An enhanced L-Shaped method for optimizing periodic-review inventory control problems modeled via two-stage stochastic programming

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Abstract. This paper presents the development of an enhanced L-Shaped method applied to an inventory management problem that considers a replenishment control system based on the periodic review (R, S) policy. We consider single-item one-echelon problems with uncertain demands and partial backorder that are modeled using two-stage stochastic programming. To enable the consideration of large-scale problems, the classical single-cut L-Shaped method and its extended multi-cut form were initially applied. Preliminary computational results indicated that the classical L-Shaped method outperformed its multi-cut counterpart, even though the former required more iterations to converge to the optimal solution. This observation inspired the development of the techniques presented for enhancing the L-Shape method in this context, which consist of the combination of a novel acceleration technique with an efficient formulation and valid inequalities for the proposed model. Numerical experiments suggest that the proposed approach significantly reduced the computational time required to solve large-scale problems.

Keywords. Stochastic programming; Inventory control; Uncertain demand; Partial backordering; L-Shaped method;

1. Introduction

The central issue regarding inventory management is to guarantee product availability for the final consumer at the lowest possible cost while subjected to a variety of particular circumstances. In that sense, inventory management plays an important role in answering key questions such as when to order, how much to order, and how much to keep as safety stock (Namit & Chen, 1999).

The literature provides different inventory control policies associated with mathematical models aiming at minimizing the total inventory management cost. Such models can be divided into two basic groups: deterministic models, which consider that all parameters are previously known, and probabilistic models, in which one or more parameters are considered uncertain. The consideration of uncertain parameters turns these models more adherent to real-world problem, at the expense of becoming more challenging in terms practical suitability due to frequent intensive computational requirements.

Considering demand uncertainty, the literature provides many inventory control policies, such as the classical systems (R, Q), (R, S), (R, s, S), (s, S) and (s, Q). In these systems, R, Q, s and S represent the review period, the fixed order quantity, ordering point and the target inventory level, respectively. In the (R, S) system considered in this paper, in each R time units (review period), a variable quantity sufficient to elevate the inventory level to position S is ordered. However, although closer to reality, these models typically have some limitations. In many of these models, for simplification purposes, the cost parameters are considered fixed along the planning horizon and the stochastic demands are approximated by known probability distributions. For instance, the main limitation of the model proposed by Hadley & Whitin (1963) in terms of uncertainty is the consideration of time-independent demand following a normal distribution. These premises limit the applicability of the model to real problems, given that the demand and other parameters can depend on many factors, such as the inherent uncertainties of the market, the cost and the season (seasonality).

According to Cunha et al. (2017), one form of relaxing the hypothesis of having a simplified model when facing stochastic demands is to use two-stage stochastic programming as a modeling framework. The two-stage structure is compatible with the previously mentioned inventory policies when used for modeling the control variables (i.e., R, S, s and Q) as the first-stage variables in each system, which represent the decisions that should be made before the uncertainties are revealed. One of the main advantages of the two-stage stochastic programming structure is that the stochastic parameters can be modeled without the need to assume any restrictive hypothesis for the stochastic phenomenon, as long as it can be approximated by a discrete set of scenarios. The suitability of two-stage stochastic programming as a framework for addressing inventory management problems has been verified in the studies performed by Fattahi et al. (2014), Cunha et al. (2017), and Dillon et al. (2017).

It is common for inventory management systems to consider complete postponement of the unmet demand (pure backorder case) or its complete loss (pure lost sales case). However, a more general approach to the unmet demand is to consider that only a fraction of the unmet demand is postponed and the remainder is lost. This consists of a partial backorder case, which is more adherent to real-world situations.

In this paper, we propose a two-stage stochastic optimization model aiming at defining optimal periodic-review (R, S) inventory control policies under demand uncertainty with the possibility of considering partial backorder. We also propose a formulation to generalize the consideration of initial inventory in a way that is perfectly adherent to the periodic review policy. Additionally, a more efficient linearization, which precludes the use of auxiliary variables (and therefore has fewer variables and constraints), is proposed to linearize the originally mixed-integer nonlinear model. These modeling aspects represent considerable improvements of the model originally proposed by Cunha et al. (2017), which only considers the pure backorder/ lost sales cases and can only take into account initial inventories by artificially expanding the planning horizon.

The proposed model can be defined as two-stage stochastic programming model with integer and continuous first-stage variables (consisting of decisions regarding when to place orders throughout the planning horizon and the definition of the target level S), where the second stage is composed of continuous variables (order quantities, postponed and unmet demand, and on hand inventories as recourse decisions). These characteristics allow one to consider a scenario-wise decomposition strategy based on Benders decomposition applied to two-stage stochastic programming, given the particular structure of this type of problem where the first-stage variables play the role of complicating variables for being the only elements that promote connections between the second-stage scenario subproblems.

Van Slyke & Wets (1969) presented the first study using Benders decomposition (1962) in a two-stage stochastic programming problem, which became known as the classical L-Shaped or single-cut L-Shaped method. Exploiting the structure of the two-stage stochastic programming models, Birge & Louveaux (1997) extended the method to a version considering multiple cuts (multi-cut L-Shaped). The computational efficiency of Benders decomposition has been widely confirmed by different studies in the literature, especially in the context of two-stage stochastic programming, as demonstrated by Costa (2005); Khodr et al. (2009), Oliveira & Hamacher (2012), and Bertsimas et al. (2013), for instance.

However, in some cases, the classical Benders decomposition (or L-Shaped method) may not present satisfactory computational efficiency. Hence, different strategies for accelerating the Benders decomposition have been proposed over the years (Côté & Laughton, 1984; Saharidis et al., 2010; Yang & Lee, 2012; Sherali & Lunday, 2013; Oliveira et al., 2014).

The present paper proposes an improved version of a Benders decomposition-based method for the proposed inventory management model through the application of a new acceleration technique combined with valid inequalities developed specifically for the proposed model that aims to improve the efficiency of the decomposition method. It is worth highlighting that the proposed acceleration technique could be straightforwardly extended to the classical version of Benders decomposition when applied to problems that share the same mathematical structure.

The main contributions of the present paper can be summarized as follows.

- proposal of a more general model than that originally proposed by Cunha et al. (2017) with the inclusion of the initial inventory and consideration of partial backorder as a generalization of both pure lost sales and pure backorder cases;

- proposal of a simpler linearization to represent the nonlinearity of the model by Cunha et al. (2017) that attains a more efficient formulation in term of the total number of variables and constraints;
- application of the L-Shaped method to the proposed model to improve the computational performance, in particular for instances that consider large numbers of scenarios and longer planning horizons;
- development of valid inequalities to improve the solution process of the proposed model, allowing the obtention of optimal solutions more efficiently in terms of computational times;
- proposal of a new acceleration technique that exploits the structure of the master problem of the L-Shaped method and is general enough to be considered in applications based on the classical Benders method (i.e., deterministic problems) that share similar mathematical structure.

This paper is structured as follows. In Section 2, we present a literature review of inventory management models and the main techniques available for accelerating Benders decomposition, which can also be applied to the L-Shaped method. In Section 3, we describe the structure of the problem. Details the proposed inventory management model are provided in Section 4. In Section 5, we present the classical L-Shaped methods and its extended multi-cut form with their application to the model proposed in Section 4. In Section 6, valid inequalities are developed and the new technique for accelerating the L-Shaped method is presented. All numerical results and analyses of the experiments performed are given in Section 7. We conclude in Section 8 with a discussion of results obtained and future research directions.

2. Literature Review

2.1. Inventory management models

Over the years, new formulations of deterministic models have been proposed with the aim to reduce the simplifications of the EOQ model proposed by Harris (1913). Most of the focus has been on increasing its generality and consequently its adequacy for real-world applications, with emphasis on the following classical models found in the inventory management literature: economic lot size model, quantity discount pricing model and economic lot size model with backorder (Zipkin, 2000), and the EOQ model with partial backorder proposed by Pentico & Drake (2009). However, the consideration of all parameters of an inventory management model as deterministic, particularly the demand, might not be a satisfactory strategy depending on the values of the costs involved in the inventory management or when considering the need for a mandatory minimum level of service, for instance.

Considering partial backorder and stochastic demand following a known probability distribution, special attention is drawn to the studies performed by Posner & Yansouni (1972), Das (1977); Moinzadeh (1989), Rabinowitz et al. (1995), Chu et al. (2001), Thangam & Uthayakumar (2008), and

Hu et al. (2014). To date, the only studies found in the literature and using two-stage stochastic programming applied to inventory management were those performed by Fattahi et al. (2014), Cunha et al. (2017), and Dillon et al. (2017).

Based on the continuous system (s, S), Fattahi et al. (2014) modeled a two-layer network with one manufacturer, one retailer, one item, uncertainty in the demand parameter, and pure lost sales and analyzed the model in its centralized and decentralized forms. Cunha et al. (2017), proposed a replenishment control and inventory model via two-stage stochastic programming, considering periodic review (R, S), one item and uncertain demand. In their model, there is no parameter related to the initial inventory, which requires the cost parameters to be considered zero in the first few planning horizon periods so that the initial inventory can be computed. Furthermore, the model proposed by Cunha et al. (2017) considers the pure lost sales case or, with some alterations of the constraints, the pure backorder case. With the adaptation of the model proposed by Cunha et al. (2017), Dillon et al. (2017) proposed a red blood cells inventory management model that considers the minimization of the operational cost, the perishability of blood, multi-periods, multi-products and uncertain demand.

2.2. Techniques for accelerating Benders decomposition

One of the major computational challenges of Benders decomposition methods is associated with the solution of its master problem, particularly when the method is applied to solve mixed-integer linear programming (MILP) problems. A form of accelerating Benders decomposition is by generating more efficient cuts, i.e., cuts or sets of cuts that, when added to the relaxed master problem (RMP), can enable convergence of the solution process within fewer iterations when compared with its original form. Some researchers propose the inclusion of an additional set of strong cuts in each iteration of the method. In this context, the studies performed by the following authors can be cited: Magnanti & Wong (1981), Papadakos (2008), Rei et al. (2009), Saharidis et al. (2010), Yang & Lee (2012), Sherali & Lunday (2013), and Oliveira et al. (2014) .

Aiming to reduce the number of iterations required for convergence, and consequently the solution time, some researchers have focused on modifying the traditional Benders decomposition master-slave relationship, often leading to satisfactory results. In this context, the recent studies are Crainic et al. (2016) and Gendron et al. (2016).

Some techniques focus on improving the solution method applied to the RMP in the whole or part of the solution process as, very often, most of the total execution time for Benders decomposition is dedicated to solving the RMP. In this context, the following studies can be cited: Benoist et al. (2002), Corréa et al. (2007), Naoum-Sawaya & Elhedhli (2013) and Pérez-Galarce et al. (2014).

In some cases, the slave problem is difficult to solve; hence, aiming to improve the computational performance of Benders decomposition, some studies focused on developing efficient strategies for its solution. In this context, the studies performed by the following authors can be cited: Zakeri et al. (2000), Cordeau et al. (2001), and Mercier et al. (2005).

When there are many subproblems to be solved at each iteration of the solution process, the use of parallel computational techniques can improve the computational performance of the method. Different studies in the literature obtain satisfactory results with the use of this technique, and the following are emphasized: Linderoth & Wright (2003), Wolf & Koberstein (2013), and Chermakani (2015).

Other frequently applied techniques found in the literature are the use of valid inequalities. The inclusion of a series of valid inequalities in the master problem restricts its domain, meaning that the method is typically able to provide better lower and upper bounds since the first iterations of the solution process, thus representing an effective method to accelerate convergence. In addition, by doing such, one could possibly eliminate infeasible solutions thus partially or completely eliminating the necessity of generating feasibility cuts. Satisfactory results obtained with the use of valid inequalities can be confirmed by many studies in the literature, which is commonly used in combination with other acceleration techniques aiming at obtaining better computational performance. In this context, the following recent studies can be cited: Saharidis et al. (2011), Tang et al. (2013), Jeihoonian et al. (2014), Lei et al. (2014), and Pishvaee et al. (2014).

This literature survey suggests that no studies that have proposed models and considered the use of the two-stage stochastic programming to support with decision making regarding replenishment and inventory control with partial backorder. Furthermore, no studies about inventory management via two-stage stochastic programming concerned with developing methods with improved computational performance when addressing large-scale problems were found.

3. Problem statement

In this study, the retailer is assumed to use the periodic replenishment inventory control system (R, S) where, particularly for the proposed model, R is defined by k and r, k denotes the period when the first order is placed, and r denotes the periodicity between orders. S, represented by s in the mathematical model proposed in the next section (by convention, all variables are represented by lowercase letters), denotes the maximum inventory level (target level) of the item. The problem consists in determining the optimal target level S, the optimal period for first order, k, and the optimal periodicity of the inventory control system of the retailer, r, for a single item.

