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Invited Review in Celebration of the 50th Anniversary of EURO

Fifty years of portfolio optimization

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ABSTRACT

The allocation of resources to alternative investment opportunities is one of the most important decisions organizations and individuals face. These decisions can be guided by building and solving portfolio optimization models that capture the salient aspects of the investment problem, including decision-makers' preferences, multiple objectives, and decision opportunities over the planning horizon. In this paper, we give a historically grounded overview of portfolio optimization which, as a field within operational research with roots in finance, is vast thanks to many decades of research and the huge diversity of problems that have been tackled. In particular, we provide a unified and therefore unique treatment that covers the full breadth of portfolio optimization problems, including, for instance, the allocation of resources to financial assets and the selection of indivisible assets such as R&D projects. We also identify opportunities for future methodological and applied research, hoping to inspire researchers to contribute to the growing field of portfolio optimization.

1. Introduction

In a broad sense, portfolio management refers to all the processes of identifying, selecting, and managing a collection of investment assets by an organization or an individual. These processes involve the screening and selection of these assets; the allocation of resources to those selected; the monitoring of portfolio performance over time; and the updating of the portfolio by introducing new assets or discontinuing earlier allocations, for instance. In this context, determining the 'best' composition of the portfolio is a crucial decision problem that can be addressed through portfolio optimization.

The origins of portfolio optimization can be traced to the pivotal contributions by Markowitz (1952). Since the establishment of this modern portfolio theory, the field of portfolio optimization has become immense. Presently, it spans a wealth of analytical and often sophisticated models that build on many quantitative disciplines. Accordingly, portfolio optimization can be approached from complementary perspectives. To begin with, pivotal contributions in decision theory, such as the representation of preferences with the Neumann–Morgenstern utility function, allow portfolio optimization problems to be addressed within the framework of expected utility maximization and its variants. Second, probabilistic and statistical studies focus on deriving probability distributions over future outcomes so that the consequences of the selected portfolio for the decision-maker can be assessed. Third, advances in computational capabilities help in formulating tractable optimization models and designing, implementing, and evaluating efficient algorithms for solving these. Fourth, concerns in leveraging effective decision support include, among others, the need to ensure that the portfolio model constitutes a valid problem representation and that interactions between the model and its users are aligned with the model's intended purpose.

As portfolio optimization originated some 70 years ago, we have not limited ourselves to the past half a century mentioned in the title. Instead, we highlight selected milestones in the evolution of portfolio optimization since its birth. This notwithstanding, most emphasis is given to developments from the past twenty years or so, as this allows us to give an account of the current state-of-the-art in portfolio optimization. We hope this account and our reflections on future opportunities inspire those who seek to benefit from portfolio optimization and wish to advance its frontiers.

In this review, written to celebrate the 50th Anniversary of the Association of European Operational Research Societies (EURO), we have aimed to provide balanced coverage of theoretical and methodological contributions to portfolio optimization. These contributions are universal; in our view, there are no major differences in how they are being leveraged in different parts of the world. However, in our coverage of applications–which by necessity is representative rather than comprehensive–we have given some precedence to European researchers and journals when making inevitable choices due to space

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limitations. We note that many European researchers are engaged both in methodological development and real applications, which fosters the fruitful interplay between theory and practice, thereby mitigating the risk of formulating unrealistic models that fail to meet the demands encountered in practice.

In terms of scope, we have purposely abstained from covering computational techniques of machine learning and artificial intelligence (AI) for portfolio optimization. On the one hand, this is because these techniques are covered in recent reviews such as Gunjan and Bhattacharyya (2023). On the other hand, while these techniques have become increasingly pervasive in finance, they are less relevant in selecting portfolios of indivisible real assets for which relevant data from the past tends to be more limited. Furthermore, most of our emphasis is mainly on what problem features can be incorporated into portfolio optimization models. For the technical details of solving these models, we refer to recent references which also include solution-oriented reviews (see, e.g., Loke et al., 2023).

The rest of this paper is structured as follows. Section 2 gives an overview of the historical origins of portfolio optimization. Section 3 outlines the main elements of optimization models for portfolio selection. Section 4 considers optimization approaches, and Section 5 presents examples from the expansive literature on applications. In Section 6, we share our perspectives on opportunities for future research and practice.

2. The origins of portfolio optimization

For this paper, *portfolio optimization* is defined at the juncture of its two constituent terms, *portfolio* and *optimization*. Historically, the etymological roots of the term *portfolio* can be traced to the Italian term *portafoglio* which refers to a case or wallet containing documents such as drawings by an artist. Over the years, this term has taken on broader meanings. In business and finance, a *portfolio* refers to any collection of assets that can be considered investments. We adopt this definition, noting, however, that the assets in the portfolio need not be financial, nor does their value have to be measured in monetary terms. For instance, built infrastructures, R&D projects, and patents constitute assets. Importantly, though, because these assets are *investments*, they entail decision opportunities that can be pursued by committing resources with the aim of receiving later benefits.

Optimization uses mathematical methods to help identify the most preferred decision alternative out of all alternatives that satisfy relevant constraints, based on a systematic evaluation with regard to the criteria that reflect the decision-maker's preferences. Consequently, portfolio optimization is concerned with alternative *portfolios of assets* to support the selection of the *preferred portfolio* consisting of those assets that contribute most to the attainment of stated objectives, based on an evaluation with regard to evaluation criteria for these objectives while satisfying the constraints that apply. There can be different kinds of constraints, both on what properties the individual assets in the portfolio may have to fulfil and how the selected portfolio is required to perform. Typically, there are uncertainties because, for instance, the benefits gained by committing resources to the selected assets are uncertain at the time of portfolio selection.

In view of the above, we define portfolio optimization as *the use of mathematical methods to support the selection of preferred portfolios of assets by accounting for the decision-maker's preferences, relevant constraints and uncertainties*. This definition is generic as it does not refer to specific application fields such as finance or R&D management. An important rationale for the present paper–which discusses the breadth of portfolio optimization in many contexts–is to highlight similarities across application areas so that synergies between these can be exploited.

Historically, though, the literature on portfolio optimization has evolved in parallel tracks, propelled by the specific characteristics in application domains where, on the one hand, investment decisions can be meaningfully represented by *continuous* decision variables (e.g., choosing the share of stocks for a financial investment portfolio) or by *discrete* variables (e.g., making 'lumpy' investments such as the construction of bridges). Portfolio optimization problems with discrete variables can be addressed with portfolio decision analysis (PDA; e.g., Liesiö et al., 2021; Salo et al., 2011). One of the key application areas of PDA is project portfolio management, which covers all the activities in selecting, monitoring, and managing project portfolios (see, e.g., Levine, 2005). Thus, there is plenty of overlap between methods for project portfolio selection (for a review, see Kandakoglu et al., 2023) and PDA. Yet the scope of PDA is broader in that the alternatives need not be 'projects' in a conventional sense. Moreover, the aims of PDA may go beyond selection decisions to support the design of multi-stage decision processes, for instance.

Next, we give a brief overview of the evolution of portfolio optimization, assuming that the reader has some knowledge of decision analysis, risk management, finance, and mathematical optimization.

2.1. Financial portfolios

The founding principles of the modern portfolio optimization theory were laid in the seminal work of Markowitz (1952) who considered the construction of portfolios of financial assets. This theory effectively assumes that investors seek to maximize the expected utility of the portfolio's return. Markowitz observed that various utility functions can be approximated by a quadratic function of an investment's expected return and variance (Markowitz, 2014). This leads to the well-known mean–variance model which can be formulated as the bi-objective quadratic optimization problem

$$\begin{aligned} &\mathrm{v-max}_{\lambda_1,\ldots,\lambda_m\in\mathcal{A}} \quad (f_1, f_2) = \Big(\sum_{j=1}^{m}\lambda_j\mu_j,\ -\sum_{j=1}^{m}\sum_{j'=1}^{m}\lambda_j\lambda_{j'}\sigma_{j,j'}\Big) \\ &\text{Subject to:} \qquad \lambda_1+\lambda_2+\cdots+\lambda_m=1, \end{aligned} \tag{1}$$

where λ_j is the proportion of capital allocated to asset j, μ_j is the expected return of asset j, $\sigma_{j,j'}$ is the covariance of the returns of assets j and j', and $\mathcal{A}$ is the set of admissible capital allocations.

Starting from the 1960s, the foundations of the mean–variance model received much attention concerning the generality of this normative theory and its consistency with the classical decision-making axioms of von Neumann and Morgenstern. Specifically, early studies in this area characterized the conditions under which the two-parameter mean–variance model can be used to analyse the choices of an expected-utility-maximizing investor (see, for instance, Hakansson, 1972; Samuelson, 1967; Tobin, 1969). Later studies presented empirical results to evaluate the quality of the approximations derived by using the mean–variance quadratic formulation to different types of utility functions (Levy & Markowitz, 1979; Pulley, 1981). While such studies have, generally, provided positive results, Samuelson (1970) showed that the mean–variance model is most useful when risk is low. In contrast, the consideration of higher moments provides improved results in the general case. Moreover, Kane (1982) noted that under a discrete-time portfolio rebalancing scheme, the mean–variance model may be a poor approximation for expected utility, and examined the properties of an enhanced model that considers skewness as an additional parameter.

