$, there is at least one passenger that wishes to travel there from another leaf node.

If the LPP does not fulfill these assumptions, i.e. in the $\mathcal{R}$ of the LPP, there exists an OD-pair involving the center node and no leaf-node-to-leaf-node OD-pairs with the same origin or the destination as the first mentioned OD-pair, the set-up can be adjusted manually to include the outlying OD-pairs while fulfilling the mentioned assumptions: for any OD-pair involving the center node and with no leaf-node-to-leaf-node OD-pairs using the same origin or destination, we include a two stop line between the origin and the destination as part of the final line plan, and reduce the budget by the cost of that line.

Since the eventual model will utilize binary frequencies and all leaf nodes are only connected to the center node in the PTN, based on our assumptions, we can disregard all origin-destination pairs involving the center node in our model: The vehicles have infinite capacity in the case of binary frequencies and, due to the structure of the PTN, any passengers paths traveling out of or into a leaf node must go through the center node. Hence, if we ensure that, in the optimal solution, a passenger can travel from leaf node a to leaf node b, we ensure that there also exists a feasible path for all passengers wishing to travel from leaf node a to the center node or from the center node to the leaf node b. Consequently, we get the following formal definition for $\mathcal{R}$.

Definition 3.2. The origin-destination set
The origin-destination set of the LPP is defined as follows:

$$\mathcal{R} := \{(s,t) \in S \times S \mid s \neq t,\ s \cdot t > 0\}$$

To optimize a line plan, we optimize the passenger routes in the given PTN using the defined line pool within budgetary constraints. This, on the other hand, requires a better formulation that tracks the optimal passenger route in the PTN, the lines the passenger uses to travel said route as well as if the node is the origin or the destination of the passenger. Hence, we convert the PTN into a directed graph $CG = (\mathcal{V}, \mathcal{E})$ referred to as a change&go network or a CG-network, as later referred to in this paper. Each path in the constructed CG-network represents traveling a specific route in the PTN using specific lines. Therefore, each path in the CG-network corresponds to a path in the PTN.

To achieve this, the node set $\mathcal{V}$ consists of the change&go nodes $\mathcal{V}_{CG}$ and the origin-destination nodes $\mathcal{V}_{OD}$. The change&go nodes $\mathcal{V}_{CG}$, or CG nodes for short, are created by attaching the PTN nodes to the lines traveling through the PTN node so that one CG node connects one PTN node to one line traveling through it. The origin-destination nodes, or OD nodes for short, are created by connecting each of the leaf PTN nodes to the notation OD to indicate that the PTN node is a possible origin or destination. Consequently, the formal definition of $\mathcal{V}$ is as follows:

Definition 3.3. Definition of $\mathcal{V}$

The node set $\mathcal{V}$ is defined as follows

$\mathcal{V} := \mathcal{V}_{CG} \cup \mathcal{V}_{OD}$

where

1° $\mathcal{V}_{OD} := \{(i, OD) \mid ((i,j) \in \mathcal{R}) \vee ((j,i) \in \mathcal{R})\}$

2° $\mathcal{V}_{CG} := \{(k, [i,j]) \mid k \in S,\ [i,j] \in \mathcal{L},\ (k=0) \vee (k=i) \vee (k=j)\}$

Similarly, the edge set $\mathcal{E}$ consists of a group of go edges $\mathcal{E}_{go}$, change edges $\mathcal{E}_{change}$, and origin-destination edges $\mathcal{E}_{OD}$, later referred to as OD edges. Here, the go edges represent traveling a direct connection between the two stations, represented by the PTN edges, in one direction using a line traveling through the two stations in the right order. Thus, the go edges are created as follows: Let the CG node be of the form $(k, [i,j]) \in \mathcal{V}_{CG}$. Now, a go edge is created to connect this CG node to CG node $(k', [i,j]) \in \mathcal{V}_{CG}$ where k' is the PTN node that follows the PTN node k in the sequence definition of the line $[i,j] \in \mathcal{L}$. Consequently, this set can be divided into sets of line edges $\mathcal{E}_l$, which represent all the edges that can be traveled using the line

l. Similarly, the line l can also be defined as a sequence of the go edges belonging to $\mathcal{E}_l$.

On the other hand, the OD edges represent getting on a line at a passenger's origin or getting off a line at their destination. Hence, the OD edges are created as follows: Let the CG node be of the form $(k, OD) \in \mathcal{V}_{OD}$. Now, an OD edge is created to connect this OD node to every CG node $(k, [i, j]) \in \mathcal{V}_{CG}$, to represent getting on a line from PTN node k. Additionally, an OD edge is created to connect each CG node of the form $(k, [i, j]) \in \mathcal{V}_{CG}$ to this OD node $(k, OD) \in \mathcal{V}_{OD}$ to represent getting off a line at the PTN node k. Since there are no OD nodes attached to the center node of the PTN, there are also no OD edges attached to any CG nodes connected to the center node of the PTN.

In contrast, the change edges represent a possible transfer between lines. All possible change edges can be made by pairing two CG nodes with the same station value but different line values. The first CG node contains the line the passenger transfers from and the second CG node contains the line the passenger transfers to. However, while solving the LPP, we are only interested in the optimal passenger routes. Therefore, we can simplify solving the LPP by eliminating all transfers that cannot belong to an optimal passenger route.

As mentioned before, each path in the CG-network corresponds to a path in the PTN. If the corresponding PTN-path is not simple, the path in the CG network must include unnecessary travel and therefore cannot represent an optimal passenger route with respect to minimal perceived travel time. Hence, we assume that passengers use routes corresponding to a simple path in the PTN. Since the PTN is a tree, there exists only one simple path between two points in the PTN. Consequently, the path in the PTN corresponding to the passenger route is fixed for any $(s, t) \in \mathcal{R}$. We mark this result as a remark as follows:

Remark 3.4. We assume that passengers only travel routes corresponding to a simple path in the PTN. Consequently, the path in the PTN corresponding to the passenger route is fixed for any $(s, t) \in \mathcal{R}$.

The simple path connecting an arbitrary the OD-pair $(s, t) \in \mathcal{R}$ is $((s, 0),\ (0, t))$. Consequently, a leaf node in the PTN can only be either the beginning or the end of a simple path corresponding to an OD-pair, and therefore, either the origin or the destination of a passenger route. Hence, any transfers made in the leaf nodes will lead to a non-optimal passenger route. Therefore, we only allow transfers in the center node. Additionally, if a passenger route uses a transfer from the line $[i, j] \in \mathcal{L}$ to line $[k, i] \in \mathcal{L}$ in the center node, it cannot correspond to a simple path in the PTN. Hence, we eliminate any transfers where the start station of the first line and the end station of the second line are the same.

Due to only including optimal passenger routes, we can also eliminate any transfers between lines with the same starting station or end station: Any passenger routes using a transfer from line $[i, j]$ to $[k, j]$ or from line $[i, k]$ to $[i, j]$, where $[i, j],\ [k, j],\ [i, k] \in \mathcal{L}$ and $k \in S \setminus \{0,\ i,\ j\}$, are longer than passenger routes only using the line $[i, j]$ to travel the same PTN edges. Hence, any route using such transfers cannot be optimal. Based on all of the principles explained above, the formal definition of $\mathcal{E}$ is as follows:

Definition 3.5. Definitions on $\mathcal{E}$

The following definitions hold:

$\mathcal{E} := \mathcal{E}_{go} \cup \mathcal{E}_{change} \cup \mathcal{E}_{OD}$

where

1° $\mathcal{E}_{l=[i,j]} := \{((p,[i,j]),(q,[i,j])) \in \mathcal{V}_{CG} \times \mathcal{V}_{CG} \mid (p = i \wedge q = 0) \vee (p = 0 \wedge q = j)\}$

2° $\mathcal{E}_{go} := \bigcup_{l \in \mathcal{L}} \mathcal{E}_l$

3° $\mathcal{E}_{change} := \{((0,[i,j]),(0,[p,q])) \in \mathcal{V}_{CG} \times \mathcal{V}_{CG} \mid p \neq i,\ q \neq i,\ q \neq j)\}$

4° $\mathcal{E}_{OD} := \{((k,OD),(k,[k,j])) \in \mathcal{V}_{OD} \times \mathcal{V}_{CG} | (k,k') \in \mathcal{R}\} \cup \{((k',[i,k']),(k',OD)) \in \mathcal{V}_{CG} \times \mathcal{V}_{OD} \mid (k,k') \in \mathcal{R}\}$

Definitions 3.3 and 3.5 and the structure of the CG-network, in general, are further demonstrated in the Appendix B, which presents the CG-network corresponding to the PTN presented in Figure 1.

Since the CG-graph is directed, we can model the possible directions of flow at each node by defining a set of incoming edges $\delta^+(v)$ and a set of outgoing edges $\delta^-(v)$ for each node $v \in \mathcal{V}$. The definition happens as follows:

Definition 3.6. Incoming and outgoing sets

If $v \in \{(i,[i,j]) \in \mathcal{V} \,| i \in S \setminus \{0\},\ j \in S \setminus \{i\}\}$, then

$$\begin{aligned} \delta^+(v) &= \{((i,\ OD),\ (i,\ [i,j]))\} \\ \delta^-(v) &= \{((i,\ [i,j]),\ (0,\ [i,j]))\} \end{aligned}$$

.

If $v \in \{(j,[i,j]) \in \mathcal{V} \,| i \in S \setminus \{j\},\ j \in S \setminus \{0\}\}$, then

$$\begin{aligned} \delta^+(v) &= \{((0,\ [i,j]),\ (j,\ [i,j]))\} \\ \delta^-(v) &= \{((j,\ [i,j]),\ (j,\ OD))\} \end{aligned}$$

.

If $v \in \{(0,[i,j]) \in \mathcal{V} \,| i,j \in S \setminus \{0\},\ i \neq j\}$, then

$$\begin{aligned} \delta^+(v) = \{&((i,\ [i,j]),\ (0,\ [i,j]))\} \cup ... \\ &...\{((0,\ [c,d]),\ (0,\ [i,j]))\}_{(c \in S \setminus \{0,j\},\, d \in S \setminus \{c,j,0\})} \\ \delta^-(v) = \{&((0,\ [i,j]),\ (j,\ [i,j]))\} \cup ... \\ &...\{((0,\ [i,j]),\ (0,\ [c,d]))\}_{(c \in S \setminus \{i,d,0\},\, d \in S \setminus \{i,0\})} \end{aligned}$$

.

If $v \in \{(i, OD) \in \mathcal{V} \,|i \in S \ \setminus \ \{0\}\}$, then

$$\delta^+(\ v\ ) = \{((i,\ [c,i]),\ (i,\ OD))\}_{(c \in S \setminus \{0,\, i\})}$$
$$\delta^-(\ v\ ) = \{((i,\ OD),\ (i,\ [i,d]))\}_{(d \in S \setminus \{0,\, i\})}$$

3.2 Original Formulation for the LPP with Minimal Perceived Travel Time

In this paper, we compare the developed formulation and its solutions to the formulation for the LPP with the minimal perceived travel time developed by Schöbel and Scholl. In this section, we shall outline the final formulation for the LPP in the PTN and CG networks as specified in Section 3.1 and the necessary definitions and explanations in the notation. For a more detailed explanation behind the reasoning and features of the formulation, please refer to the paper of Schöbel and Scholl. [8]

For the formulation, we shall make the following assumptions: we assume that the total available budget for the line plan B is known, the cost of establishing the line C_l is known for every line $l \in \mathcal{L}$, the travel time between two stations t_{uv} is known for every edge $(u, v) \in E$, the number of passengers wishing to travel from station s to t, i.e. quantity w_{st} is known for each origin-destination pair $(s, t) \in \mathcal{R}$, and passengers will only travel to their destination using simple paths in PTN and CG. To determine the most time-efficient path, we determine an edge cost c_e for each edge to represent the time cost of passengers choosing to take that path. As proposed by Schöbel and Scholl [8], the edge cost is set as follows

$$c_e = \begin{cases} 0, & e \in \mathcal{E}_{OD} \\ k_1 t_{uv}, & e = ((u,l),(v,l)) \in \mathcal{E}_{go} \\ k_2, & e \in \mathcal{E}_{change} \end{cases} \tag{1}$$

where k_1 and k_2 are positive parameters that can be used to scale the time costs for transferring when compared to driving in a line.