For the supply chain structure presented in Figure 1, a finite planning horizon is considered. It is composed of a discrete number of periods p, which may represent days, weeks or months. We assume that the orders received at the start of a period can be consumed in the same period. The orders are always placed at the start of each period and should be placed every r periods, from period k, and a variable quantity sufficient to raise the inventory level to the S position is ordered.

The proposed model considers partial backorder. The maximum backorder value that can be delayed (with the loss of the remainder) in a certain period and scenario is given by a fraction β of the unmet demand, where $\beta \in [0,1]$. However, with a simple modification of the model, discussed in the next section, a minimum demand level can be guaranteed if necessary.

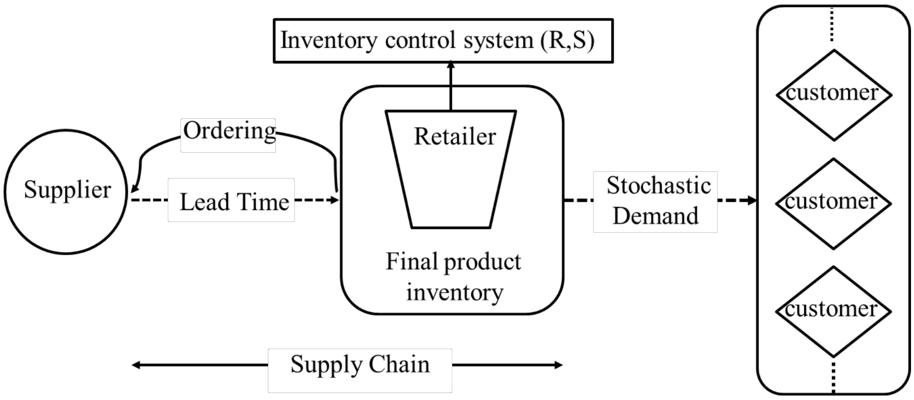

Figure 1- Supply chain structure

The proposed mathematical model is formulated via two-stage stochastic programming (Birge & Louveaux, 1988; Birge & Louveaux, 1997). This modeling framework enables the representation of the stochastic demand through discrete scenarios and their respective probabilities. Hence, any stochastic process can be represented, as long as it is possible to obtain discrete samples from it, which enables one to consider aspects such as seasonality or general correlations.

4. Proposed mathematical model

In relation to the proposed model, some premises are considered in an attempt to simplify the present context. However, it is important to highlight that the model could be straightforwardly extended to incorporate more general situations. The premises are the following:

1. the upper bound $\bar{S}$ of target level s is also a physical restriction on the inventory capacity. In addition, the capacities and costs of alternative means of transportation between the supplier and the retailer are not considered;
2. the backorders do not have specific delivery time limits, but because it is a cost minimization problem, the backorders are considered to be satisfied as soon as possible, aiming to minimize the costs associated with the demand fulfillment delay;
3. the lead time is constant and pre-determined, but it could be easily indexed in time, as could the backorder rate;
4. although the costs are indexed by period, we consider them constant throughout the planning horizon, without loss of generality;

5. for simplification purposes, the proposed model considers a single item. Its extension to consider multiple products is straightforward.

The mathematical notation and definitions of sets, indexes, parameters and variables of the proposed model can be found in Appendix A.

4.1. First-stage problem

The first-stage problem comprises the decisions concerning control variables, which should be made prior to observing the actual realization of the uncertainty in the system. In the present study, the first-stage variables are the review periodicity decisions R, the period in which the first order is placed K and the target level S. The first-stage problem is given by

$$\min_{s,v,u} \sum_{p} CF^{p}\, v^{p} + E_{\Omega}[Q(v,s,\xi)] \tag{4.1}$$

subject to

$$\sum_{r,k} u^{r,k} = 1 \tag{4.2}$$

$$\sum_{r,k} W_{p}^{r,k}\, u^{r,k} = v^{p} \qquad \forall p \tag{4.3}$$

$$s \leq \bar{S} \tag{4.4}$$

$$u^{r,k} \in \{0,1\} \qquad \forall r, \forall k \tag{4.5}$$

$$v^{p} \in \{0,1\} \qquad \forall p. \tag{4.6}$$

$$s \geq 0 \tag{4.7}$$

In the objective function (4.1), the first term represents the ordering cost, and $E_{\Omega}[Q(v,s,\xi)]$ represents the expected value of the cost associated with the second-stage problem. Constraint (4.2) indicates the existence of exactly one pair of values for the cycle size R and for the period of the first order ($R = r$ and $K = r$, when $u^{r,k} = 1$). Constraint (4.3) indicates the periods in which the orders are placed in the planning horizon as a function of the choice of values for R and K. Constraint (4.4) establishes the upper bound of the variable that represents the target inventory level (capacity restriction). In (4.5) and (4.6), the domains of the binary first-stage variables (that determine when the orders will be placed) are presented. In (4.7), the variable representing the target level S is defined as continuous and nonnegative.

4.2. Second-stage problem

The second-stage problem aims to minimize the inventory management costs, lost demands and backorders throughout the planning horizon, given the selection of v^{p} (i.e. R and K) and S (s) and the observed scenario ξ. In the second stage, the second-stage or recourse variables are determined, which represent the dynamics of inventory management after the demand is observed, aiming at minimizing the total cost.

$$Q(v, s, \xi) = \min_{a,f,i,it,l,q} \sum_{p} [H^p\, i(\xi)^p + B^p f(\xi)^p + BA^p l(\xi)^p] \quad (4.8)$$

subject to

$$a(\xi)^p + f(\xi)^p + l(\xi)^p = D(\xi)^p + l(\xi)^{p-1} \qquad \forall p \quad (4.9)$$

$$(1-\beta) l(\xi)^p - \beta f(\xi)^p \leq 0 \qquad \forall p \quad (4.10)$$

$$II + i(\xi)^{p-1} + q(\xi)^{p-TE} = i(\xi)^p + a(\xi)^p \qquad \forall p = 1 \quad (4.11)$$

$$i(\xi)^{p-1} + q(\xi)^{p-TE} = i(\xi)^p + a(\xi)^p \qquad \forall p \geq 2 \quad (4.12)$$

$$II + it(\xi)^{p-1} + q(\xi)^p = it(\xi)^p + D(\xi)^p - f(\xi)^p \qquad \forall p = 1 \quad (4.13)$$

$$it(\xi)^{p-1} + q(\xi)^p = it(\xi)^p + D(\xi)^p - f(\xi)^p \qquad \forall p \geq 2 \quad (4.14)$$

$$q(\xi)^p - (s - it(\xi)^{p-1} - II) \leq \bar{S}(1 - v^p) \qquad \forall p = 1 \quad (4.15)$$

$$q(\xi)^p - (s - it(\xi)^{p-1}) \leq \bar{S}(1 - v^p) \qquad \forall p \geq 2 \quad (4.16)$$

$$q(\xi)^p - (s - it(\xi)^{p-1} - II) \geq \bar{S}(v^p - 1) \qquad \forall p = 1 \quad (4.17)$$

$$q(\xi)^p - (s - it(\xi)^{p-1}) \geq \bar{S}(v^p - 1) \qquad \forall p \geq 2 \quad (4.18)$$

$$q(\xi)^p \leq \bar{S} v^p \qquad \forall p \quad (4.19)$$

$$a(\xi)^p, i(\xi)^p, it(\xi)^p, l(\xi)^p, q(\xi)^p, f(\xi)^p \geq 0 \qquad \forall p. \quad (4.20)$$

In the objective function (4.8), the total inventory ($H^p i(\xi)^p$), lost sales ($B^p f(\xi)^p$) and backorder costs ($BA^p l(\xi)^p$) are considered. Constraint (4.9) represents the quantities of the demand met, lost and in backorder in each period for each scenario ξ. Constraint (4.10) guarantees that the backorder will be, at most, a fraction β of the total unmet demand in each period for each scenario ξ. Constraints (4.11) and (4.12) represent the inventory balance from one period to the next in each scenario ξ. Constraints (4.13) and (4.14) represent the balance of the inventory positions (on-hand stock plus orders in transit) from one period to the next in each scenario ξ. The set of constraints (4.15) to (4.19) define the quantities of the item to be ordered at the start of each period for scenario ξ and represent the linearization of constraints (4.21) and (4.22), which are modified from the original model proposed by Cunha et al. (2017) to consider the initial inventory parameter in the formulation. Finally, in (4.20), the second-stage variables are defined as continuous and nonnegative.

$$q(\xi)^p = (s - it(\xi)^{p-1} - II) v^p \qquad \forall p = 1 \quad (4.21)$$

$$q(\xi)^p = (s - it(\xi)^{p-1}) v^p \qquad \forall p \geq 1. \quad (4.22)$$

Constraints (4.15) to (4.19) proposed in this paper are an alternative representation of the ordered quantities of the item throughout the planning horizon in each scenario ξ, differently from Cunha et al. (2017), without using any additional auxiliary variable. In the Appendix B we present a discussion of the validity of the proposed linearization.

In contexts where one need to guarantee a minimum demand level to be met (i.e., where the backorder percentage has a lower bound), it is enough to rewrite constraint (4.10) as

$$(1-\beta)l(\xi)^p - \beta f(\xi)^p \geq 0 \qquad \forall p. \qquad (4.23)$$

It is important to emphasize that, because constraint (4.10), setting $\beta = 0$, implies in $l = 0$, i.e., the pure lost sales case is considered. On the other hand, setting $\beta = 1$ and replacing (4.10) with (4.23) in the proposed model, results in $f = 0$ thus enforcing the pure backorder case.

It is worth highlighting that making $x_1 = (s - it(\xi)^{p-1} - II)$ in $p = 1$ and $x_1 = (s - it(\xi)^{p-1})$ for $p \geq 2$, implies in both cases that $-\overline{S} \leq x_1 \leq \overline{S}$. Hence, some values of the first-stage variables may result in $x_1 < 0$, i.e., $q(\xi)^p < 0$, which makes the second-stage problem infeasible. Therefore, the proposed model is not of relatively complete resources. This issue will be addressed later on when we present the development of valid inequalities for the problem.

5. Classical and Extended L-Shaped Formulation

5.1. Classical L-Shaped method

For a brief explanation of the L-Shaped method by Van Slyke & Wets (1969), consider the following problem:

$$\min_{x,y,z} C_1^T x + C_2^T y + \sum_{\xi} Pr(\xi)\, q(\xi)^T z(\xi) \qquad (5.1)$$

subject to

$$Ax + By \leq c \qquad (5.2)$$

$$Tx + Jy + V(\xi)z(\xi) \leq h \qquad \forall \xi \qquad (5.3)$$

$$x \in \{0,1\}, y \geq 0 \qquad (5.4)$$

$$z(\xi) \geq 0 \qquad \forall \xi, \qquad (5.5)$$

where $Pr(\xi)$ is the probability of each scenario ξ, and the parameter $C_1, C_2, q(\xi), A, B, T, J, h, c$ and $V(\xi)$ are known matrices in the domain of real numbers with compatible dimensions.

Problem (5.1)-(5.5) represents a two-stage stochastic linear programming problem with the first stage composed of binary and continuous variables. The objective function of the first-stage problem is composed by the first two terms of the objective function (5.1) added to the expected value of the second-stage problem and is subject to restriction constraint (5.2) and the definitions in (5.4). The second-stage or primal slave problem (PSP) is formed by the third term of the objective function (5.1), constraint (5.3) and the definition in (5.5).

Decomposing problem (5.1)-(5.5), into the relaxed master problem (RMP) and PSP, the dual slave problem (DSP) is

$$\max_{u} T(\bar{x}, \bar{y}) = \max_{u} E_{\Omega}[M(\bar{x}, \bar{y}, \xi)] = \max_{u} \sum_{\xi} Pr(\xi)[u(\xi)^T (h - T\bar{x} - J\bar{y})] \quad (5.6)$$

subject to

$$V^T(\xi) u(\xi) \leq q(\xi) \qquad \forall \xi \quad (5.7)$$

$$u(\xi) \leq 0 \qquad \forall \xi, \quad (5.8)$$

where $u(\xi)$ is the dual variable associated and the bars in $\bar{x}, \bar{y}$ in (5.6) denote that the respective variables are fixed and their values originate from the solutions of the RMP. The RMP is given by

$$\min_{x,y,z}. \; C_1^T x + C_2^T y + m \quad (5.9)$$

subject to

$$Ax + By \leq c \quad (5.10)$$

$$\sum_{\xi} Pr(\xi)[(u(\xi)^T)^b (h - Tx - Jy)] \leq m \qquad \forall b \in \mathrm{B} \subseteq U_V \quad (5.11)$$

$$\sum_{\xi} Pr(\xi)[(u(\xi)^T)^i (h - Tx - Jy)] \leq 0 \qquad \forall i \in \mathrm{I} \subseteq U_R, \quad (5.12)$$

$$x \in \{0,1\}, y \geq 0, m \in \mathbb{R}, \quad (5.13)$$

where each b and i is related to a solution $u(\xi)^b \in B$ and $u(\xi)^i \in I$, and B and I are, respectively, the subsets of extreme points (U_V) and extreme rays (U_R) of polyhedron U, which is defined by (5.7) and (5.8).