Beyond studies on the foundations of the mean–variance model, the basis for alternative measures of risk beyond variance was laid in the 1970s and 1980s. Specifically, the theory of stochastic dominance was introduced in the context of portfolio selection (Bawa, 1978). The foundations of downside risk measures were established (Fishburn, 1977), which spawned an active research area on coherent risk measures over the next decades (Artzner et al., 1999). Moreover, new risk measures were introduced, particularly measures that can be optimized through linear programming formulations such as the Gini mean difference model of Yitzhaki (1982) and the mean-absolute deviation model of Konno and Yamazaki (1991). Further early developments that have impacted the evolution of portfolio optimization include the formulation of multi-period and dynamic optimization approaches (Hakansson, 1971; Karatzas et al., 1987) as well as the introduction of the multicriteria and multiobjective methodologies for portfolio selection (Colson & De Bruyn, 1989; Martel et al., 1988).

A more extensive overview of the foundations and early development of portfolio optimization theory for financial investments can be found, for instance, in Elton and Gruber (1997) and Markowitz (1999).

2.2. Portfolio optimization of indivisible real assets

Although the early roots of using mathematical optimization to select a portfolio of real indivisible assets developed separately from the seminal work of Markowitz (1952), the close methodological links between these two problem types were soon recognized. That is, while (Asher, 1962) applied linear programming models to optimize the allocation of resources to a portfolio of R&D projects without referencing Markowitz's work, the survey paper of Weingartner (1966) already connected the research streams on R&D project selection, financial portfolio optimization (Markowitz, 1952), and capital rationing (Lorie & Savage, 1955). Weingartner presented the classical knapsack formulation for optimizing a portfolio of investment alternatives (projects), which can be written in the general form as

$$\max_{\lambda\in\{0,1\}^m} \sum_{j=1}^{m} \lambda_j v^j \qquad A\lambda \leq B, \tag{2}$$

where v^j is the net present value (NPV) of the jth investment alternative, and matrix A and vector B represent relevant portfolio constraints. These constraints can include, for instance, resource constraints (e.g., budget) as well as logical dependencies between projects (mutual exclusivity of projects, contingent projects). Building on the work of Markowitz (1952), Weingartner (1966) also suggested measuring risk as the variance in the portfolio payoff, which led to the quadratic integer programming problem

$$\max_{\lambda\in\{0,1\}^m} \sum_{j=1}^{m} \lambda_j v^j - \alpha \sum_{j=1}^{m}\sum_{j'=1}^{m} \lambda_j \lambda_{j'} \sigma_{j,j'} \qquad A\lambda \leq B, \tag{3}$$

where α is a predefined parameter capturing the level of risk-aversion.

In the 1980s, several articles in *European Journal of Operational Research* reported real applications of portfolio optimization models for selecting R&D projects (e.g., Golabi, 1985; Lootsma et al., 1986; Weber et al., 1990). In these applications, estimates about the project benefits were obtained by carrying out a multiattribute/criteria evaluation instead of stating these benefits as a single estimate of monetary value. Specifically, the project values in model (2) were derived from an additive multiattribute utility/value function

$$v^j = \sum_{i=1}^{n} w_i v_i(x_{ji}), \tag{4}$$

where w_i and v_i are the importance weight and attribute-specific value function, respectively, for the ith attribute and x_{ji} is the performance of the jth project candidate with regard to the ith attribute. Computationally, this extension is straightforward as the optimization problem (2) can be solved with integer linear programming (ILP). However, it is not obvious that adding up the projects' multiattribute values gives the correct overall portfolio value in the objective function. This concern was addressed by Golabi et al. (1981) who were the first to apply multiattribute value and utility theory (MAVT/MAUT; Dyer & Sarin, 1979; Keeney & Raiffa, 1976; Krantz et al., 1971) to build an axiomatic foundation for portfolio value/utility functions.

A more thorough exposition of the early developments of portfolio optimization, with an emphasis on the use of decision analytic approaches to inform choices from a discrete set of alternatives, can be found in the book on portfolio decision analysis by Salo et al. (2011).

3. Dimensions of portfolio optimization problems

To highlight the diversity of portfolio optimization problems, we next elaborate on the salient dimensions of portfolio problems. As such, these dimensions are useful in that they can be helpful, for example, in assessing how similar different problems are.

3.1. Divisibility of assets

The decision opportunities afforded by assets may vary depending on whether the investments are *divisible* (i.e., it is possible to make very small investments that are meaningfully represented by continuous decision variables) or *indivisible* (i.e., the investments are lumpy, as is the case with go/no-go projects).

In a typical setting for optimizing portfolios of financial assets, the composition of the portfolio is defined by asset weights λ_j which indicate what proportion of the available capital is invested in the jth asset. If the assets are perfectly divisible, all allocations with $0 \leq \lambda_j \leq 1$ are admissible (assuming that there are no possibilities for taking a short position, in which case λ_j would be negative). Investments into real assets typically correspond to decisions in which the asset is either included in or excluded from the portfolio. This can be represented by the binary variable $\lambda_j \in \{0,1\}$. In the portfolio optimization problem, there may be both divisible and indivisible assets at the same. This gives rise to hybrid settings where, for instance, a company is considering investments into financial instruments along with capital investments into real assets.

3.2. Decision-makers and stakeholders

There may be one or several DMs whose preferences need to be represented in the portfolio optimization model, as exemplified by the case studies Vilkkumaa, Salo et al. (2014) and Fasth et al. (2020). In addition, there may be stakeholders who are not empowered to act as decision-makers but whose values and interests need to be accounted for by developing multicriteria models for decision support (for an overview, see Salo et al., 2021).

3.3. Decision criteria

The performance of a portfolio, as measured by its contribution to the attainment of decision objectives, may involve one or more decision criteria. For instance, net present value (NPV) is often employed as a criterion in assessing the attractiveness of projects that are expected to be profitable. In recent years, non-financial factors have gained considerable interest, exemplified by environmental, social, and governance criteria for socially responsible investments (Ballestero et al., 2012; Utz et al., 2015). Typically, several criteria are needed to capture the DMs' preferences.

3.4. Planning horizon

Portfolio optimization problems differ with regard to the length of the time horizon over which the relevant decision consequences of the portfolio are considered. Problems with longer time horizons tend to involve more uncertainty, leading to questions about how these uncertain future consequences, such as later benefits, will be considered in decision-making. A portfolio may also contain assets with different maturities. In this case, all the consequences for some assets may be realized before the end of the planning horizon, which may give rise to a reinvestment problem. It is also possible that not all investment opportunities are available or known at the beginning of the planning horizon. As a result, there is a need to decide what share of available resources is committed to the presently available investment opportunities, on the one hand, and how much is reserved for the investment opportunities that may be available later, on the other hand.

3.5. Timing of decisions

Portfolio problems differ in terms of when it is possible to make decisions that change the portfolio's composition, for instance, by investing in further assets, altering the level of earlier investments, or discontinuing investments (see, e.g., Angelelli et al., 2008; Gustafsson & Salo, 2005; Vilkkumaa et al., 2015). Here, one can distinguish between single-period, multi-period, and continuous portfolio optimization problems. These categories are not mutually exclusive as there may be assets about which decisions can be made on a continuous basis, whilst decisions about others can be made only at specific points in time.

In finance, dynamic optimization models (i.e., multi-period and continuous) help design portfolio management strategies that account for transaction costs and changing market conditions such as price trends. Moreover, the multi-period setting employs discrete-time models in which investment decisions can be taken only at specific points in time while continuous portfolio optimization assumes that the portfolio can be rebalanced at any time.

The intertwined decisions in the processes of asset screening (i.e., what is the set of assets to which resources can be allocated) and capital allocation (i.e., how are resources to be allocated to these assets) can be approached in different ways. These processes can be carried out in separate stages (e.g., first focusing on the evaluation of the assets and then proceeding to asset allocation; see Pendaraki et al., 2005; Xidonas et al., 2023). Alternatively, they can be considered jointly by combining both stages through integrated portfolio optimization, as illustrated by models for cardinality-constrained portfolio optimization (Bertsimas & Shioda, 2009; Woodside-Oriakhi et al., 2011). Moreover, real-world portfolio optimization is often required to provide allocations that satisfy round lot requirements (Mansini et al., 2015). In this case, the optimization result must specify the exact number of units for the investment as fractional investments are inadmissible.

3.6. Uncertainties and incomplete information

There can be uncertainties about essentially all aspects of the portfolio optimization problem, including the characteristics of individual assets; their interrelationships and consequences over the planning horizon; the availability of resources that can be invested; and the preferences that guide the formulation of decision criteria and their evaluation. Here, we use the term *uncertainty* as referring to the inability to specify with certainty the value of some parameter that appears in the portfolio optimization model, such as the decision consequences associated with selecting a given portfolio.[1]

Uncertainties can be addressed through several approaches, most notably by using standard probability theory to model parameters as random variables or by characterizing the lack of complete information by placing bounds on what parameter values are viewed as possible. This latter approach leads to an optimization model that accommodates incomplete information and can be explored, for example, to derive robust decision recommendations, which are justified in view of all parameter values within the stated bounds. Overall, there is a broad range of approaches to modelling uncertainties, including the use of theories with non-additive expected utilities (see, e.g., Gilboa, 2009)

Importantly, uncertainties give rise to *risk* in that the realized consequences of the portfolio *ex-post* may be worse than what was expected *ex-ante* when the portfolio was selected. Overall, risk is a central concept in investment decisions, as evidenced by the wide range of risk measures and the burgeoning literature on coherent risk measures (Artzner et al., 1999; Lim et al., 2011; Szegö, 2002). Beyond the traditional use of variance as a measure of risk, popular risk measures include semi-variance (Markowitz et al., 1993), higher-order risk measures (e.g., skewness and kurtosis, Kerstens et al., 2011; Kim et al., 2014), mean absolute deviation (Konno & Yamazaki, 1991), drawdown measures (Chekhlov et al., 2005), value-at-risk (Jorion, 2006) and conditional value-at-risk (Rockafellar & Uryasev, 2002). Examples of further financial performance measures include stochastic dominance criteria (Ogryczak & Ruszczyński, 1999; Roman et al., 2006), the Omega ratio (Kapsos, Zymler et al., 2014), the Gini mean difference index (Mansini et al., 2014), as well as risk-adjusted performance indicators (e.g., Sharpe, Sortino, etc.).