To model the paths of the passengers in the CG with integer programming, we define variable $x^e_{st} \in \{0, 1\}$ to represent whether the edge e is used while passengers travel from s to t, variable $y_l \in \{0, 1\}$ for $l \in \mathcal{L}$ to represent whether the line is included in the chosen line plan. We assume the passengers to always look for the shortest path meaning that, as stated by Schöbel and Scholl [8], "$x^e_{st} = 1$ if and only if edge e is used on the shortest dipath" when traveling from node (s, OD) to node (t, OD) in CG, and similarly variable $y_l = 1$ "if and only if line l is chosen to be in the line concept". Moreover, to set up a flow problem constraint ensuring that all passengers take uninterrupted paths, we define parameter $\theta \in \mathbf{Z}^{|\mathcal{V}|\times|\mathcal{E}|}$ to be the node-arc-incidence matrix of CG, variable $x_{st} \in \{0, 1\}^{|\mathcal{E}|}$ to be vector containing the values of x^e_{st} corresponding to the columns of θ, and parameter $b_{st} \in \mathbf{Z}^{|\mathcal{V}|}$ to be the vector containing the flow values as defined by

$$b_{st}^{i} = \begin{cases} 1, & i = (s, OD) \\ -1, & i = (t, OD) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Using, these variables Schöbel and Scholl present the following formulation for the LPP with minimal perceived travel time in [8], which they label as LPMT1. The formulation is defined below.

$$\begin{aligned}
\text{(LPMT1)} \quad & \min \sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}} w_{st}\, c_e\, x_{st}^{e} && && (3)\\
s.t. \quad & \sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}_l} x_{st}^{e} \;\le\; |\mathcal{R}||\mathcal{E}_l|\, y_l && \forall\, l \in \mathcal{L} && (4)\\
& \theta x_{st} = b_{st} && \forall\, (s,t) \in \mathcal{R} && (5)\\
& \sum_{l\in\mathcal{L}} C_l y_l \;\le\; B && && (6)\\
& x_{st}^{e} \in \{0,1\} && \forall\, (s,t) \in \mathcal{R},\ e \in \mathcal{E} && (7)\\
& y_l \in \{0,1\} && \forall\, l \in \mathcal{L} && (8)
\end{aligned}$$

As stated by Schöbel and Scholl, constraint (4) ensures that line $l \in \mathcal{L}$ is included in the line concept if any of the edges belonging to it are used for any $(s, t) \in \mathcal{R}$, constraint (5) ensures that the paths of all passengers are uninterrupted, and constraint (6) ensures that the establishment costs of the line plan remain within the given budget. The objective function (3), naturally, minimizes the perceived travel time since the objective function sums the time costs of all edges of CG used by the passengers in the available line plan and, therefore, minimizes the total time spent by the passengers both while traveling on the lines as well as transferring by minimizing this sum.[8]

The formulation above is developed using binary frequencies, meaning that we assume the transportation vehicles to have infinite capacity. Consequently, Schöbel and Scholl mention that the formulation does not take into account the possible capacity limits or the number of vehicles traveling on the line. Furthermore, the formulation includes an implicit assumption that all passengers take the same route in CG. Schöbel and Scholl provide another formulation utilizing integer frequencies which considers capacity limitations and the number of traveling vehicles in the paper [8]. However, the details of this formulation are not necessary for this paper.

3.3 The Compact Formulation For the LPP with Minimal Perceived Travel Time

In this section, we outline our model's reasoning and formulation in the PTN as described in Section 3.1. The main idea is to provide a model that also considers the transfers between lines while minimizing the total perceived travel time, and requires less computations to solve, at least in non-trivial cases. Therefore, we aim to create a model that records the necessary passenger behavior with fewer variables than LPMT1, while still being consistent with the structure of the PTN and CG defined in Section 3.1.

As the basis for our model, we use the Remark 3.4. Additionally, we assume that all lines use the same vehicles, meaning that the travel time of a go-edge in the CG is not dependent on the line, only on the stations the passengers traveling between. Therefore, the total driving time of a path in CG for any $(s,t) \in \mathcal{R}$ depends only on the corresponding path in the PTN. Since, as stated in the Remark 3.4, that path is fixed in PTN for any $(s,t) \in \mathcal{R}$, the total driving time of a passenger in CG is also fixed for any $(s,t) \in \mathcal{R}$. Therefore, the perceived travel time for any set of OD-pairs $\mathcal{R}$ is affinely dependent on the number of transfers made by the passengers. Based on the definition of the transfer-edge set $\mathcal{E}_{change}$, transferring is only possible in the center node. Therefore, the model needs to monitor the behavior of the passengers only at this node. Additionally, the model needs to monitor which lines are included in the line plan.

We assume that all vehicles used in the network have infinite capacity and consequently, all passengers traveling from the same origin to the same destination take the same path. To ensure that the model monitors the necessary behavior, let us define that variable $x_{direct}^{(i,0),(0,j)} \in \{0,1\}$, where $i,j \in S \setminus \{0\}$ and $i \neq j$, represents whether any passengers take the direct line from node i to j, and variable $x_{transfer}^{(i,0),(0,j)} \in \{0,1\}$, where $i,j \in S \setminus \{0\}$ and $i \neq j$, represents whether passengers transfer on their way from i to j. Additionally, to monitor the selected line plan, let us define variable $z_l \in \{0,1\}$, where $l \in \mathcal{L}$, to represent whether the line l is included in the line plan. Consequently, $z_l = 1$ if and only if line l is part of the line plan. Let us also assume that the cost of creating line $l \in \mathcal{L}$, C_l, the budget for the line plan, B, and the number of passengers wishing to travel from s to t, w_{st}, where $(s,t) \in \mathcal{R}$, are known.

Using these variables and known parameters, we can model the LPP problem with minimal perceived travel time using the formulation defined below. We shall later refer to this formulation as the compact formulation.

Definition 3.7. The compact formulation

$$\min \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot x_{transfer}^{(s,0),(0,t)} \tag{9}$$

$$\text{s.t.} \quad x_{direct}^{(i,0),(0,j)} + x_{transfer}^{(i,0),(0,j)} = 1 \qquad \forall\, (i,j) \in \mathcal{R} \tag{10}$$

$$x_{direct}^{(i,0),(0,j)} \leq z_{[i,j]} \qquad \forall\, [i,j] \in \mathcal{L} : (i,j) \in \mathcal{R} \tag{11}$$

$$\sum_{k\neq i} z_{[i,k]} \geq 1 \qquad \forall\, i \in S \setminus \{-1,\, 0\} : (i,j) \in \mathcal{R} \tag{12}$$

$$\sum_{k\neq j} z_{[k,j]} \geq 1 \qquad \forall\, j \in S \setminus \{-1,\, 0\} : (i,j) \in \mathcal{R} \tag{13}$$

$$x_{transfer}^{(i,0),(0,j)} \leq \sum_{i\neq k\neq j} z_{[i,k]} \qquad \forall\, (i,j) \in \mathcal{R} \tag{14}$$

$$x_{transfer}^{(i,0),(0,j)} \leq \sum_{i\neq k\neq j} z_{[k,j]} \qquad \forall\, (i,j) \in \mathcal{R} \tag{15}$$

$$\sum_{l\in\mathcal{L}} C_l \cdot z_l \leq B \tag{16}$$

$$x_{direct}^{(i,0),(0,j)},\ x_{transfer}^{(i,0),(0,j)} \in \{0,1\} \qquad \forall\, (i,j) \in \mathcal{R} \tag{17}$$

$$z_l \in \{0,1\} \qquad \forall\, l \in \mathcal{L} \tag{18}$$

In this model, constraint (10) ensures that all passengers are taking the same path and that passengers are traveling from node $i \in S \setminus \{0\}$ to node $j \in S \setminus \{0\}$ if origin-destination pair $(i,j) \in \mathcal{R}$ exists. Constraint (11), on the other hand, ensures that the three-stop line $[i,j] \in \{[i,j] \in \mathcal{L} \mid i \cdot j \neq 0\}$ is included in the line plan if any passengers are traveling directly between those nodes. Constraint (16), on the other hand, ensures that the establishment costs of the resulting line plan do not exceed the budget available.

Furthermore, constraint (12) ensures that if there is an origin-destination pair where leaf node $i \in S \setminus \{0\}$ is the origin station, at least one line, that originates from that origin station, has to be included in the line plan. Similarly, constraint (13) ensures that if there is an origin-destination pair where the leaf node $i \in S \setminus \{0\}$ is the destination, then at least one line, that terminates at that station, has to be included in the line plan. This means that all passengers wishing to travel from a leaf node to the center node or from the center node to a leaf node have to have at least one line they can use to do so. Moreover, all passengers wishing to travel from the leaf node to an leaf node will first drive from their origin to the center node and then from the center node to their destination, due to the structure of the PTN. Consequently, these constraints ensure that all passengers have at least a combination of lines they can transfer between to complete their journey.

Where constraints (12) and (13) ensure that all origin-destination pairs have at least one possible traveling path in the selected line plan, constraints (14) and (15) correspondingly ensure that those paths are feasible in the CG network defined in section 3.1. In practice, this means that if $x_{transfer}^{(i,0),(0,j)} = 1$ for some $(i,j) \in \mathcal{R}$, constraint (14) ensures that at least one line starting from node i, other than the direct line $[i, j] \in \mathcal{L}$, is included in the line plan, and constraint (15) ensures that at least one line ending to node j, other than the direct line $[i, j] \in \mathcal{L}$, is included in the line plan. Hence, constraints (14) and (15) ensure that if passengers transfer on their way from node $i \in S \setminus \{0\}$ to $j \in S \setminus \{0\}$, at least one passenger route, which does not involve using the direct line $[i, j]$, must be possible in the selected line plan. In this case, the line plan may or may not include the direct line $[i, j]$, however, if it is included it is interpreted not to be used.

4 Analysis of the LPMT1 and the Compact Formulation

In this section, we show that the compact formulation satisfies the objective of this thesis: We shall show that the compact formulation is a valid and simpler alternative to the LPMT1 formulation. First, we shall show that the compact formulation and the LPMT1 are equivalent and, therefore, solve the same problem. Afterward, we shall compare the complexity of the problems by comparing the approximate sizes of the formulations, and then make the necessary conclusions.

4.1 The Proof of Equivalence of LPMT1 and the Compact Formulation

For our compact formulation to even be a valid alternative to the LPMT1 formulation introduced by Schöbel and Scholl [8], we must show that these formulations solve the same LPP, i.e. these two formulations are equivalent. We do this by proving that, an optimal solution of the compact formulation can be constructed from any optimal solution of the LPMT1 formulation and an optimal solution of the LPMT1 formulation can be constructed from any optimal solution of the compact formulation. In this section, we shall complete this proof in two phases. First, we prove that there exists a corresponding feasible solution in the compact solution for every optimal solution in the LPMT1 formulation and vice versa. In the second phase, we show that if the initial solution is optimal, then the corresponding solution in the other formulation is also optimal.

We only construct the proof of equivalence for the binary frequency formulation created by Schöbel and Scholl. This is done since the binary frequency formulation is the only sensible comparison: The compact formulation relies heavily on the features and assumptions of the binary frequency such as the assumption of infinite vehicle capacity. Hence, it is reasonable to assume that it is not possible to construct a proof of equivalence between the compact formulation and any formulation utilizing integer frequencies, due to the incompatible assumptions.

4.1.1 Proof for the Existence of a Corresponding Feasible Solution

In this section, we introduce the proof that for any optimal solution to one formulation, there exists a feasible solution in the other.

First, we formulate the proof for the existence of a corresponding feasible solution in the LPMT1 for any optimal solution of the compact formulation.

Lemma 4.1. *For any feasible solution of the compact formulation, we can construct a feasible solution of LPMT1.*

Proof:

Let $(x_{direct}, x_{tranfer}, z)$ with $x_{direct} = (x_{direct}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)}$, $(x_{transfer}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)}$, and $z = (z_l)_{(l \in \mathcal{L})}$ be a feasible solution of the compact formulation.

For any $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$, let us set for origin-destination pair $(s,t) \in \mathcal{R}$

$$x_{st}^e = \begin{cases} 1, & e = ((s,\ [s,t]),\ (0,\ [s,t])) \\ 1, & e = ((0,\ [s,t]),\ (t,\ [s,t])) \\ 1, & e = ((s,\ OD),\ (s,\ [s,t])) \\ 1, & e = ((t,\ [s,t]),\ (t,\ OD)) \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

For any $x_{transfer}^{(i,\,0),\,(0,\,j)} = 1$, constraints (14) and (15) ensure that $\exists\, p, q \in S \setminus \{i,j\}$: $z_{[i,p]} = 1,\ z_{[q,j]} = 1$. Therefore, if $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$, let us set for origin-destination pair $(s,t) \in \mathcal{R}$.

$$x_{st}^e = \begin{cases} 1, & e = ((s,\ [s,p]),\ (0,\ [s,p])) \\ 1, & e = ((0,\ [q,t]),\ (t,\ [q,t])) \\ 1, & e = ((s,\ OD),\ (s,\ [s,p])) \\ 1, & e = ((t,\ [q,t]),\ (t,\ OD)) \\ 1, & e = ((0,\ [s,p]),\ (0,\ [q,t])) \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

for some fixed $p, q \in S \setminus \{s,t\}$, where $z_{[s,p]} = 1$ and $z_{[q,t]} = 1$.
Additionally, set for line $l \in \mathcal{L}$, $y_l = z_l$.

Let us now show that a corresponding solution $(x_{st},\ y)$ with $x_{st} = (x_{st}^e)_{(e \in \mathcal{E},\,(s,t) \in \mathcal{R})}$ and $y = (y_l)_{(l \in \mathcal{L})}$ constructed with the method described above is a feasible solution to the LPMT1 formulation described in the Section 3.2.

Let us show that the corresponding solution fulfills constraint (6):
Based on constraint (16), we get

$$\sum_{l \in \mathcal{L}} z_l \leq B$$

We set that for the corresponding solution $y_l = z_l,\ l \in \mathcal{L}$.
Therefore,

$$\sum_{l \in \mathcal{L}} y_l = \sum_{l \in \mathcal{L}} z_l \leq B$$

and consequently the corresponding solution fulfills the LPMT1 constraint (6).