Inequalities (5.11) and (5.12) are not explicitly defined constraints. Instead, they are implicitly defined by a finite number of Benders cuts (constraints of the type (5.14) or (5.15) are generated with information from the solutions of the DSP or from the dual information provided by the solution of the PSP).

A schematic description of the classical L-Shaped (or single-cut L-Shaped method) is given in Figure 2. According to Figure 2, in the iterative process that governs the L-Shaped method, at each iteration, one solution is obtained for the RMP, which is considered as an input parameter in the DSP (or PSP). In turn, the DSP (or PSP) is solved to obtain the dual information that is used for the generation of a so-called Benders cut that will be added to the RMP. In case the DSP has an optimal solution, the generated cut is of type (5.11), known as an optimality cut. However, if the DSP is unbounded (which implies in the infeasibility of the PSP), cut (5.12) is generated; this is known as a feasibility cut.

There is a constraint (5.11) and (5.12) in the complete master problem for each extreme point and extreme ray obtained from the DSP, respectively. However, to obtain the optimal solution, only a small number of these cuts are expected to be added to the RMP until the optimal solution (presuming that the feasible region of the original problem is not empty) is obtained.

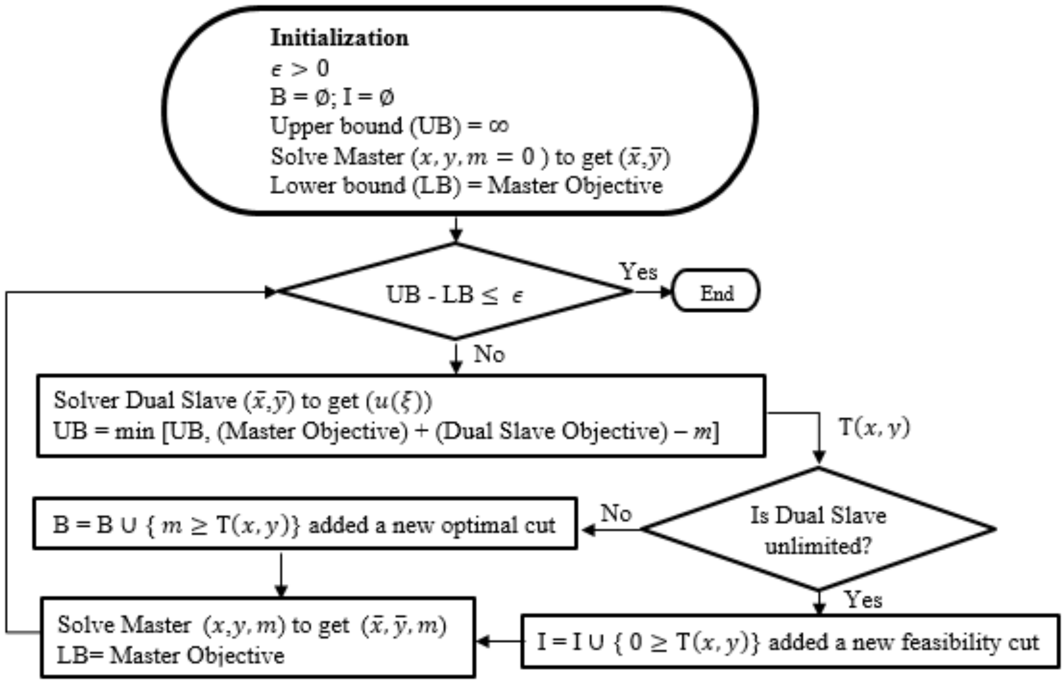

Figure 2- Flowchart of single-cut L-Shaped method

5.2. Extended L-Shaped method

According to Birge & Louveaux (1997), the structure of two-stage stochastic problems allows the addition of multiple cuts to the RMP at each iteration, each cut being related to an individual scenario. According to their findings, the use of this method can provide substantial improvement in the convergence properties of the method. The main difference between the multi-cut and single-cut L-Shaped methods is in the master problem formulation, which can be conveniently adapted to the multi-cut L-Shaped structure as follows:

$$\min_{x,y,z} C_1^T x + C_2^T y + \sum_{\xi} Pr(\xi) m(\xi) \tag{5.14}$$

subject to

$$Ax + By \leq c \tag{5.15}$$

$$[(u(\xi)^T)^b (h - Tx - Jy)] \leq m(\xi) \qquad \forall \xi, \forall b \in B \subseteq U_V \tag{5.16}$$

$$[(u(\xi)^T)^i (h - Tx - Jy)] \leq 0 \qquad \forall \xi, \forall i \in I \subseteq U_R \tag{5.17}$$

$$x \in \{0,1\}, y \geq 0 \tag{5.18}$$

$$m(\xi) \in \mathbb{R} \qquad \forall \xi. \tag{5.19}$$

A flowchart for the multi-cut L-Shaped method would be similar to that shown in Figure 2, being the main difference the multiple Benders cuts that would be added to the RMP after each solution of the DSP.

5.3. Classical and extended L-Shaped formulations of the proposed model

In the DSP of the proposed model, the complicating variables s and v^p are and set to $\bar{s}$ and $\bar{v}^p$ (i.e., they have fixed values), whereas α, γ, σ, π, μ, ρ, and ω are dual variables associated with constraints (4.9), (4.10), (4.11-4.12), (4.13-4.14), (4.15-4.16), (4.17-4.18) and (4.19), respectively. Therefore, the slave problem can be formulated as

$$E(\bar{v},\bar{s},\xi) = \max_{\alpha,\gamma,\sigma,\pi,\mu,\rho,\omega} \sum_p [D(\xi)^p\,(\alpha(\xi)^p + \pi(\xi)^p) + \bar{S}\bar{v}^p\,\omega(\xi)^p + \bar{S}\,(\bar{v}^p - 1)(\rho(\xi)^p - \mu(\xi)^p) + \bar{s}(\,\rho(\xi)^p + \mu(\xi))] + [\pi(\xi)^{p=1} + \sigma(\xi)^{p=1} + \mu(\xi)^{p=1} + \rho(\xi)^{p=1}] \quad (5.20)$$

subject to

$$\pi(\xi)^{p+1} - \pi(\xi)^p + \mu(\xi)^{p+1} + \rho(\xi)^{p+1} \leq 0 \qquad \forall\xi, \forall p \qquad (5.21)$$

$$\sigma(\xi)^{p+1} - \sigma(\xi)^p \leq H^p \qquad \forall\xi, \forall p \qquad (5.22)$$

$$-\sigma(\xi)^p + \alpha(\xi)^p \leq 0 \qquad \forall\xi, \forall p \qquad (5.23)$$

$$\pi(\xi)^p + \alpha(\xi)^p - \beta\,\gamma(\xi)^p \leq B^p \qquad \forall\xi, \forall p \qquad (5.24)$$

$$\pi(\xi)^p + \sigma(\xi)^{p+TE} + \omega\,(\xi)^p + \mu(\xi)^p + \rho(\xi)^p \leq 0 \qquad \forall\xi, \forall p \qquad (5.25)$$

$$\alpha(\xi)^p - \alpha(\xi)^{p+1} + (1-\beta)\,\gamma(\xi)^p \leq BA^p \qquad \forall\xi, \forall p \qquad (5.26)$$

$$\alpha(\xi)^p, \pi(\xi)^p, \sigma(\xi)^p \in \mathbb{R} \qquad \forall\xi, \forall p \qquad (5.27)$$

$$\rho(\xi)^p \geq 0 \qquad \forall\xi, \forall p \qquad (5.28)$$

$$\omega(\xi)^p, \mu(\xi)^p, \gamma(\xi)^p \leq 0 \qquad \forall\xi, \forall p. \qquad (5.29)$$

The formulation of the single-cut L-Shaped RMP for the proposed model is given by

$$\min_{v,u,s} \sum_p CF^p\,v^p + m \qquad (5.30)$$

subject to

$$\sum_{r,k} u^{r,k} = 1 \qquad (5.31)$$

$$\sum_{r,k} W^{p,r,k}\,u^{r,k} = v^p \qquad \forall p \qquad (5.32)$$

$$0 \leq s \leq \bar{S} \qquad (5.33)$$

$$\left(\sum_\xi Pr(\xi)E(v,s,\xi)\right)^b \leq m \qquad \forall b \in B \subseteq U_V \qquad (5.34)$$

$$\left(\sum_\xi Pr(\xi)E(v,s,\xi)\right)^i \leq 0 \qquad \forall i \in I \subseteq U_R \qquad (5.35)$$

$u^{r,k} \in \{0,1\}$ $\quad \forall r, \forall k$ (5.36)

$v^p \in \{0,1\}$ $\quad \forall p.$ (5.37)

$m \geq 0$ (5.38)

Therefore, the single-cut L-Shaped formulation (SF) is composed of RMP (5.30)-(5.38) and DSP (5.20)-(5.29). In the multi-cut L-Shaped formulation (MF), the DSP is also given by formulation (5.20)-(5.29). The main change is in the master problem, which receives one cut for each scenario at each iteration, i.e., multiple cuts are added to the master problem at each iteration of the MF, and it is composed of (5.39)-(5.43). One should notice that the DSPs are independent scenario-wise and can be solved independently in parallel, which could result in further improvements in the computational performance of the proposed technique. The formulation of the multi-cut L-Shaped relaxed master problem for the proposed model is provided next:

$$\min_{v,u,s} \sum_{p} CF^p \, v^p + \sum_{\xi} Pr(\xi) m(\xi) \tag{5.39}$$

subject to

$E(v,s,\xi)^b \leq m(\xi)$ $\quad \forall \xi, \forall b \in B \subseteq U_V$ (5.40)

$E(v,s,\xi)^i \leq 0$ $\quad \forall \xi, \forall i \in I \subseteq U_R$ (5.41)

$m(\xi) \geq 0$ $\quad \forall \xi$ (5.42)

(5.31), (5.32), (5.33), (5.36), (5,37) (5.43)

6. Accelerating the L-Shaped Method

6.1. Valid inequalities for the proposed model

An efficient method for accelerating Benders decomposition is to incorporate additional information about the PSP in the RMP. This enhancement could be achieved by adding inequalities that explore specific characteristics of the PSP, limiting the solution space of the RMP variables without preventing the obtention of optimal solutions for the problem. Aiming at improving the computational performance of the SF and MF, the following valid inequalities are proposed. All proofs for the propositions presented in this section are available in Appendix B.

Proposition 1. *The inequalities*

$$s \geq II - \sum_k \left(DK(\xi)^k \sum_r u^{r,k} \right) \qquad \forall \xi \qquad (6.1)$$

make the proposed model of relatively complete recourse, where

$$DK(\xi)^k = \sum_{p=1}^{k-1} D(\xi)^p \qquad \forall \xi, \forall k \geq 2$$

$$DK(\xi)^k = 0. \qquad \forall \xi, \forall k = 1 \qquad (6.2)$$

Proposition 2. *The inequalities*

$$m(\xi) \geq \sum_{p,r,k} [(H^p CII(\xi)^p - MCK3^p CFII(\xi)^p) WK1^{p,k} u^{r,k}] + \sum_{p,r,k} [MCK2^{p,r,k} D(\xi)^p WK2^{p,k} u^{r,k}] \qquad \forall \xi \qquad (6.3)$$

are valid lower-bound inequalities for $m(\xi)$ *of the RMP (5.39)-(5.43) in MF, where*

$$CII(\xi)^p = max\big((II - D(\xi)^p + CII(\xi)^{p-1}), 0\big) \qquad \forall p = 1 \qquad (6.4)$$

$$CII(\xi)^p = max\big((-D(\xi)^p + CII(\xi)^{p-1}), 0\big) \qquad \forall p \geq 2 \qquad (6.5)$$

$$CFII(\xi)^p = min\big((II - D(\xi)^p + CII(\xi)^{p-1}), 0\big) \qquad \forall p = 1 \qquad (6.6)$$

$$CFII(\xi)^p = min\big((-D(\xi)^p + CII(\xi)^{p-1}), 0\big) \qquad \forall p \geq 2 \qquad (6.7)$$

$$WK1^{p,k} = 1 \qquad \forall k, \forall p \leq TE + k - 1$$

$$WK1^{p,k} = 0 \qquad \forall k, \forall p > TE + k - 1 \qquad (6.8)$$

$$WK2^{p,k} = 1 - WK1^{p,k} \qquad \forall p, \forall k \qquad (6.9)$$

$$WYK^{p,r,k} = \big(WHK^{p,r,k} + WYK^{p-1,r,k}\big) WHK^{p,r,k} \qquad \forall p, \forall r, \forall k \qquad (6.10)$$

$$WHK^{p,r,k} = 1 - W^{(p-TE),r,k} \qquad \forall p, \forall r, \forall k \qquad (6.11)$$

$$MCK2^{p,r,k} = min[WYK^{p,r,k} H^p, B^p, (B^p(1-\beta) + BA^p \beta)] \qquad \forall p, \forall r, \forall k \qquad (6.12)$$

$$MCK3^p = min[B^p, (B^p(1-\beta) + BA^p \beta)]. \qquad \forall p \qquad (6.13)$$

Remark 1. S*imilarly, it can be said that (6.14) is a valid lower-bound inequality of the RMP (5.30)-(5.38) of the SF.*

$$m \geq \sum_\xi Pr(\xi) \left\{ \sum_{p,r,k} \{(H^p CII(\xi)^p - MCK3^p FII(\xi)^p) WK1^{p,k} u^{r,k}\} + \sum_{p,r,k} [MCK2^{p,r,k} D(\xi)^p WK2^{p,k} u^{r,k}] \right\} \qquad (6.14)$$

Proposition 3. *The inequalities*

$$\sum_{r,(k|k\geq MK)} u^{r,k} = 1 \tag{6.15}$$

are valid constraints that limits the selection of the period to place the first order, where

$$DKS(\xi)^p = \sum_{p=1}^{p} D(\xi)^p \qquad \forall p, \forall \xi \tag{6.16}$$

$$DKM(\xi)^p = II - DKS(\xi)^p \qquad \forall p, \forall \xi \tag{6.17}$$

$$MK = [Min(\, p|DKM(\xi)^p < 0)] - TE \tag{6.18}$$

The proposed enhanced model, denoted by E-SF, implies that (6.1), (6.14), and (6.15) are included in the RMP of the SF. Similarly, E-MF implies that (6.1), (6.3), and (6.15) are included in the RMP in MF.