Even if the uncertain estimates about the consequences of assets in the optimization model are unbiased, the performance of the selected portfolio *ex-post* will, on average, fall short of what one would expect based on the *ex-ante* estimates. Thus, just like problems of choosing one of many alternatives, portfolio optimization problems exhibit the optimizer's curse phenomenon (Smith & Winkler, 2006) in that those assets whose performance has been overestimated tend to have a higher chance of being selected. This phenomenon can be mitigated by employing systematic debiasing techniques (see, e.g., Kettunen & Salo, 2017; Vilkkumaa, Liesiö et al., 2014) or robust satisficing (Long et al., 2022).

The consequences of a portfolio may be influenced by factors such as the future direction of the economy that are uncertain when the portfolio is selected. Here, one may distinguish between *exogenous* uncertainties associated with random variables that do not depend on the selected portfolio; and *endogenous* uncertainties which are impacted by this portfolio. From the modelling perspective, exogenous uncertainties are easier to accommodate as there is no causal relationship between them and the portfolio selection. For instance, Contingent Portfolio Programming (Gustafsson & Salo, 2005) helps select project portfolios whose consequences depend on scenarios whose probabilities are not impacted by project selections. In the case of endogenous uncertainties, in contrast, there is a need to characterize these causal relationships by specifying conditional probabilities that depend on the selections (see, e.g. Salo et al., 2022; Vilkkumaa et al., 2018). This may call for a significant elicitation effort.

Established by researchers working on decision-aiding methodologies, the dominance-based rough set approach supports portfolio decisions by permitting comparisons between probability distributions and considering the investor's preferences over time (Greco et al., 2010, 2013). There are also further approaches for representing uncertainties and varying degrees of belief, most notably fuzzy set theory. In this paper, however, we do not cover approaches based on fuzzy numbers, partly because these cannot be readily integrated with probabilistic representations that are compatible with statistical methods and the expected utility theory. Recent advances in using non-additive expected utility theories for portfolio selection are discussed by Baker et al. (2020). An overview of alternative representations of uncertainty in multicriteria problems is given in Durbach and Stewart (2012).

3.7. Constraints

Depending on the problem, portfolio optimization models may feature many kinds of constraints:

- *Logical constraints* represent logical relationships that the selected assets in a feasible portfolio must fulfil. Such constraints arise, for instance, in selecting projects of which some may be prerequisites for others.
- *Resource constraints* arise from limitations concerning the availability of different types of resources, such as the size of the available budget, that must be committed to implement the selected portfolio.
- *Risk constraints* place bounds on the degree of variability that the consequences of feasible portfolios are allowed to have.

[1] We do not consider uncertainties that pertain to non-mathematical aspects such as whether or not all relevant stakeholders have been identified and consulted in formulating the portfolio model.

- *Strategic constraints* represent requirements concerning the contribution the selected portfolio has to make on one or more strategically important decision criteria. They may also represent requirements on the composition of a portfolio to ensure, for example, an acceptable balance across different kinds of assets. In financial portfolios, strategic constraints may also limit the number of different kinds of assets in the portfolio (cardinality-constrained portfolios), round lots, transaction costs, or other elements that are integral to the investment process.

One can distinguish between *hard* constraints, which must not be violated, and *soft* constraints, which may be violated to some extent, albeit at a penalty. Moreover, the availability of resources need not be given exogenously, as the completion of some projects may generate resources that can be deployed to make subsequent investments into other assets (see, e.g., Champion et al., 2023).

4. Solution approaches for portfolio optimization problems

4.1. Bi-objective optimization

In finance, bi-objective models for exploring the risk-return trade-off have attracted the most attention. The general form of these models can be expressed in two alternative ways:

$$\left.\begin{array}{ll} \min & \mathrm{Risk}(\lambda) \\ \text{s.t.:} & \mathrm{Return}(\lambda) \geq \varrho \\ & \lambda \in \mathcal{A} \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{ll} \max & \mathrm{Return}(\lambda) - \alpha \mathrm{Risk}(\lambda) \\ \text{s.t.:} & \lambda \in \mathcal{A} \end{array}\right\} \quad (6)$$

where $\mathrm{Return}(\lambda)$ and $\mathrm{Risk}(\lambda)$ are the return and risk functions defined by the asset weights $\lambda = (\lambda_1, \ldots, \lambda_m)$. In formulation (5), the risk-return trade-off is defined through the return constraint $\mathrm{Return}(\lambda) \geq \varrho$, where ϱ denotes the expected return that the investor seeks to achieve. On the other hand, in formulation (6), the risk weight $\alpha > 0$ represents the relative importance of risk in the objective.

Different efficient portfolios can be obtained by changing the parameters ϱ and α. If the set $\mathcal{A}$ is convex, both formulations trace out all efficient portfolios as the parameters ϱ in (5) and α in (6) are varied (but there is no direct correspondence between the portfolios that are generated for different values of these two parameters). However, if the set $\mathcal{A}$ is non-convex, the formulation (5) outperforms the weighted model (6) in the sense that the latter may fail to identify unsupported efficient portfolios.

The exact formulation of the general forms (5) and (6) depends on how the return and risk functions are defined as well as the other dimensions of the portfolio optimization problem as outlined in Section 3. The resulting models are expressed as linear, quadratic, or non-linear optimization problems whose structure depends very much on the adopted risk measure. For instance, while variance is a quadratic function of assets' weights, other risk measures, such as the mean absolute deviation and conditional value-at-risk, can be written in a linear form. More complex risk measures, such as skewness and kurtosis, lead to non-linear optimization problems. Some constraints may lead to mixed-integer optimization problems. For instance, imposing constraints on the number of kinds of assets requires the introduction of binary variables $\mathbf{z} = (z_1, \ldots, z_m)$ to indicate whether an asset is included in the portfolio or not, and the addition of the constraints $\lambda \geq \epsilon \mathbf{z}$ and $z_1 + \cdots + z_m \leq K$, where ϵ is the minimum weight for the assets in the portfolio and K is the pre-specified maximum number of assets in the portfolio.

Portfolio optimization problems with non-linear or combinatorial features give rise to computational challenges. This has motivated the development of a wide range of special solution algorithms, heuristics, and evolutionary approaches. Algorithms based on analytical approaches are effective for solving problems with a few hundred assets (Cesarone et al., 2013; Graham & Craven, 2021). Bertsimas and Cory-Wright (2022) present an algorithm based on the cutting-plane method that scales up well for larger instances, too. Heuristics and meta-heuristics, too, have been extensively used for finding high-quality portfolio allocations. Maringer (2005) presents the main contributions and application contexts for such algorithms, whereas Erwin and Engelbrecht (2023) provide a comprehensive list of various algorithms and an overview of the literature covering the past three decades.

Another strong line of research has focused on the uncertainty of the inputs required to implement portfolio optimization models. Optimization under uncertainty can be addressed through two main paradigms: stochastic programming and robust optimization. In stochastic programming, uncertainty is described through probability distributions, permitting the generation of solutions that satisfy constraints in a probabilistic sense. Robust optimization, on the other hand, is more appropriate when there is a lack of knowledge about probabilities and a need to consider possible realizations of the uncertain parameters in a pre-specified set (Bertsimas et al., 2011). Moreover, robust optimization models are computationally easier to solve.

Instead of relying on point estimates for input parameters (e.g., expected returns and covariances), the robust optimization framework assumes that the inputs belong to uncertainty sets around the point estimates. For instance, a simple box uncertainty set $\mathcal{U} = \{\boldsymbol{\mu} \mid |\mu_i - \widehat{\mu}_i| \leq \delta_i\}$ assumes that the vector of unknown returns $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ belongs to an area around the vector of estimates $\widehat{\boldsymbol{\mu}}$, which is typically defined through historical data. The parameters $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m)$ define the size of the uncertainty set. However, this straightforward way of defining the uncertainty set focuses solely on returns and ignores the correlations between the assets. Other types of uncertainty sets have been used to address this shortcoming, such as ellipsoidal and mixture distribution uncertainty sets (Fabozzi et al., 2010). The framework of robust optimization is applicable in the context of various bi-objective models based on different risk-return measures, beyond the mean–variance model, such as the Sharpe ratio (Chakrabarti, 2021), value-at-risk (Ghaoui et al., 2003; Zhu & Fukushima, 2009), and the Omega ratio (Kapsos, Christofides et al., 2014). Multi-period models have also been proposed (Gülpınar & Rustem, 2007; Ling et al., 2020). A bibliographic overview of this area for financial portfolios can be found in Xidonas et al. (2020).

