A three-index formulation for the Truck and Trailer Vehicle Routing Problem

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Abstract

The truck and trailer routing problem (TTRP) is a vehicle routing problem in which a fleet of trucks stationed at a central depot must serve a set of customers. The fleet includes also a number of trailers that cannot move autonomously but can be pulled by the trucks. Some customers can only be served by a truck due to accessibility constraints while other customers may be served by either a truck or a truck-trailer combined vehicle. A truck can uncouple its trailer and leave it at a customer location while serving other customers.

We define the TTRP mathematically using a graph that has a node for each customer as well as the depot. We present two mixed-integer linear programming formulations based on commodity flow and using three-index indicator variables. The first, directed, formulation defines the indicator variables for the directed arcs of the graph while the second, undirected, formulation defines them for the undirected edges of the graph. We improve the latter formulation with some valid inequalities, implement it using a branch-and-cut algorithm and study its computational behavior. The algorithm is able to solve to optimality synthetic problem instances with up to 20 customers.

Keywords vehicle routing problem, truck and trailer routing problem, branch-and-cut

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Tiivistelmä

Perävaunullisten kuorma-autojen reitittämiseen liittyvä ongelma truck and trailer routing problem (TTRP) on reititysongelma, jossa keskusvarikolta käsin toimivan kuorma-autokannan on palveltava asiakasjoukkoa. Autokantaan kuuluu myös useita perävaunuja, joita kuorma-autot voivat vetää. Joitakin vaikeasti saavutettavia asiakkaita voidaan palvella vain kuorma-autolla, kun taas muita asiakkaita voidaan palvella joko kuorma-autolla tai kuorma-auton ja perävaunun muodostamalla ajoneuvoyhdistelmällä. Perävaunu voidaan irrottaa kuorma-autosta ja jättää asiakkaan luo siksi aikaa, kun auto palvelee muita asiakkaita.

Tämä työ määrittää TTRP:n matemaattisesti käyttäen graafia, jonka jokainen solmu vastaa asiakasta tai varikkoa. Työssä muotoillaan hyödykkeiden virtaukseen perustuva lineaarinen sekalukuoptimointitehtävä, joka käyttää kolmen indeksin indikaattorimuuttujia. Muotoilusta esitetään kaksi versiota. Ensimmäinen, suunnattu, muotoilu määrittelee indikaattorimuuttujat jokaiselle graafin suunnatulle kaarelle, kun taas toinen, suuntaamaton, muotoilu määrittelee ne graafin suuntaamattomille särmille. Jälkimmäistä muotoilua parannetaan valideilla epäyhtälöillä, toteutetaan branch-and-cut -algoritmi sen ratkaisemiseksi ja testataan sitä laskennallisesti. Algoritmi löytää optimaalisen ratkaisun keinotekoisiin testiongelmiin, joissa on korkeintaan 20 asiakasta.

Avainsanat reititys, TTRP, kokonaislukuoptimointi

Contents

1 Introduction

The truck and trailer vehicle routing problem (TTRP) is an extension of the classic vehicle routing problem (VRP), first studied by Dantzig and Ramser (1959). In the VRP, a fleet of homogeneous vehicles must satisfy the demands of a set of customers. Each customer's demand must be satisfied by one visit of a vehicle. The capacity of each vehicle is limited, therefore a single vehicle can serve only a subset of the customers. The fleet is stationed at a central depot, where the route taken by each vehicle must start and end. The objective of VRP is to minimize the total travel costs of the fleet of vehicles.

In TTRP, each vehicle is a truck that can pull a trailer. All trucks and trailers have a fixed capacity. Some customers, known as vehicle customers, can be reached by a full combination vehicle that consists of a truck and a trailer, while others, known as truck customers, can only be visited by trucks. This could be because of accessibility constraints such as being situated on a mountain, in an area with lacking road infrastructure, or in a city center with a ban on large vehicles. The trucks are allowed to leave their trailer behind at a vehicle customer location while they serve truck customers.

Figure 1 depicts a solution to a TTRP instance with four truck customers, 16 vehicle customers, two trucks and two trailers. The depot, vehicle customers, and truck customers are depicted by a gray square, red dots, and blue triangles, respectively. Solid arrows between two locations denote a path that is traversed by a full vehicle while dashed arrows denote paths traversed by a truck without a trailer. Customers are labeled with their demands and paths with the amount of goods carried in the vehicle along that path.

In this thesis, we formulate TTRP as a mixed-integer linear programming problem using commodity flows and three-index indicator variables. We present two different formulations, one using directed and one using undirected indicator variables. The undirected formulation is improved by adding valid inequalities to it. Finally, we perform computational experiments with the undirected formulation.

This thesis is organized as follows. Section 2 reviews the existing literature on the applications of TTRP, problems related to it, and solution methods developed for it. Section 3 presents the two formulations of TTRP developed by us. Section 4 describes the computational experiments. We give our conclusions in Section 5.

2 Literature review

There is some literature on the real-world applications TTRP and problems closely related to it. In Semet and Taillard (1993), a heterogeneous fleet of trucks and trailers supply a Swiss chain of grocery stores, and some stores cannot receive a shipment that uses a trailer. Gerdessen (1996) mentions two applications: the Dutch dairy industry delivering dairy products into crowded city centers with trucks while trailers are parked elsewhere, and the distribution of compound animal feed in rural areas with narrow roads. Hoff and Løkketangen (2007) discusses milk collection in

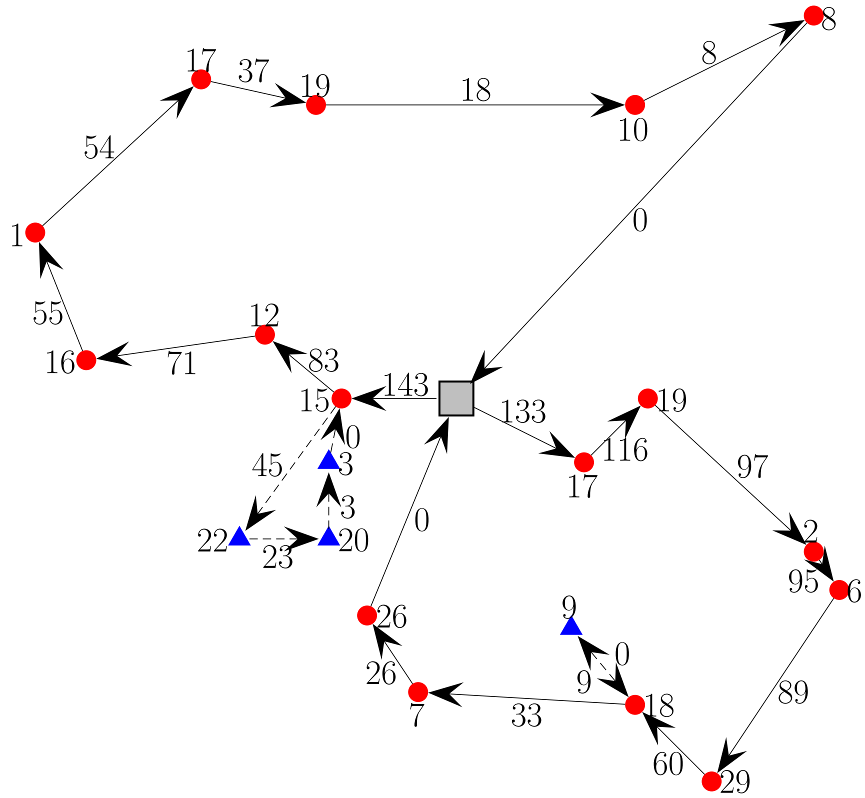

Figure 1: A solution for a TTRP instance with 20 customers.

rural areas of western Norway, where most farms are small and cannot be accessed by a large combined vehicle. Therefore, a truck will leave its trailer at a parking place while it collects milk from farms. Caramia and Guerriero (2010b) and Drexl (2012) are also concerned with milk collection.

There is a variety of vehicle routing problems that incorporate the concept of using trucks and detachable trailers but differ slightly from the TTRP or extend it in some way.

In the vehicle routing problem with trailers (VRPT) (Gerdessen (1996)), there is no clear group of truck customers. Instead, each customer has a maneuvering time that represents how much longer it takes to visit that customer with a full vehicle compared to visiting with a truck. Additionally, each trailer is parked exactly once and all customers have the same demand.

The capacitated m-ring-star problem (CmRSP) (Baldacci et al. (2007)) forms sets of rings that pass through customer locations and a central depot, and then assigns each unvisited customer to a single customer on a ring. In the two-echelon capacitated vehicle routing problem (2E-CVRP) (Gonzalez-Feliu et al. (2008), Jepsen et al. (2013)), a central depot uses large capacity vehicles to serve satellite locations which in turn use small capacity vehicles to serve customers.