6.2. Proposed acceleration technique

The novel acceleration technique presented in this section considers two master problems in its formulation, the RMP and an auxiliary RMP (ARMP), and has two versions, depending on the solution structure of these problems. When the RMP and the ARMP have the same cut structure, i.e., when both of them are based on single-cut L-Shaped or multi-cut L-Shaped formulations, we refer to it as being a "pure" version or form of the proposed technique. In turn, when the solution structures are different, it corresponds to what we call the "hybrid" version.

The development of the technique is inspired by its application to mixed-integer stochastic linear programming problems. Nevertheless, it is worth mentioning its straightforward extension to deterministic MILP problems. The technique is focused on reducing the solution time through the reduction of the solution time of the RMP. The driver for its development is the fact that when the SF and MF were applied to the inventory management model presented in Section 4, preliminary experiment showed that the SF had the best performance in terms of solution time. According to the computational results that are presented in the following section, the solution times of the RMP are significantly increased when multiple cuts are inserted per iteration of the MF, which causes its performance to be inferior to SF, despite fewer iterations being required to obtain the optimal solutions.

6.2.1. Pure form of the proposed acceleration technique

The flowchart of the proposed acceleration technique in its pure version with single-cut L-Shaped structure is presented in Figure 3. It is important to notice that in the pure versions, the RMP and ARMP are similar problems, where the only difference is the fact that the integer variables have fixed values in the ARMP.

According to Figure 3, the initial values of the lower and upper bound (LB and UB, respectively) are initially determined. After the solution of the RMP, the LB parameter takes the value of Z_{RMP} (value of the objective function of RMP), and the variable x takes the value obtained in the solution of the RMP. Next, one needs to check if the primary stop condition is satisfied and, if not, the current values of the variables x are fixed in $\bar{x}$ in the ARMP – turning it into a continuous linear programming problem –, the parameter $\widehat{UB}$ takes on the current value of the parameter UB, and the parameter LB_A takes the current value of the parameter LB. Next, the satisfaction of the auxiliary stop condition is verified, and if it is unsatisfied, a process called the auxiliary cycle is initiated. During this auxiliary cycle, using the classical L-Shaped method with ARMP and DSP, aims to determine the optimal solution y^* given the value of $\bar{x}$; this solution is named the optimal ARMP solution and represented by $(\bar{x}, y^*)$. The auxiliary cycle ends when the auxiliary stop condition is satisfied (i.e., $UB - LB_A \leq \epsilon$) or when the ARMP becomes infeasible. Once the cycle ends, if $UB < \widehat{UB}$, the optimal solution of the ARMP $(\bar{x}\ y^*)$ is certainly found and the parameters (X, Y) take on their respective values. Otherwise, if $UB \ = \widehat{UB}$ and the problem does not have multiple first-stage solutions, the auxiliary cycle ended before obtaining the optimal solution for the ARMP in relation to the current $\bar{x}$; therefore, a solution $(\bar{x}, y)$ is obtained.

After the end of the auxiliary cycle, the integer variables are unfixed, and the RMP is solved, obtaining a solution (x, y). Next, it is verified whether the primary stop condition is satisfied and, if it is the case, the optimal solution of the problem is obtained and is equal to (X, Y). If the primary stop condition is not satisfied, the current integer values obtained in the solution of the RMP are fixed in $\bar{x}$ in the ARMP, LB_A takes on the current value of LB, and a new auxiliary cycle is started. Hence, the auxiliary cycles and the new solutions of the RMP are obtained sequentially and alternately until the optimal solution of the problem is found.

The flowchart of the pure multi-cut technique is similar to that presented in Figure 3, where the main difference is the generation and addition of multiple optimality or feasibility cuts to the RMP and ARMP from solutions obtained by the DSP per iteration. When the pure versions are applied in the proposed model, have the pure single-cut formulation (PSF) and pure multi-cut formulation (PMF). Notice that both can be enhanced with the proposed valid inequalities. In that case, E-PSF, means (6.1), (6.14) and (6.15) are added into the RMP and ARMP; analogously, in E-PMF, (6.1), (6.3) and (6.15) are added into the RMP and ARMP.

A key factor of the proposed technique is the maintenance of the smaller upper bound in both stop conditions, making the auxiliary cycles more efficient. Considering the auxiliary cycles, if the finding of solution $(\bar{x}, y^*)$ depends on having a larger value for Z_{ARMP} when compared with the current LS (considering a minimization problem without loss of generality), the auxiliary cycle ends before obtaining y^* and a solution $(\bar{x}, y)$ is obtained. Therefore, the number of iterations of the auxiliary cycle decreases.

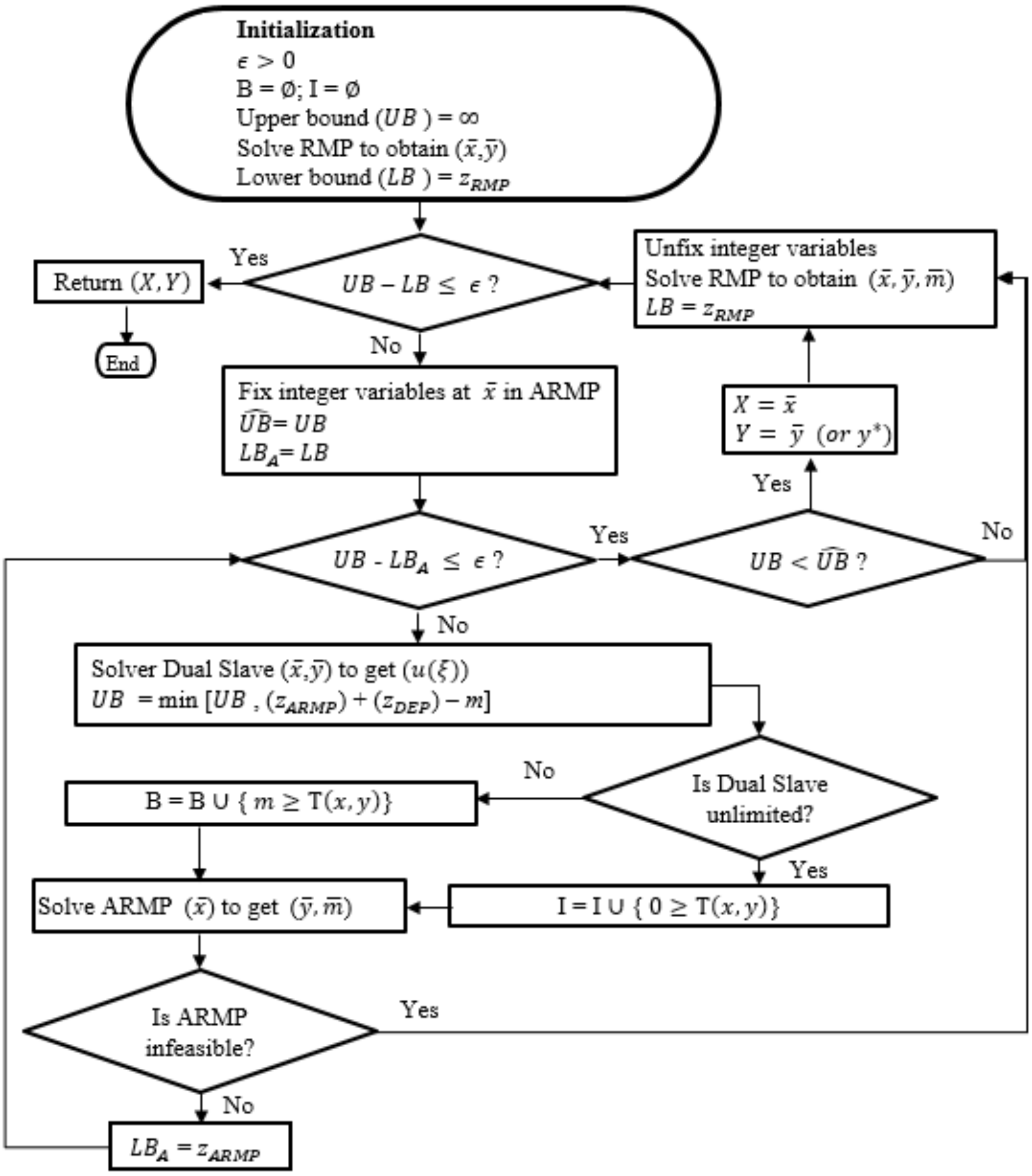

Figure 3- Flowchart of the acceleration technique proposed in the pure version and with a single-cut L-Shaped structure

To illustrate the typical behavior of the proposed method, Figure 4 displays the values UB, LB e LB_A obtained when applying the pure single-cut L-Shaped form of the method to a generic instance of the problem at hand. The yellow, red and black dots represent the values of LB_A when an optimal solution $(\bar{x}, y^*)$ of the ARMP is obtained, the values of LB when a solution (x, y) of the RMP is obtained and the values of LB_A when a solution $(\bar{x}, y)$ of the ARMP is obtained, respectively. As expected, due to the logic of the proposed technique, Figure 4 indicates that whenever a solution $(\bar{x}, y^*)$ is obtained, the value of UB decreases and the optimal solution of the RMP will always be the last solution $(\bar{x}, y^*)$ obtained in the auxiliary cycle. It should be emphasized that a solution $(\bar{x}, y^*)$ can be obtained without a concurrent decrease in the value of UB during the auxiliary cycle, in case the problem presents multiple first-stage solutions. It is also observed that an auxiliary cycle always occurs between the two consecutive solutions of the RMP and that the values of LB related to solutions (x, y) of the RMP have a typical Benders decomposition behavior, i.e., as the iterations are performed, they are equal to or greater than the previous value.

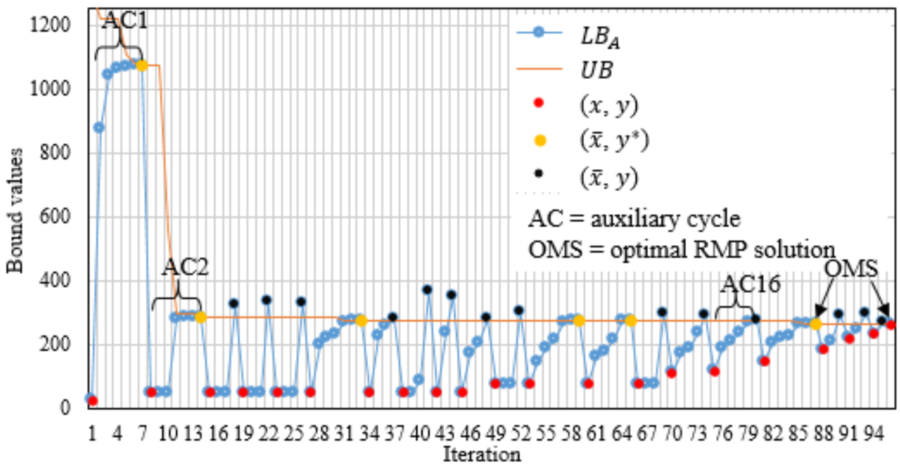

Figure 4-Behavior of bounds obtained using the pure single-cut L-Shaped form

6.2.2. Convergence of the method

Essentially, by inserting an additional cycle that considers a simplified relaxed master problem (the ARMP) the proposed acceleration technique modifies the natural sequence in which the RMP and the DSP are solved in the classic Benders decomposition. Nevertheless, one can show that convergence is not hindered by using the proposed acceleration technique. We summarize the analysis of the algorithm convergence in the three following propositions, which are valid for the pure forms, the hybrid form and the modified hybrid form of the technique, as presented in the next subsections. Detailed proofs for these are provided in Appendix C.