Recently, there has been growing interest in distributionally robust optimization (DRO) which combines elements from both stochastic and robust optimization. DRO captures uncertainties about outcomes (Wiesemann et al., 2014) by assuming that the probability distribution belongs to an ambiguity set. This leads to results that are less conservative than those of other robust approaches (Goh & Sim, 2010). Mohajerin Esfahani and Kuhn (2018) and Postek et al. (2016) develop computationally tractable formulations for DRO problems under mild assumptions. They also apply the DRO approach to portfolio optimization using the conditional value-at-risk risk measure. Chen et al. (2023) present a similar approach by employing using mean absolute deviation as a measure of risk.

The frameworks of robust optimization and DRO are mainly data-driven in that they consider uncertainties about model parameters on which there is relevant historical data or which can be simulated. Nevertheless, the formulation of portfolio optimization models often requires that information about the decision-maker's preferences is also specified. This information may also involve many kinds of uncertainties (e.g., incompleteness, vagueness, etc.). We elaborate on this area of research in Sections 4.3 and 4.4.

4.2. Multi-objective approaches

The multi-faceted nature of portfolio selection calls for a sufficiently comprehensive framework that spans multiple decision criteria for characterizing the performance of alternative portfolios. Specifically, approaches that account for multiple conflicting objectives help investors and portfolio managers explore different investment options in view of their preferences. Their relevance for portfolio optimization has been discussed extensively by Aouni et al. (2018) and Steuer et al. (2007).

Multiobjective optimization (MOO) models incorporate different investment decision criteria in identifying efficient (i.e., non-dominated) portfolios. The MOO theory provides solution procedures based on ε-constraint formulations, scalarising functions, and goal programming (GP) (for an overview of such approaches, see Miettinen, 1999). Each of these approaches transforms a MOO problem into a single-objective formulation that can be solved through standard solvers. In the ε-constraint approach, one of the objectives is optimized, while imposing constraints on the others. A simple example is the bi-objective risk-return model (5) which has been applied to financial portfolio optimization by Xidonas and Mavrotas (2014). The ε-constraint approach works well with a few objectives (i.e., two or three), but identifying the efficient frontier becomes more complex in many-objective problems. This limitation is addressed by other approaches, including those that use scalarising functions to combine the objectives into a single measure to be optimized.

The simplest scalarising function is the weighted sum. As noted in our elaboration of Eq. (1), a major limitation of weighted sum formulations is that they may not identify all efficient portfolios when the optimization model is non-convex due to integer-valued decision variables, for example. Other options are also available, such as approaches that focus on identifying portfolios that minimize deviations from reference values (points) specified by the decision-maker to define the investment objectives. An example of such a scalarising function using the maximum values $f_1^*, \dots, f_n^*$ of the objectives as the reference point (assuming the objectives are in a maximization form) is the augmented weighted Chebyshev metric

$$\max_{i=1\dots,n} [w_i(f_i(\lambda) - f_i^*)] + \theta \sum_{i=1}^{n} (f_i(\lambda) - f^*)$$

where $w_1, \dots, w_n \geq 0$ are the weights associated with the objectives and θ is a small positive constant for ensuring that the minimization will not suggest a weakly efficient portfolio. Pavlou et al. (2019) employ this approach to construct financial portfolios combining three performance criteria: return, mean absolute deviation, and conditional value-at-risk. Dächert et al. (2022) use a similar Chebychev metric for strategic asset allocation for an insurance company.

GP formulations are also based on optimizing the deviations from a set of predefined target values on the decision criteria. Compared with scalarisation models, GP provides additional flexibility in modelling and treating deviations (e.g., the prioritization of the goals), formulating the investor's goals, and specifying the corresponding target levels. For instance, Ballestero et al. (2012) build a GP model to combine financial and environmental, social and governance (ESG) goals, whereas Tamiz et al. (2013) and Tamiz and Azmi (2019) propose an extended framework that accounts for goals related to the macroeconomic environment, regional factors, as well as accounting and stock market performance ratios. A review of related GP applications can be found in Aouni et al. (2014) and Colapinto et al. (2019).

MOO approaches, such as those mentioned above, can, in principle, be applied in an *a priori* or *a posteriori* setting (Hirschberger et al., 2013; Hwang & Masud, 1979). In the first case, the investor's preferences are incorporated in a MOO model to obtain a unique solution, i.e., the portfolio that best matches the investor's preferences subject to the constraints on the investment policy. A central part of these approaches is the modelling of preferences, for example, by developing decision models based on value/utility function representations, outranking relations, or decision rules. Sub-Section 4.3–4.6 discuss these approaches.

A posteriori implementations, on the other hand, focus on the identification of all non-dominated portfolios from which the investor can then choose the most preferred one. This calls for efficient computational procedures for MOO. As with approaches for the bi-objective case, analytical and algorithmic methodologies have been proposed. In the MOO setting, analytical methods for portfolio optimization do not incorporate complex non-linearities or combinatorial features. For instance, Qi and Steuer (2020), Qi et al. (2017) derive the analytical algebraic solution for the efficient surface in portfolio optimization problems with at least three objectives, assuming that variance is the main measure of risk. For complex instances that require the solution of computationally intensive models, multiobjective evolutionary algorithms (MOEAs) have been extensively used. Reviews by Ponsich et al. (2013) and Metaxiotis and Liagkouras (2012) on the applications of MOEAs in financial portfolio optimization indicate that such approaches are typically used to address the complexity due to non-linear performance measures and the introduction of realistic constraints. Commonly, there are two to five objectives that are considered as performance measures. Except for return and variance, these objectives include value-at-risk, expected shortfall, social responsibility, skewness, and the Sharpe ratio. The constraints mainly focus on the composition and diversification of the portfolios (e.g., cardinality constraints, bounds on the asset weights), and transactions-related issues (e.g., transaction costs, round lots, turnover constraints).

As in the case of bi-objective optimization approaches, handling uncertainty has attracted interest in the MOO context. Based on the framework for bi-objective models, the first framework for data-driven robust optimization can be found in Fliege and Werner (2014) who introduce the robust Pareto frontier and illustrate how it can be applied to portfolio optimization using an ellipsoidal uncertainty set. MOO formulations based on worst-case scenarios have also been considered. For instance, Xidonas et al. (2017) use a minimax regret criterion to model uncertainties concerning the investor's risk-return attitude (i.e., the weights of risk-return criteria). Caçador et al. (2020) use a similar minimax regret model to consider the uncertainty with respect to the covariance matrix of asset returns, combined with a genetic algorithm to solve the resulting non-linear optimization model.

An alternative path to address uncertainties in multi-objective models for portfolio optimization has arisen in stochastic programming. For example, Abdelaziz et al. (2007) present a chance-constrained compromise programming formulation to model the uncertainties the returns of financial assets and their systematic risk (β coefficient). Masmoudi and Ben Abdelaziz (2017) present a similar approach while Aouni et al. (2013) employ a stochastic GP approach to model the portfolio selection process for venture capital investments, using criteria related to the return and risk of the investment, as well as the survival rate and intellectual capital rate. Bravo et al. (2010) employ a stochastic GP for the expected utility of portfolio returns.

4.3. Incomplete information on multiattribute preferences

Multiattribute portfolio models require the specification of several parameters that reflect the decision-maker's preferences. For instance, using the additive multiattribute value function (4) to produce project values $v^1, \dots, v^m$ requires that the attribute weights $w_1, \dots, w_n$ and the attribute-specific performances (scores) $v_i(x_{ji})$ are elicited for each investment candidate j. Arguably, such an elicitation process may require a significant cognitive effort, which makes it difficult to ensure that the parameter estimates are reliable.

Here, preference disaggregation and learning techniques can be useful. Disaggregation methods use techniques such as ordinal regression to infer decision models that are as consistent as possible with holistic judgements provided by the decision-maker (Jacquet-Lagrèze & Siskos, 2001). These approaches are not restricted to the elicitation of value function models as they can be applied together with various decision models, including outranking relations and decision rules (cf. Section 4.6). Pendaraki et al. (2005) employ such an approach to infer an additive value function model for evaluating and selecting mutual funds in order to construct fund-of-funds portfolios. Ehrgott et al. (2009) use a similar approach in an integrated framework in which a value function is first inferred from the investor's holistic judgements, followed by constructing the value-maximizing portfolio through optimization.

In a typical disaggregation setting, the inference process yields point estimates for the parameters of a decision model. However, these estimates involve uncertainties due to the subjectiveness in the inputs provided by the decision-maker (i.e., holistic judgements). It is, therefore, advisable to complement decision recommendations based on the identification of the optimal portfolio with global sensitivity analyses focusing on how the optimal portfolio would change in response to changes in the parameter values.

These concerns have motivated the development of portfolio optimization approaches that explicitly capture incomplete information about multiattribute preferences. Technically, these approaches often rely on set inclusion in which parameters are constrained to a set of feasible values that is constructed from the stated preference information. Decision recommendations are then generated based on the identification of non-dominated portfolios, i.e., those feasible portfolios for which there does not exist another feasible portfolio that has a greater value for all allowed parameter values, or decision rules (e.g., maximin, minimax-regret). For instance, Liesiö et al. (2007) consider a set of attribute weights $W \subset \mathbb{R}^n_+$ defined through a system of linear constraints corresponding to the decision-makers' preference statements and intervals $v_j(x_{ji}) \in [\underline{v}_{ji}, \overline{v}_{ji}]$ for the investments alternatives' attribute-specific values (scores). Arguably, allowing the scores to take on any values within these intervals leads to conservative decision recommendations, which has motivated the development of models that limit the number of decision alternatives whose scores deviate from their most likely value in the spirit of robust optimization (Fliedner & Liesiö, 2016; Hassanzadeh et al., 2014).