Let us show that the corresponding solution fulfills the LPMT1 constraint (5):
The constraint (5) can be reformulated as follows:

$$\forall\ (s,t) \in \mathcal{R}:$$

$$\sum_{e \in \delta^+(\ v\ )} x^e_{st} - \sum_{e \in \delta^-(\ v\ )} x^e_{st} = 0 \qquad \forall\ v \in \mathcal{V} \setminus \{(s, OD), (t, OD)\} \quad (21)$$

$$\sum_{e \in \delta^+(\ v\ )} x^e_{st} - \sum_{e \in \delta^-(\ v\ )} x^e_{st} = -1 \qquad v = (s, OD) \quad (22)$$

$$\sum_{e \in \delta^+(\ v\ )} x^e_{st} - \sum_{e \in \delta^-(\ v\ )} x^e_{st} = 1 \qquad v = (t, OD) \quad (23)$$

where $\delta^+(\ v\ )$ is the set of incoming edges connected to node $v \in \mathcal{V}$, $\delta^-(\ v\ )$ is the set of outgoing edges connected to node $v \in \mathcal{V}$. Therefore, if the corresponding solution fulfills constraints (21),(22), and (23) for all $(s,t) \in \mathcal{R}$, then the corresponding solution fulfills LPMT1 constraint (5). Let us prove these constraints for all $(s,t) \in \mathcal{R}$ by proving that they are fulfilled for an arbitrary $(s,t) \in \mathcal{R}$.

Using the incoming and outgoing flow sets in Definition 3.6, we can separate constraint (21) into the following subcases:

(21) :

$$x_{st}^{((i,\,OD),\,(i,\,[i,j]))} - x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} = 0, \qquad v = (i,\ [i,j]) \quad (21.1)$$

$$x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} - x_{st}^{((j,\,[i,j]),\,(j,\,OD))} = 0, \qquad v = (j,\ [i,j]) \quad (21.2)$$

$$\sum_{c \in S \setminus \{0,\,i\}} x_{st}^{((i,\,[c,i]),\,(i,\,OD))} - \sum_{d \in S \setminus \{0,\,i\}} x_{st}^{((i,\,OD),\,(i,\,[i,d]))} = 0, \quad v = (i, OD), \quad s \neq i \neq t \quad (21.3)$$

$$\sum_{c,d \in S \setminus \{0,\,j\},\,c \neq i} x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} + x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} \ldots$$

$$\ldots - \sum_{c,d \in S \setminus \{0,\,i\},\,d \neq j} x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} - x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} = 0, \qquad v = (0, [i,j]) \quad (21.4)$$

where $i, j \in S \setminus \{0\}$ and $i \neq j$.

Note that, for arbitrary $(s,t) \in \mathcal{R}$, $x_{st}^{((i,\,OD),\,(i,\,[i,j]))} = x_{st}^{((i,\,[i,j]),\,(j,\,[i,j]))}$ and $x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} = x_{st}^{((j,\,[i,j]),\,(j,\,OD))}$ for all $i, j \in S \setminus \{0\}$ where $i \neq j$, by definitions (19) and (20). Similarly, by definitions (19) and (20), for arbitrary $(s,t) \in \mathcal{R}$, $x_{st}^{((i,\,[c,i]),\,(i,\,OD))} = 0$ and $x_{st}^{((i,\,OD),\,(i,\,[i,d]))} = 0$ for all $i \in S \setminus \{0,\ s,\ t\}$, and $c, d \in S \setminus \{0,\ i\}$. Consequently, constraint equations (21.1), (21.2), and (21.3) hold for all $i, j \in S \setminus \{0\}$ where $i \neq j$ and $(s,t) \in \mathcal{R}$.

Finally, due to constraint (10), we need to analyse the constraint (21.4) only when $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$ and when $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$, for an arbitrary $(s,t) \in \mathcal{R}$:

Let $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$ for an arbitrary $(s,t) \in \mathcal{R}$. Based on the construction rule (19),

$$x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} = x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} = \begin{cases} 1, & i = s \wedge j = t \\ 0, & \text{otherwise} \end{cases}$$

and $x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} = 0$ and $x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} = 0$ for all approapriate values of of c and d. Hence, the constraint (21.4) is fulfilled for all values of $i, j \in S \setminus \{0\}$ where $i \neq j$, when $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$ for arbitrary $(s,t) \in \mathcal{R}$.

Let $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$, for an arbitrary $(s,t) \in \mathcal{R}$. Now, based on construction rule (20),

$$x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} = \begin{cases} 1, & i = s \wedge j = p \\ 0, & \text{otherwise} \end{cases}$$

$$x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} = \begin{cases} 1, & i = q \wedge j = t \\ 0, & \text{otherwise} \end{cases}$$

$$x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} = \begin{cases} 1, & c = s \wedge d = p \wedge i = q \wedge j = t \\ 0, & \text{otherwise} \end{cases}$$

$$x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} = \begin{cases} 1, & i = s \wedge j = p \wedge c = q \wedge d = t \\ 0, & \text{otherwise} \end{cases}$$

for fixed $p, q \in S \setminus \{s,\ t\}$, where $z_{[s,p]} = 1$ and $z_{[q,t]} = 1$.

Consequently, if $i = s$ and $j = p$, then

$$\begin{aligned}
&\sum_{c,d \in S \setminus \{0,\, j\},\, c \neq i} x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} + x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} \dots \\
&\dots - \sum_{c,d \in S \setminus \{0,\, i\},\, d \neq j} x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} - x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} \\
=&\sum_{c,d \in S \setminus \{0,\, p\}, c \neq s} 0 + 1 - \sum_{c,d \in S \setminus \{0,\, s\},\, d \neq p} x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} - 0 \\
=&0 + 1 - 1 - 0 = 0
\end{aligned}$$

if $i = q$ and $j = t$, then

$$\begin{aligned}
&\sum_{c,d \in S \setminus \{0,\, j\},\, c \neq i} x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} + x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} \dots \\
&\dots - \sum_{c,d \in S \setminus \{0,\, i\},\, d \neq j} x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} - x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} \\
=&\sum_{c,d \in S \setminus \{0,\, t\},\, c \neq q} x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} + 0 - \sum_{c,d \in S \setminus \{0,\, q\},\, d \neq t} 0 - 0 \\
=&1 + 0 - 0 - 1 = 0
\end{aligned}$$

and otherwise

$$\begin{aligned}
&\sum_{c,d \in S \setminus \{0,\, j\},\, c \neq i} x_{st}^{((0,\,[c,d]),\,(0,\,[i,j]))} + x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} \dots \\
&\dots - \sum_{c,d \in S \setminus \{0,\, i\},\, d \neq j} x_{st}^{((0,\,[i,j]),\,(0,\,[c,d]))} - x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} \\
=&0 + 0 - 0 - 0 = 0
\end{aligned}$$

meaning that the constraint (21.4) holds for all $i, j \in S \setminus \{0\}$ where $i \neq j$ and $(s,t) \in \mathcal{R}$.

Since all of the subcases of constraint (21) hold for arbitrary $(s,t) \in \mathcal{R}$, the corresponding solution satisfies the constraint (21) for an arbitrary $(s,t) \in \mathcal{R}$.

Using the incoming and outgoing sets in Definition 3.6 and the fact that, based on construction rules (19) and (20), $x_{st}^{((i,\,[c,i]),\,(i,\,OD))} = 0$ for all $i \in S \setminus \{t\}$ and $x_{st}^{((i,\,OD),\,(i,\,[i,d]))} = 0$ for all $i \in S \setminus \{s\}$, for any $(s,t) \in \mathcal{R}$, we can simplify the constraints (22) and (23) to the forms

$$-\sum_{d \in S \setminus \{0,\, s\}} x_{st}^{((s,\,OD),\,(s,\,[s,d]))} = -1 \tag{22.1}$$

$$\sum_{c \in S \setminus \{0,\, t\}} x_{st}^{((t,\,[c,t]),\,(t,\,OD))} = 1 \tag{23.1}$$

Due to constraint (10), we need to analyze the simplified forms (22.1) and (23.1) only when $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$ and when $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$, for an arbitrary $(s,t) \in \mathcal{R}$. Based on the construction rules (19) and (20), both when $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$ and when

$x_{transfer}^{(s,0),\,(0,t)} = 1$, there exists exactly one value of d, for which $x_{st}^{((s,\,OD),\,(s,\,[s,d]))} = 1$, and exactly one value of c, for which $x_{st}^{((t,\,[c,t]),\,(t,\,OD))} = 1$. In both cases, $x_{st}^{((s,\,OD),\,(s,\,[s,d]))} = 0$ and $x_{st}^{((t,\,[c,t]),\,(t,\,OD))} = 1$ for all other values of c and d.

Consequently, simplified forms (22.1) and (23.1) hold in both subcases. Therefore, the corresponding solution satisfies the constraints (22) and (23) for an arbitrary $(s,t) \in \mathcal{R}$.

Since we have shown the corresponding solution to satisfy the constraints (21), (22), and (23) for an arbitrary $(s,t) \in \mathcal{R}$, we have proven that the corresponding solution fulfills the constraint (5) in the LPMT1.

Let us show that the corresponding solution fulfills the LPMT1 constraint (4):

Since $x_{st}^e \in \{0,\ 1\}$ for all $(s,t) \in \mathcal{R}$ and $e \in \mathcal{E}_l$, then

$$\sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}_l} x_{st}^e \leq |\mathcal{R}|\,|\mathcal{E}_l|$$

for all $l \in \mathcal{L}$.

Consequently, for any values of y_l, $l \in \mathcal{L}$, where $y_l = 1$, the constraint (4) is fulfilled.

Now, let $y_l = 0$, for an arbitrary $l = [i,j] \in \mathcal{L}$. By the construction rules of the corresponding solution, $z_l = y_l = 0$, meaning that due to the constraint (11)

$$x_{direct}^{(i,0),\,(0,j)} \leq z_{l=[i,j]} = 0$$

Therefore, it must be that in the original feasible solution $x_{direct}^{(i,0),\,(0,j)} = 0$ and, by constraint (10), $x_{transfer}^{(i,0),\,(0,j)} = 1$ for $l = [i,j]$. Based on the construction rule (20), $x_{st}^e = 1$ only if $e \in \mathcal{E}_{l'}$ with $z_{l'} = 1$. Since $z_l = 0$, $x_{st}^e = 0$ for all $e \in \mathcal{E}_l$, and , consequently,

$$\sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}_l} x_{st}^e = \sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}_l} 0 = 0 \leq |\mathcal{R}|\,|\mathcal{E}_l| \cdot 0 = |\mathcal{R}|\,|\mathcal{E}_l| \cdot y_l$$

Hence, constraint (4) is fulfilled for any values of y_l, $l \in \mathcal{L}$ where $y_l = 0$, meaning that the constraint (4) is fulfilled for all values of y_l, $l \in \mathcal{L}$

Since we have shown that the corresponding solution fulfills all of the constraints of the LPMT1 formulation, we have proven that the corresponding solution is a feasible solution to the LPMT1 formulation described in Section 3.2. Thus, we have proven Lemma 4.1 to be true.

Similarly , we formulate the proof for the existence of a corresponding solution in the compact formulation for an optimal solution in the LPMT1. Since the structure of the proof is very similar to the proof of Lemma 4.1, we only provide the algorithm for constructing the corresponding solution in this section. The complete proof for Lemma 4.2 can be found in the Appendix A.

Lemma 4.2. *For any optimal solution of the LMPT1 formulation, we can construct a feasible solution of the compact formulation defined in Definition 3.7.*

Proof: The proof for Lemma 4.2 can be found in the Appendix A.

In the proof, the corresponding solution in the compact formulation is defined from the optimal solution of LPMT1 as follows:

For any $(s,t) \in \mathcal{R}$, let us set

$$x_{direct}^{(s,\,0),\,(0,\,t)} = \begin{cases} 1, & x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = 1 \wedge x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} = 1 \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

and

$$x_{transfer}^{(s,\,0),\,(0,\,t)} = \begin{cases} 1, & \exists\, p,q \in S \setminus \{0,\ s,\ t\} :\ x_{st}^{((0,\,[s,p]),\,(0,\,[q,t]))} = 1 \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

Additionally, let us set $z_l = y_l$ for lines $l \in \mathcal{L}$.

4.1.2 Proof of the optimality in both directions

In this subsection, we shall prove that the corresponding solution in one of the formulations is optimal if the original solution in the other formulation is optimal.

Let us annotate that the objective function (3) of the LPMT1 formulation is

$$f(x) = f\,(\,(x_{st}^e)_{(\,(s,t)\in\mathcal{R},\,e\in\mathcal{E})}\,) = \sum_{(s,t)\in\mathcal{R}}\sum_{e\in\mathcal{E}} w_{st}\, c_e\, x_{st}^e, \quad x_{st}^e \in \{0,\ 1\} \tag{30}$$

and the objective function (9) of the compact formulation is

$$g(x) = g(\,(x_{transfer}^{(s,\,0),\,(0,\,t)})_{(s,t\in S\,\setminus\,\{0\},\,s\,\neq\,t)}\,) = \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \tag{31}$$

Let us first simplify the objective function (30) using the fact that the driving time is set for every $(s,t) \in \mathcal{R}$ due to the tree-structure of the PTN:

In Section 3.2, it is defined that travel time between the stations u and v, t_{uv} i.e. the time it takes to drive from station u to station v, does not depend on the line taken. Therefore, the total driving time associated with origin-destination pair $(s,t) \in \mathcal{R}$ depends only on the passenger's path from s to t in the PTN-network, not the CG-network. As mentioned in Remark 3.4, the passenger's path in the PTN-network is fixed for every $(s,t) \in \mathcal{R}$. Consequently, the total driving time is also fixed for every $(s,t) \in \mathcal{R}$. Let the total travel time for $(s,t) \in \mathcal{R}$ be marked with constant T_{st}.