Proposition 4. *All auxiliary cycles of the proposed technique will always converge to an optimal solution* $(\bar{x}, y^*)$ *or solution* $(\bar{x}, y)$ *of the ARMP.*

Proposition 5. *The proposed acceleration technique does not compromise convergence properties of the classical Benders decomposition.*

Proposition 6. *The solution obtained by the proposed technique and the classical Benders method will be the same when the same tolerance value* ϵ *is used. Additionally, the optimal solution of the problem will always be the last optimal solution* $(\bar{x}, y^*)$ *of the ARMP obtained in the auxiliary cycle.*

It is trivial to verify that multiple first-stage solutions do not prevent convergence of the proposed technique.

6.2.3. Hybrid form of the proposed acceleration technique

At its hybrid form, the ARMP has a multi-cut L-Shaped structure and the RMP has the single-cut L-Shaped structure for the optimality cuts and the multi-cut L-Shaped structure for the feasibility cuts, i.e., they are problems with different structures. Thus, the benefit is that the auxiliary cycles require fewer iterations for convergence because they are in their multi-cut L-Shaped form and because the ARMP is a simplified linear problem with only continuous variables. It is worth highlighting that many possible alternative configurations are possible in this context. We opted for this particular one based on preliminary experiments with all possible combinations.

The RMP will receive the same cuts added to the ARMP during the local cycles but single cuts in the case of optimality cuts and in the form of multiple cuts (one cut for each scenario) in the case of feasibility cuts, thus guaranteeing that the ARMP and the RMP have the same feasibility region.

Nevertheless, when the cuts are combined to be inserted in the single-cut L-Shaped form in the RMP, information from the solution of the DSP could be lost in comparison to the cuts in the multi-cut L-Shaped form, which could lead the RMP to revisit previously found integer solutions during the solution process. To avoid this undesired effect, it is necessary to include "locally optimal cuts" (LOC) in the RMP at the end of each auxiliary cycle.

Considering the current auxiliary cycle and that $C^b_{n,j}$ and TM^b are parameters where $b = 0$ (initial value) and immediately after the end of each auxiliary cycle $b = b + 1$, the result is

$$C^b_{n,j} = \bar{x}_{n,j}$$
$$TM^b = \left(\sum\nolimits_{\xi} Pr(\xi) m(\xi)\right), \tag{6.19}$$

where $\bar{x}_{n,j}$ is the value of $x_{n,j}$ (integer variables of the RMP that are fixed in the ARMP) at the end of the auxiliary cycle and $m(\xi)$ takes on the value obtained at the end of the current auxiliary cycle. Hence, the following LOC is added in the RMP:

$$\left(TM^b\right) \frac{\left[\left(\sum_{i,n} C^b_{n,j} x_{n,j}\right) - \left(\sum_{i,n} C^b_{n,j}\right) + 1\right]}{N} \leq m \qquad \Big| \sum_{i,n} \bar{x}_{n,j} > 0 \tag{6.20}$$

$$\left(TM^b\right)\left(1 - \sum\nolimits_{i,n} x_{n,j}\right) \leq m \qquad \Big| \sum_{i,n} \bar{x}_{n,j} = 0. \tag{6.21}$$

According to definition (6.20), for any different values of the integer variables in relation to the value obtained at the end of the current auxiliary cycle, the left side of the equation is null or negative. When the value is equal to that obtained at the end of the current auxiliary cycle, m is at least equal to the value of TM^b (Z_{ARMP} at the end of the current auxiliary cycle). A similar analysis can be performed for (6.21). It should be emphasized that (6.20) and (6.21) cover different types of problems with binary variables; however, they are not general equations, and it may be necessary to develop a LOC with a different format but following a similar logic in relation to (6.20) and (6.21).

The LOC developed for the hybrid version of the proposed technique and specific for the proposed model can be verified in (6.22).

$$m \geq \sum_{r,k} LO^{r,k,b} u^{r,k}, \tag{6.22}$$

where the parameter $LO^{r,k,b}$ is given by

$$LO^{r,k,b} = \left[\sum_{\xi} (Pr(\xi)\, m(\xi)) \right] u^{r,k} \qquad \forall k, \forall r, \tag{6.23}$$

with $m(\xi)$ and $u^{r,k}$ in (6.23) assuming the values obtained in the solution of the ARMP at the end of each auxiliary cycle. In addition, the single-cut and multi-cut hybrid formulation (SMHF) can be enhanced, referred to as E-SMHF, when (6.1), (6.14) and (6.15) are added to the RMP, and (6.1), (6.3) and (6.15) are added to the ARMP.

The flowchart of the hybrid version of the technique is represented in Figure 5. As previously mentioned, in the hybrid version, the ARMP and RMP are problems with different solution structures. Therefore, it is recommended that the definition of the upper bound, in this case, is performed as described in Figure 5 to avoid convergence problems. In addition, the RMP and ARMP should receive feasibility cuts in the multi-cut form. This stops infeasible solutions for the ARMP from being obtained in the RMP and its respective Benders cuts from being inserted in the ARMP.

6.2.4. Modified hybrid form of the proposed acceleration technique

In the RMP, the variable s does not appear in any of the constraints nor is part of the objective function. Hence, the variable s can be removed from the RMP, causing the integer variables to be the only decision variables; thus, only the LOC is used. Therefore, this modified single-cut and multi-cut hybrid formulation (MSMHF) entails solving a simpler integer problem as the RMP, which could represent a reduction in the solution time.

In its enhanced version, E-MSMHF, the RMP is formed by (5.30)-(5.32), (5.36)-(5.38), (6.14), (6.15), in addition to the valid inequalities (6.1), which are used after each solution of the RMP to guarantee that only optimality cuts will be generated in all iterations. Basically, this change consists of starting the search by the auxiliary optimal solution, always with the maximum value between the minimum permitted as a function of $u^{r,k}$ in (6.1) and a certain value selected (in this case, the average value of its bounds was selected). Additionally, this procedure can decrease the number of iterations required for the convergence, considering that the values obtained for the variable s, solving the RMP, are distant from the value obtained in the optimal solution (usually extreme values). The pseudocode of E-MSMHF is presented in Appendix D.

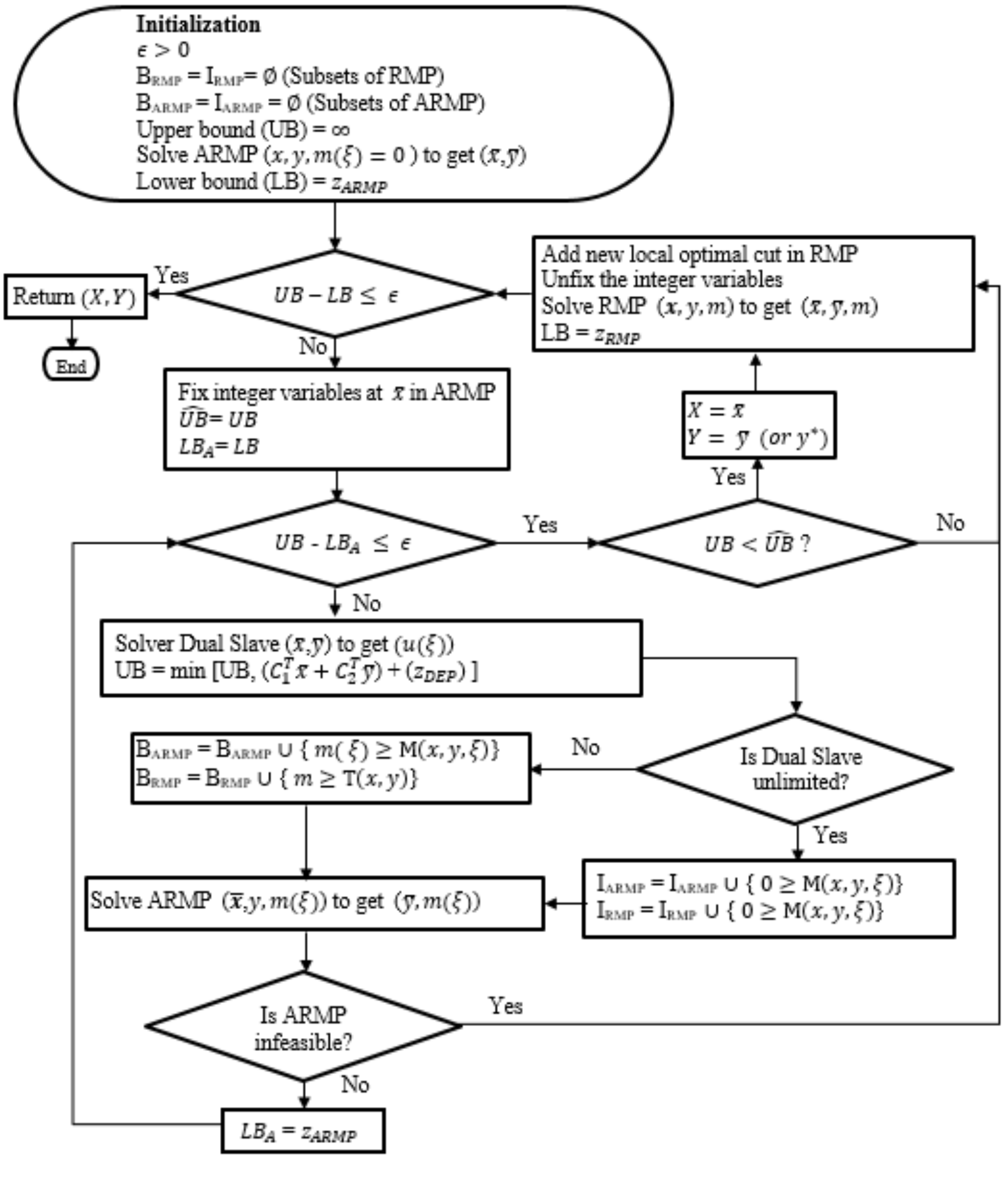

Figure 5- Flowchart of the proposed acceleration technique in its hybrid version

7. Computational Experiments

All models proposed in this study were implemented using the AIMMS 3.14 software package and solved using CPLEX 12.6. All experiments were performed on an Intel i7-4510 2.0 GHz processor with 8G RAM. Table 1 contains the instance classes, which are defined by distinct set sizes their respective sizes in terms of number of variables and constraints. The scenarios for the demand were randomly generated following a normal distribution with the mean of 50 and variance of 75. For each of these classes, we generated a total of ten instances. The parameters $B^p, BA^p, CF^p, H^p, \overline{S},\ \beta$ and II

are set to 25, 12, 25, 0.2, 500, 0.7, and 265, respectively. The lead time is equivalent to two time periods. We used as stop criteria a solution time limit (TL) of one hour (3600 seconds) and a tolerance (ϵ) of 10^{-5}. The CPU time, number of iterations, and relative optimality gap (given by $\frac{UB-LB}{UB}$) in Tables 2, 3, 5, 6, and 8 are presented, with their mean (A), largest (L) and smallest (S) values.

Table 2 presents the results obtained in terms of CPU time, number of iterations, and relative optimality gap value using the SF and MF, and solving the deterministic equivalent problem ("Full Problem") directly using CPLEX.

Table 1- Classes of the instances used in the present study and their respective impact on problem size

Class	Size of sets				Number of variables			Number of constraints
	$\|\Omega\|$	$\|P\|$	$\|R\|$	$\|K\|$	Binary	Continuous ($\beta = 0$)	Continuous ($\beta > 0$)	
C1	50	36	10	10	136	9.001	10.801	12.638
C2	50	72	10	10	172	18.001	21.601	25.274
C3	100	72	10	10	172	36.001	43.201	50.474
C4	100	72	20	10	272	36.001	43.201	50.474
C5	250	72	20	10	272	90.001	108.001	126.074
C6	250	72	20	20	472	90.001	108.001	126.074
C7	500	72	20	20	472	180.001	216.001	252.074
C8	500	90	20	20	490	225.001	270.001	315.092
C9	250	180	20	20	580	225.001	270.001	315.182
C10	500	180	20	20	580	450.001	540.001	630.182
C11	250	365	20	20	765	456.251	547.501	639.117
C12	500	365	20	20	765	912.501	1.095.001	1.277.867

According to Table 2, the total CPU time for all formulations increases with the increase in the number of scenarios, periods, periodicities, and periods available for placing the first order, as expected. Additionally, solving the full problem achieves smaller mean CPU times for C1 only. However, from C2, SF consistently provided the smallest mean CPU time in relation to MF and Full Problem, and was always capable to return the optimal solution within the time limit. From C7, only SF was able to obtain the optimal solution within the limit of one hour. It is important to highlight that the number of iterations of MF is consistently smaller than that of SF. Nonetheless, the difference between the mean solution times of SF and MF increase as the instances get larger.