Vilkkumaa, Salo et al. (2014) extend the set inclusion approach to group decision-making by modelling incomplete information about the relative importance of the decision-makers in the group as well as the individual preferences of each group member. Liesiö and Punkka (2014) consider incomplete information on the baseline value that specifies the value of not selecting an investment alternative into the portfolio (Clemen & Smith, 2009; Golabi et al., 1981). Set inclusion has also been used to capture incomplete information about the costs of investment alternatives through interval-valued coefficients in the portfolio constraint $A\lambda \leq B$ in model (2) (Liesiö et al., 2008; Lourenço et al., 2012). Furthermore, portfolio optimization helps guide the allocation of resources to decision-making units (DMUs) whose performance in converting these inputs to outputs can be examined with efficiency analysis. A key insight from such analyses is that allocating more resources to DMUs with high efficiency scores while withdrawing resources from those with low scores may lead to a less efficient portfolio of DMUs, suggesting that resource allocation decisions should be approached through portfolio optimization rather than extrapolation from efficiency analysis (Liesiö et al., 2020a).

Computationally, many of the above approaches lead to a multiple objective zero–one linear programming (MOZOLP) problem from which the set of non-dominated portfolios is computed. When the investment alternatives' attribute-specific consequences or costs are interval-valued, the optimization problem has interval-valued objective function coefficients. Exact algorithms for this problem by Liesiö et al. (2007, 2008) are based on the algorithms of Villarreal and Karwan (1981) for solving MOZOLP problems with point-estimate objective function coefficients. The exact algorithm by Argyris et al. (2011) can identify the set of those non-dominated portfolios that are optimal for some attribute weights (cf. supported efficient solutions). Approximate algorithms for the interval-valued MOZOLP problem have been developed by Mild et al. (2015). Challenges in identifying all non-dominated solutions have also motivated the development of heuristic evolutionary algorithms (Doerner et al., 2006; Gutjahr et al., 2010) as well as interactive methods that identify some of the non-dominated portfolios based on interactive process eliciting decision-maker's preference information (see, e.g., Hassanzadeh et al., 2014).

4.4. Incomplete information on risk-preferences and state probabilities

The decision-maker's risk attitude needs to be modelled with an appropriate risk measure or a utility function to incorporate risk aversion into portfolio optimization. However, the specification of the exact functional form for the utility function may be time-consuming because there is a need to complete a lengthy process consisting of a series of preference elicitation questions. Thus, it can be helpful to provide decision recommendations that can be justified in view of different assumptions about risk preferences (e.g., a range of risk-aversion levels).

In this setting, the concept of stochastic dominance (SD) helps compare the outcome distributions for decision alternatives (in our case portfolios of assets), making it possible to determine whether or not one of them is preferred to the other for every utility function belonging to a specific class of functions. Concretely, if the outcome distribution dominates another in the sense of first-order stochastic dominance (FSD; Quirk & Saposnik, 1962), the former portfolio has a higher expected utility than the latter for any increasing utility function. Second-order stochastic dominance (SSD; Hadar & Russell, 1969) holds if the expected utility is higher for all increasing and concave utility functions. SSD is particularly relevant for portfolio optimization because it is compatible with the preferences of risk-neutral and risk-averse decision-makers.

The most popular approach for incorporating stochastic dominance into portfolio optimization is to introduce constraints to ensure that the optimal portfolio dominates a predefined benchmark distribution (e.g., return distribution of the market portfolio in case of financial assets). Kuosmanen (2004) develop MILP models to handle FSD constraints. Because the incorporation of SSD constraints does not require binary decision variables, it leads to LP models as demonstrated by Dentcheva and Ruszczynski (2003), Post (2003), and Kuosmanen (2004). More recently, Post and Kopa (2013) and Kallio and Dehghan Hardoroudi (2019) propose constraints for higher orders of stochastic dominance to model preferences consistent with smaller subsets of increasing concave utility functions. Portfolio optimization models with stochastic dominance constraints under incompletely specified state probabilities are developed by Dupacova and Kopa (2014) and Liesiö et al. (2020b).

When incompletely defined risk preferences are accounted for through stochastic dominance constraints, there is a need to define a benchmark portfolio (or a distribution) that the optimal portfolio must dominate. Yet the resulting decision recommendations can be sensitive to the selection of this benchmark. This issue could be circumvented by identifying all feasible portfolios that are not stochastically dominated, albeit this would be computationally more challenging than the incorporation of stochastic dominance constraints. Still, identifying all stochastically non-dominated portfolios by solving multiple objective optimization problems has attracted attention. Liesiö and Salo (2012) and Vilkkumaa et al. (2018) solve all non-dominated portfolios of indivisible investment alternatives in view of incomplete probability and risk-preference information in scenario planning. More general approaches for identifying the set of non-dominated portfolios by solving multiobjective optimization problems have been developed by Liesiö et al. (2023).

4.5. Non-additive multiattribute value and utility functions

In practice, the additive-linear portfolio value function is the most widely used preference model for optimizing portfolios of indivisible investment alternatives evaluated with regard to multiple attributes. In this function, the portfolio value is expressed as the sum of the multiattribute values of the alternatives that are included in the portfolio. Formally, substituting the additive multiattribute-value/utility function

(4) into the objective function of portfolio optimization problem (2) yields the linear-additive portfolio optimization model

$$\max_{\lambda\in\{0,1\}^m} \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} w_i v_i(x_{ji}), \qquad A\lambda \le B. \tag{7}$$

It is rather straightforward to use this model. To elicit preferences, it suffices to specify an additive multiattribute value function that aggregates the consequences of each investment alternative into a single-dimensional value. Moreover, the optimal portfolio can be identified with any standard zero–one linear programming (or ILP) algorithm. The potential shortcoming of model (7) is that it cannot capture non-linearities in the overall portfolio value. For instance, the model assumes that an alternative's additional contribution to this portfolio value is the same (i.e., constant), regardless of the other alternatives the portfolio contains.

These concerns have motivated research on non-additive portfolio value/utility functions. Already Golabi et al. (1981) proposed that the multiplicative multiattribute utility function should be employed (instead of the additive utility function) to model the utilities of individual investments in (7); this still preserves the ZOLP structure of the optimization problem. Grushka-Cockayne et al. (2008) develop non-additive portfolio value functions and corresponding MILP formulations to support selecting a portfolio of improvement actions in air traffic management. Argyris et al. (2014) consider preferences represented by any concave multiattribute value function and develop MILP formulations to build portfolio optimization approaches in which the decision-maker's preferences are interactively elicited. Liesiö (2014) and Liesiö and Vilkkumaa (2021) establish axiomatic foundations for families of symmetric multi-linear portfolio value and utility functions. They also develop tailored implicit enumeration algorithms as well as ILP and MILP formulations to identify the optimal portfolio under such value/utility functions.

4.6. Other preference models

While value/utility function models have been used extensively to model preferences in portfolio optimization, a smaller strand of literature developed primarily by European researchers has explored other approaches to preference modelling, such as outranking methods and dominance-based decision rules. The main advantage of these approaches is that they are flexible in modelling preference structures that do not fit the standard normative framework of utility theory.

The first application of outranking models to portfolio optimization was presented by Martel et al. (1988) on using ELECTRE I and II methods to evaluate financial portfolios based on the comparison of their risk-return profiles. Concretely, the ELECTRE I method identifies portfolios that best match the decision-maker's preferences, whereas ELECTRE II ranks the portfolios from the best to the worst. The ELECTRE I method has been employed by Perez Gladish et al. (2007) as a component of a three-stage methodology for financial portfolio selection that combines multivariate statistical techniques for modelling returns, a fuzzy multiobjective optimization model for portfolio construction, and the ELECTRE I method for portfolio selection. Apart from portfolio selection, outranking models have supported the initial screening of assets, which is an integral part of the portfolio optimization process. For example, Xidonas et al. (2009) employ the ELECTRE Tri method to sort stocks into performance categories.

Approaches to guide investments into real indivisible assets have also been proposed based on preference models incorporating outranking relations. For instance, Vetschera and de Almeida (2012) present optimization models in which the principles of the PROMETHEE V method are adapted to construct project portfolios. Kandakoglu et al. (2022) combine PROMETHEE methods both with Stochastic Multicriteria Acceptability Analysis (SMAA, Lahdelma et al., 1998) and an integer programming model to generate a cluster of maintenance projects under resource constraints. Balderas et al. (2022) and Fernandez et al. (2015) present hybrid methodologies that integrate outranking approaches founded on ELECTRE methods with multi-objective portfolio optimization models.

Greco et al. (2013) propose a further hybrid approach to financial portfolio optimization. Using the dominance-based rough set approach (DRSA), they capture investors' risk-return preferences and identify efficient portfolios with the mean–variance model. The resulting decision rules from DRSA serve as inputs to an iterative and interactive optimization process that identifies the portfolio that best matches the investors' preferences. DRSA has also been applied to optimize portfolios of real assets. For example, Barbati et al. (2018) formulate a multi-objective optimization problem whose criteria indicate how many projects in the portfolio attain specified reference levels, whereafter DRSA is used to determine the optimum portfolio.

Table 1
Recent examples of applications of financial portfolio optimization.