Now, we can denote that

$$\sum_{((u,\,l),\,(v,\,l))\,\in\,\mathcal{E}_{go}} t_{uv}\cdot x_{st}^{((u,\,l),\,(v,\,l))} = T_{st} \qquad \forall\,(s,t)\in\mathcal{R} \tag{32}$$

Using this denotation we can simplify the objective functions of feasible solutions to the LPMT1 formulation from function (30) the following way:

$$
\begin{aligned}
f(x) &= \sum_{(s,t)\in\mathcal{R}} \sum_{e\in\mathcal{E}} w_{st} c_e x_{st}^e \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(\sum_{((u,l),(v,l))\in\mathcal{E}_{go}} k_1 \cdot t_{vu} \cdot x_{st}^{((u,l),(v,l))} + \sum_{e\in\mathcal{E}_{change}} k_2 \cdot x_{st}^e \Big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(k_1 \cdot \sum_{((u,l),(v,l))\in\mathcal{E}_{go}} t_{vu} \cdot x_{st}^{((u,l),(v,l))} + k_2 \cdot \sum_{e\in\mathcal{E}_{change}} x_{st}^e \Big) \quad \mid (32) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(k_1 \cdot T_{st} + k_2 \cdot \sum_{e\in\mathcal{E}_{change}} x_{st}^e \Big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \sum_{e\in\mathcal{E}_{change}} x_{st}^e
\end{aligned} \tag{33}
$$

It is also important to note that in an optimal LPMT1-solution, for all $i \in S\setminus\{0, s\}$ or $j \in S \setminus \{0, t\}$ where $(s,t) \in \mathcal{R}$, $x_{st}^{((0,[i,p]),(0,[q,j]))} = 0$, as mentioned in Section **??**. Consequently, the objective function of any optimal solutions of the LPMT1 formulation can be further simplified to the following form:

$$
f(x_{optimal}) = \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \sum_{b,\, c\in S\setminus\{0,\, s,\, t\}} x_{st}^{((0,[s,b]),(0,[c,t]))} \Big) \tag{34}
$$

where parameters w_{st}, k_1, k_2, and T_{st} do not depend on the decision variables. The simplification is provided in detail in Appendix C.

Let us determine the relationship between the values of the objective functions of two corresponding solutions. The resulting relationships are explained in Lemmas 4.3 and 4.4 below. Since the structure of the proof for Lemmas 4.3 and 4.4 is very similar, we provide the detailed proof only to Lemma 4.3 in this section as a demonstration. The detailed proof to Lemma 4.4 can be found in the Appendix D.

Lemma 4.3. *Let* $((x_{direct}^{(i,0),(0,j)})_{(i,j\in S\setminus\{0\},\, i\neq j)},\ (x_{transfer}^{(i,0),(0,j)})_{(i,j\in S\setminus\{0\},\, i\neq j)},\ (z_l)_{(l\in\mathcal{L})})$ *be an optimal solution to the compact formulation described in Definition 3.7, and let* $((x_{st}^e)_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})},\ (y_l)_{(l\in\mathcal{L})})$ *be its corresponding feasible solution in the LPMT1 formulation as defined in the proof of Lemma 4.1. Now*

$$
\begin{aligned}
f(x) &= f\big((x_{st}^e)_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})} \big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g\big((x_{transfer}^{(s,0),(0,t)})_{(s,t\in S\setminus\{0\},\, s\neq t)} \big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g(x)
\end{aligned} \tag{35}
$$

Proof. Let arbitrary $(s,t) \in \mathcal{R}$. Now if $x_{transfer}^{(s,0),(0,t)} = 1$, due to constraints (14) and (15), there exists $p, q \in S \setminus \{s,t\}$ such that $z_{[s,p]} = z_{[q,t]} = 1$. Based on conversion rule (20),

$$x_{st}^{((0,\,[a,b]),\,(0,\,[c,d]))} = \begin{cases} 1, & a = s \wedge b = p \wedge c = q \wedge d = t \\ 0, & \text{otherwise} \end{cases}$$

for fixed $p, q \in S \setminus \{s, t\}$, where $z_{[s,p]} = 1$ and $z_{[q,t]} = 1$.

Consequently, when $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$,

$$\begin{aligned} k_2 \cdot w_{st} \cdot \sum_{e \in \mathcal{E}_{change}} x_{st}^e &= k_2 \cdot w_{st} \cdot \sum_{a \in S \setminus \{0\}} \sum_{d \in S \setminus \{0,a\}} \sum_{b,\,c \in S \setminus \{0,\,a,\,d\}} x_{st}^{((0,\,[a,b]),\,(0,\,[c,d]))} \\ &= k_2 \cdot w_{st} \cdot (0 + x_{st}^{((0,\,[s,p]),\,(0,\,[q,t]))}) \\ &= k_2 \cdot w_{st} \cdot 1 = k_2 \cdot w_{st} \cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \end{aligned}$$

On the other hand, if $x_{transfer}^{(s,\,0),\,(0,\,t)} = 0$, due to constraint (10), $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$. Consequently, based on conversion rule (19), $x_{st}^e = 0$ for all $e \in \mathcal{E}_{change}$.

Hence, when $x_{transfer}^{(s,\,0),\,(0,\,t)} = 0$,

$$\begin{aligned} k_2 \cdot w_{st} \cdot \sum_{e \in \mathcal{E}_{change}} x_{st}^e &= k_2 \cdot w_{st} \cdot \sum_{e \in \mathcal{E}_{change}} 0 \\ &= k_2 \cdot w_{st} \cdot 0 = k_2 \cdot w_{st} \cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \end{aligned}$$

Thus, for arbitrary $(s, t) \in \mathcal{R}$,

$$k_2 \cdot w_{st} \cdot \sum_{e \in \mathcal{E}_{change}} x_{st}^e = k_2 \cdot w_{st} \cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \tag{36}$$

Consequently, from simplified form (33) we get

$$\begin{aligned} f(x) &= \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot \sum_{e \in \mathcal{E}_{change}} x_{st}^e \quad \mid (36) \\ &= \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \quad \mid (31) \\ &= \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g(x) \end{aligned}$$

which matches to equation (35). Hence, lemma 4.3 is true.

□

Lemma 4.4. *Let* $((x_{st}^e)_{((s,t) \in \mathcal{R},\, e \in \mathcal{E})},\ (y_l)_{(l \in \mathcal{L})})$ *be an optimal solution to the LPMT1 formulation and* $((x_{direct}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)},\ (x_{transfer}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)},\ (z_l)_{(l \in \mathcal{L})})$ *be its corresponding solution in the compact formulation, both as defined in the proof of Lemma 4.2. Now, the objective function of the corresponding solution is*

$$\begin{aligned} g(x) &= g(\ (x_{transfer}^{(s,\,0),\,(0,\,t)})_{(s,t \in S \setminus \{0\},\, s \neq t)}\ ) \\ &= \frac{1}{k_2}(f\ (\ (x_{st}^e)_{(\ (s,t) \in \mathcal{R},\, e \in \mathcal{E})_{optimal}}\ ) - \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}) \\ &= \frac{1}{k_2}(f(x_{optimal}) - \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}) \end{aligned} \tag{37}$$

Proof: The proof is provided in the Appendix D.

Let us now investigate the optimality of the corresponding solutions:

Theorem 4.5. *Let x be an optimal solution of either the compact formulation or the LPMT1 formulation and y be the feasible corresponding solution in the other formulation, constructed from x. Then solution y must also be optimal.*

Proof. **Let us first prove** that a corresponding solution in the compact formulation is optimal if the initial solution is optimal in the LPMT1 formulation by disproving the antithesis:

Let $x = ((x_{direct}^{(i,0),(0,j)})_{(i,j \in S \setminus \{0\}, i \neq j)}, (x_{transfer}^{(i,0),(0,j)})_{(i,j \in S \setminus \{0\}, i \neq j)}, (z_l)_{(l \in \mathcal{L})})$ be a corresponding feasible solution in the compact formulation, and $y = ((x_{st}^e)_{((s,t) \in \mathcal{R}, e \in \mathcal{E})}, (y_l)_{(l \in \mathcal{L})}$ be the initial solution in LPMT1. From lemma 4.4, we get that the value of the objective function of the solution x is

$$g(x) = \frac{1}{k_2}(f(y_{optimal}) - \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} = \frac{1}{k_2}(f(y) - \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}$$

The antithesis is that even if

$$f(y) = \min f(x)$$

then there exists an optimal solution in the compact formulation

$$x' = ((x_{direct}^{(i,0),(0,j)'})_{(i,j \in S \setminus \{0\}, i \neq j)}, (x' = ((x_{direct}^{(i,0),(0,j)'})_{(i,j \in S \setminus \{0\}, i \neq j)}, ((z_l')_{(l \in \mathcal{L})})$$

for which

$$g(x') = g((x_{transfer}^{(i,0),(0,j)'})_{(i,j \in S \setminus \{0\}, i \neq j)}) < g((x_{transfer}^{(i,0),(0,j)})_{(i,j \in S \setminus \{0\}, i \neq j)}) = g(x)$$

Using lemma 4.1, we get that there exists a feasible solution y' in LPMT1 that corresponds to the optimal solution x'. From lemma 4.3, we get that the value of the objective function of this solution is of the form

$$f(y') = \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g(x)$$

If $g(x') < g(x)$, then

$$\begin{aligned} f(x') &= \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g(x') \\ &< \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot g(x) \\ &< \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \frac{1}{k_2}(f(y) - \sum_{(s,t) \in \mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}) \\ &= f(y) \end{aligned}$$

which would mean that $f(y) \neq \min f(x)$. Thus, the antithesis is proven false, and consequently, the original clause must be true.

Let us then prove that a corresponding feasible solution in the LPMT1 formulation is optimal if the initial solution is optimal in the compact formulation:

Let $y = ((x^e_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})}, (y_l)_{(l\in\mathcal{L})}$ be a feasible corresponding solution in the LPMT1 formulation, and $x = ((x^{(i,0),\,(0,j)}_{direct})_{(i,j\in S\setminus\{0\},\, i\neq j)},\ (x^{(i,0),\,(0,j)}_{transfer})_{(i,j\in S\setminus\{0\},\, i\neq j)},$ $(z_l)_{(l\in\mathcal{L})})$ be the initial solution in the compact formulation. From lemma 4.3, we get that the value of the objective function of the solution y is

$$f(y) = \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st} + k_2\cdot g(x)$$

The antithesis is that even if

$$g(x) = \min g(x)$$

there exists an optimal solution

$$y' = ((x^{e'}_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})}, (y'_l)_{(l\in\mathcal{L})})$$

for which

$$f(y') = f(x^{e'}_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})}) < f((x^{e}_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})}) = f(y)$$

Using lemma 4.2, we get that there exists a feasible solution x' in the compact formulation that corresponds to y'. From lemma 4.4, we get that the objective function value of this solution is of the form

$$g(x') = \frac{1}{k_2}(f(x'_{optimal}) - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) = \frac{1}{k_2}(f(y') - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st})$$

If $f(y') < f(y)$, then we get

$$\begin{aligned}
g(x') &= \frac{1}{k_2}(f(y') - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) \\
&< \frac{1}{k_2}(f(y) - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) \\
&< \frac{1}{k_2}(\sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st} + k_2\cdot g(x) - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) \\
&= g(x)
\end{aligned}$$

which would mean that $g(x) \neq \min g(x)$. Thus, the antithesis is proven to be false, and consequently, the original clause must be true. Thus, the corresponding solution is shown to be optimal both when the initial solution is an optimal solution of either the LPMT1 formulation or the compact formulation. Thus, Theorem 4.5 must be true. □

Based on Theorem 4.5, we can construct an optimal solution for the compact formulation from any optimal solution of the LPMT1 formulation and vice versa. Thus, we can assume that these formulations are equivalent.

4.2 The Comparison of sizes for LPMT1 and the Compact formulation

The objective of the thesis was to develop a simpler alternative for the LPMT1 formulation. Since the compact formulation defined in Section 3.3 was shown to be equivalent to the LPMT1 formulation in the previous subsection, it is a good candidate for the simpler alternative. In this subsection, we shall investigate this further by calculating and comparing the approximate sizes for both formulations.

To simplify the calculations of the approximate sizes, wedetermine the approximate sizes of the origin-destination set $\mathcal{R}$ and the linepool $\mathcal{L}$, as well as the PTN-graph $(S,\ E)$ and CG-graph $(\mathcal{V},\ \mathcal{E})$ as they are defined in Section 3.1. We will approximate the size as a function of the number of the leaf stations, using the Big $\mathcal{O}$ notation.