Table 2- Computational results with the application of the CPLEX and L-Shaped methods

Class	Full Problem			SF									MF								
	CPU time [s]			CPU time [s]			Iteration			Gap (%)			CPU time [s]			Iteration			Gap (%)		
	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S
C1	20	23	17	58	64	46	191	196	184	-	-	-	203	246	181	139	151	131	-	-	-
C2	96	116	83	95	102	88	196	212	183	-	-	-	289	527	211	147	172	136	-	-	-
C3	344	399	296	177	197	157	197	212	188	-	-	-	614	1094	439	144	169	129	-	-	-
C4	420	606	342	232	245	224	244	283	231	-	-	-	1097	2200	770	162	189	153	-	-	-
C5	2862	3600	2176	561	614	511	250	276	240	-	-	-	2725	3600	2182	153	160	143	0.1	0.6	-
C6	2734	3588	1908	580	665	538	255	336	223	-	-	-	3563	3600	3200	147	157	123	0.2	0.8	-
C7	3600	3600	3600	1171	1562	1010	257	343	224	-	-	-	3600	3600	3600	109	120	90	0.6	0.9	0.5
C8	3600	3600	3600	1383	1603	1254	260	353	226	-	-	-	3600	3600	3600	105	118	64	0.6	0.9	0.5

Table 3 shows the effects of using the valid inequalities in the improved on the solution time and number of iterations of the L-Shaped formulations. Additionally, the results obtained with SF, which represents the formulation with the best computational performance in Table 2, are complemented with classes from C9 to C12.

As can be seen in Table 3, there is substantial improvement in terms of the number of iterations and CPU time for E-SF and E-MF formulations when compared with SF. Considering the largest CPU time for E-SF and E-MF, only in C12 E-MF was not able to obtain the optimal solution within the time limit. However, its mean solution time in this instance is 1074s, which is significantly less than one hour. In addition, one can note that in C12, SF does not obtain the optimal solution within the time limit in any of the tests performed. It is important to highlight that with the use of the valid inequalities in the E-SF an E-MF formulations, the increase of |R| from 10 to 20 (C3 to C4) and the increase of |K| from 10 to 20 (C5 to C6) do not have a significant impact on the solution time, as in SF. Additionally, in E-SF and E-MF, the values of the mean, largest and smallest iteration numbers remained the same.

Table 3- Effects of the use of valid inequalities in the L-Shaped formulations

Class	SF									E-SF						E-MF					
	CPU time [s]			Iteration			Gap (%)			CPU time [s]			Iteration			CPU time [s]			Iteration		
	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S
C1	58	64	46	191	196	184	-	-	-	12	14	9	41	44	37	15	18	12	29	31	24
C2	95	102	88	196	212	183	-	-	-	21	32	18	42	46	40	24	36	20	29	33	28
C3	177	197	157	197	212	188	-	-	-	39	43	33	43	50	41	51	88	38	30	35	27
C4	232	245	224	244	283	231	-	-	-	40	44	34	43	50	41	54	92	40	30	35	27
C5	561	614	511	250	276	240	-	-	-	99	108	89	45	52	42	139	209	109	29	35	26
C6	580	665	538	255	336	223	-	-	-	106	113	95	45	52	42	151	226	115	29	35	26
C7	1171	1562	1010	257	343	224	-	-	-	215	228	197	46	53	43	333	561	252	30	34	27
C8	1383	1603	1254	260	353	226	-	-	-	251	269	213	45	54	38	374	672	286	29	36	27
C9	1271	1790	1098	259	381	225	-	-	-	232	292	214	47	57	43	323	856	209	31	40	28
C10	2518	3349	2173	264	382	225	-	-	-	469	584	417	48	64	44	707	2140	403	32	44	27
C11	2546	3600	2200	263	363	234	0.0	0.0	-	450	727	397	48	76	42	648	3153	319	32	56	27
C12	3600	3600	3600	193	236	177	0.6	0.9	0.2	881	1333	773	48	72	43	1073	3600	683	30	40	27

The percentage decrease in the number of iterations and the speed-up factor when E-MF are compared with SF and MF and when E-SF is compared with SF and E-MF are presented in Table 4. The obtained values were calculated using the average values of Tables 2 and 3. When compared with the SF and MF, the speed-up factor of the E-MF formulation is defined as the ratio of the solution times of SF to E-MF and of MF to E-MF, respectively. Similarly, the speed-up factors of E-SF are obtained when its solution times are compared with those of SF and E-MF.

According to Table 4, the performance of E-MF and E-SF is better in relation to SF, with speed-up factor values between 3.5 and 4.3 and between 4.5 and 5.8, respectively. The use of the valid inequalities in E-MF and E-SF promotes significant reductions in the mean number of iterations. When E-MF is compared with SF and MF, the percentage of mean decrease ranges from 85% to 89% and from 79% to 81%, respectively.

Table 4 - Comparison of E-MF with SF and MF and of E-SF with SF and E-MF

Class	E-MF				E-SF			
	Speed-up factor		Iteration reduction (%)		Speed-up factor		Iteration reduction (%)	
	SF	MF	SF	MF	SF	E-MF	SF	E-MF
C1	3.9	13.5	85	79	4.8	1.3	79	-41
C2	4.0	12.0	85	80	4.5	1.1	79	-45
C3	3.5	12.0	85	79	4.5	1.3	78	-43
C4	4.3	20.3	88	81	5.8	1.4	82	-43
C5	4.0	19.6	88	81	5.7	1.4	82	-55
C6	3.8	23.6	89	80	5.5	1.4	82	-55
C7	3.5	-	88	-	5.4	1.5	82	-53
C8	3.7	-	89	-	5.5	1.5	83	-55
C9	3.9	-	88	-	5.5	1.4	82	-52
C10	3.6	-	88	-	5.4	1.5	82	-50
C11	3.9	-	88	-	5.7	1.4	82	-50
C12	-	-	-	-	-	1.2	-	-60

Furthermore, the efficiency of the multi-cut strategy is notably improved with the use of the valid inequalities, where E-MF presents mean speed-up factors between 12.0 and 23.6 when compared with MF. However, although the percentage of mean decrease of E-SF ranges from -60% to -41%, when compared with E-MF (i.e., when E-SF needs on average 41% to 60% more iterations in relation to E-MF), the performance of the single-cut approach is still better than the multi-cut, where E-SF presents mean speed-up factors of 1.1 to 1.5 when compared with E-MF.

Table 5 presents the values of the total time spent on the solution of the RMP and its percentages in relation to the total solution time (M/T), for SF, MF, E-MF, and E-SF. The results in Table 5 indicate that iteratively adding multiple Benders cuts to the RMP of MF and E-MF makes them significantly harder to solve when compared with SF and E-SF. This difference is the key factor underlying the improved performance of the single-cut method in comparison to the multi-cut method. The mean percentage of time spent on the solutions of the RMP ranged from 79% to 91% and from 50% to 62% in MF and E-MF, respectively. In turn, in the SF and E-SF formulations, these ranges are only from 3% to 16% and from 5% to 17%, respectively.

The use of valid inequalities has direct impact on the reduction of the number of iterations, which in turn implicates that smaller master problems (i.e. with less cuts) have to be solved. According to Table 5, this effect is determinant in the decrease of the total time spent on solving the RMP in the E-MF formulation. However, for E-SF, the differences in the mean solution time of the RMP in relation to the SF are significantly smaller. Therefore, one could infer that the reduction in the total solution time of E-SF is mainly due to the reduction in the number of times that the master problem is solved.

Table 6 displays the effects on the CPU time and on the number of iterations of the formulations implemented with the proposed acceleration techniques combined with valid inequalities. According to the maximum solution times of all formulations presented in Table 6, the optimal solution was obtained within the established time limit in all tests performed.

Table 5 - Total solution times of the RMP for the SF, MF, E-MF and E-SF

Class	SF - Master Problem						MF - Master Problem						E-SF - Master Problem						E-MF - Master Problem					
	CPU time [s]			M/T (%)			CPU time [s]			M/T (%)			CPU time [s]			M/T (%)			CPU time [s]			M/T (%)		
	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S
C1	9	10	8	16	18	15	167	203	146	82	85	80	2	2	2	17	20	16	8	10	6	53	59	48
C2	12	16	10	13	17	11	227	464	152	79	88	71	3	4	2	14	17	11	12	25	8	50	69	40
C3	12	16	10	7	10	6	492	976	317	80	89	72	4	5	3	10	12	8	27	65	16	53	73	41
C4	29	41	25	13	17	10	960	2071	634	88	94	81	4	6	4	10	14	10	29	68	17	54	74	44
C5	31	41	25	6	8	5	2390	3411	1820	88	93	83	8	9	7	8	9	7	77	147	48	55	71	44
C6	50	95	35	9	14	6	3238	-	2849	91	-	89	14	16	13	13	15	12	90	166	56	60	73	48
C7	56	116	37	5	8	3	-	-	-	-	-	-	25	30	24	12	14	11	208	430	134	62	77	52
C8	61	128	40	4	8	3	-	-	-	-	-	-	26	34	21	10	13	9	226	529	137	60	79	48
C9	83	237	49	7	13	4	-	-	-	-	-	-	18	27	15	8	10	7	181	69	74	56	81	35
C10	87	246	51	3	7	2	-	-	-	-	-	-	33	52	28	7	9	6	414	1777	158	59	83	39
C11	150	565	80	6	16	3	-	-	-	-	-	-	28	69	21	6	10	5	366	2672	79	56	85	24
C12	-	-	-	-	-	-	-	-	-	-	-	-	45	97	37	5	7	4	539	3036	174	50	81	26

Furthermore, once again the increase of the |R| from 10 to 20 (C3 to C4) and the |K| from 10 to 20 (C5 to C6) did not have significant impact on the solution time of the improved formulations, with very similar corresponding mean values. Considering the mean CPU times, E-PSF and E-PMF can be considered to have similar computational performance in the tested cases, with E-PMF performing slightly better in most cases. Considering the mean, largest and smallest CPU time values, the performance of E-SMHF is clearly better than E-PSF and E-PMF. Similarly, the computational performance of E-MSMHF is superior to that of E-SMHF.

Table 6 - Effects of the use of the proposed acceleration techniques combined with valid inequalities

Class	E-PSF						E-PMF						E-SMHF						E-MSMHF					
	CPU time [s]			Iteration			CPU time [s]			Iteration			CPU time [s]			Iteration			CPU time [s]			Iteration		
	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S
C1	11	13	8	41	44	37	10	12	8	29	31	24	8	9	6	29	31	24	7	8	5	24	27	22
C2	18	20	15	42	46	40	16	19	15	29	33	28	13	15	11	30	34	28	11	13	10	25	30	24
C3	37	41	31	43	50	41	34	41	28	30	35	27	26	31	24	30	36	27	21	25	20	25	32	23
C4	38	44	31	43	50	41	34	42	29	30	35	27	27	29	24	30	36	27	22	25	20	25	32	23
C5	97	110	85	45	52	42	91	121	76	29	35	26	66	72	59	30	36	26	57	63	50	26	31	23
C6	97	107	86	45	52	42	94	130	82	29	35	26	68	73	63	30	36	26	57	64	51	26	31	23
C7	200	216	178	46	53	43	197	293	163	30	34	27	140	154	127	30	35	27	115	126	109	26	30	23
C8	236	256	202	45	54	38	237	385	193	29	36	27	163	179	152	30	37	27	139	154	124	26	33	24
C9	219	273	201	47	56	43	216	481	154	31	39	27	151	187	138	32	40	28	127	165	106	27	38	22
C10	460	540	407	48	64	44	449	972	310	32	43	27	319	445	257	33	46	27	267	375	223	29	42	24
C11	432	728	375	48	82	42	417	1457	267	32	57	27	302	591	242	32	59	27	258	511	204	29	55	23
C12	851	1214	741	48	69	43	881	3265	554	32	57	27	600	1187	500	32	59	28	515	1031	439	29	54	24

In Table 7, we present the percentage decrease in the number of iterations and the speed-up factor when the E-PSF is compared with SF and E-SF, when E-PMF is compared with SF and E-MF; when E-SMHF is compared with SF, E-PSF and E-PMF; and when E-MSMHF is compared with SF and E-SMHF. The obtained values were calculated using the average values of Table 6.

The values of the speed-up factors observed in Table 7 corroborate with the analyses of the results in Table 6. Additionally, it can be seen that E-MSMHF is the formulation with the best computational performance among those tested, reaching speed-up factors from 8.3 to 10.5 in relation to the SF and from 1.1 to 1.2 in relation to E-SMHF. The second best computational performance is the E-SMHF, which reaches speed-up factors values from 6.8 to 8.6, from 1.4 to 1.5, and from 1.2 to 1.5 in relation to SF, E-FPS and PMF, respectively. It is also possible to notice that E-PSF and E-PMF overperform E-SF and E-MF, respectively.