Field	Authors	Methodology
Mutual funds	Tamiz et al. (2013)	GP
	Utz et al. (2014)	Inverse MO
	Zhou et al. (2018)	DEA
Indices	Abid et al. (2023)	GP
Bonds	Li (2019)	DP
	Drenovak et al. (2021)	MOEA
Cryptocurrencies	Brauneis and Mestel (2019)	MV
	Hashemkhani Zolfani et al. (2022)	MCDA
	Maghsoodi (2023)	MCDA
	Youssef et al. (2023)	GP
Commodities	Al Janabi et al. (2017)	NLP
	Wang et al. (2022)	MV
Derivatives	Zymler et al. (2013)	RO
	Fu et al. (2014)	DP
	Gülpınar and Çanakoğlu (2017)	RO
Insurance	Dächert et al. (2022)	MO

CP: compromise programming, DP: dynamic programming, FMO: fuzzy multiobjective optimization, GP: goal programming, MCDA: multicriteria decision analysis, MINLP: mixed-integer non-linear programming, MO: multiobjective optimization, MOEA: multiobjective evolutionary algorithm, NLP: non-linear programming, MV: mean–variance, RO: robust optimization.

5. Applications, empirical results, and discussion

To a significant extent, portfolio optimization has evolved through efforts to tackle challenges in building and solving models for real-life applications. In effect, the literature on the applications of portfolio optimization is so voluminous that we cannot review it comprehensively. Rather, we seek to offer insights based on reported applications and recent review papers.

5.1. Financial portfolios

Most applications of portfolio optimization models for financial assets are data-driven in that the required modelling parameters are estimated from historical data. In effect, while most studies have focused on equity portfolios for which historical data are readily available, other investment contexts have also been considered. Table 1 gives a representative list of applications from the past decade that highlights the variety of assets, including mutual funds, equity indices, bonds, cryptocurrencies, commodities, derivatives, as well as insurance portfolios that have been considered in portfolio optimization. Beyond the standard mean–variance model, the solution approaches include multiobjective optimization and GP, robust optimization, dynamic and stochastic programming, multi-criteria decision analysis, and non-linear programming (for details, we refer to these papers).

While most applications in Table 1 are founded on a data-driven approach, some approaches incorporate the investor's preferences into the analysis with a constructive scheme. For instance, Ehrgott et al. (2004) extends the risk-return framework into an enriched hierarchy of five objectives (12-month and 3-year return, annual dividend, volatility, Standard and Poor's Star Ranking) which are aggregated to form an investor-specific additive value function. The recommended portfolio allocation is then determined by maximizing the value function using metaheuristics. However, because this value function can be challenging to construct, Ehrgott et al. (2009) extend this approach by adopting the UTADIS preference disaggregation method (Doumpos & Zopounidis, 2002) in which the value function can be inferred from decision examples. Implementing these approaches can be expedited with decision support systems that provide data management capabilities as well as decision modelling and reporting tools (Samaras et al., 2008; Xidonas et al., 2021, 2011).

Such constructive approaches to financial portfolio optimization are becoming increasingly important due to the growing interest in incorporating non-financial objectives into the decision process. Many of these objectives are linked to sustainability principles such as environmental, social, and governance factors (ESG) which have been addressed through goal programming approaches (Abid et al., 2023; Ballestero et al., 2012) and multiobjective formulations (Gasser et al., 2017; Methling & von Nitzsch, 2020; Utz et al., 2014, 2015), for instance. In effect, the vast market on sustainability-linked investments creates a major opportunity for using these models to set up socially responsible investment (SRI) strategies that comply with ESG principles (see, e.g., Chen et al., 2021). These models can also be used in an explanatory framework, as noted by Utz et al. (2014). For example, the recent multiobjective formulations by Steuer and Utz (2023) suggest that the investments of ESG mutual funds fall short of their claimed sustainability policies.

Questions pertaining to the timing of portfolio decisions are addressed, for example, by Pham et al. (2022) who develop a distributionally robust portfolio optimization approach in a continuous time setting in the presence of ambiguity about expected returns as well as correlations between asset returns. Topaloglou et al. (2008) develop discrete-time models for optimizing international stock and bond portfolios through a stochastic programming approach. Because multiperiod models for portfolio optimization can pose computational challenges, advanced computational procedures and solution platforms can be useful. For instance, Östermark (2017) propose a computational approach which combines a genetic algorithm with a time-series module in a massively parallel processing platform.

While there has been plenty of interest in continuous optimization approaches, their practical usefulness has been questioned. For instance, Kolm et al. (2014) note that it can be difficult to estimate model parameters (e.g., expected returns and covariances over multiple periods), to overcome computational challenges caused by complex optimization problems, and to incorporate real-world features at large. Moreover, Elton and Gruber (1997) criticize the common assumption that returns are independent over time. They also argue that although continuous-time approaches have (mostly) confirmed the results of discrete multi-period results by employing more realistic assumptions, they have not offered new significant insights. In effect, the empirical evidence provided by Carroll et al. (2017) suggests that adjusting the composition of the portfolio will hinder performance if the transaction costs are high.

Beyond multi-period optimization schemes, trading methodologies and approaches that explicitly account for the timing of investments are also relevant. For instance, the hybrid approach of Chen and Wang (2015) combines a portfolio optimization model for capital allocation with a genetic algorithm for trading decisions. Décamps et al. (2005) develop an optimal stopping approach to help decide when to invest in a project whose current value is known but whose evolution over time is unknown. De Gennaro Aquino et al. (2023) extend the mean–variance model to a broader set of investment options in which the investor can invest in available investment opportunities or, alternatively, explore new investments that will become available in the future.

Overall, the formulation of more elaborate models and methodological approaches leads to multi-faceted questions about how these models and approaches compare with each other and, importantly, what benefits can be gained by adopting new tools as opposed to following simpler investment strategies. In response to these questions, Table 2 provides a representative list of comparative studies on portfolio optimization approaches for investments in financial assets.

Focusing on bi-objective models, DeMiguel et al. (2009) compare the naïve allocation (i.e., equal weights for all assets) with fourteen models within the mean–variance framework. Their empirical results from the US equity market show that the naïve approach is quite competitive with bi-objective optimization models in terms of its risk-adjusted return (e.g., the Sharpe ratio) and turnover, particularly when the number of assets is large, there is limited historical data, and the idiosyncratic risk is low. In a similar comparative analysis, Cesarone et al. (2020) examine the stability properties of the simple risk diversification strategies and optimization approaches. They conclude that the risk parity strategy (in which the portfolio is chosen so that the assets contribute equally to the total risk) provides the most stable results. However, the risk-adjusted performance of this strategy was often inferior to optimization-based models. Georgantas et al. (2021) examine the stability of portfolio optimization models by employing robust optimization approaches. Based on an analysis of bi-objective models and their robust counterparts, they conclude that the latter has provided superior risk-adjusted performance, particularly during the 2007–2009 financial crisis. Bi-objective approaches are also considered by Mansini et al. (2003) who focus on models that can be formulated as LP problems, based on different risk-metrics such as the mean absolute deviation, the Gini mean difference (GMD) index, and conditional value-at-risk (CVaR). They report that the LP models provide more stable asset allocations than the mean–variance model, whereas GMD and CVaR performed better in their out-of-sample returns.

In a similar study, Ramos et al. (2023) cover a more extensive arsenal of LP solvable models taking transaction costs also into consideration. They employed several performance metrics in comparing the results of the optimization models with the Standard & Poor's 100 index. Although the benchmark index's performance was inferior to the portfolios constructed with optimization approaches, they note that no conclusive finding can be drawn as to the relative performance of these models when multiple performance metrics are considered. This suggests that approaches that accommodate multiple criteria can be well-aligned with the many facets in assessing portfolio performance.

Multiobjective models have been considered by Anagnostopoulos and Mamanis (2011), Ceren and Köksalan (2014), and Pavlou et al. (2019), among others. Anagnostopoulos and Mamanis (2011) compare five MOEAs in cardinality-constrained mean–variance optimization, focusing on the computational performance of algorithms and the quality of solutions. The strength Pareto evolutionary algorithm (SPEA2) performed best of the tested algorithms. Ceren and Köksalan (2014) used a data set from the Istanbul stock exchange to compare bi-objective and multiobjective models that covered four criteria (return, variance, liquidity, CVaR). They gave much attention to the conflicting nature of the objectives and the added value they bring into the portfolio optimization process. Yet a limitation of this analysis was that it was based on in-sample results.

Focusing on discrepancies between in-sample and out-of-sample performance, Pavlou et al. (2019) examined three bi-objective models and a three-objective formulation using Standard & Poor's 500 US index data. The mean–variance formulation gave the most robust results of the three bi-objective models. Nevertheless, the multiobjective model with three objectives (return, mean absolute deviation, CVaR) outperformed the bi-objective formulations in terms of its out-of-sample robustness.

Table 2
A representative list of comparative studies on financial portfolio optimization.

Author	Objectives (beyond return)	Model type	Other features
Mansini et al. (2003)	Variance, MAD, Maximum loss, GMD, CVaR	LP	
Angelelli et al. (2008)	MAD, CVaR	MILP	Heuristics, CCP
DeMiguel et al. (2009)	Variance	QP	
Anagnostopoulos and Mamanis (2011)	Variance	MIQP	MOEAs, CCP
Ceren and Köksalan (2014)	Variance, CVaR, Liquidity	LP/QP & MOO	
Pavlou et al. (2019)	Variance, MAD, CVaR	LP/QP, MOO	
Cesarone et al. (2020)	Variance, Semi-MAD, CVaR, Maximum loss, Omega ratio	LP/QP	
Georgantas et al. (2021)	Variance, CVaR, Omega ratio	LP/QP	RO
Ramos et al. (2023)	Expected/maximum loss, Expected loss deviation, Shortfall deviation risk, Expectile VaR, Deviation expectile VaR	LP	Transaction costs, CCP

CCP: Cardinality constrained portfolios, GMD: Gini mean difference, LP: Linear programming, MILP: Mixed integer LP, QP: Quadratic programming, VaR: Value-at-risk.