Let n be the number of leaf stations in the PTN, meaning leaf nodes that represent physical stations. Now, due to the structure of the PTN defined in Section 3.1, the size of the PTN-graph, the linepool, and the OD-set can be approximated as follows:

$$
\begin{aligned}
&|S| = |\{-1,\ 0,\ 1,\ \dots,n\}| = n+2 && \Rightarrow \mathcal{O}(n)\\
&|E| = |\{\{0,j\} \mid j \in S \ \setminus \ \{0\}\}| = |S \setminus \{0\}| = n+1 && \Rightarrow \mathcal{O}(n)\\
&|\mathcal{L}| = (|S \setminus \{0\}|)(|S \setminus \{0\}| - 1) = n^2 + n && \Rightarrow \mathcal{O}(n^2)\\
&|\mathcal{R}| \leq |S \setminus \{-1,\ 0\}| \cdot (|S \setminus \{-1,\ 0\}| - 1) = n^2 - n && \Rightarrow \mathcal{O}(n^2)
\end{aligned}
$$

Similarly, the size of the CG-graph is approximated step by step as follows:

$$
\begin{aligned}
&|\mathcal{V}_{OD}| \leq |S \setminus \{-1,\ 0\}| = n && \Rightarrow \mathcal{O}(n)\\
&|\mathcal{V}_{CG}| = 3 \cdot |\mathcal{L}| = 3n^2 + 3n && \Rightarrow \mathcal{O}(n^2)\\
&|\mathcal{V}| = |\mathcal{V}_{CG}| + |\mathcal{V}_{OD}| = \mathcal{O}(n^2) + \mathcal{O}(n) && \Rightarrow \mathcal{O}(n^2)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathcal{E}_l| = 2 && \Rightarrow \mathcal{O}(1)\\
&|\mathcal{E}_{go}| = \sum_{l \in \mathcal{L}} |\mathcal{E}_l| = 2 \cdot |\mathcal{L}| && \Rightarrow \mathcal{O}(n^2)\\
&|\mathcal{E}_{change}| = |S \setminus \{0\}| \cdot (|S \setminus \{0\}| - 1) \cdot (|S \setminus \{0\}| - 1) \cdot (|S \setminus \{0\}| - 3) = n^4 - n^2 && \Rightarrow \mathcal{O}(n^4)\\
&|\mathcal{E}_{OD}| \leq 2 \cdot |\mathcal{V}_{OD}| \cdot |\{(k, [k,j]) \in \mathcal{V}_{CG} \mid k \text{ is fixed}\}| = 2n \cdot \mathcal{O}(n^2) && \Rightarrow \mathcal{O}(n^3)\\
&|\mathcal{E}| = |\mathcal{E}_{go}| + |\mathcal{E}_{change}| + |\mathcal{E}_{OD}| = \mathcal{O}(n^2) + \mathcal{O}(n^4) + \mathcal{O}(n^3) && \Rightarrow \mathcal{O}(n^4)
\end{aligned}
$$

Utilizing these calculations, we determine the approximation for the number of variables in each formulation, as a function of the number of leaf stations n. The results are summarized below as follows: Table 1 shows the approximations for the number of variables in each variable group present in the formulations as well as the relevant information used to calculate it. Table 2 shows the approximation for the number of each type of constraint present in the LPMT1 formulation. Table 3 presents the same information for the compact formulation. The calculations used to determine the results in Tables 1, 2, and 3, can be found in the Appendix E.

	The variable group	Exact amount	Size approximation
LPMT1	x^e_{st}	$\lvert\mathcal{R}\rvert \cdot \lvert\mathcal{E}\rvert$	$\mathcal{O}(n^6)$
	y_l	$\lvert\mathcal{L}\rvert$	$\mathcal{O}(n^2)$
Total number			$\mathcal{O}(n^6)$
Compact	$x^{(s,\,0),\,(0,\,t)}_{direct}$	$\lvert\mathcal{R}\rvert$	$\mathcal{O}(n^2)$
	$x^{(s,\,0),\,(0,\,t)}_{transfer}$	$\lvert\mathcal{R}\rvert$	$\mathcal{O}(n^2)$
	z_l	$\lvert\mathcal{R}\rvert$	$\mathcal{O}(n^2)$
Total number			$\mathcal{O}(n^2)$

Table 1: The approximations for the number of variables, categorized by formulation and the variable group.

Constraint Reference	Exact Amount	Size Approximation
(4)	$\lvert\mathcal{L}\rvert$	$\mathcal{O}(n^2)$
(5)	$\lvert\mathcal{V}\rvert \cdot \lvert\mathcal{R}\rvert$	$\mathcal{O}(n^4)$
(6)	1	$\mathcal{O}(1)$
Total number		$\mathcal{O}(n^4)$

Table 2: The approximations for the number of each type of constraint in the LPMT1 formulation.

Constraint References	Exact Amount	Size Approximation
(10), (14), (15)	$\lvert\mathcal{R}\rvert$	$\mathcal{O}(n^2)$
(11)	$\min(\lvert\mathcal{L}\rvert,\ \lvert\mathcal{R}\rvert)$	$\mathcal{O}(n^2)$
(12)	$\lvert\{i \in S \setminus \{-1,\ 0\} :\ (i,j) \in \mathcal{R}\}\rvert$	$\mathcal{O}(n)$
(13)	$\lvert\{j \in S \setminus \{-1,\ 0\} :\ (i,j) \in \mathcal{R}\}\rvert$	$\mathcal{O}(n)$
(16)	1	$\mathcal{O}(1)$
Total number		$\mathcal{O}(n^2)$

Table 3: The approximations for the number of each type of constraint in the compact formulation.

From Table 1 and by comparing Tables 2 and 3, we can see that both the number of variables and the number of constraints grow significantly faster in the LPMT1 formulation than in the compact formulation. Thus, we can conclude that the compact formulation is significantly smaller than the LPMT1. This indicates that the compact formulation should also be simpler and more efficient to solve than LPMT1, as desired. However, further research is required to verify this conclusion empirically.

5 Summary and Conclusions

In this thesis, we utilized the features of trees to create a formulation for the line planning problem that minimizes the total travel time of all passengers traveling in a star-shaped station network: Based on the fact that, in tree networks, there exists only one simple path between two nodes, we were able to show that, in trees, the total travel time depends only on the number of transfers made. Consequently, we used this result to create the compact formulation introduced in Definition 3.7, where the total travel time is minimized by minimizing transfers.

As aimed in this thesis, the developed compact formulation acts as a simpler alternative to the formulation developed by Schöbel and Scholl in [8] and introduced in this thesis in Section 3.2: In Section 4.1, the compact formulation is shown to be equivalent to the formulation developed by Schöbel and Scholl, in this paper referred to as LPMT1. Additionally, the compact formulation is shown to be significantly smaller than the corresponding version of the LPMT1 formulation in Section 4.2, which indicates that the compact formulation should be also easier to solve. However, more research should be done to corroborate this empirically.

Since the compact formulation is a simpler alternative to the LPMT1 formulation, the computation times of certain line planning problems could be reduced by replacing the LPMT1 formulation with the compact formulation. Due to the significant difference in size, the compact formulation could still be usable in some instances when the corresponding LPMT1 formulation has grown too large to be solved within reasonable computation times. However, the scope of application and therefore the potential of the compact formulation is limited: While constructing the compact formulation, we relied heavily on the specific features of the star-shaped tree. Thus, the formulation cannot be generalized even to most trees without heavy restructuring.

Despite its limited scope of application, the compact formulation can potentially be useful in several instances. This is because certain station networks, where multiple chains of nodes connect a single central transfer hub, can be modeled as a star: If we assume that the lines start and terminate at the endpoints of the different branches originating from the hub, each leaf node of the star can be set to represent one entire branch using the assumptions and reasoning shown in Section 3.1. Some real-world transit systems, such as Stockholm's metro system Tunnelbana and Frankfurt's S-Bahn commuter rail system, can be modeled as this type of network under certain assumptions. Thus, the compact formulation can be used to create approximate line plans for existing transit systems despite not fully representing them.

Beyond its limited applications and benefits, the compact formulation demonstrates that the special features of trees can be leveraged to construct simpler alternatives to the generally applicable formulations. This affirms the potential for further research on generalizing the compact formulation or the methods used in this thesis to other types of trees. The resulting generalized models could be used to accurately model a wider range of existing and potential traffic systems. Further research should also be done on comparing the actual computation times of compact formulation on different algorithms and to verify if the difference in computation times is significant enough to warrant a shift in methodologies.

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A The Detailed Proof for the Existence of a Feasible Solution in LPMT1

In this Appendix, we show detailed proof for Lemma 4.2 stated below as Lemma A.1.

Lemma A.1. *For any optimal solution of the LMPT1 formulation, we can construct a feasible solution of the compact formulation defined in Definition 3.7.*

Proof:

Let $(x,\, y)$ with $x = (x^e_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})}$ and $y = (y_l)_{(l\in\mathcal{L})}$ be an optimal solution to the LPMT1 formulation. Since $(x,\, y)$ is an optimal solution, for all $i \in S \setminus \{s,\, 0\}$ or $j \in S \setminus \{t,\, 0\}$, $x^{((i,\,l),\,(j,\,l))}_{st} = 0$ where $l \in \mathcal{L}$, $x^{((0,\,[i,p]),\,(0,\,[q,j]))}_{st} = 0$, $x^{((i,\,OD),\,(i,\,l))}_{st} = 0$, and $x^{((j,\,l),\,(j,\,OD))}_{st} = 0$. The reasoning for this is as follows:

Since we are only interested in the optimal solutions, Remark 3.4 applies. Due to this and the shape of the PTN, the fixed simple paths in the PTN, which correspond to the routes traveled by the passengers, are determined as follows: for $(s,t) \in \mathcal{R}$, the fixed route is $(\,(s,0),\ (0,t)\,)$. Since driving edges $((i,\ l),\ (j,\ l))$, where $l \in \mathcal{L}$ and $i \in S \setminus \{s,\ 0\}$, or $j \in S \setminus \{t,\ 0\}$, do not correspond to any of the PTN edges listed above, they cannot be included in the route the passenger takes from s to t. Thus, for all $i \in S \setminus \{s,\ 0\}$ or $j \in S \setminus \{t,\ 0\}$, $x^{((i,\,l),\,(j,\,l))}_{st} = 0$. Similarly, OD edges $((i,\ OD),\ (i,\ l))$ for all $i \in S \setminus \{s,\ 0\}$ or $((j,\ l),\ (j,\ OD))$ for all $j \in S \setminus \{t\}$, where $l \in \mathcal{L}$, cannot be used to access the starting and the ending node of the fixed PTN route. Hence, $x^{((i,\,OD),\,(i,\,l))}_{st} = 0$, and $x^{((j,\,l),\,(j,\,OD))}_{st} = 0$ for all $i \in S \setminus \{s,\ 0\}$, and $j \in S \setminus \{t,\ 0\}$.

Additionally, any CG-path that includes multiple transfers done in the same PTN-node cannot be the quickest alternative: Since the passengers only transfer on the center node, and all lines travel through the center node, a passenger can always access the line they plan to take out of the node with up to one transfer. Therefore, any transfers made in the optimal solution must be between a line the passenger is taking into the center node and a line the passenger is taking out of the center node. Since the lines are simple paths in the PTN, this together with the construction of the fixed PTn paths means that, the line the passengers are transferring out of must originate in s, and the line the passengers are transferring into must end at t, for all $(s,t) \in \mathcal{R}$. Thus, for all $i \in S \setminus \{s,\ 0\}$ or $j \in S \setminus \{t,\ 0\}$, $x^{((0,\,[i,p]),\,(0,\,[q,j]))}_{st} = 0$.

For any $(s,t) \in \mathcal{R}$, let us set

$$x^{(s,\,0),\,(0,\,t)}_{direct} = \begin{cases} 1, & x^{((s,\,[s,t]),\,(0,\,[s,t]))}_{st} = 1 \wedge x^{((0,\,[s,t]),\,(t,\,[s,t]))}_{st} = 1 \\ 0, & \text{otherwise} \end{cases} \tag{A1}$$

and

$$x^{(s,\,0),\,(0,\,t)}_{transfer} = \begin{cases} 1, & \exists\, p, q \in S \setminus \{0,\ s,\ t\} :\ x^{((0,\,[s,p]),\,(0,\,[q,t]))}_{st} = 1 \\ 0, & \text{otherwise} \end{cases} \tag{A2}$$

Additionally, let us set $z_l = y_l$ for lines $l \in \mathcal{L}$.

Let us now show that a corresponding solution $(x_{direct},\, x_{tranfer},\, z)$ with $x_{direct} = (x_{direct}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)}$, $(x_{transfer}^{(i,\,0),\,(0,\,j)})_{(i,j \in S \setminus \{0\},\, i \neq j)}$, and $z = (z_l)_{(l \in \mathcal{L})}$ constructed with the method described above is a feasible solution to the compact formulation described in the Section 3.3.