Table 8 presents the values of the total time spent on the solution of the RMP and ARMP, and their percentages in relation to the total solution time (M/T), for the E-PSF, E-PMF, E-SMHF and E-MSMHF. According to Table 8, there is a significant reduction in the mean values of the total time spent to solve the master problems when compared to the values in Table 5, specifically in the E-PMF and E-SMHF. In Table 5, the mean percentage in relation to the total CPU time of E-MF ranges from 50% to 62%, whereas Table 8 presents the mean variations from 25% to 38% and from 6% to 13% in E-PMF and E-SMHF, respectively.

Table 7- Comparison of computational performance with formulations with proposed acceleration techniques

Class	E-PSF				E-PMF				E-SMHF						E-MSMHF			
	Speed-up factor		Iteration reduction (%)		Speed-up factor		Iteration reduction (%)		Acceleration Factor			Iteration reduction (%)			Speed-up factor		Iteration reduction (%)	
	SF	E-SF	SF	E-SF	SF	E-MF	SF	E-MF	SF	E-PSF	E-PMF	SF	E-PSF	E-PM	SF	E-SMH	SF	E-SMHF
C1	5.3	1.1	79	0	5.8	1.5	85	0	7.3	1.4	1.3	85	29	0	8.3	1.1	87	17
C2	5.3	1.2	79	0	5.9	1.5	85	0	7.3	1.4	1.2	85	29	-3	8.6	1.2	87	17
C3	4.8	1.1	78	0	5.2	1.5	85	0	6.8	1.4	1.3	85	30	0	8.4	1.2	87	17
C4	6.1	1.1	82	0	6.8	1.6	88	0	8.6	1.4	1.3	88	30	0	10.5	1.2	90	17
C5	5.8	1.0	82	0	6.2	1.5	88	0	8.5	1.5	1.4	88	33	-3	9.8	1.2	90	13
C6	6.0	1.1	82	0	6.2	1.6	89	0	8.5	1.4	1.4	88	33	-3	10.2	1.2	90	13
C7	5.9	1.1	82	0	5.9	1.7	88	0	8.4	1.4	1.4	88	35	0	10.2	1.2	90	13
C8	5.9	1.1	83	0	5.8	1.6	89	0	8.5	1.4	1.5	88	33	-3	9.9	1.2	90	13
C9	5.8	1.1	82	0	5.9	1.5	88	0	8.4	1.5	1.4	88	32	-3	10.0	1.2	90	16
C10	5.5	1.0	82	0	5.6	1.6	88	0	7.9	1.4	1.4	88	31	-3	9.4	1.2	89	12
C11	5.9	1.0	82	0	6.1	1.6	88	0	8.4	1.4	1.4	88	33	0	9.9	1.2	89	9
C12	-	1.0	-	0	-	1.2	-	-7	-	1.4	1.5	-	33	0	-	1.2	-	9

According to the percentage decrease in the number of iterations in Table 7, the E-PSF has the same mean number of iterations of E-SF, and E-PMF has practically the same mean number of iterations of E-MF, except that in C12, where E-PMF requires 7% more iterations. The E-SMHF presents similar number of iterations in relation to the E-PMF. Hence, the reductions obtained in the solution times of Table 6 are mainly due to the reductions in the total time spent on solving the master problems, which is precisely the expected effect of the proposed acceleration techniques. Furthermore, as expected, this effect is smaller in E-PSF in relation to E-SF, and significantly larger in E-PMF and E-SMHF in relation to E-MF.

Finally, according to the decrease in the number of iterations and the best computational performance of E-MSMHF in relation to SMHF, the strategy with the removal of variable s from the RMP in the hybrid formulation was successful, especially in the larger classes, in which significantly faster solutions can be obtained, considering the mean computational time.

Table 8 – Sum of the total solution times of the RMP and ARMP for the E-PSF, E-PMF, E-SMHF and E-MSMHF

Class	E-PSF– Masters						E-PMF – Masters						E- SMHF – Masters						E-MSMHF - Masters					
	CPU time [s]			M/T (%)			CPU time [s]			M/T (%)			CPU time [s]			M/T (%)			CPU time [s]			M/T (%)		
	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S	A	L	S
C1	1	1	1	9	9	6	3	4	2	30	33	26	1	1	1	13	13	10	1	1	1	14	14	10
C2	1	1	1	6	7	4	4	9	3	25	47	18	1	1	1	8	11	7	1	1	1	9	9	6
C3	1	2	1	3	5	3	9	18	6	26	43	19	2	3	1	8	11	6	1	2	1	5	8	5
C4	2	2	1	5	6	3	9	19	6	26	44	21	2	3	2	7	12	7	1	2	1	5	9	5
C5	3	4	2	3	4	2	29	62	19	32	51	24	5	8	4	8	12	6	3	5	2	5	9	4
C6	4	6	4	4	6	4	34	69	22	36	53	27	7	11	6	10	15	9	4	6	3	7	10	6
C7	8	11	8	4	6	3	73	166	49	37	57	28	15	23	13	11	15	9	8	12	7	7	11	6
C8	9	13	8	4	5	3	90	235	51	38	61	25	16	27	13	10	15	8	9	17	7	6	11	5
C9	6	11	5	3	5	2	77	350	25	36	73	16	10	24	7	7	13	5	6	16	3	5	9	3
C10	11	21	9	2	4	2	160	695	59	36	71	17	22	65	13	7	15	5	13	41	7	5	11	3
C11	9	29	6	2	4	1	134	968	29	32	66	10	17	83	8	6	14	3	10	51	4	4	10	2
C12	14	41	10	2	3	1	323	2310	66	37	71	11	34	183	15	6	15	3	21	120	8	4	12	2

8. Conclusions

In this paper, we propose an inventory management model with periodic review policy *(R, S)* and partial backorder with one layer, one item and uncertain demand, formulated via two-stage stochastic programming. The proposed model extends that originally proposed by Cunha et al. (2017) as the proposed formulation allows for the consideration of initial inventory in a more natural manner and considers the possibility of partial backorder, with the possibility of being simply converted into pure lost sales or pure backorder cases, being thus more general. Additionally, a more efficient exact linearization was applied to the model, which provides a formulation with less constraints and variables in comparison to the model developed by Cunha et al. (2017).

According to the numerical results obtained, although the SF always requires more iterations than the MF to converge and obtain the optimal solutions, the performance of the SF was notably better than MF and the full problem formulation. It should be highlighted that the MF was computationally inferior to the full problem formulation.

To allow the efficient solution of larger problems, valid inequalities were developed along with a new acceleration technique containing three variants: pure single-cut L-Shaped version, pure multi-cut L-Shaped version and a hybrid version that uses both the multi-cut L-Shaped and single-cut L-Shaped structures. Additionally, it was possible to develop a specific hybrid formulation modified for the type of model proposed in the present study due to the particular characteristic of its master problem.

Among the forms of the proposed acceleration technique, the modified hybrid version combined with the set of valid inequalities, E-MSMHF stood out computationally, resulting in the best computational performance among the performed tests. On average, E-MSMHF obtained solutions from 8.3 to 10.5 times faster than SF.

The results obtained support the conclusion that the more realistic problems – considering, for instance, more items, demand simulations with more complex stochastic processes and requiring a large number of scenarios and long planning horizons for the control system (*R, S*) – can be efficiently addressed by the proposed approach. In addition, the proposed acceleration technique is sufficiently general to be applied to other problems in the literature that share a similar structure, including applications based on the classical Benders method (i.e., deterministic problems).

As future studies, application of the acceleration technique proposed in the present study to other models, including MILP models without continuous variables in the master problem, is suggested to verify whether the computational performance is improved and, if possible, to combine it with other existing techniques, aiming at maximizing the computational performance of the models that would be tested. Additionally, we propose the application of the improved inventory management model in real situations, comparing the results obtained with those of models already existing in the literature, aiming to observe their advantages, disadvantages and limitations.

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Appendix A – Model notation

Sets

P	–	Time periods;
Ω	–	Scenarios;
R	–	Review periods, where $\tau \subset P$;
K	–	Periods for the first order, where $K \subset P$;
$\lvert\cdot\rvert$	–	Size of the set $\cdot$;

Indices

$p \in P$	–	Time period;
$\xi \in \Omega$	–	Scenario;
$r \in \mathrm{R}$	–	Review period;
$k \in K$	–	Period for the first order;

Parameters

B^p	–	Cost of unmet demand per unit of item in period p;
BA^p	–	Backorder cost per unit of item in period p;
CF^p	–	Fixed ordering cost in period p;
H^p	–	Inventory cost per unit of item in period p;
$D(\xi)^p$	–	Demand for item in scenario ξ and period p;
β	–	Backorder rate; $\beta \in [0,1]$.
II	–	Initial inventory;
$\overline{S}$	–	Upper bound for stock level of the item;
$Pr(\xi)$	–	Probability of scenario ξ;
TE	–	Lead time;
$W^{p,r,k}$	–	Auxiliary parameter that indicates the period of the order depending on the value of k e r; $W^{p,r,k} \in \{0,1\}$; $r = 1, \dots, \lvert R\rvert$; $p = 1, \dots, \lvert P\rvert$; $k = 1, \dots, \lvert K\rvert$;

where $W^{p,r,k}$ is represented by the following matrix**:**

		k														
		1					2					3 …				
		r					*r*					*r* …				
		1	2	3	4	…	1	2	3	4	…	1	2	3	4	…
	1	1	1	1	1	…	0	0	0	0	…	0	0	0	0	…
	2	1	0	0	0	…	1	1	1	1	…	0	0	0	0	…
	3	1	1	0	0	…	1	0	0	0	…	1	1	1	1	…
p	4	1	0	1	0	…	1	1	0	0	…	1	0	0	0	…
	5	1	1	0	1	…	1	0	1	0	…	1	1	0	0	…
	6	1	0	0	0	…	1	1	0	1	…	1	0	1	0	…
	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	

Variables

$a(\xi)^p$	–	Quantity of demand met in scenario ξ and period p;
$f(\xi)^p$	–	Quantity of unmet demand in scenario ξ and period p;
$i(\xi)^p$	–	Final inventory in stock at the end of each period p, in each scenario ξ;
$it(\xi)^p$	–	Total inventory position (in stock plus backorders) at the end of each period, in each scenario ξ;
$l(\xi)^p$	–	Backorder quantity in scenario ξ and period p;
$q(\xi)^p$	–	Ordered quantity of item at the start of each period, in each scenario ξ.
s	–	Target inventory level for the item throughout the time horizon;
v^p	–	Indicates the existence or absence of order for item in period p; $v^p \in \{0,1\}$.
$u^{r,k}$	–	Auxiliary variable for determination of cycle size R and period k in which the first order is placed; $u^{r,k} \in \{0,1\}$.

Appendix B – Linearization of variable product

The exact linearization of $y = x_1 x_2$, when $0 \leq x_1 \leq M$ and x_2 is a binary variable is given by the following set of constraints

$$y \leq M x_2 \tag{B.1}$$

$$y \leq x_1 \tag{B.2}$$

$$y \geq x_1 + M(x_2 - 1) \tag{B.3}$$

$$y \geq 0. \tag{B.4}$$

Cunha et al. (2017) use this artifice to linearize individualy the product of variables in constraint (B.5), which requires the inclusion of the auxiliary variables $iti(\xi)^p$ $itiv(\xi)^p$ and sv^p, the parameter Big - M, ITI, and a set of $5 \times |P| \times |\Omega| + 3 \times |P|$ constraints in place of (9.5).

$$q(\xi)^p = (s - it(\xi)^{p-1})v^p \qquad \forall p \tag{B.5}$$

However, considering $-M \leq x_1 \leq M$ and x_2 a binary variable, the exact linearization of $y = x_1 x_2$ is given by

$$y \leq M x_2 \tag{B.6}$$

$$y \leq x_1 + M(1 - x_2) \tag{B.7}$$

$$y \geq x_1 + M(x_2 - 1) \tag{B.8}$$

$$y \geq -M x_2. \tag{B.9}$$

The validity of the constraints (B.6) - (B.9) can be verified in Table B.1 where all possible combinations of values for x_1 and x_2 are considered. Therefore, setting $x_1 = (s - it(\xi)^{p-1} - II)$ when $p = 1$ and $x_1 = (s - it(\xi)^{p-1})$ when $p \geq 2$, in both cases, implies $-\overline{S} \leq x_1 \leq \overline{S}$. Hence, for $y = q(\xi)^p$ and $x_2 = v^p$, applying the linearization (B.6)-(B.8), in (4.21) and (4.22), being (B.9) disconsidered because $q(\xi)^p \geq 0$, we obtain the set of constraints (4.15)-(4.20), which does not require auxiliary variables, and consists of $3 \times |P| \times |\Omega|$ constraints and include the parameter referring to the initial inventory, without loss of quality in the linearization.