5.2. Portfolio decision analysis for allocating resources to indivisible assets

The recent survey paper Liesiö et al. (2021) reviews recent developments in PDA by examining 146 papers from 2006 to 2019 with an emphasis on the features of the decision model, its solution approach, and, where applicable, the primary application area. In particular, optimization was the solution approach in 82% of the papers, highlighting the close connections between PDA and portfolio optimization. Mohagheghi et al. (2019) review the literature at the intersection of project management and project portfolio optimization. They cover 148 papers published from 1993 to 2018, based on the selection of evaluation criteria, representation of uncertainties, and choice of the solution approach. Like Liesiö et al. (2021), they conclude that the highest number of papers in this area have been published in the *European Journal of Operational Research*.

Given these comprehensive reviews on optimization models for selecting portfolios of indivisible alternatives, we do not seek to cover the wealth of research in this area. Instead, we give an overview of representative case studies to illustrate the breadth of application areas and the diversity of methodological approaches. The selection of these papers in Table 3 is geared towards relatively recent examples which better reflect state-of-the-art capabilities. For most (but not for all) of these papers, we summarize the problem and the approach taken to address it.

Defence: In their review of 54 papers, Harrison et al. (2020) point to the unique challenges in applying portfolio optimization in defence because, for example, planning horizons are often long and the consequences of deploying military systems depend on the actions of counterparties. This notwithstanding, optimization is helpful as it provides a structured framework within which relevant inputs can be synthesized to inform the design and operation of defence systems. The usefulness of such a framework is demonstrated by Kangaspunta et al. (2012) who present how combat models can be integrated with optimization models to identify portfolios of cost-effective weapons systems in achieving military goals. Davendralingam and DeLaurentis (2015), in turn, consider how the development of military capabilities in System of Systems (SoS) can be guided through robust portfolio optimization, focusing on choices from alternative architectural designs. Their approach is illustrated with a naval warfare scenario in which portfolios of viable systems are generated from a candidate list of available systems.

R&D Management: Gouglas and Marsh (2021) describe how the Coalition of Epidemic Preparedness Innovations (CEPI) allocated some US$ 140M in 2017 to technology platforms to accelerate the development of effective vaccines for epidemic infectious diseases caused by some previously unknown pathogen. In this case study, 16 platform projects were assessed within a valuation framework that accounted for project-level and portfolio-level considerations. Estimates about seven factors contributing to each platform's success probability were elicited from experts and then combined using a multiplicative model. Preference information was elicited through discrete choice experiments. Finally, Monte Carlo simulation and extensive sensitivity analyses were employed to assess the performance of feasible portfolios. In retrospect, this application has been enormously impactful as six selected projects served as platforms for developing COVID-19 vaccines that helped combat the global pandemic.

Another instructive example from R&D management is given by Kloeber (2011) who gives a good overview of what approaches are commonly employed by large pharmaceutical companies. Possibilities for adopting interactive approaches are illustrated by Stummer and Heidenberger (2003) who propose a three-phase approach that starts by screening the project proposals that are worth further evaluation. In the second phase, all non-dominated portfolios are computed by solving a multiobjective integer linear programming model whose constraints include interdependencies, strategic requirements, and resource constraints. In the third phase, the decision-maker can set aspiration levels for the objectives in order to explore the set of feasible portfolios iteratively until a satisfying portfolio is identified.

Energy and climate change: Electricity retailers are exposed to financial risks when they procure electric energy from the spot market before delivering it to meet their customers' demands. Rocha and Kuhn (2012) show how retailers can hedge against these risks with a portfolio of electricity derivatives constructed by solving a multistage stochastic mean–variance optimization model. Computational challenges are tackled by employing linear decision rules to restrict the recourse decisions so that the optimal portfolio can be obtained from a convex quadratic program.

Fleischhacker et al. (2019) describe a case study on using portfolio optimization to promote distributed energy resources and to implement energy efficiency measures in Linz, Austria. Building on available open-source models and clustering algorithms, they equip the stakeholders with tools for calculating the capabilities and restrictions of the local energy system with regard to two partly conflicting criteria, i.e., costs

Table 3
Selected examples of optimizing the allocation of resources to real assets.

Field	Authors	Methodologies
Defence	Harrison et al. (2020)	Survey of methods
	Kangaspunta et al. (2012)	Simulation, cost-efficiency analysis
	Davendralingam and DeLaurentis (2015)	Robust portfolio optimization
R&D management	Gouglas and Marsh (2021)	Stochastic dominance, Monte Carlo simulation
	Kloeber (2011)	Risk analysis, pipeline optimization
	Stummer and Heidenberger (2003)	Multiobjective optimization, aspiration levels
	Noro and Dias (2023)	Bi-objective optimization
Energy and climate change	Rocha and Kuhn (2012)	Multi-stage stochastic optimization
	Fleischhacker et al. (2019)	Scenarios, clustering algorithms
	Baker et al. (2020)	Robust PDA, belief dominance
Urban and environmental management	Lahtinen et al. (2017)	Survey of methods, application of RPM
	Fasth et al. (2020)	Conflict constrained optimization
Maintenance	Mild et al. (2015)	RPM with approximative algorithms
	Sacco et al. (2019)	RPM
	Mancuso et al. (2017)	Fault trees, Bayesian belief networks
Supply chains	Sawik (2019)	System dynamics, dynamic optimization
Built infrastructures	Barbati et al. (2023)	MCDA, PROMETHEE, ELECTRE Tri
	Roberti et al. (2017)	AHP, multiobjective optimization
Human resources	Gutjahr (2015)	Survey of methods
	Gutjahr et al. (2010)	Bi-level optimization
	Stummer and Kiesling (2009)	Multicriteria screening, aspiration levels

AHP: Analytic Hierarchy Process; RPM: Robust Portfolio Modelling; MCDA: Multicriteria Decision Analysis.

and carbon emissions; they also use scenario planning (Bunn & Salo, 1993) to examine possible lock-in effects of existing infrastructure and future developments. The results alert to the importance of the steady transformation of local energy systems in reaching economically sustainable goals.

Motivated by the recognition that long planning horizons may involve ‘deep’ uncertainties that cannot be captured by completely specified probability distributions, Baker et al. (2020) present an approach based on robust portfolio decision analysis to guide R&D investments into energy technologies. Specifically, they identify which portfolios of R&D investments are robust in view of the beliefs expressed by three different expert groups on how R&D investments will impact the future technological capabilities that can be leveraged to mitigate climate change. This application is also instructive as it demonstrates how portfolio optimization can be integrated with integrated assessment models (IAM).

Urban and environmental management: In general, applications of portfolio optimization for real investments typically involve multiple criteria; they may also involve investment alternatives that are rather different from each other. Focusing on environmental management, Lahtinen et al. (2017) note the need to construct portfolios consisting of many kinds of actions (e.g., energy saving measures, investments in renewables, educational projects, development and adoption of new technologies, regulation policies) so that when these actions are implemented together, they are jointly effective in contributing to multiple decision objectives such as mitigating greenhouse gas emissions, fostering biodiversity, and minimizing costs. As a result, the construction of such portfolios of actions can be usefully guided by portfolio optimization. In support of this claim, Lahtinen et al. (2017) provide pointers to several environmental management applications in which portfolio optimization techniques, such as quadratic programming and multiple objective integer optimization, have been effective in providing decision support.

Urban planning affects the quality of life within local communities over different spatial scales and time horizons. It also involves many stakeholder groups which may have conflicting interests. To manage possible conflicts, Fasth et al. (2020) construct a decision analytic framework in which stakeholders assess urban planning actions and estimate the weights of the criteria employed in this assessment. For each action, a conflict index and the overall value are calculated. All Pareto efficient action portfolios are computed from an optimization model in which different levels of conflict are treated as a constraint. Sensitivity analysis is an integral part of this framework, illustrated with data from the municipality of Upplands Vasby, Sweden.

Maintenance: Mild et al. (2015) describe how the Finnish Transport Agency (FTA) has embraced portfolio optimization in its annual planning processes when selecting dozens of bridge maintenance projects from hundreds of project candidates. Specifically, this selection decision is guided by the Core Index values of an RPM model which accounts for multiple criteria, project interdependencies, uncertainties concerning project consequences, and financial and other relevant constraints.

Mancuso et al. (2017) employ portfolio optimization to identify cost-effective safety measures for the airlock subsystem of the CANDU nuclear power plant. Towards this end, they convert the fault tree representation of the subsystem into a Bayesian belief network as a framework for assessing the consequences of different portfolios of safety measures. In particular, they show that the optimization approach leads to considerably lower residual risk for different cost levels than the selection of safety measures based on conventional risk importance measures.

Sacco et al. (2019) develop a risk-based maintenance framework to help decision-makers select optimal maintenance plans to reduce the severity and likelihood of failures in a high-pressure natural gas pipeline in Great Britain. The results indicate, for example, that if there are no spatial constraints on budget allocation across zones, the selected maintenance actions will be concentrated only on some critical zones such as Scotland and the southernmost part of England. However, if predefined maintenance budgets are apportioned to different areas, the selected portfolio may be sub-optimal with a smaller risk reduction over the maintenance horizon.