In the single truck and trailer routing problem with satellite depots (STTRPSD) (Villegas et al. (2010)), a single truck-and-trailer combination vehicle serves the demands of a set of customers. The vehicle must leave its trailer behind at a satellite depot when visiting customers.

The relaxed truck and trailer routing problem (RTTRP) (Lin et al. (2010)) does not limit the number of trucks and trailers that can be used. The truck and trailer routing problem with time windows (TTRPTW) (Lin et al. (2011)) gives each customer a time window during which it must be served.

The generalized truck and trailer routing problem (GTTRP) (Drexl (2007), Drexl (2011)) demands that trailers are parked at specific transshipment locations. Each transshipment location and customer has a time window, and the vehicles may have different capacities, accessibility constraints and operating costs. The vehicle routing problem with trailers and transshipments (VRPTT) (Drexl (2012), Drexl (2014)) extends the GTTRP further by allowing a trailer that has been left at a transshipment location by one truck to be picked up by another truck.

Since the TTRP is NP-hard, most of the solution methods presented in the literature are heuristics. The earliest heuristics by Chao (2002) and Scheuerer (2006) used tabu search. Later methods include mathematical programming and local search (Caramia and Guerriero (2010a)), simulated annealing (Lin et al. (2010)), and a metaheuristic using both local search and large neighborhood search (Derigs et al. (2013)). The hybrid metaheuristic by Villegas et al. (2013) is considered the current state-of-the art method.

The two-commodity flow formulation of Bartolini and Schneider (2020) is the only work we are aware of that presents a formulation of the type of TTRP considered in this thesis and solves it to optimality using an exact algorithm. Some exact solution methods have been proposed for other problems related to the TTRP. These include the GTTRP (Drexl (2011)), VRPTT (Drexl (2014)), STTRPSD (Belenguer et al. (2016)) and TTRPTW (Parragh and Cordeau (2017), Rothenbächer et al. (2018)).

3 Mathematical formulations

This section defines the TTRP formally and presents two mathematical formulations for it.

3.1 Problem description and notation

We define the TTRP on an undirected complete graph $G = (V, E)$, where $V = \{0, \ldots, n\}$ and $E = \{\{i, j\} | i, j \in V, i < j\}$. Each node $i \in \{1, \ldots, n\}$ represents a *customer* with a positive *demand* q_i, and the node 0 represents a *depot* that stations a fleet of m_t trucks and m_r trailers. The capacity of each truck is Q_t and the capacity of each trailer is Q_r. A trailer is non-autonomous and must be pulled by a truck. Each truck can either move on its own without a trailer or pull a single trailer, in which case it has a capacity $Q_t + Q_r$ and is known as a *full vehicle*. Let $V_c = \{1, \ldots, n\}$ be the set of all customer nodes. It is partitioned into two sets V_v and V_t. Nodes in $V_v = \{1, \ldots, n_v\}$ correspond to n_v customers that can be reached by either a truck or a full vehicle, called *vehicle customers*. Nodes in $V_t = \{n_v + 1, \ldots, n\}$ correspond to n_t *truck customers* that can only be reached by a truck. Note that the total amount of customers is $n = n_v + n_t$. We define also the set of vehicle customers plus the depot, $V_v^0 = \{0\} \cup V_v$. Each edge $\{i, j\} \in E$ has an associated weight c_{ij} that represents the cost of traversal between i and j with either a truck or a full vehicle. To simplify notation, we define also the symmetric cost $c_{ji} = c_{ij}$. The costs follow the triangle inequality: $c_{ij} + c_{jk} \geq c_{ik}$ $\forall i, j, k \in V$.

The objective of the TTRP problem is to satisfy the demand q_i of each customer i by sending out vehicles from the depot. The total traversal costs of all vehicles are minimized while the capacities of the vehicles may not be exceeded, and no truck customer may be served by a full vehicle.

The path of a full vehicle, starting from the depot, visiting vehicle customers and ending back at the depot, is known as a *vehicle route*. At any vehicle customer k along its a route, the full vehicle can decouple its trailer and leave it parked, turning the full vehicle into a truck without a trailer. The truck can then visit and serve other customers using its capacity Q_t before returning to k, reattaching the trailer and continuing along its vehicle route. This type of path that uses only the truck is called a *truck route* starting at node k. We assume that load can be transferred between the trailer and the truck while the trailer is parked. Therefore, multiple truck routes can start at the same vehicle customer k. Truck routes may also start at the depot.

In addition to the notation introduced before, we will also define sets of directed arcs of the graph G. Let $A = \{(i, j) | i, j \in V,\ i \neq j\}$ be the set of all arcs. The cost of traversing arc $(i, j) \in A$ is c_{ij}. Let $A_v = \{(i, j) \in A | i, j \in V_v\}$ be the set of arcs between vehicle customers, and let $A_v^0 = \{(i, j) \in A | i, j \in V_v^0\}$ be the set of arcs that can be traversed by a full vehicle.

3.2 Three-index directed formulation

This section presents a formulation of the TTRP which is an extension of the one-commodity flow formulation for the capacitated Vehicle Routing Problem (Gavish and Graves (1978), Gouveia (1995)). We extend the formulation of Gavish and Graves (1978) by considering two sets of flow variables instead of only one: one set for flows along vehicle routes and one for flows along truck routes.

We define the following variables: binary variables ξ_{ij} that take the value 1 if and only if a vehicle route uses the arc $(i,j) \in A_v^0$, binary variables ζ_{ij}^k that take the value 1 if and only if a truck route starting from $k \in V_v^0$ uses the arc $(i,j) \in A$, continuous variables x_{ij} and y_{ij} specifying the amount of load transferred along an arc (i,j) by a vehicle route or truck route, respectively, and binary variables v_k taking the value 1 if and only if vehicle customer $k \in V_v$ is served by a vehicle route.

The TTRP can be formulated as follows:

$$(F1) \qquad \min \sum_{(i,j)\in A_v^0} c_{ij}\xi_{ij} + \sum_{k\in V_v^0} \sum_{(i,j)\in A} c_{ij}\zeta_{ij}^k \tag{1}$$

$$\text{subject to} \quad \sum_{(j,i)\in A_v^0} \xi_{ji} = v_i, \quad \forall i \in V_v \tag{2}$$

$$\sum_{k\in V_v^0\setminus\{i\}} \sum_{(j,i)\in A} \zeta_{ji}^k = 1 - v_i, \quad \forall i \in V_v \tag{3}$$

$$\sum_{k\in V_v^0} \sum_{(j,i)\in A} \zeta_{ji}^k = 1, \quad \forall i \in V_t \tag{4}$$

$$\sum_{(j,i)\in A_v^0} \xi_{ji} = \sum_{(i,j)\in A_v^0} \xi_{ij}, \quad \forall i \in V_v^0 \tag{5}$$

$$\sum_{(j,i)\in A} \zeta_{ji}^k = \sum_{(i,j)\in A} \zeta_{ij}^k, \quad \forall i \in V,\ k \in V_v^0 \tag{6}$$

$$\sum_{(0,j)\in A_v^0} \xi_{0j} \le m_r, \tag{7}$$

$$\sum_{(0,j)\in A_v^0} \xi_{0j} + \sum_{(0,j)\in A} \zeta_{0j}^0 \le m_t, \tag{8}$$

$$\zeta_{0j}^k = \zeta_{j0}^k = 0, \quad \forall k \in V_v,\ j \in V_c \tag{9}$$

$$\zeta_{kj}^k \le v_k, \quad \forall k \in V_v,\ j \in V \tag{10}$$

$$x_{ij} \le (Q_t + Q_r)\xi_{ij}, \quad \forall (i,j) \in A_v^0 \tag{11}$$

$$y_{ij} \le Q_t \sum_{k\in V_v^0} \zeta_{ij}^k, \quad \forall (i,j) \in A \tag{12}$$

$$\sum_{(j,i)\in A} y_{ji} - \sum_{(i,j)\in A} y_{ij} = q_i, \quad \forall i \in V_t \tag{13}$$

$$\sum_{(j,i)\in A_v^0} x_{ji} - \sum_{(i,j)\in A_v^0} x_{ij} + \sum_{(j,i)\in A} y_{ji} - \sum_{(i,j)\in A} y_{ij} = q_i, \quad \forall i \in V_v \tag{14}$$

$$x_{j0} = 0, \quad \forall (j,0) \in A_v^0 \tag{15}$$

$$y_{j0} = 0, \quad \forall (j,0) \in A \tag{16}$$

$$y_{ji} \le Q_t(1 - v_i), \quad \forall i \in V_v,\ \forall (j,i) \in A \tag{17}$$