As an optimal and therefore feasible solution of LPMT1, the initial solution fulfills the constraint (5). As mentioned in Section **??**, this constraint can be written using the incoming and outgoing edges as constraints (21), (22), and (23). Based on the definitions of the incoming and outgoing edge sets described in Definition 3.6, the constraint (21) can be separated and simplified into subcases represented in (21.1), (21.2), (21.3), and (21.4). Based on all of these subcases, and since for all $i \in S \setminus \{s, 0\}$ or $j \in S \setminus \{t, 0\}$, $x_{st}^{((i,\,l),\,(j,\,l))} = 0$, $x_{st}^{((0,\,[i,p]),\,(0,\,[q,j]))} = 0$, $x_{st}^{((i,\,OD),\,(i,\,l))} = 0$, as well as $x_{st}^{((j,\,l),\,(j,\,OD))} = 0$, if the initial solution is feasible in the LPMT1, then the following holds

$$\forall (s,t) \in \mathcal{R}:$$
$$\forall\, i, j \in S \setminus \{0\} : i \neq j \; :$$

$$x_{st}^{((i,\,OD),\,(i,\,[i,j]))} = x_{st}^{((i,\,[i,j]),\,(0,\,[i,j]))} \tag{A3.1}$$

$$x_{st}^{((0,\,[i,j]),\,(j,\,[i,j]))} = x_{st}^{((j,\,[i,j]),\,(j,\,OD))} \tag{A3.2}$$

$$x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} \tag{A3.3}$$

$$x_{st}^{((s,\,[s,j]),\,(0,\,[s,j]))} = \sum_{c \in S \setminus \{0,\,s,\,t\}} x_{st}^{((0,\,[s,j]),\,(0,\,[c,t]))}, \;\; j \neq t \tag{A3.4}$$

$$\sum_{d \in S \setminus \{0,\,s,\,t\}} x_{st}^{((0,\,[s,d]),\,(0,\,[i,t]))} = x_{st}^{((0,\,[i,t]),\,(t,\,[i,t]))}, \;\; i \neq s \tag{A3.5}$$

$$\sum_{d \in S \setminus \{0,\,s\}} x_{st}^{((s,\,OD),\,(s,\,[s,d]))} = 1 \tag{A3.6}$$

$$\sum_{c \in S \setminus \{0,\,t\}} x_{st}^{((t,\,[c,t]),\,(t,\,OD))} = 1 \tag{A3.7}$$

Let us first show that the corresponding solution fulfills the constraint (10):

Based on equation (A3.3), $x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} = 1$ or $x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} = 0$. If $x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} = 1$, based on conversion rule (A1), $x_{direct}^{(s,\,0),\,(0,\,t)} = 1$.

On the other hand, based on the equations (A3)

$$x_{st}^{((s,\,OD),\,(s,\,[s,t]))} = x_{st}^{((s,\,[s,t]),\,(0,\,[s,t]))} = x_{st}^{((0,\,[s,t]),\,(t,\,[s,t]))} = x_{st}^{((t,\,[s,t]),\,(t,\,OD))} = 1$$

Consequently, we get from equation (A3.6)

$$\begin{aligned}
& \sum_{d\in S \setminus \{s\}} x_{st}^{((s,\, OD),\, (s,\, [s,d]))} = 1 \\
\Leftrightarrow \quad & x_{st}^{((s,\, OD),\, (s,\, [s,t]))} + \sum_{d\in S \setminus \{s,t\}} x_{st}^{((s,\, OD),\, (s,\, [s,d]))} = 1 \quad | x_{st}^{((s,\, OD),\, (s,\, [s,t]))} = 1 \\
\Leftrightarrow \quad & 1 + \sum_{j\in S \setminus \{s,t\}} x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 1 \\
\Leftrightarrow \quad & \sum_{j\in S \setminus \{s,t\}} x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 0
\end{aligned}$$

Additionally, we get from equation (A3.7)

$$\begin{aligned}
& \sum_{i\in S \setminus \{t\}} x_{st}^{((t,\, [i,t]),\, (t,\, OD))} = 1 \\
\Leftrightarrow \quad & x_{st}^{((t,\, [s,t]),\, (t,\, OD))} + \sum_{i\in S \setminus \{s,\, t\}} x_{st}^{((t,\, [i,t]),\, (t,\, OD))} = 1 \quad | x_{st}^{((t,\, [s,t]),\, (t,\, OD))} = 1 \\
\Leftrightarrow \quad & 1 + \sum_{i\in S \setminus \{s,\, t\}} x_{st}^{((t,\, [i,t]),\, (t,\, OD))} = 1 \\
\Leftrightarrow \quad & \sum_{i\in S \setminus \{s,\, t\}} x_{st}^{((t,\, [i,t]),\, (t,\, OD))} = 0
\end{aligned}$$

Consequently, $x_{st}^{((s,\, OD),\, (s,\, [s,p]))} = 0$ and $x_{st}^{((t,\, [q,t]),\, (t,\, OD))} = 0$ for all $p, q \in S \setminus \{s,\, t\}$. Thus, based on the equations (A3), for all $p, q \in S \setminus \{s,\, t\}$,

$$\sum_{c\in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,p]),\, (0,\, [c,t]))} = x_{st}^{((s,\, [s,p]),\, (0,\, [s,p]))} = x_{st}^{((s,\, OD),\, (s,\, [s,p]))} = 0$$

and

$$\sum_{d\in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,d]),\, (0,\, [q,t]))} = x_{st}^{((0,\, [q,t]),\, (t,\, [q,t]))} = x_{st}^{((t,\, [q,p]),\, (t,\, OD))} = 0$$

Therefore, it must be that $x_{st}^{((0,\, [s,p]),\, (0,\, [q,t]))} = 0$ for all $p, q \in S \setminus \{s,\, t\}$. Based on conversion rule (A2), this means that $x_{transfer}^{(s,\, 0),\, (0,\, t)} = 0$. Thus, if $x_{st}^{((s,\, [s,t]),\, (0,\, [s,t]))} = x_{st}^{((0,\, [s,t]),\, (t,\, [s,t]))} = 1$, then $x_{direct}^{(s,\, 0),\, (0,\, t)} + x_{transfer}^{(s,\, 0),\, (0,\, t)} = 1 + 0 = 1$ meaning that contraint (10) is fulfilled in this subcase.

If $x_{st}^{((s,\, [s,t]),\, (0,\, [s,t]))} = x_{st}^{((0,\, [s,t]),\, (t,\, [s,t]))} = 0$, then based on conversion rule (A1), $x_{direct}^{(s,\, 0),\, (0,\, t)} = 0$. Additionally, based on equations (A3),

$$x_{st}^{((s,\, OD),\, (s,\, [s,t]))} = x_{st}^{((s,\, [s,t]),\, (0,\, [s,t]))} = x_{st}^{((0,\, [s,t]),\, (t,\, [s,t]))} = x_{st}^{((t,\, [s,t]),\, (t,\, OD))} = 0$$

and

$$\begin{aligned}
&\sum_{d \in S \setminus \{0,\, s\}} x_{st}^{((s,\, OD),\, (s,\, [s,d]))} = 1 \\
\Leftrightarrow \quad x_{st}^{((s,\, OD),\, (s,\, [s,t]))} + &\sum_{d \in S \setminus \{0,\, s,\, t\}} x_{st}^{((s,\, OD),\, (s,\, [s,d]))} = 1 \quad |x_{st}^{((s,\, OD),\, (s,\, [s,t]))} = 0 \\
\Leftrightarrow \quad &\sum_{d \in S \setminus \{0,\, s,\, t\}} x_{st}^{((s,\, OD),\, (s,\, [s,d]))} = 1
\end{aligned}$$

Consequently, since $x_{st}^{e} \in \{0,\ 1\}$, there must exist $p \in S \setminus \{s,\ t\}$ for which $x_{st}^{((s,\, OD),\, (s,\, [s,p]))} = 1$. Let us fix $p \in S \setminus \{s,\ t\}$ such that $x_{st}^{((s,\, OD),\, (s,\, [s,p]))} = 1$. Thus, we get from equations (A3),

$$\sum_{c \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,p]),\, (0,\, [c,t]))} = x_{st}^{((s,\, [s,p]),\, (0,\, [s,p]))} = x_{st}^{((s,\, OD),\, (s,\, [s,p]))} = 1$$

Since $x_{st}^{e} \in \{0,\ 1\}$, this means that for the fixed $p \in S \setminus \{s,\ t\}$, there must exist $q \in S \setminus \{s,\ t\}$ for which $x_{st}^{((0,\, [s,p]),\, (0,\, [q,t]))} = 1$.

Consequently, there must exist $p, q \in S \setminus \{s,\ t\}$ for which $x_{st}^{((0,\, [s,p]),\, (0,\, [q,t]))} = 1$. Based on the conversion rule (20), this means that $x_{transfer}^{(s,\, 0),\, (0,\, t)} = 1$.

Thus, if $x_{st}^{((s,\, [s,t]),\, (0,\, [s,t]))} = x_{st}^{((0,\, [s,t]),\, (t,\, [s,t]))} = 0$, then $x_{direct}^{(s,\, 0),\, (0,\, t)} + x_{transfer}^{(s,\, 0),\, (0,\, t)} = 0 + 1 = 1$, meaning that constraint (10) is fulfilled also in this subcase. Consequently, the corresponding solution fulfills the constraint (10).

Let us now show that the corresponding solution fulfills the constraint (11):

We have set $z_l = y_l$ for all $l \in \mathcal{L}$. Therefore, since $x_{direct}^{(i,\, 0),\, (0,\, j)} \in \{0,\ 1\}$, if $y_{[i,j]} = 1$ in the initial solution, the inequality $x_{direct}^{(i,\, 0),\, (0,\, j)} \leq z_{[i,j]}$ holds.

Thus, the constraint (11) is fulfilled if $y_{[i,j]} = 1$.

If $y_{[i,j]} = 0$, since $x_{st}^{e} \in \{0,\ 1\}$, then by constraint (4) of the initial solution,

$$\begin{aligned}
\sum_{(s,t) \in \mathcal{R}} \sum_{e \in \mathcal{E}_{[i,j]}} x_{st}^{e} \leq 0 \quad &\Leftrightarrow \quad x_{st}^{((i,\, [i,j]),\, (0,\, [i,j]))} + x_{st}^{((0,\, [i,j]),\, (j,\, [i,j]))} = 0,\ \forall\, (s,t) \in \mathcal{R} \\
&\Leftrightarrow \quad x_{st}^{((i,\, [i,j]),\, (0,\, [i,j]))} = x_{st}^{((i,\, [i,j]),\, (0,\, [i,j]))} = 0,\ \forall\, (s,t) \in \mathcal{R}
\end{aligned}$$

Consequently, $x_{ij}^{((i,\, [i,j]),\, (0,\, [i,j]))} = x_{ij}^{((i,\, [i,j]),\, (0,\, [i,j]))} = 0$. Therefore, $x_{direct}^{(i,\, 0),\, (0,\, j)} = 0$ based on construction rule (A1), meaning that

$$0 = x_{direct}^{(i,\, 0),\, (0,\, j)} \leq z_{[i,j]} = y_{[i,j]} = 0$$

Hence, constraint (11) is also fulfilled when $y_{[i,j]} = 0$ in the initial solution.

Thus, constraint (11) is fulfilled for all $[i, j] \in \mathcal{L}$ for which $(i, j) \in \mathcal{R}$.

Let us now show that the corresponding solution fulfills constraints (12) and (13):

Let us prove by contraposition that constraint (12) holds for arbitrary $s \in S \setminus \{-1,\, 0\}$ and $p \in S \setminus \{-1,\, 0,\, s\}$, and constraint (13) holds for arbitrary $t \in S \setminus \{-1,\, 0\}$ and $q \in S \setminus \{-1,\, 0,\, t\}$, where $(s,p), (q,t) \in \mathcal{R}$: Since $z_l \in \{0,\, 1\}$ for all $l \in \mathcal{L}$, this happens by showing that if there exists $s,\, t \in S \setminus \{-1,\, 0\}$, $p \in S \setminus \{-1,\, 0,\, s\}$, and $q \in S \setminus \{-1,\, 0,\, t\}$, for which $(s,p),\ (q,t) \in \mathcal{R}$, in the corresponding solution such that

$$\sum_{k\neq s} z_{[s,k]} < 1 \quad \text{or} \qquad\qquad \sum_{k\neq t} z_{[k,t]} < 1$$

then the initial solution cannot be an optimal solution in the LPMT1 formulation.