Table B.1- All possibles products of $y = x_1 x_2$ for $-M \leq x_1 \leq M$ and $x_2 \in \{0,1\}$

x_1	x_2	$x_1 x_2$	Constraints imply	
$-M \leq x_1 \leq M$	0	0	$y \leq 0$ $y \leq x_1 + M$ $y \geq x_1 - M$ $y \geq 0$	$y = 0$
$-M \leq x_1 \leq M$	1	x_1	$y \leq M$ $y \leq x_1$ $y \geq x_1$ $y \geq -M$	$y = x_1$

Appendix C – Proofs

Proposition 1.

Proof. Parameter $DK(\xi)^k$ denotes the sum of the demands since the first period of the planning horizon to the period prior to all possible periods for first order ($k \in K$), in each scenario. Hence, the expression $\sum_k \big(DK(\xi)^k \sum_r u^{r,k}\big)$ represents the sum of the demands since the first period of the planning horizon to the period prior to the first period determined in the solution of the RMP, in each scenario. Therefore, $II - \sum_k \big(DK(\xi)^k \sum_r u^{r,k}\big)$ indicates the inventory in stock, which originates from the initial inventory in the period of the first order, in each scenario. Thus, (6.1) guarantees that variable s will not assume values smaller than $II - \sum_k \big(DK(\xi)^k \sum_r u^{r,k}\big)$, which guarantees nonnegative values for variables $q(\xi)^p$, due to constraints (4.21) and (4.22). Hence, when (6.1) is included in the RMP (5.30)-(5.38) and (5.39)-(5.43) (the SF and MF, respectively), they become of relatively complete resources (meaning that for any feasible first-stage solution, there is at least one feasible second-stage solution), and thus only optimality cuts are inserted in the RMP during the solution process. Similarly, the insertion of (6.1) in (4.1)-(4.7) also makes the proposed formulation of relatively complete resources. ■

Proposition 2.

Proof. Consider the division of the planning horizon into two time intervals. The interval $1 \leq p \leq TE + k - 1$, composed by the first period and all other periods until the eve of the first delivery. This is represented in parameter $WK1^{p,k}$ in (6.8). The interval $TE + k \leq p \leq |P|$, when orders can be received, is represented in the parameter $WK2^{p,k}$ in (6.9). In both cases, the parameter takes value 1 when within their specified range, and 0 otherwise.

Parameter $CII(\xi)^p$ in (6.4) and (6.5) denotes the inventory quantity in stock, from the initial inventory, that will not be used in the period p in each scenario. Parameter $CFII(\xi)^p$ denotes the quantity of the demand unmet by the inventory in stock, which originates from the initial inventory, for period p in each scenario. Parameter $MCK3$ in (6.13) represents the minimum cost value, between the loss and the partial backorder of the unmet demand. Therefore, for each value of the first-stage variable $u^{r,k}$, the minimum cost of the second-stage problem, in the interval $1 \leq p \leq TE + k - 1$ is given by $\sum_{p,r,k}\{(H^p CII(\xi)^p - MCK3^p CFII(\xi)^p) WK1^{p,k} u^{r,k}\}$, i.e., it is the minimum cost obtained by holding only the inventory that originates from the initial inventory II.

Parameter $WHK^{p,r,k}$ in (6.11) has the value of 1 in the periods without the delivery of orders throughout the planning horizon, where 0 is the value assumed in the remaining periods. Hence, the parameter $WYK^{p,r,k}$ defined in (6.10) denotes in each period p the accumulated number of periods since the last delivery, for each value of r and k. Therefore, the minimum inventory cost considering the total demand to be met and the absence of backorders, in the interval $TE + k \leq p \leq |P|$, is $\sum_{p,r,k}[WYK^{p,r,k}H^{p}D(\xi)^{p}WK2^{p,k}u^{r,k}]$, for each value of the first-stage variable $u^{r,k}$.

Parameter $MCK2^{p,r,k}$ in (6.12) determines, in each period p, the lowest cost among the inventory cost, lost sales and partial backorder for each value of r and k. Therefore, in the interval $TE + k \leq p \leq |P|$, the minimum inventory management cost in the second-stage problem is given by $\sum_{p,r,k}[MCK2^{p,r,k}D(\xi)^{p}WK2^{p,k}u^{r,k}]$ for each value of the first-stage variable $u^{r,k}$. Therefore, as both parcels are considered at their minimal value and $m(\xi)$ corresponds to a lower approximation of the original second-stage cost function, (6.3) is a valid lower bound inequality of the RMP (5.39)-(5.43) of the MF. ■

Proposition 3.

Proof. Parameter $DKS(\xi)^{p}$ defined in (6.16) assumes, in each period, the value of the demand in period p plus the sum of demands of all previous periods for each scenario. Therefore, parameter $DKM(\xi)^{p}$ in (6.17) represents the remaining inventory in stock in relation to the initial inventory II in each period and scenario. In the optimal solution, the period for the first order k should take on a value so that the first delivery occurs in a period with negative $DKM(\xi)^{p}$ for at least one scenario. In the *(R, S)* system, if the first delivery occurs in a period with positive $DKM(\xi)^{p}$ for all scenarios, it would not be part of the optimal solution because it would incur in unnecessary inventory cost. Hence, defining $\min.(\mathrm{p} \mid DKM(\xi)^{p} < 0)$ as the minimum value in period p, where the parameter $DKM(\xi)^{p}$ is negative in at least one scenario, the parameter MK in (6.18) is a lower bound on k. Therefore, the inequality (6.15) is a valid restriction that limits the selection of the period for placing the first order and can be inserted in the RMP of the SF and MF. ■

Proposition 4.

Proof. Each auxiliary cycle is fundamentally an application of Benders decomposition to a continuous linear problem cosidering two modifications: the value of the initial UB at each cycle is the smallest value obtained in the previous auxiliary cycle and the Benders cuts generated in each cycle are accumulated in both the ARMP and RMP.

The maintenance of the value of UB from the previous cycle results in two possibilities of convergence for the auxiliary cycle: by obtaining the optimal solution $(\bar{x}, y^{*})$ or the solution $(\bar{x}, y)$. If at the end of the auxiliary cycle, the current values of $\bar{x}$ produce a solution y^{*} whose value of the objective function of the ARMP (LB_{A}) is smaller than $\widehat{UB}$ (for the case of minimization without loss of generality), the auxiliary cycle converges when $0 \leq LS - LI \leq \epsilon$, and the new solution $(\bar{x}, y^{*})$ of the ARMP, which by definition is better than the previous solutions, will be obtained. In this case, at the end of the auxiliary cycle, $UB < \widehat{UB}$ and $(\bar{x}, y^{*})$ are stored in (X, Y).

Otherwise, as the solutions of the ARMP during the auxiliary cycle generate a monotonically increasing sequence of lower bounds LB_A, convergence of the auxiliary cycle will eventually occur and solution ($\bar{x}$,y) will be obtained. In this case, at the end of the auxiliary cycle, we have that $UB = \widehat{UB}$. It is important to observe that the accumulation of Benders cuts in the ARMP and RMP approximate them to the equivalent deterministic problem. However, when feasibility cuts are added to the ARMP, this may cause infeasibilities due to the setting of a new value for $\bar{x}$ at each auxiliary cycle, which changes its feasible region. Nevertheless, as observed in the flowchart shown in of Figure 3, this is circumvent by ending the auxiliary cycle when the unfeasibility of the ARMP is verified, continuing the iterative solution process. In this case, at the end of the auxiliary cycle, $UB = \widehat{UB}$ and the solution obtained by solving the ARMP will be of the type $(\bar{x}, y)$. ■

Proposition 5.

Proof. The auxiliary cycles between the solutions of the ARMP and the DSP can be regarded as auxiliary problems solved with the intention of accelerating the Benders decomposition process. Additionally, according to the logic of the technique presented in Figure 3, whenever the RMP is solved, the obtained solution is fixed in the DSP and one Benders cut is added to the RMP. Next, a number of additional cuts, from the solutions obtained in the auxiliary cycle – which always converge, according to Proposition 4 – are generated and added to the RMP. Such cuts improve the current approximation without making the RMP infeasible. Then, the RMP is solved and the primary stop condition is verified; if the condition is not satisfied, the iterative process continues. Therefore, the algorithm will converge after a finite number of iterations because in the worst-case scenario, after obtaining all the optimal solutions $(\bar{x}, y^*)$ of the ARMP – whose total quantity is a finite number smaller or equal to the number of possible x values – in the auxiliary cycle and after the insertion of the Benders cuts in the RMP, the optimal solution of the problem will be found. Hence, the proposed acceleration technique will always converge whenever the conditions for the classical Benders to converge are fulfilled. ■

Proposition 6.

Proof. Considering a minimization problem without loss of generality. At the end of an auxiliary cycle, if a better value of UB ($UB < \widehat{UB}$) is obtained, then an optimal solution $(\bar{x}, y^*)$ of the ARMP is obtained. Suppose that (x^*, y^*) is the optimal solution of the problem that would be obtained using classical Benders decomposition. When $\bar{x} = x^*$ at the end of the auxiliary cycle, the optimal solution ($\bar{x}$, y^*) of the ARMP – which in this case is the optimal solution of the problem – will always be obtained because UB, whose values obtained throughout the iterative process form a monotonically decreasing sequence, will reach its minimum value at the end of the auxiliary cycle. Therefore, as long as the RMP is being solved without a solution that satisfies $\bar{x} = x^*$, the value of UB will not reach the minimum value, and a solution (x^*, y) can always be obtained by the RMP such that $UB - LB > \epsilon$. Hence, the primary stop condition will only be satisfied after the RMP obtains a solution (x^*, y), and consequently $\bar{x} = x^*$ is fixed in the ARMP. Thus, at the end of the auxiliary cycle an optimal solution $(\bar{x}, y^*)$ of the ARMP, or specifically in this case, the optimal solution (x^*, y^*) of the RMP equivalent to the solution that would be obtained in the classical Benders decomposition will be obtained.

When the ARMP obtains the solution $(\bar{x}, y^*)$ = (x^*, y^*), UB will reach its minimum value and the parameters (X, Y) take on the values of (x^*, y^*), given that $UB < \widehat{UB}$, at the end of the auxiliary cycle. Hence, all the other solutions obtained by ARMP at the end of the subsequent auxiliary cycles are of the type $(\bar{x}, y)$ and at the end of the cycles $UB = \widehat{UB}$ and therefore the last value stored in (X, Y) will always be the optimal solution (x^*, y^*) of the problem. ■

Appendix D – Pseudocode for E-MSMHF

Start

Read P, Ω, τ, K //size of sets//
Read B^p, BA^p, CF^p, H^p //cost parameters//
Read $D(\xi)^p$, $Pr(\xi)$ //uncertainty parameters//
Read β, II, $\overline{S}$, TE, $W^{p,r,k}$, ϵ //other parameters//
Relaxed Master (5.30, 5.31, 5.32, 5.36, 5.37, 5.38, 6.14, 6.15)
Auxiliary Relaxed Master (5.39-5.43, 6.1, 6.3, 6.15)
Dual Slave (5.20-5.29)
$UB \leftarrow \infty$
Solve Auxiliary Relaxed Master
$LB \leftarrow$ Auxiliary Relaxed Master Objective
While $(UB - LB > \epsilon)$ do //general stop condition//
    $\bar{s} \leftarrow s$; $\bar{v}^p \leftarrow v^p$ // $\bar{s}$ and $\bar{v}^p$ are used in Dual Slave solution//
    Fix integer variables ($u^{r,k}$, v^p)
    $\widehat{UB} \leftarrow UB$ // $\widehat{UB}$ is used for the storing optimal auxiliary solutions//
    $LB_A \leftarrow LB$ // LB_A is used to differentiate the auxiliary stop condition//
    While $(UB - LB_A > \epsilon)$ do // auxiliary stop condition//
        Solve Dual Slave
        Add (5.40) in Auxiliary Relaxed Master
        $UB \leftarrow$ min[UB, $\sum_p CF^p\, v^p$ + Dual Slave Objective]
        Solve Auxiliary Relaxed Master
        $LB_A \leftarrow$ Auxiliary Relaxed Master Objective
        $\bar{s} \leftarrow s$ // $\bar{s}$ is used in Dual Slave solution //
    End (While)
    If $UB < \widehat{UB}$ Then $(X, Y) \leftarrow (v^p, s^*)$ //stores the optimal auxiliary solution
    Add (6.20) in Relaxed Master //LOC//
    Unfix ($u^{r,k}$, v^p)
    Solve Relaxed Master // provides only v^p for PED solution//
    $LB \leftarrow$ Relaxed Master Objective
    $\bar{s} \leftarrow$ max $[(II - \sum_k (DK(\xi)^k \sum_r u^{r,k})), \bar{S}/2]$ // it's used in Dual Slave solution //
End (While)
End