$$\xi_{ij} \in \{0,1\}, \quad \forall (i,j) \in A_v^0 \tag{18}$$

$$\zeta_{ij}^k \in \{0,1\}, \quad \forall (i,j) \in A,\ k \in V_v^0 \tag{19}$$

$$x_{ij} \ge 0, \quad \forall (i,j) \in A_v^0 \tag{20}$$

$$y_{ij} \ge 0, \quad \forall (i,j) \in A \tag{21}$$

$$v_i \in \{0,1\}, \quad \forall i \in V_v \tag{22}$$

The objective function (1) minimizes the cost of all arcs used by vehicle or truck routes. Constraints (2) and (3) ensure that each vehicle customer is visited by either a vehicle route, or a truck route coming from the depot or another vehicle customer. In a similar fashion, constraints (4) ensure that each truck customer is visited by exactly one truck route. Constraints (5) state that a vehicle route that arrives at a customer also leaves that customer. In the case $i = 0$, constraints (5) impose that for each vehicle route leaving the depot, another vehicle route returns to it. Constraints (6) are similar to (5) but concern truck routes: each truck route arriving at a customer must have a corresponding route leaving that customer, and these routes must have the same starting point. When $i = k$, (6) ensure that the number of truck routes that start from a location is equal to the number of truck routes ending there. Constraints (7) impose an upper bound on the number of full vehicles used, (8) on the number of trucks used. Constraints (9) prevent a truck route that starts at a vehicle customer from visiting the depot. Constraints (10) state that a truck route can start from a vehicle customer only if that customer is served by a vehicle route. Constraints (11) and (12) define the capacities of a full vehicle and a truck without a trailer, respectively. Constraints (13) are flow conservation constraints for the truck customers. They state that the flow of goods that arrives at a truck customer must be reduced exactly by the customer's demand. Constraints (14) are flow conservation constraints for the vehicle customers. If a vehicle customer is served by a truck route, then (14) become identical to (13). Otherwise, if the vehicle customer is served by a vehicle route, then (14) also take into account the flows of the truck routes traversing it. Constraints (15) and (16) impose boundary conditions for the flows: each full vehicle or truck that returns to the depot must be empty. Similarly, constraints (17) state that if a vehicle customer is the starting point of some truck routes, the truck routes ending at that vehicle customer must have zero flow.

We will now informally show that a solution $(\overline{\xi}, \overline{\zeta}, \overline{v}, \overline{x}, \overline{y})$ of (F1) corresponds to a solution of the TTRP.

We define the following sets of vehicle and truck arcs: $\overline{R}_v = \{(i, j) \in A_v^0 | \overline{\xi}_{ij} = 1\}$ and $\overline{R}_t^k = \{(i, j) \in A | \overline{\zeta}_{ij}^k = 1\}$, $\forall k \in V_v^0$.

According to (2), for each vehicle customer i with $\overline{v}_i = 1$, there must exist exactly one arc $(j, i) \in \overline{R}_v$. On the other hand, (2) and the definition of $\overline{R}_v$ impose that $\overline{R}_v$ cannot contain any arc (j, i) where i is a vehicle customer with $\overline{v}_i = 0$, or a truck customer.

Because of (5), for each arc $(j, i) \in \overline{R}_v$, $i \in V_v^0$, there exists exactly one arc $(i, k) \in \overline{R}_v$. Also, if there are no arcs $(j, i) \in \overline{R}_v$, there can be no arcs $(i, k) \in \overline{R}_v$. Together with (2) this means that for each vehicle customer $i \in V_v$ with $v_i = 1$, there is exactly one arc in $\overline{R}_v$ entering i and one arc leaving i. Moreover, there are no arcs in $\overline{R}_v$ that start or end at any truck customer, or vehicle customer i with $\overline{v}_i = 0$.

Putting these observations together, we see that the arcs in $\overline{R}_v$ form a collection of directed cycles that visit vehicle customers i with $\overline{v}_i = 1$. Each such customer is visited by exactly one cycle defined by arcs in $\overline{R}_v$. The cycles may also visit the depot, which is the only location where they are allowed to overlap.

We can carry out a similar deduction for the sets $\overline{R}_t^k$. From (4) and (6) we see that for each truck customer $i \in V_t$ there must be exactly one $k \in V_v^0$ for which there are arcs $(j,i) \in \overline{R}_t^k$ and $(i,k) \in \overline{R}_t^k$. Because of (3), the same holds for vehicle customers i that have $\overline{v}_i = 0$. According to (3), arcs of $\overline{R}_t^k$ can only go through a vehicle customer i with $\overline{v}_i = 1$ if $i = k$. Altogether, arcs of $\overline{R}_t^k$ form a set of directed cycles that visit truck customers and vehicle customers that have $\overline{v}_i = 0$. Each of those customers is visited by exactly one cycle defined by arcs in $\overline{R}_t^k$, for exactly one value of k. A cycle in $\overline{R}_t^k$ can also go through the vehicle customer k, and this is the only node at which two cycles in $\overline{R}_t^k$ may overlap.

All cycles defined by arcs in $\overline{R}_t^k$ must visit the vehicle customer k. This is ensured by constraints (13) and (14). Let $V(\overline{R}_t^k)$ be the set of nodes that are incident to arcs in $\overline{R}_t^k$: $V(\overline{R}_t^k) = \{i \in V | \exists j \in V : (i,j) \in \overline{R}_t^k \vee (j,i) \in \overline{R}_t^k\}$. For every truck customer i in $V(\overline{R}_t^k)$ the incoming and outgoing flows $\overline{y}_{ji}$ and $\overline{y}_{ij}$ satisfy (13). We have seen that all the vehicle customers visited by arcs in $\overline{R}_t^k$ must have $\overline{v}_i = 0$, which means they cannot be visited by any arc in $\overline{R}_v$. This, together with (11), implies that all the flow variables $\overline{x}_{ij}$ and $\overline{x}_{ji}$ in (14) are zero, so (14) becomes equivalent to (13). Because of these constraints, the flow defined by variables $\overline{y}_{ij}$ that goes through a customer must decrease by the demand of that customer, and since all demands are greater than zero, each cycle defined by arcs in $\overline{R}_t^k$ must visit the starting point k where flow conservation is not enforced.

A similar result can be obtained for $\overline{R}_v$ by using the vehicle route flows $\overline{x}_{ij}$. Each vehicle customer i on a cycle defined by arcs in $\overline{R}_v$ has $\overline{v}_i = 1$. Using (17), (14) becomes

$$\sum_{(j,i) \in A_v^0} x_{ji} - \sum_{(i,j) \in A_v^0} x_{ij} = q_i + \sum_{(i,j) \in A} y_{ij}$$

which is similar to (13) but with the addition of the term $\sum_{(i,j) \in A} y_{ij}$, which represents the flow outgoing from customer i on all truck routes defined by arcs in $\overline{R}_t^i$. In other words, the demand of customer i is augmented with the demands of the customers visited by the truck routes that start from i. Therefore, any cycle defined by the arcs in $\overline{R}_v$ must go through the depot where flow conservation is not enforced.

Consider a truck route leaving from a vehicle customer k. We have seen that the amount of flow leaving k on that route is reduced by q_i at every customer i the route visits. Because of (17), the flow that comes back to k must be 0. This means that the flow leaving k is equal to the demand of all the customers visited by the route. According to (12) and because $\sum_{k \in V_v^0} \overline{\zeta}_{ij}^k \in \{0,1\}$, this flow cannot be larger than the capacity of the truck.

Using (15) and (11), we get a similar result for $\overline{x}_{ij}$. In this case, the demand of a vehicle customer i on a vehicle tour C defined by arcs in $\overline{R}_v$ is augmented with all the demands of customers visited by the truck routes going out from i. Therefore, the flow going out of the depot on C must carry exactly the amount of load that is demanded by the customers visited by it plus all its truck routes. This amount cannot be larger than the capacity of the full vehicle because of (11).

Additionally, (7) and (8) limit the number of arcs outgoing from the depot so that a feasible number of trucks and trailers is used.

In conclusion, $\overline{R}_v$ and $\overline{R}_t^k$ form sets of valid vehicle and truck routes, and the flows of goods on these routes also behave correctly.

3.3 Three-index undirected formulation

Formulation (F1) contains a very large number of variables, and most of them are three-index binary variables (ζ_{ij}^k). Therefore we can reduce the number of variables by defining variables ζ_{ij}^k and ξ_{ij} on edges instead of arcs. The flow variables x_{ij} and y_{ij} will still be defined on arcs.