Let us first assume that the initial solution is an optimal solution of the LPMT1 and fix that for $s,\, t \in S \setminus \{-1,\, 0\}$, $p \in S \setminus \{-1,\, 0,\, s\}$, $q \in S \setminus \{-1,\, 0,\, t\}$, where $(s,p),\ (q,t) \in \mathcal{R}$,

$$\sum_{k\neq s} z_{[s,k]} < 1 \quad \text{or} \qquad\qquad \sum_{k\neq t} z_{[k,t]} < 1$$

Since we have set that $z_l = y_l$ for all $l \in \mathcal{L}$, and $y_l \in \{0,\, 1\}$, we get that for these fixed $s, t \in S \setminus \{-1,\, 0\}$

$$\begin{aligned}
&& \sum_{k\neq s} z_{[s,k]} &< 1 \quad \text{or} && & \sum_{k\neq t} z_{[k,t]} &< 1\\
&& \sum_{k\neq s} y_{[s,k]} &< 1 \quad \text{or} && & \sum_{k\neq t} y_{[k,t]} &< 1\\
&\Leftrightarrow & \sum_{k\neq s} y_{[s,k]} &= 0 \quad \text{or} && \Leftrightarrow & \sum_{k\neq t} y_{[k,t]} &= 0\\
\Leftrightarrow\quad & \forall\, k \in S \setminus \{0,\, s\}: & y_{[s,k]} &= 0 \quad \text{or} & \Leftrightarrow\quad & \forall\, k \in S \setminus \{0,\, t\}: & y_{[k,t]} &= 0
\end{aligned}$$

By constraint (4) of the LPMT1, since $x^e_{st} \in \{0,\, 1\}$, we get that

$$\begin{aligned}
& \forall\, k \in S \setminus \{0,\, s\}: && \forall\, k \in S \setminus \{0,\, t\}:\\
& \quad \sum_{(s',t')\in\mathcal{R}} \sum_{e\,\in\,\mathcal{E}_{[s,k]}} x^e_{s't'} \leq 0 && \quad \sum_{(s',t')\in\mathcal{R}} \sum_{e\,\in\,\mathcal{E}_{[k,t]}} x^e_{s't'} \leq 0\\
\Leftrightarrow\quad & \sum_{e\,\in\,\mathcal{E}_{[s,k]}} x^e_{sp} \leq 0 & \Leftrightarrow\quad & \sum_{e\,\in\,\mathcal{E}_{[k,t]}} x^e_{qt} \leq 0\\
\Leftrightarrow\quad & x_{sp}^{((s,\,[s,k]),\,(0,\,[s,k]))} + x_{sp}^{((0,\,[s,k]),\,(k,\,[s,k]))} \leq 0 & \Leftrightarrow\quad & x_{qt}^{((k,\,[k,t]),\,(0,\,[k,t]))} + x_{qt}^{((0,\,[k,t]),\,(t,\,[k,t]))} \leq 0\\
\Leftrightarrow\quad & x_{sp}^{((s,\,[s,k]),\,(0,\,[s,k]))} = 0 & \Leftrightarrow\quad & x_{qt}^{((0,\,[k,t]),\,(t,\,[k,t]))} = 0
\end{aligned}$$

We have previously, shown that any optimal solution to the LPMT1 fulfills the equations (A3). Hence, using equations (A3.1) and (A3.2), we get

$$\sum_{d\in S\setminus\{0,\,s\}} x_{sp}^{((s,\,OD),\,(s,\,[s,d]))} = \sum_{d\in S\setminus\{0,\,s\}} x_{sp}^{((s,\,[s,d]),\,(0,\,[s,d]))} = \sum_{d\in S\setminus\{0,\,s\}} 0 = 0$$

or

$$\sum_{c\in S\setminus\{0,\,t\}} x_{qt}^{((t,\,[c,t]),\,(t,\,OD))} = \sum_{c\in S\setminus\{0,\,t\}} x_{qt}^{((0,\,[c,t]),\,(t,\,[c,t]))} = \sum_{c\in S\setminus\{0,\,t\}} 0 = 0$$

This, in turn, conflicts with equation (A3.6) or equation (A3.7), meaning that the initial solution cannot be an optimal solution in the LPMT1 formulation.

Thus, the corresponding solution fulfills the constraints (12) and (13).

Let us show that the corresponding solution fulfills constraints (14) and (15):
Since $z_l \in \{0,\ 1\}$ for all $l \in \mathcal{L}$, if $x_{transfer}^{(s,\,0),\,(0,\,t)} = 0$ for an arbitrary $(s,t) \in \mathcal{R}$, then

$$x_{transfer}^{(s,\,0),\,(0,\,t)} = 0 \le \sum_{s\,\neq\,k\,\neq\,t} z_{[s,k]} \quad \text{and} \quad x_{transfer}^{(s,\,0),\,(0,\,t)} = 0 \le \sum_{s\,\neq\,k\,\neq\,t} z_{[k,t]}$$

Therefore, constraints (14) and (15) are fulfilled for all $(s,t) \in \mathcal{R}$ where $x_{transfer}^{(s,\,0),\,(0,\,t)} = 0$.

Contrarily, if $x_{transfer}^{(s,\,0),\,(0,\,t)} = 1$ for an arbitrary $(s,t) \in \mathcal{R}$ then, by contruction rule (A2), there exists $p, q \in S \setminus \{0,\ s,\ t\}$ for which $x_{st}^{((0,\,[s,p]),\,(0,\,[q,t]))} = 1$. As a natural consequence, we get based on equations (A3.4) and (A3.5)

$$x_{st}^{((s,\,[s,p]),\,(0,\,[s,p]))} = \sum_{c\in S\setminus\{0,\,s,\,t\}} x_{st}^{((0,\,[s,p]),\,(0,\,[c,t]))} \ge x_{st}^{((0,\,[s,p]),\,(0,\,[q,t]))} = 1$$

and

$$x_{st}^{((0,\,[q,t]),\,(t,\,[q,t]))} = \sum_{d\in S\setminus\{0,\,s,\,t\}} x_{st}^{((0,\,[s,d]),\,(0,\,[q,t]))} \ge x_{st}^{((0,\,[s,p]),\,(0,\,[q,t]))} = 1$$

Since $x_{st}^{e} \in \{0,\ 1\}$, this means that $x_{st}^{((s,\,[s,p]),\,(0,\,[s,p]))} = 1$ and $x_{st}^{((0,\,[q,t]),\,(t,\,[q,t]))} = 1$. By the constraint (4) of the initial solution, we thus get

$$|\mathcal{R}|\,|\mathcal{E}_{[s,p]}| \cdot y_{[s,p]} \ge \sum_{(s',t')\in\mathcal{R}} \sum_{e\in\mathcal{E}_{[s,p]}} x_{s't'}^{e} \ge \sum_{e\in\mathcal{E}_{[s,p]}} x_{st}^{e} \ge x_{st}^{((s,\,[s,p]),\,(0,\,[s,p]))} = 1$$

and

$$|\mathcal{R}|\,|\mathcal{E}_{[q,t]}| \cdot y_{[q,t]} \ge \sum_{(s',t')\in\mathcal{R}} \sum_{e\in\mathcal{E}_{[q,t]}} x_{s't'}^{e} \ge \sum_{e\in\mathcal{E}_{[q,t]}} x_{st}^{e} \ge x_{st}^{((0,\,[q,t]),\,(t,\,[q,t]))} = 1$$

Consequently, since $x_{st}^{e} \in \{0,\ 1\}$, $y_{[s,p]} = 1$ and $y_{[q,t]}$. Additionally, since we have set that $z_l = y_l$ for all $l \in \mathcal{L}$, we get

$$\sum_{s\,\neq\,k\,\neq\,t} z_{[s,k]} \ge z_{[s,p]} = y_{[s,p]} = 1 = x_{transfer}^{(s,\,0),\,(0,\,t)}$$

and

$$\sum_{s\,\neq\,k\,\neq\,t} z_{[k,t]} \ge z_{[q,t]} = y_{[q,t]} = 1 = x_{transfer}^{(s,\,0),\,(0,\,t)}$$

Therefore, the constraints (14) and (15) are fulfilled for all $(s,t) \in \mathcal{R}$ where $x_{transfer}^{(s,0),(0,t)} = 1$.

Thus, the corresponding solution fulfills the constraints (14) and (15).

Let us finally prove that the corresponding solution fulfills constraint (16):
We have assumed that the cost of creating line $l \in \mathcal{L}$, C_l, and the budget, B, are known and, therefore, remain constant for the different formulations. Additionally, we set $z_l = y_l$ for all $l \in \mathcal{L}$. Consequently, based on these assumptions and from constraint (6) of the initial feasible solution we get

$$\sum_{l \in \mathcal{L}} C_l \cdot z_l = \sum_{l \in \mathcal{L}} C_l \cdot y_l \leq B$$

Therefore, the corresponding solution fulfills the constraint (16).

Thus, the constructed corresponding solution is a feasible solution to the compact formulation, and, consequently, any optimal solution of the LPMT1 formulation has a corresponding feasible solution in the compact formulation.

B The Change&Go Network with Three Tangible Leaf Nodes in the PTN

In this appendix, Figure B1 presents the change&go network corresponding to the PTN shown in Figure 1.

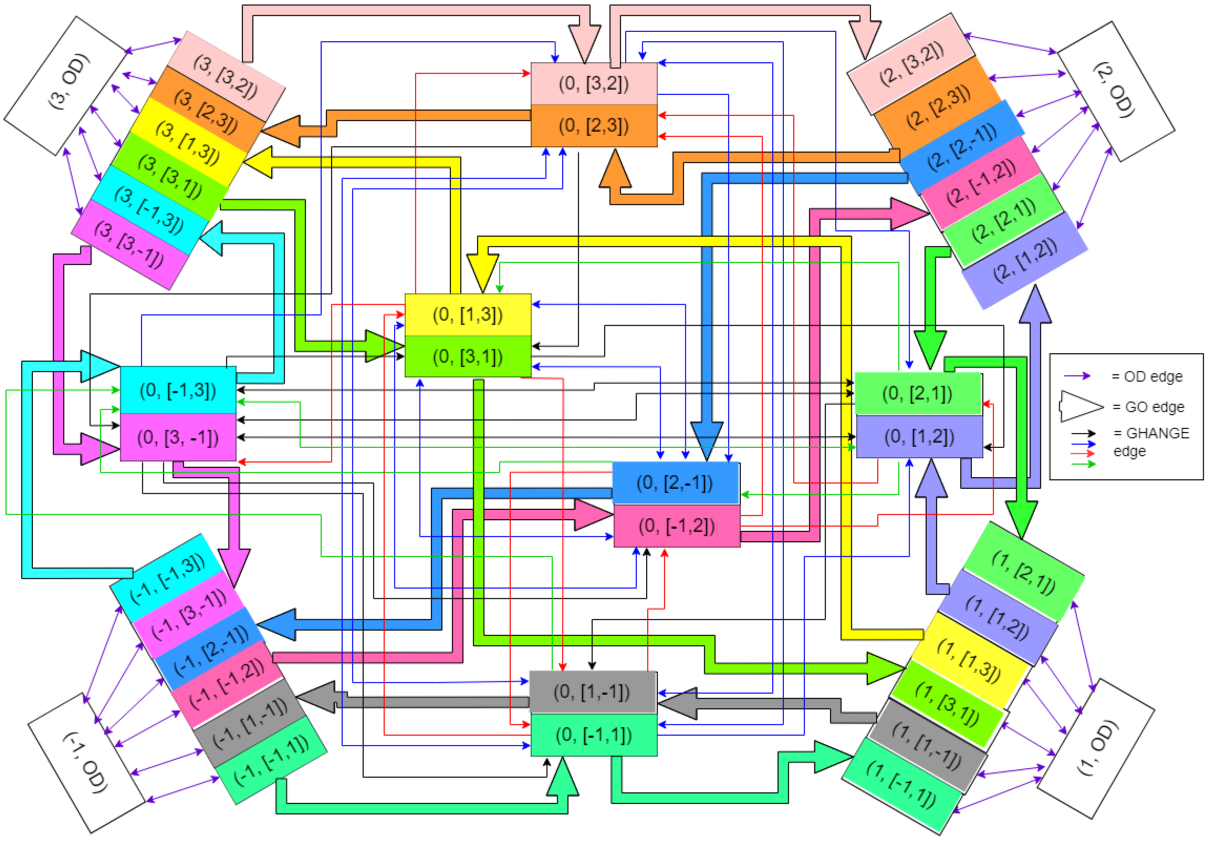

Figure B1: The change&go network when the corresponding PTN is a star-shaped tree with three actual leaf nodes, as shown in Figure 1.

C The Simplification of Equation (33)

In this appendix, we shall show the simplification of the transformed objective function of the LPMT1 presented in equation (33) into the simplified form represented in equation (34). The basis of the simplification is that in an optimal LPMT1-solution, for all $i \in S \setminus \{0,\ s\}$ or $j \in S \setminus \{0,\ t\}$ where $(s,t) \in \mathcal{R}$, $x_{st}^{((0,\ [i,p]),\ (0,\ [q,j]))} = 0$, as mentioned in Appendix **??**.

The simplification happens as follows:

$$
\begin{aligned}
f(x_{optimal}) &= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \sum_{e\,\in\,\mathcal{E}_{change}} x_{st}^{e} \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(\sum_{a\in S\setminus\{0\}} \sum_{d\in S\setminus\{0\}} \ldots \\
&\quad \ldots \sum_{b,\ c\in S\setminus\{0,\ a,\ d\}} x_{st}^{((0,\ [a,b]),\ (0,\ [c,d]))} \Big)\Big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(\sum_{a\in S\setminus\{0,\ s\}} \sum_{d\in S\setminus\{0\}} \ldots \\
&\quad \ldots \sum_{b,\ c\in S\setminus\{0,\ a,\ d\}} x_{st}^{((0,\ [a,b]),\ (0,\ [c,d]))} + \sum_{d\in S\setminus\{0\}} \sum_{b,\ c\in S\setminus\{0,\ s,\ d\}} x_{st}^{((0,\ [s,b]),\ (0,\ [c,d]))} \Big)\Big)\ldots \\
&\quad \ldots \Big|\ \ x_{st}^{((0,\ [i,p]),\ (0,\ [q,j]))} = 0,\ \forall\, i \in S \setminus \{0,\ s\} \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(\sum_{a\in S\setminus\{0,\ s\}} \sum_{d\in S\setminus\{0\}} \sum_{b,\ c\in S\setminus\{0,\ a,\ d\}} 0 \ldots \\
&\quad \ldots + \sum_{d\in S\setminus\{0,\ t\}} \sum_{b,\ c\in S\setminus\{0,\ s,\ d\}} x_{st}^{((0,\ [s,b]),\ (0,\ [c,d]))} + \sum_{b,\ c\in S\setminus\{0,\ s,\ t\}} x_{st}^{((0,\ [s,b]),\ (0,\ [c,t]))} \Big)\Big)\ldots \\
&\quad \ldots \Big|\ \ x_{st}^{((0,\ [i,p]),\ (0,\ [q,j]))} = 0,\ \forall\, j \in S \setminus \{0,\ t\} \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \Big(0 + \sum_{d\in S\setminus\{0,\ t\}} \sum_{b,\ c\in S\setminus\{0,\ s,\ d\}} 0 \ldots \\
&\quad \ldots + \sum_{b,\ c\in S\setminus\{0,\ s,\ t\}} x_{st}^{((0,\ [s,b]),\ (0,\ [c,t]))} \Big)\Big) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + k_2 \cdot \Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \sum_{b,\ c\in S\setminus\{0,\ s,\ t\}} x_{st}^{((0,\ [s,b]),\ (0,\ [c,t]))} \Big)
\end{aligned}
$$

where parameters w_{st}, k_1, k_2, and T_{st} do not depend on the decision variables.