In order to use variables defined on edges, we define an extended graph $\overline{G} = (\overline{V}, \overline{E})$ which is obtained by introducing duplicate nodes of the depot and the vehicle customers. In this extended graph, the vehicle routes start at the depot but end at the duplicate node of the depot, which is called the "ending depot". Similarly, all truck routes starting from a vehicle customer must end at the corresponding duplicate node. In the following, we first introduce the undirected formulation, and then discuss in detail the reasons for which the extended graph is used.

We introduce some additional notation because of the duplication. For each $i \in V_v^0$ we have a duplicate $i' = i + n + 1$. Particularly, $0' = n + 1$ is the duplicate of the depot. We call the original depot the starting depot and its duplicate node the ending depot. Let $\overline{V}_v = \{i' | i \in V_v\}$ be the set of duplicate vehicle customers and $\widetilde{V}_v = \{0\} \cup V_v \cup \{n+1\}$ be the set of all nodes that can be visited by a full vehicle. Now, $\overline{V} = \{0\} \cup V_v \cup V_t \cup \{n+1\} \cup \overline{V}_v = \{0, \ldots, n_v + n + 1\}$ is the set of all vertices in the extended graph.

Next, we define the set $\overline{E} = \{\{i,j\} | i,j \in \overline{V}, i < j\} \setminus \{\{i,j\} | i \in V_v^0, i' = j\} \setminus \{\{0,j\} | j \in \overline{V}_v\} \setminus \{\{i,j\} | i,j \in \overline{V}_v \cup \{n+1\}\}$ containing all edges of the extended graph. Edges between two duplicate nodes, edges between the depot and a duplicate vehicle customer, and edges between a node and its duplicate are not included in this set since they will never be traversed by a vehicle in our model. We define also the set $E_v = \{\{i,j\} \in \overline{E} | i,j \in V_v\}$ containing all edges between vehicle customers, and the set $\widetilde{E}_v = \{\{i,j\} \in \overline{E} | i,j \in \widetilde{V}_v\}$ containing all edges that can be used by a full vehicle.

We define additional sets of arcs similarly: $\overline{A} = \{(i,j),(j,i) | \{i,j\} \in \overline{E}\}$ is the set of all arcs and $\widetilde{A}_v = \{(i,j) \in \overline{A} | i,j \in \widetilde{V}_v\}$ the set of all arcs that can be traversed by a full vehicle.

Let $\overline{\delta}(i) = \{\{j,k\} \in \overline{E} | i = j \vee i = k\}$, $\widetilde{\delta}_v(i) = \overline{\delta}(i) \cap \widetilde{E}_v$, and $\delta(i) = \overline{\delta}(i) \cap E$ be the sets of edges of $\overline{E}$, $\widetilde{E}_v$, and E, respectively, that are incident to node $i \in \overline{V}$. If $l = \{i,j\}$, we use the notation $\xi_l = \xi_{ij}$ and $\zeta_l^k = \zeta_{ij}^k$.

We extend the costs c_{ij} to the extended graph. For the edge $\{i,j\} \in \overline{E}$ where $i \in V$ and $j \in \overline{V}_v \cup \{n+1\}$, we set $c_{ij} = c_{il}$, where $l' = j$.

The variables ζ_{ij}^k and ξ_{ij} are now defined on edges, but retain their previous meaning: ξ_{ij} is equal to one if and only if the edge $\{i,j\} \in \widetilde{E}_v$ is used in a vehicle route. ζ_{ij}^k is equal to one if and only if the edge $\{i,j\} \in \overline{E}$ is used in a truck route that starts from $k \in V_v^0$. We also introduce additional binary variables s_i^k, $i \in V_c$, $k \in V_v^0$, that have the value 1 if and only if the customer i is served by a truck route that starts from the node k. For vehicle customers i that are served by vehicle routes, we have $s_i^i = 1$.

The undirected formulation is the following:

$$(F2) \qquad \min \sum_{\{i,j\}\in\widetilde{E}_v} c_{ij}\xi_{ij} + \sum_{k\in V_v}\sum_{\{i,j\}\in\overline{E}} c_{ij}\zeta_{ij}^k \tag{23}$$

$$\text{subject to} \sum_{l\in\widetilde{\delta}_v(i)} \xi_l = 2v_i, \quad \forall i \in V_v \tag{24}$$

$$\sum_{k\in V_v^0\setminus\{i\}}\sum_{l\in\overline{\delta}(i)} \zeta_l^k = 2(1-v_i), \quad \forall i \in V_v \tag{25}$$

$$\sum_{k\in V_v^0} s_i^k = 1, \quad \forall i \in V_c, \tag{26}$$

$$\sum_{l\in\overline{\delta}(i)} \zeta_l^k = 2s_i^k, \quad \forall i \in V_c,\ k \in V_v^0 \setminus \{i\} \tag{27}$$

$$\sum_{\{0,j\}\in\widetilde{E}_v} \xi_{0j} = \sum_{\{j,n+1\}\in\widetilde{E}_v} \xi_{j(n+1)}, \tag{28}$$

$$\sum_{l\in\overline{\delta}(i)} \zeta_l^i = \sum_{l\in\overline{\delta}(i')} \zeta_l^i, \quad \forall i \in V_v^0 \tag{29}$$

$$\sum_{\{0,j\}\in\widetilde{E}_v} \xi_{0j} \le m_r, \tag{30}$$

$$\sum_{\{0,j\}\in\widetilde{E}_v} \xi_{0j} + \sum_{\{0,j\}\in\overline{E}} \zeta_{0j}^0 \le m_t, \tag{31}$$

$$\zeta_l^i \le v_i, \quad \forall i \in V_v,\ l \in \overline{\delta}(i) \tag{32}$$

$$x_{ij} \le (Q_t + Q_r)\xi_{ij}, \quad \forall \{i,j\} \in \widetilde{E}_v \tag{33}$$

$$x_{ji} \le (Q_t + Q_r)\xi_{ij}, \quad \forall \{i,j\} \in \widetilde{E}_v \tag{34}$$

$$y_{ij} \le Q_t \sum_{k\in V_v^0} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E} \tag{35}$$

$$y_{ji} \le Q_t \sum_{k\in V_v^0} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E} \tag{36}$$

$$\sum_{(j,i)\in\overline{A}} y_{ji} - \sum_{(i,j)\in\overline{A}} y_{ij} = q_i, \quad \forall i \in V_t \tag{37}$$

$$\sum_{(j,i)\in\widetilde{A}_v} x_{ji} - \sum_{(i,j)\in\widetilde{A}_v} x_{ij} + \sum_{(j,i)\in\overline{A}} y_{ji} - \sum_{(i,j)\in\overline{A}} y_{ij} = q_i, \quad \forall i \in V_v \tag{38}$$

$$x_{j(n+1)} = 0, \quad \forall (j, n+1) \in \widetilde{A}_v \tag{39}$$

$$y_{ji'} = 0, \quad \forall i \in V_v^0,\ (j, i') \in \overline{A} \tag{40}$$

$$\zeta_{0j}^k = 0, \quad \forall \{0,j\} \in \overline{E},\ k \in V_v \tag{41}$$

$$\zeta_{ij}^k = 0, \quad \forall \{i,j\} \in \overline{E},\ k \in V_v^0,\ j \in \overline{V}_v \cup \{n+1\},\ j \ne k' \tag{42}$$

$$\xi_{ij} \in \{0,1\}, \quad \forall \{i,j\} \in \widetilde{E}_v \tag{43}$$

$$\zeta_{ij}^k \in \{0,1\}, \quad \forall \{i,j\} \in \overline{E},\ k \in V_v^0 \tag{44}$$

$$x_{ij} \ge 0, \quad \forall (i,j) \in \widetilde{A}_v \tag{45}$$

$$y_{ij} \geq 0, \quad \forall (i,j) \in \overline{A} \tag{46}$$
$$v_i \in \{0,1\}, \quad \forall i \in V_v \tag{47}$$
$$s_i^k \in \{0,1\}, \quad \forall i \in V_c,\ k \in V_v^0 \tag{48}$$

The objective function (23) minimizes the total cost of edges used in the solution. Constraints (24) state that the degree of a vehicle customer with respect to vehicle routes is either 0 or 2, depending on whether it is served by a vehicle route or not. Equations (25) enforce a similar constraint with respect to the truck routes. Constraints (26) state that each customer must be served by either a vehicle route or a truck route. Constraints (27) ensure that a truck route goes through a customer if and only if the customer is served by that truck route. Constraints (28) impose that the number of vehicle routes that leave the starting depot must be equal to the number of vehicle routes that arrive at the ending depot. Constraints (29) define a similar constraint for the truck routes, imposing that the number of truck routes that leave a vehicle customer or the depot must be equal to the number of truck routes that end at its duplicate node. Constraints (30) and (31) limit the number of trailers and trucks used, respectively. Constraints (32) ensure that truck routes can only begin at vehicle customers that are served by a vehicle route. Equations (33) and (34) are capacity constraints for vehicle routes, (35) and (36) for truck routes. Equations (37) and (38) are flow conservation constraints for truck customers and vehicle customers, respectively. Constraints (39) state that no flow can enter the ending depot and (40) state the same for duplicated vehicle customers. Constraints (41) prevent truck routes that start at a vehicle customer from visiting the depot, and (42) allow truck routes to arrive at a duplicate vehicle customer only if they originate from the corresponding vehicle customer. Finally, (43)–(48) define the domains of the decision variables.

3.4 Analysis of the undirected formulation

Defining the binary variables on edges instead of arcs raises the problem of modeling the fact that some edges can be traversed twice. Consider an edge $\{0, j\} \in E, j \in V_v$. This edge is traversed twice in a vehicle route if j is the only customer that is visited by the route. Yet, if the variable associated with this edge is binary, the edge is counted only once when calculating the degree of j. Therefore j violates the constraint that the degree of a customer node must be two. This problem occurs also for edges that are used in a truck route that serves only one customer.

A straightforward approach for dealing with edges that are traversed twice would be to use variables that can also take the value two on those edges that can be used twice. To be more precise, the definitions of the variables ξ_{ij} and ζ_{ij}^k would be

$$\xi_{ij} \in \{0,1\}, \quad \forall i,j \in V_v \tag{49}$$
$$\xi_{0j} \in \{0,1,2\}, \quad \forall j \in V_v \tag{50}$$
$$\zeta_{ij}^k \in \{0,1\}, \quad \forall k \in V_v^0,\ \{i,j\} \in E,\ k \neq i,j \tag{51}$$
$$\zeta_l^k \in \{0,1,2\}, \quad \forall k \in V_v^0,\ l \in \delta(k) \tag{52}$$

In this case, (49) define variables ξ for edges that are not adjacent to the depot, while (50) define variables ξ for edges that are adjacent to the depot and therefore might need to take the value 2. Similarly, (52) define variables ζ for edges with one endpoint which is the origin of a truck route, and (51) define them for the remaining edges.

This simple way of modeling single customer routes is however incorrect in our case, that is, if variables ξ and ζ are defined on edges and (F1) is changed accordingly by replacing constraints (18)–(19) with (49)–(52), the resulting formulation is incorrect. Specifically, is admits integer solutions which are infeasible as they contain vehicle or truck routes that start from the depot, or a vehicle customer, but do not return back to it. This is shown in Figure 2, which depicts a solution given by the formulation for an instance with seven vehicle customers and three truck customers. The fleet consists of three trucks of capacity 50 and two trailers of capacity 50. In Figure 2, the depot is represented as a gray square, vehicle customers as red dots and truck customers as blue triangles. Edges that are part of a vehicle route are drawn as solid lines and edges that are part of a truck route as dashed lines. Customers and edges are labeled with their demands and flows, respectively.

From Figure 2, we see that two routes can start at the depot, meet at a customer and end there. Note that both routes service the last customer, which violates the rules of the TTRP. This incorrect solution satisfies all customer degree constraints, vehicle capacity constraints and flow conservation constraints. The binary edge variables contain no information about the direction of the flows on those edges, which makes this configuration possible.

This is the reason we use the extended graph $\overline{G}$ in (F2). In this graph we enforce that each route starting at the depot, or a vehicle customer, must end at the corresponding duplicate. In this way the edge between the starting node and the first visited customer is always distinct from the one between the last customer visited and the duplicate of the starting node. Therefore the degree of a customer that is the only one visited by a route is also two.

On the other hand, if duplication of the depot and vehicle customers is used, each route is represented as a path starting from a node i and ending at the corresponding duplicate i', therefore infeasible solutions as that shown in Figure 2 are easily prevented by imposing that the number of edges incident with i is equal to the number of edges incident with its duplicate i'. Formulation (F2) enforces this with equations (28) and (29).

It is worth mentioning that (F2) allows solutions where two opposite flows y_{ij} and y_{ji} or x_{ij} and x_{ji} are non-zero at the same time. This is however not a major issue, and the formulation (F2) remains correct because any feasible solution (ξ, ζ, v, s, x, y) where two non-zero opposite flows exist on edge $\{i, j\}$ can be transformed into an equivalent solution $(\xi, \zeta, v, s, x', y')$ with the same cost where only one of the corresponding arcs (i, j) and (j, i) has positive flow. We obtain the equivalent flows x' and y' by setting

$$x'_{ij} = x_{ij} - x_{ji}, \quad \forall (i,j) \in \widetilde{A}_v,\ x_{ij} \geq x_{ji} \tag{53}$$

$$x'_{ji} = 0, \quad \forall (i,j) \in \widetilde{A}_v,\ x_{ij} \geq x_{ji} \tag{54}$$

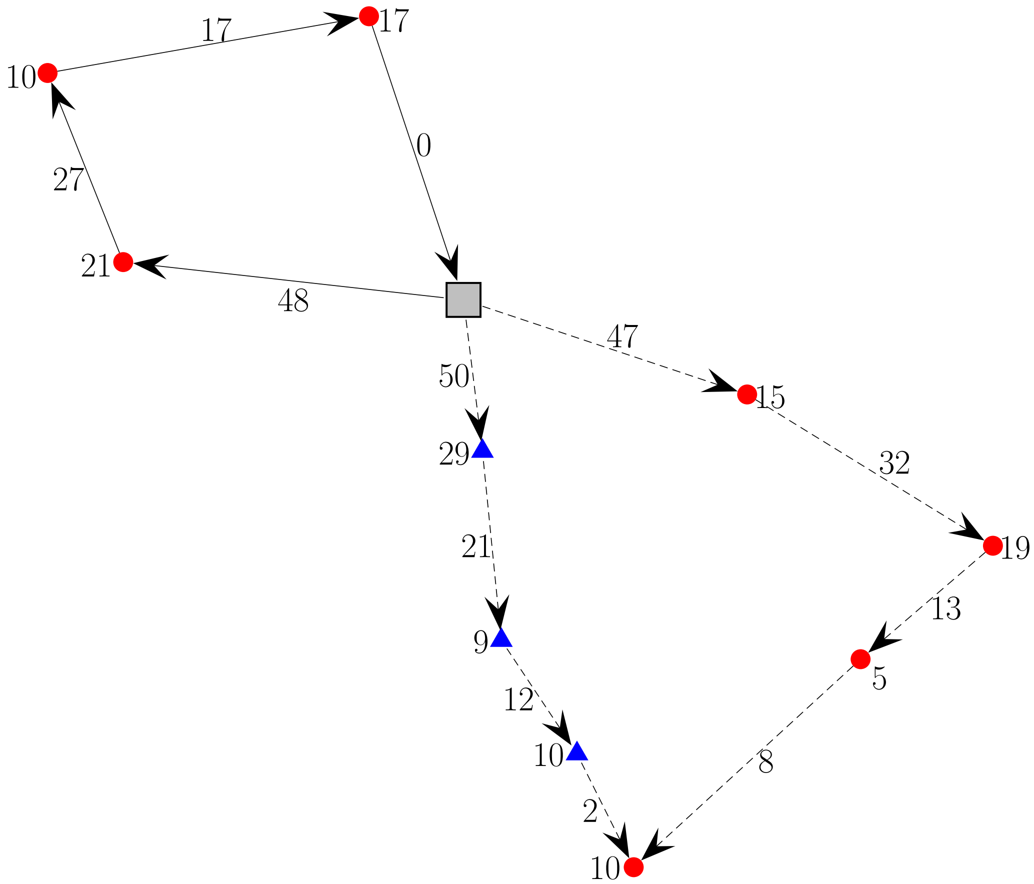

Figure 2: An incorrect solution for a TTRP instance.

$$y'_{ij} = y_{ij} - y_{ji}, \quad \forall (i,j) \in A,\ y_{ij} \geq y_{ji} \tag{55}$$

$$y'_{ji} = 0, \quad \forall (i,j) \in A,\ y_{ij} \geq y_{ji} \tag{56}$$

Finally, Table 1 shows the numbers of different variables in formulations (F1) and (F2) with respect to the number of customers n and number of vehicle customers n_v. As n and n_v grow, the total variable count of (F1) is approximately $n_v n^2$, while that of (F2) is approximately $\frac{1}{2}n_v n^2 + n_v^2 n$. Therefore (F2) reduces the variable count of (F1) only when $n_v < \frac{1}{2}n$. If more than half of the customers are vehicle customers, (F2) has more variables than (F1).