D The Detailed Proof for the Lemma 4.4

In this appendix, we show the detailed proof for the Lemma 4.4 used in Section 4.1.2. The particular Lemma is introduced as D.1 below.

Lemma D.1. *Let* $((x^{e}_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})},\ (y_l)_{(l\in\mathcal{L})})$ *be an optimal solution to the LPMT1 formulation and* $((x^{(i,\,0),\,(0,\,j)}_{direct})_{(i,j\in S\,\setminus\,\{0\},\,i\neq j)},\ (x^{(i,\,0),\,(0,\,j)}_{transfer})_{(i,j\in S\,\setminus\,\{0\},\,i\neq j)},\ (z_l)_{(l\in\mathcal{L})})$ *be its corresponding solution in the compact formulation, both as defined in Appendix* **??**. *Now, the objective function of the corresponding solution is*

$$\begin{aligned} g(x) &= g(\ (x^{(s,\,0),\,(0,\,t)}_{transfer})_{(s,t\in S\,\setminus\,\{0\},\,s\neq t)}\ ) \\ &= \frac{1}{k_2}(f\ (\ (x^{e}_{st})_{(\ (s,t)\in\mathcal{R},\, e\in\mathcal{E})_{optimal}}\ ) - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) \\ &= \frac{1}{k_2}(f(x_{optimal}) - \sum_{(s,t)\in\mathcal{R}} w_{st}\cdot k_1\cdot T_{st}) \end{aligned} \tag{D1}$$

Proof. Let arbitrary $(s,t)\in\mathcal{R}$. Now, based on conversion rule (A2),

$$x^{(s,\,0),\,(0,\,t)}_{transfer} = \begin{cases} 1, & \exists\, p,\ q\in S\ \setminus\ \{s,\ t\}:\ x^{((0,\,[s,p]),\,(0,\,[q,t]))}_{st} = 1 \\ 0, & \text{otherwise} \end{cases}$$

By equations (A3), for $i,\ j\in S\ \setminus\ \{s,\ t\}$,

$$\begin{aligned} &\sum_{c\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,j]),\,(0,\,[c,t]))}_{st} = x^{((s,\,OD),\,(s,\,[s,j]))}_{st} \in \{0,\ 1\} \\ &\sum_{d\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,d]),\,(0,\,[i,t]))}_{st} = x^{((t,\,[i,t]),\,(t,\,OD))}_{st} \in \{0,\ 1\} \\ &\sum_{j\in S\,\setminus\,\{0,\,s\}} x^{((s,\,OD),\,(s,\,[s,j]))}_{st} = 1 \\ &\sum_{i\in S\,\setminus\,\{0,\,t\}} x^{((t,\,OD),\,(t,\,[i,t]))}_{st} = 1 \end{aligned}$$

Thus, if there exists $p,\ q\in S\ \setminus\ \{s,\ t\}$ for which $x^{((0,\,[s,p]),\,(0,\,[q,t]))}_{st} = 1$,

$$\begin{aligned} &\sum_{c\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,p]),\,(0,\,[c,t]))}_{st} > 0 && \sum_{d\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,d]),\,(0,\,[q,t]))}_{st} > 0 \\ \Leftrightarrow &\sum_{c\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,p]),\,(0,\,[c,t]))}_{st} = 1 && \Leftrightarrow \sum_{d\in S\,\setminus\,\{0,\,s,\,t\}} x^{((0,\,[s,d]),\,(0,\,[q,t]))}_{st} = 1 \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{j \in S \setminus \{0,\, s\}} x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 1 && \sum_{i \in S \setminus \{0,\, t\}} x_{st}^{((t,\, OD),\, (t,\, [i,t]))} = 1 \\
\Leftrightarrow\;& 1 + \sum_{j \in S \setminus \{0,\, s,\, p\}} x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 1 \quad \Leftrightarrow && 1 + \sum_{i \in S \setminus \{0,\, t,\, q\}} x_{st}^{((t,\, OD),\, (t,\, [i,t]))} = 1 \\
\Leftrightarrow\;& \sum_{j \in S \setminus \{0,\, s,\, p\}} x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 0 \quad \Leftrightarrow && \sum_{i \in S \setminus \{0,\, t,\, q\}} x_{st}^{((t,\, OD),\, (t,\, [i,t]))} = 0
\end{aligned}
$$

meaning that for all $j \in S \setminus \{s,\, p\}$ and $i \in S \setminus \{t,\, q\}$,

$$
\begin{aligned}
\sum_{c \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,j]),\, (0,\, [c,t]))} &= x_{st}^{((s,\, OD),\, (s,\, [s,j]))} = 0 \\
\sum_{d \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,d]),\, (0,\, [i,t]))} &= x_{st}^{((t,\, [i,t]),\, (t,\, OD))} = 0
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_{b,\, c \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,b]),\, (0,\, [c,t]))} &= \sum_{c \in S \setminus \{9,\, s,\, t\}} x_{st}^{((0,\, [s,p]),\, (0,\, [c,t]))} + \sum_{b \in S \setminus \{0,\, s,\, t,\, p\}} \cdots \\
&\cdots \sum_{c \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,b]),\, (0,\, [c,t]))} \\
&= 1 + \sum_{b \in S \setminus \{0,\, s,\, t,\, p\}} 0 \\
&= 1 = x_{transfer}^{(s,\, 0),\, (0,\, t)}
\end{aligned}
$$

If $x_{st}^{((0,\, [s,p]),\, (0,\, [q,t]))} = 0$ for all $p, q \in S \setminus \{s,\, t\}$, then

$$
\begin{aligned}
\sum_{b,\, c \in S \setminus \{0,\, s,\, t\}} x_{st}^{((0,\, [s,b]),\, (0,\, [c,t]))} &= \sum_{0,\, b,\, c \in S \setminus \{s,\, t\}} 0 \\
&= 0 = x_{transfer}^{(s,\, 0),\, (0,\, t)}
\end{aligned}
$$

Therefore, for arbitrary $(s, t) \in \mathcal{R}$,

$$
x_{transfer}^{(s,\, 0),\, (0,\, t)} = \sum_{b,\, c \in S \setminus \{s,\, t\}} x_{st}^{((0,\, [s,b]),\, (0,\, [c,t]))} \tag{D2}
$$

Consequently, from the objective function (31), we get

$$
\begin{aligned}
g(x) &= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot x_{transfer}^{(s,\,0),\,(0,\,t)} \quad \mid (\text{D2}) \\
&= \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot \sum_{b,\,c\in S\setminus\{0,\,s,\,t\}} x_{st}^{((0,\,[s,b]),\,(0,\,[c,t]))} \\
&= \frac{1}{k_2}\Big(\sum_{(s,t)\in\mathcal{R}} \sum_{b,\,c\in S\setminus\{0,\,s,\,t\}} k_2 \cdot w_{st} \cdot x_{st}^{((0,\,[s,b]),\,(0,\,[c,t]))}\Big) \\
&= \frac{1}{k_2}\Big(\sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st} + \sum_{(s,t)\in\mathcal{R}} \sum_{b,\,c\in S\setminus\{0,\,s,\,t\}} k_2 \cdot w_{st} \cdot x_{st}^{((0,\,[s,b]),\,(0,\,[c,t]))} \ldots \\
&\ldots - \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}\Big) \quad \mid (34) \\
&= \frac{1}{k_2}\Big(f(x_{optimal}) - \sum_{(s,t)\in\mathcal{R}} w_{st} \cdot k_1 \cdot T_{st}\Big)
\end{aligned}
$$

which matches to the equation (D1). This means that lemma 4.4 is true.

$\square$

E The Complete Calculations of the Approximations For the Number of Variables And Constraints

This appendix includes the complete calculations used to calculate the number of different groups of variables and different variables in the LPMT1 formulation and the compact formulation. The number of variables and the constraints using the big $\mathcal{O}$ notation. and the approximation is given as a function of n, which represents the number of leaf stations in the underlying PTN-graph. The results of this appendix are summarized in the tables 1, 2, and 3 of the Section 4.2.

Utilizing the LPMT1 formulation defined in Section 3.2 and the size approximations calculated in Section 4.2, we can define the size approximations for the number of variables in a specific group as follows:

$$\begin{aligned}
x^e_{st}: \quad & |(x^e_{st})_{((s,t)\in\mathcal{R},\, e\in\mathcal{E})} = |\mathcal{R}|\cdot|\mathcal{E}| = \mathcal{O}(n^2)\cdot\mathcal{O}(n^4) \Rightarrow \quad & \mathcal{O}(n^6)\\
y_l: \quad & |(y_l)_{(l\in\mathcal{L})}| = |\mathcal{L}| = n^2+n \Rightarrow \quad & \mathcal{O}(n^2)
\end{aligned}$$

The size approximations for the number of a certain type of constraints (4) and (6) can be calculated similarly as below:

$$\begin{aligned}
(4): \quad & |\mathcal{L}| = n^2+n \Rightarrow \quad & \mathcal{O}(n^2)\\
(6): \quad & 1 \Rightarrow \quad & \mathcal{O}(1)
\end{aligned}$$

From these numbers, we can determine the approximation for the total number of variables in the LPMT1 formulation is

$$\mathcal{O}(n^2)+\mathcal{O}(n^6)=\mathcal{O}(n^6)$$

The size approximation for the flow constraint (5) is slightly more complex: The constraint is presented in matrix multiplication form where the incidence matrix $\theta \in \mathbf{Z}^{|\mathcal{V}|\times|\mathcal{E}|}$, the variable vector $x_{st} \in \{0,\ 1\}^{|\mathcal{E}|}$, and the parameter $b \in \mathbf{Z}^{\mathcal{V}}$. Therefore, the number of equations contained in one matrix multiplication is in fact $|\mathcal{V}|$. Consequently, the approximation for the number of flow constraints is as follows:

$$(5): \quad |\mathcal{V}|\cdot|\mathcal{R}| = \mathcal{O}(n^2)\cdot\mathcal{O}(n^2) \;\Rightarrow \quad \mathcal{O}(n^4)$$

Thus, the approximation for the total number of constraints in the LPMT1 formulations is

$$\mathcal{O}(n^2)+\mathcal{O}(1)+\mathcal{O}(n^4)=\mathcal{O}(n^4)$$

Similarly, we can calculate the approximation for the number of variables in a specific group of the compact formulation utilizing the definition of the compact formulation in Definition 3.7 and the size approximations calculated in Section 4.2:

$$\begin{aligned}
x^{(s,\,0),\,(0,\,t)}_{direct}: \quad & |(x^{(s,\,0),\,(0,\,t)}_{direct})_{((s,t)\in\mathcal{R})} = |\mathcal{R}| \Rightarrow \quad & \mathcal{O}(n^2)\\
x^{(s,\,0),\,(0,\,t)}_{transfer}: \quad & |(x^{(s,\,0),\,(0,\,t)}_{transfer})_{((s,t)\in\mathcal{R})} = |\mathcal{R}| \Rightarrow \quad & \mathcal{O}(n^2)\\
z_l: \quad & |(z_l)_{(l\in\mathcal{L})}| = |\mathcal{L}| = n^2+n \Rightarrow \quad & \mathcal{O}(n^2)
\end{aligned}$$

Based on these approximations, we can determine the approximation for the total number of variables in the compact formulation as follows:

$$\mathcal{O}(n^2) + \mathcal{O}(n^2) + \mathcal{O}(n^2) = \mathcal{O}(n^2)$$

The approximations for the number of each constraint in the compact formulation can be calculated similarly, as demonstrated below:

$$\begin{aligned}
(10): &\quad & |\mathcal{R}| = n^2 - n \Rightarrow &\quad \mathcal{O}(n^2)\\
(11): &\quad & \min(|\mathcal{L}|,\ |\mathcal{R}|) = |\mathcal{R}| = n^2 - n \Rightarrow &\quad \mathcal{O}(n^2)\\
(12): &\quad & |\{i \in S \ \setminus\ \{-1,\ 0\} :\ (i,j) \in \mathcal{R}\}| \le |S| - 2 = n \Rightarrow &\quad \mathcal{O}(n)\\
(13): &\quad & |\{j \in S \ \setminus\ \{-1,\ 0\} :\ (i,j) \in \mathcal{R}\}| \le |S| - 2 = n \Rightarrow &\quad \mathcal{O}(n)\\
(14): &\quad & |\mathcal{R}| = n^2 - n \Rightarrow &\quad \mathcal{O}(n^2)\\
(15): &\quad & |\mathcal{R}| = n^2 - n \Rightarrow &\quad \mathcal{O}(n^2)\\
(16): &\quad & 1 \Rightarrow &\quad \mathcal{O}(1)
\end{aligned}$$

Consequently, we can calculate the approximation for the number of constraints in the compact formulation as below:

$$4 \cdot \mathcal{O}(n^2) + 2 * \cdot \mathcal{O}(n) + \mathcal{O}(1) = \mathcal{O}(n^2)$$