Table 1: Variable counts of the two TTRP formulations

Variable	(F1)	(F2)
ξ	$n_v(n_v+1)$	$\frac{1}{2}n_v(n_v+1)+n_v$
ζ	$(n_v+1)n(n+1)$	$(n_v+1)(\frac{1}{2}n(n+1)+n_v(n-1)+n)$
v	n_v	n_v
s	–	$n(n_v+1)$
x	$n_v(n_v+1)$	$n_v(n_v+1)+2n_v$
y	$n(n+1)$	$n(n+1)+2n_v(n-1)+2n$
Total	$(n_v+2)(n^2+n)+2n_v^2+3n_v$	$(n_v+3)(\frac{1}{2}n^2+n_v(n-1)+\frac{3}{2}n)$ $+n(n_v+1)+\frac{3}{2}n_v^2+\frac{11}{2}n_v$

3.5 Improvements to the undirected formulation

The LP relaxation of formulation (F2) can be strengthened by a simple observation: if an arc (i,j) is used in a solution, the flow going over that arc must be at least q_j and at most $Q-q_i$, where Q is the capacity limit of the flow. This is a well-known way of strengthening the flow bounds originally proposed by Gavish (1982). In our case, however, this strengthening is less straightforward due to the fact that (F2) is defined on an undirected graph, and there are both vehicle and truck routes. To simplify notation, we define the demands $q_i = 0\ \forall i \in \{0\} \cup \overline{V}_v \cup \{n+1\}$.

We impose upper bounds for the flows by replacing (33)–(36) with

$$x_{ij} \leq (Q_t + Q_r - q_i)\xi_{ij}, \quad \forall \{i,j\} \in \tilde{E}_v \tag{57}$$
$$x_{ji} \leq (Q_t + Q_r - q_j)\xi_{ij}, \quad \forall \{i,j\} \in \tilde{E}_v \tag{58}$$
$$y_{ij} \leq (Q_t - q_i) \sum_{k \in V_v^0} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E},\ i \in V_t \tag{59}$$
$$y_{ji} \leq (Q_t - q_j) \sum_{k \in V_v^0} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E},\ j \in V_t \tag{60}$$
$$y_{ij} \leq Q_t\zeta_{ij}^i + (Q_t - q_i) \sum_{k \in V_v^0 \setminus \{i\}} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E},\ i \in V_v^0 \tag{61}$$
$$y_{ji} \leq Q_t\zeta_{ij}^j + (Q_t - q_j) \sum_{k \in V_v^0 \setminus \{j\}} \zeta_{ij}^k, \quad \forall \{i,j\} \in \overline{E},\ j \in V_v^0 \tag{62}$$

Furthermore, we add the lower bounds

$$x_{ij} + x_{ji} \geq \min(q_i, q_j)\xi_{ij}, \quad \forall \{i,j\} \in \tilde{E}_v \tag{63}$$
$$x_{0j} \geq q_j\xi_{0j}, \quad \forall \{0,j\} \in \tilde{E}_v \tag{64}$$
$$y_{ij} + y_{ji} \geq \min(q_i, q_j) \sum_{k \in V_v^0} \zeta_{ij}^k, \quad \forall (i,j) \in E,\ i,j \in V_t \tag{65}$$
$$y_{ij} + y_{ji} \geq q_j\zeta_{ij}^i + \min(q_i, q_j) \sum_{k \in V_v^0 \setminus \{i\}} \zeta_{ij}^k, \quad \forall \{i,j\} \in E,\ i \in V_v^0,\ j \in V_t \tag{66}$$
$$y_{ij} + y_{ji} \geq q_i\zeta_{ij}^j + \min(q_i, q_j) \sum_{k \in V_v^0 \setminus \{j\}} \zeta_{ij}^k, \quad \forall \{i,j\} \in E,\ j \in V_v^0, i \in V_t \tag{67}$$
$$y_{ij} + y_{ji} \geq q_j\zeta_{ij}^i + q_i\zeta_{ij}^j + \min(q_i, q_j) \sum_{k \in V_v^0 \setminus \{i,j\}} \zeta_{ij}^k, \quad \forall \{i,j\} \in E,\ i,j \in V_v^0 \tag{68}$$

Inequalities (57) and (58) impose that a flow that leaves a node on a vehicle route has been reduced by the demand of that node. Inequalities (59) and (60) impose similar limits for truck route flows that leave truck customers. Inequalities (61) and (62) extend (59) and (60) to vehicle customers, with the addition that the flow that leaves node i on a truck route can use the full capacity Q_t if i is the starting node of that truck route.

Inequalities (63) state that if the edge $\{i, j\}$ is used in a vehicle route, the flow on it must satisfy at least the smaller of the demands of its endpoints. Since the variable ξ_{ij} does not tell us which way the flow is going, we must constrain both x_{ij} and x_{ji} simultaneously, and cannot tell which of the two demands must be satisfied, forcing us the use their minimum. Inequalities (64) cover the special case where the edge is incident to the depot. In this case, we know that the flow is going out from the depot. Inequalities (65) translate (63) to truck routes between two truck customers. Inequalities (66) and (67) extend (65) to the case where one of the nodes i and j can be the starting point of a truck route. If a truck route starts from i and uses the edge $\{i, j\}$, the flow y_{ij} is equal to at least the demand q_j. Inequalities (68) cover the case where both i and j are potential truck route starting points.

4 Computational experiments

4.1 Test instances

The test instances we use are derived from the 21 instances of Chao (2002). Since the original instances have too many customers for our solution method, they were reduced in size by choosing subsets of vehicle and truck customers. This resulted in 79 instances with the number of customers varying between 10 and 30. For more information on the size reduction process, see Bartolini and Schneider (2020).

Table 2 lists the properties of the test instances. It lists the identifier of the instance, the number of customers (n), the number of truck customers (n_t), the ratio of truck customers to all customers ($\frac{n_t}{n}$), the number of available trucks (m_t), the number of available trailers (m_r), the truck capacity (Q_t), the trailer capacity (Q_r), the combined demand of all customers ($\sum_{i \in V_c} q_i$), the ratio of total demand to total capacity ($\frac{\sum_{i \in V_c} q_i}{m_t Q_t + m_r Q_r}$), the combined demand of truck customers ($\sum_{i \in V_t} q_i$), and the ratio of truck customer demand to total truck capacity ($\frac{\sum_{i \in V_t} q_i}{m_t Q_t}$). Note that the last ratio can be greater than one since each truck is allowed to refill to capacity by transferring load from its trailer between truck routes.

Table 2: Dimensions of the test instances.

Name	n	n_t	$\frac{n_t}{n}$	m_t	m_r	Q_t	Q_r	$\sum_{i \in V_c} q_i$	$\frac{\text{demand}}{\text{capacity}}$	$\sum_{i \in V_t} q_i$	truck $\frac{\text{demand}}{\text{capacity}}$
10-01-0-A	10	3	0.3	1	1	50	150	145	0.725	48	0.96
10-01-0-B	10	3	0.3	3	2	50	50	145	0.58	48	0.32
10-01-0-C	10	3	0.3	4	4	30	30	145	0.604	48	0.4
10-01-1-A	10	2	0.2	1	1	50	150	133	0.665	30	0.6
10-01-1-B	10	2	0.2	3	2	50	50	133	0.532	30	0.2

10-01-1-C	10	2	0.2	4	4	30	30	133	0.554	30	0.25
10-02-0-A	10	4	0.4	1	1	50	150	145	0.725	69	1.38
10-02-0-B	10	4	0.4	3	2	50	50	145	0.58	69	0.46
10-02-0-C	10	4	0.4	4	4	30	30	145	0.604	69	0.575
10-02-1-A	10	4	0.4	1	1	50	150	133	0.665	52	1.04
10-02-1-B	10	4	0.4	3	2	50	50	133	0.532	52	0.347
10-02-1-C	10	4	0.4	4	4	30	30	133	0.554	52	0.433
10-03-0-A	10	7	0.7	1	1	50	150	145	0.725	113	2.26
10-03-0-B	10	7	0.7	3	2	50	50	145	0.58	113	0.753
10-03-0-C	10	7	0.7	4	4	30	30	145	0.604	113	0.942
10-03-1-A	10	8	0.8	1	1	50	150	133	0.665	109	2.18
10-03-1-B	10	8	0.8	3	2	50	50	133	0.532	109	0.727
10-03-1-C	10	8	0.8	4	4	30	30	133	0.554	109	0.908
15-10-0-A	15	5	0.333	1	1	50	350	208	0.52	71	1.42
15-10-0-B	15	5	0.333	3	3	100	50	208	0.462	71	0.237
15-10-0-C	15	5	0.333	6	2	50	50	208	0.52	71	0.237
15-10-1-A	15	3	0.2	1	1	50	350	220	0.55	34	0.68
15-10-1-B	15	3	0.2	3	3	100	50	220	0.489	34	0.113
15-10-1-C	15	5	0.333	6	2	50	50	208	0.52	71	0.237
15-11-0-A	15	9	0.6	1	1	50	350	208	0.52	136	2.72
15-11-0-B	15	9	0.6	3	3	100	50	208	0.462	136	0.453
15-11-0-C	15	9	0.6	6	2	50	50	208	0.52	136	0.453
15-11-1-A	15	6	0.4	1	1	50	350	220	0.55	82	1.64
15-11-1-B	15	6	0.4	3	3	100	50	220	0.489	82	0.273
15-11-1-C	15	6	0.4	6	2	50	50	220	0.55	82	0.273
15-12-0-A	15	9	0.6	1	1	50	350	196	0.49	133	2.66
15-12-0-B	15	9	0.6	3	3	100	50	196	0.436	133	0.443
15-12-0-C	15	9	0.6	6	2	50	50	196	0.49	133	0.443
15-12-1-A	15	11	0.733	1	1	50	350	308	0.77	200	4
15-12-1-B	15	11	0.733	3	3	100	50	308	0.684	200	0.667
15-12-1-C	15	11	0.733	6	2	50	50	308	0.77	200	0.667
20-07-0-A	20	14	0.7	1	1	100	300	276	0.69	200	2
20-07-0-B	20	14	0.7	3	3	100	100	276	0.46	200	0.667
20-07-0-C	20	14	0.7	2	2	100	100	276	0.69	200	1
20-07-0-D	20	14	0.7	5	3	50	50	276	0.69	200	0.8
20-07-1-A	20	13	0.65	1	1	100	300	307	0.767	193	1.93
20-07-1-B	20	13	0.65	3	3	100	100	307	0.512	193	0.643
20-07-1-C	20	13	0.65	2	2	100	100	307	0.767	193	0.965
20-07-1-D	20	13	0.65	5	3	50	50	307	0.767	193	0.772
20-08-0-A	20	2	0.1	1	1	100	300	276	0.69	49	0.49
20-08-0-B	20	2	0.1	3	3	100	50	276	0.613	49	0.163
20-08-0-C	20	2	0.1	2	2	100	50	276	0.92	49	0.245
20-08-0-D	20	2	0.1	5	3	50	50	276	0.69	49	0.196
20-08-1-A	20	4	0.2	1	1	100	300	276	0.69	54	0.54
20-08-1-B	20	4	0.2	3	3	100	50	276	0.613	54	0.18
20-08-1-C	20	4	0.2	2	2	100	50	276	0.92	54	0.27
20-08-1-D	20	4	0.2	5	3	50	50	276	0.69	54	0.216
20-09-0-A	20	9	0.45	1	1	100	300	276	0.69	139	1.39
20-09-0-B	20	9	0.45	3	3	100	100	276	0.46	139	0.463
20-09-0-C	20	9	0.45	2	2	100	50	276	0.92	139	0.695
20-09-0-D	20	9	0.45	5	3	50	50	276	0.69	139	0.556
20-09-1-A	20	8	0.4	1	1	100	300	276	0.69	106	1.06
20-09-1-B	20	8	0.4	3	3	100	100	276	0.46	106	0.353
20-09-1-C	20	8	0.4	2	2	100	50	276	0.92	106	0.53

20-09-1-D	20	8	0.4	5	3	50	50	276	0.69	106	0.424
30-10-0-A	30	6	0.2	1	1	100	600	404	0.577	79	0.79
30-10-0-B	30	6	0.2	3	3	100	100	404	0.673	79	0.263
30-10-0-D	30	6	0.2	5	3	75	75	404	0.673	79	0.211
30-10-1-A	30	7	0.233	1	1	100	600	528	0.754	120	1.2
30-10-1-B	30	7	0.233	3	3	100	100	528	0.88	120	0.4
30-10-1-C	30	7	0.233	4	4	100	50	528	0.88	120	0.3
30-11-0-A	30	16	0.533	1	1	100	600	404	0.577	254	2.54
30-11-0-B	30	16	0.533	3	3	100	100	404	0.673	254	0.847
30-11-0-D	30	16	0.533	5	3	75	75	404	0.673	254	0.677
30-11-1-A	30	15	0.5	1	1	100	600	528	0.754	242	2.42
30-11-1-B	30	15	0.5	3	3	100	100	528	0.88	242	0.807
30-11-1-C	30	15	0.5	4	4	100	50	528	0.88	242	0.605
30-12-0-A	30	19	0.633	1	1	100	600	404	0.577	270	2.7
30-12-0-B	30	19	0.633	3	3	100	100	404	0.673	270	0.9
30-12-0-D	30	19	0.633	5	3	75	75	404	0.673	270	0.72
30-12-1-A	30	23	0.767	1	1	100	600	528	0.754	376	3.76
30-12-1-B	30	23	0.767	3	3	100	100	528	0.88	376	1.25
30-12-1-C	30	23	0.767	4	4	100	50	528	0.88	376	0.94
30-12-1-D	30	23	0.767	5	3	75	75	528	0.88	376	1

4.2 Results

We solved the formulation (F2) with IBM ILOG CPLEX version 12.5, called from C++. We used the CPLEX branch-and-cut algorithm with default settings and added the valid inequalities described in Section 3.5. We ran the experiments on a computer running Windows 7 with a 3.2GHz CPU and 8GB RAM.

We ran the branch-and-cut algorithm for each instance, terminating it once optimality was reached or 2 hours had elapsed. Table 3 lists the results of the experiments on each test instance. The column LB_0 reports the initial lower bound of the objective function given by an LP relaxation at the root node of the branch-and-cut algorithm. LB_1 is the lower bound after the addition of the valid inequalities, LB_{end} is the lower bound at termination and UB is the best upper bound. The ratios of LB_0, LB_1, and LB_{end} to UB are also listed, denoted by $\%LB_0$, $\%LB_1$, and $\%LB_{\text{end}}$, respectively. In the cases where the ratio $\%LB_{\text{end}}$ is 100.0, the solution that corresponds to UB is optimal, and UB is listed in bold. Finally, the table reports the running time of the algorithm (in seconds) and the number of nodes in the branch-and-cut tree.

The algorithm solves to optimality all instances with 10 and 15 customer, 17 of 24 instances with 20 customers, and no instances with 30 customers.

The average optimality gap for 20-customer instances is approximately 26.8% at the root node without the valid inequalities, 15.7% with them, and 1.9% at termination. The average optimality gaps for 30-customer instances are 33.3% at the root node without the valid inequalities, 23.5% with them, and 15.0% at termination.

Our formulation performs worse than the more sophisticated two-commodity flow approach of Bartolini and Schneider (2020), which solves most 30-customer instances within two hours. On the other hand, our method can solve larger problems than the VRPTT solution algorithms of Drexl (2014), which solve instances with up to 16 locations. This is unsurprising since the VRPTT is considerably more general and complicated than the TTRP.

Table 3: Results of the computational experiments.

Name	LB_0	$\%LB_0$	LB_1	$\%LB_1$	LB_{end}	$\%LB_{\text{end}}$	UB	time	nodes
10-01-0-A	126.651	66.8	148.633	78.4	189.665	100.0	**189.665**	4.992	2118
10-01-0-B	135.177	70.3	157.742	82.0	192.349	100.0	**192.349**	8.314	5687
10-01-0-C	158.508	75.5	179.120	85.3	210.045	100.0	**210.045**	2.106	1859
10-01-1-A	167.544	80.9	195.869	94.6	207.029	100.0	**207.029**	1.575	395
10-01-1-B	172.486	84.7	195.839	96.2	203.609	100.0	**203.609**	0.889	109
10-01-1-C	194.467	76.1	207.041	81.1	255.408	100.0	**255.408**	16.629	9493
10-02-0-A	126.651	60.3	154.186	73.4	209.975	100.0	**209.975**	2.698	3417
10-02-0-B	135.476	69.9	162.246	83.7	193.786	100.0	**193.786**	5.023	3167
10-02-0-C	162.367	71.3	193.876	85.1	227.764	100.0	**227.764**	2.464	2077
10-02-1-A	182.242	67.9	210.506	78.5	268.279	100.0	**268.279**	4.617	3711
10-02-1-B	181.583	78.7	199.670	86.5	230.870	100.0	**230.870**	2.371	2854
10-02-1-C	209.580	73.2	226.563	79.1	286.308	100.0	**286.308**	12.495	8581
10-03-0-A	137.030	54.5	215.682	85.7	251.619	100.0	**251.619**	1.669	4201
10-03-0-B	143.880	72.0	171.237	85.7	199.911	100.0	**199.911**	2.527	2970
10-03-0-C	175.840	69.9	216.662	86.1	251.724	100.0	**251.724**	0.951	672
10-03-1-A	186.358	57.4	315.852	97.3	324.504	100.0	**324.504**	0.452	458
10-03-1-B	185.738	72.0	207.348	80.4	258.026	100.0	**258.026**	4.04	7096
10-03-1-C	220.778	68.1	278.069	85.8	323.984	100.0	**323.984**	149.7	2621
Average		70.5		84.7		100.0		4.184	
15-10-0-A	197.001	66.4	237.647	80.1	296.643	100.0	**296.643**	30.466	7418
15-10-0-B	215.619	72.0	246.082	82.2	299.330	100.0	**299.330**	534.909	124707
15-10-0-C	240.078	69.8	280.636	81.5	344.165	100.0	**344.165**	504.943	118600
15-10-1-A	179.527	68.8	234.385	89.9	260.813	100.0	**260.813**	20.966	3056
15-10-1-B	195.491	72.6	238.947	88.7	269.447	100.0	**269.447**	45.442	8877
15-10-1-C	240.078	69.8	280.636	81.5	344.165	100.0	**344.165**	496.268	118600
15-11-0-A	202.840	61.1	254.057	76.5	332.251	100.0	**332.251**	11.388	3989
15-11-0-B	221.299	72.0	253.466	82.4	307.505	100.0	**307.505**	150.197	59965
15-11-0-C	251.895	69.0	284.716	78.0	364.916	100.0	**364.916**	195.561	86423
15-11-1-A	181.253	63.4	236.236	82.6	285.986	100.0	**285.986**	47.377	10300
15-11-1-B	196.911	71.2	243.551	88.0	276.654	100.0	**276.654**	67.126	16676
15-11-1-C	215.133	69.2	262.506	84.4	311.042	100.0	**311.042**	117.577	26298
15-12-0-A	196.679	53.1	233.775	63.1	370.570	100.0	**370.570**	1901.49	1134028
15-12-0-B	207.015	80.2	227.881	88.3	258.041	100.0	**258.041**	36.098	15228
15-12-0-C	234.367	72.1	258.754	79.6	325.088	100.0	**325.088**	1439.1	533345
15-12-1-A	278.953	56.7	341.425	69.4	491.892	100.0	**491.892**	946.484	455860
15-12-1-B	251.440	73.4	292.497	85.4	342.470	100.0	**342.470**	15.631	9297
15-12-1-C	320.238	71.8	366.453	82.2	445.785	100.0	**445.785**	306.634	138727
Average		68.5		81.3		100.0		381.537	
20-07-0-A	256.669	64.2	286.489	71.6	385.739	96.4	400.033	7200.08	1218470
20-07-0-B	285.259	80.7	305.712	86.5	353.604	100.0	**353.604**	681.19	198598
20-07-0-C	285.303	75.7	311.670	82.7	376.703	100.0	**376.703**	2797.29	527369
20-07-0-D	363.135	74.5	392.409	80.5	431.219	88.4	487.697	7200.71	5771695
20-07-1-A	271.555	67.5	306.356	76.2	402.183	100.0	**402.183**	1233.76	274203
20-07-1-B	275.130	74.6	301.753	81.8	368.954	100.0	**368.954**	585.625	126694
20-07-1-C	284.662	68.3	350.510	84.1	416.677	100.0	**416.677**	489.497	108573
20-07-1-D	339.697	76.5	372.663	83.9	443.943	100.0	**443.943**	4097.74	813981

20-08-0-A	234.082	83.1	256.889	91.2	281.536	100.0	**281.536**	62.228	3458
20-08-0-B	265.465	81.9	281.505	86.8	324.324	100.0	**324.324**	579.525	45096
20-08-0-C	265.465	81.9	280.421	86.5	324.324	100.0	**324.324**	526.001	35237
20-08-0-D	296.704	79.6	305.953	82.1	372.736	100.0	**372.736**	1950.61	105357
20-08-1-A	193.287	67.2	254.932	88.6	287.572	100.0	**287.572**	185.687	19972
20-08-1-B	218.670	71.4	274.183	89.5	306.330	100.0	**306.330**	429.733	35108
20-08-1-C	218.670	71.4	274.406	89.6	306.330	100.0	**306.330**	93.568	5268
20-08-1-D	244.091	71.4	299.948	87.8	341.730	100.0	**341.730**	474.256	33038
20-09-0-A	244.728	67.0	286.420	78.4	334.452	91.5	365.507	7200.11	1208946
20-09-0-B	271.927	77.6	293.575	83.7	330.532	94.3	350.562	7201.28	793583
20-09-0-C	289.524	73.1	312.594	79.0	378.872	95.7	395.872	7201.38	589184
20-09-0-D	332.228	73.8	359.583	79.9	415.112	92.3	449.896	7201.88	754450
20-09-1-A	194.641	67.7	259.012	90.1	287.572	100.0	**287.572**	64.693	12758
20-09-1-B	209.253	70.9	274.132	92.9	295.027	100.0	**295.027**	65.067	9662
20-09-1-C	219.045	69.9	277.112	88.4	313.536	100.0	**313.536**	329.425	43725
20-09-1-D	249.668	66.7	307.006	82.0	361.021	96.4	374.545	7201.53	782504
Average		73.2		84.3		98.1		2710.54	
30-10-0-A	275.924	73.4	317.421	84.4	345.327	91.8	375.977	7200.6	86534
30-10-0-B	300.494	74.1	347.605	85.7	357.892	88.3	405.487	7205.23	471721
30-10-0-D	315.957	75.6	348.860	83.5	385.772	92.3	417.862	7202.73	141884
30-10-1-A	267.288	68.0	323.446	82.3	349.086	88.8	392.945	7201.1	99120
30-10-1-B	299.676	67.9	347.728	78.7	384.537	87.1	441.584	7205.54	149745
30-10-1-C	321.210	63.8	371.152	73.7	383.364	76.2	503.270	7200.22	523744
30-11-0-A	281.095	67.5	331.543	79.6	383.030	91.9	416.736	7200.43	178917
30-11-0-B	313.922	67.4	353.645	76.0	367.895	79.0	465.559	7200.11	953236
30-11-0-D	336.148	70.5	369.986	77.6	389.126	81.6	476.769	7200.13	989450
30-11-1-A	276.319	63.3	330.324	75.7	371.335	85.1	436.489	7200.15	1078106
30-11-1-B	313.459	67.6	370.304	79.9	414.258	89.4	463.598	7205.01	265633
30-11-1-C	333.353	61.3	385.848	71.0	396.829	73.0	543.826	7200.16	844375
30-12-0-A	285.590	65.2	330.455	75.4	406.124	92.7	438.055	7202.88	315159
30-12-0-B	322.575	71.5	356.680	79.1	381.031	84.5	451.118	7200.13	1231159
30-12-0-D	346.288	72.7	379.260	79.6	436.531	91.6	476.402	7203.5	358332
30-12-1-A	303.247	56.2	362.338	67.2	455.641	84.5	539.163	7203.64	787476
30-12-1-B	375.052	57.6	432.134	66.4	507.325	78.0	650.586	7200.08	1961982
30-12-1-C	395.449	57.9	440.896	64.5	489.937	71.7	683.429	7200.08	1843480
30-12-1-D	396.097	65.3	442.489	73.0	532.888	87.9	606.417	7207.2	598467
Average		66.7		76.5		85.0		7202.05	

5 Conclusions

We developed a one-commodity flow mixed-integer linear programming formulation of the truck and trailer routing problem (TTRP). This formulation contains a large amount of three-index indicator variables that are defined for directed arcs between locations. We attempted to reduce the number of variables by developing a second formulation in which the indicator are instead defined for undirected edges. In cases where vehicle customers are a minority of all customers, the latter, undirected, formulation contains fewer variables than the former, directed, formulation. We discussed the rationale behind the undirected formulation and introduced a set of valid inequalities to strengthen its linear programming relaxations.

We implemented a branch-and-cut algorithm to solve the undirected formulation. We evaluated the computational performance of the algorithm using a set of test instances. The algorithm can solve to optimality problems with up to 20 customers.

The performance of our solution method could be improved by using the directed formulation when the majority of customers are vehicle customers, which increases the number of variables in the undirected formulation. Further, more types of valid inequalities could be developed for both formulations.

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