Supply chains: Sawik (2019) proposes a dynamic optimization model to assist a manufacturer in choosing a portfolio of suppliers based on indicators reflecting their financial stability, production stability, product quality, and cost. The selection of suppliers and the allocation of orders to them are guided by a combined simulation–optimization framework with real-time monitoring of supply risk indicators. Specifically, a system dynamics model is first employed to assess the effect of supply risk indicators on a manufacturer’s profit over a planning horizon. Then, a portfolio optimization model is solved to determine the optimal order allocation to suppliers in view of the manufacturer’s risk propensity. Thus, the manufacturer can re-balance its supply portfolio in response to early changes in supply risk indicators to ultimately gain higher expected profits with lower risks.

Built infrastructures: Barbati et al. (2023) present an application of portfolio optimization to foster cultural heritage by prioritizing twenty project interventions on historical buildings in Naples, Italy.

Specifically, they propose an approach called Priority Based Portfolio Selection which employs multiple criteria sorting and ranking methods, such as PROMETHEE and ELECTRE TRI, to compare projects based on the consideration of qualitative and quantitative criteria. Then, feasible portfolios of projects satisfying the resource and logical constraints are identified before the final recommendations are derived through portfolio optimization. Although this approach has been initially developed for prioritizing conservation projects, it is generic enough to be applied in different contexts.

Another example related to historical buildings is presented by Roberti et al. (2017) who evaluate possible retrofit actions such as external and internal envelope insulation, airtightness improvement, windows replacement, and ventilative cooling. They use the Analytic Hierarchy Process (Saaty, 1980) to derive a conservation score for each action based on the judgements that are elicited from ten experts. This conservation score is combined with the energy needs for heating and cooling as well as thermal comfort in a multiobjective optimization model whose solution gives the optimal portfolio retrofits for a mediaeval building in Italy. This portfolio allows for a four-fold reduction in energy needs at a high thermal comfort level.

Human resources: Gutjahr et al. (2010) develop a multi-objective optimization model to guide the selection of projects considering both economic benefits, on the one hand, and the impacts that alternative project assignments have on the evolution of competencies, on the other hand. Pareto-optimal solutions with regard to these two criteria are obtained by decomposing the problem into (i) a master problem for portfolio selection and (ii) a slave problem for assigning personnel to the work packages of selected projects. Linearized formulations are provided to solve the slave problem efficiently, and experimental results are presented for synthetically generated test instances as well as for real data from a software-intensive organization. Related themes in managing human resources are addressed by Stummer and Kiesling (2009) who view human capital not only as an input resource required for conducting research but also as the output of pursuing that research. Specifically, they present a multi-criteria decision support system (MCDSS) with financial and non-financial objectives and constraints. In this system, the set of Pareto-efficient solutions is first determined so that the decision-maker can interactively explore and filter this set. More generally, Gutjahr (2015) reviews project selection models that incorporate skills development by learning and/or forgetting.

5.3. Discussion

The above summary of applications demonstrates the breadth of problems that can be tackled through portfolio optimization. Still, one may ask what kinds of decision problems are particularly amenable to portfolio optimization and what caveats one should be aware of in seeking to benefit from it.

For starters, portfolio optimization has been immensely impactful in finance. Its profusion has been expedited by good access to data resources and, in many cases, apparent clarity concerning relevant decision criteria (e.g., expected return, volatility). Moreover, because major financial commitments are at stake, considerable benefits can be expected by moving from a sub-optimal portfolio to the optimal one. It is also usually possible to *validate* the benefits of portfolio optimization *ex-post*, for example by examining the financial outcomes and associated consequences for the decision-maker that would have materialized if the investment decision had been different from the one recommended by portfolio optimization. Still, even if there is abundant historical data on financial assets, the financial environment continues to evolve dynamically, which limits the value of the information that back-testing validation procedures provide for future investment decisions (Reschenhofer et al., 2020).

An example of another field with influential penetration of portfolio optimization is pharmaceutical R&D (cf. Gouglas & Marsh, 2021) in which projects advance through a well-defined stage-gate structure, guided by evaluation criteria that focus on the assessment of financial aspects and the demonstration of safety, and efficacy (see Hesarsorkh et al., 2021; Kloeber, 2011). Yet in the selection of 'far-out' innovation projects, it is often much less clear what the projects are expected to achieve. Moreover, the experts may not be able to assess projects with much confidence, and it is practically impossible to ascertain *ex-post* what would have happened if another portfolio of innovation projects had been selected. These challenges may explain why the development of advanced quantitative methods for selecting innovation projects has had a limited impact on practice in settings where the innovation process is unstructured, unpredictable, and ambiguous (Si et al., 2022).

In view of the above, it is easier to justify the use of portfolio optimization in problems in which there are concrete benefits from introducing a structured decision process with a well-defined time horizon, clarity about the investment alternatives and the ability to elaborate a sufficiently comprehensive set of decision criteria with regard to which the alternatives can be evaluated with a reasonable level of confidence. Conversely, if significant difficulties are encountered in building a validated optimization model, then using numerical optimization results for making investment choices may not be warranted. Still, attempts to build such a model may help expose, for example, limitations in data resources or conflicting perceptions about the 'right' problem scope. An awareness of such obstacles can be instructive in suggesting preparatory activities that need to be completed before a useful optimization model can be built. When making efforts towards this end, one should remain attentive to any concerns that may have been inadvertently neglected or purposely omitted in specifying the model.

Many benefits of portfolio optimization stem from its ability to provide recommendations for portfolio selection based on a systematic comparison within a broad set of comparable alternatives. Intuitively, one might think that choosing this portfolio of 'best' alternatives from a truly comprehensive set of alternatives would be better than pooling the results for optimizing smaller subsets containing these same alternatives. Yet several reasons suggest that the temptation to extend the scope of portfolio models in terms of time horizons or different kinds of alternatives should be resisted:

- If the set from which a portfolio is selected contains very different kinds of investment alternatives, it may be harder to specify tangible evaluation criteria that can be meaningfully interpreted and consistently applied across all these alternatives. Therefore, depending on the context, one may aspire to make comparisons across alternatives that are sufficiently 'similar' so that they can be legitimately compared.
- In many cases, extending the time horizon over which the portfolio is optimized will lead to more uncertainties about the consequences of the investment alternatives, which may erode the credibility of the results. Also, the ensuing reinvestment problem must be recognized and addressed if the assets are not of the same maturity.
- Constructing an optimization model in order to select the portfolio from a very large set of alternatives may involve indirect costs, for example, due to greater administrative efforts or the need to postpone decisions on some alternatives to create such a set. Thus, it is pertinent to assess if the benefits of optimizing for this comparatively larger set outweigh the costs of building such a set in the first place. Furthermore, when new investment alternatives arise continually, the decision process needs to be designed in view of expectations concerning what future alternatives may become available. For example, Vilkkumaa et al. (2015) present multi-stage simulation–optimization models to support the reallocation of resources to new alternatives.

Models of portfolio optimization can be built to tackle new problems or to establish decision-making practices in which such models are solved recurrently, even routinely, with the aim of gaining enduring benefits. Developing and adopting such practices can be seen as a learning process which may take time. Yet it can be expedited by distilling 'lessons learned' from using portfolio models for decision support (see, e.g., Mild et al., 2015) and by paying attention to their behavioural fit within the organization (Luoma, 2016).

We also stress that there are application areas in which mastering the mathematical aspects of portfolio optimization does not suffice for providing effective decision support. In environmental decision-making (Lahtinen et al., 2017) and urban planning (Fasth et al., 2020), for instance, there are often stakeholder groups whose interests must be accounted for, because the recommended policy actions may be met with much resistance unless the stakeholder groups are engaged in the decision process. In such situations, competencies in the design and facilitation of participatory decision processes are crucial.

6. Opportunities for future research and practice

Here, we offer reflections on timely opportunities in portfolio optimization for continued research and applied work. Many of these opportunities are linked to introducing new modelling features to extend the range of problems that can be approached with portfolio optimization. Although we do not delve into technicalities, we note that many of these opportunities are linked to methodological and algorithmic challenges that need to be tackled.

- *Building integrated models with a broader span of criteria and uses*: In finance, concerns such as economic, social, and environmental sustainability have become increasingly important, giving rise to questions about how these kinds of inherently qualitative criteria can be best incorporated into models of portfolio optimization. This can be challenging, as the criteria must be adequately defined and the corresponding data resources must be of sufficiently high quality. Conversely, models for selecting portfolios of real assets can be extended by including financial assets and performance measures. Thus, for example, budgets need not be viewed as fixed constraints but as decision variables that can be optimized. Overall, there is much potential in developing integrated approaches, as demonstrated, for instance, by Tinoco et al. (2018) who optimize business project portfolios in view of economic, social, and environmental aspects.
- *Establishing interfaces to non-numerical data sources*: Much of the online qualitative textual information is relevant to portfolio optimization. For example, written comments on financial news convey information about investor sentiment. For optimization purposes, however, such information has to be quantified, which creates a need for tools capable of exploring qualitative data resources and converting such resources into numerical inputs (see, e.g., Yu et al., 2022). Such tools can also provide useful information about stakeholders' values and preferences. Thus, they extend the range of problems that can be tackled with portfolio optimization.
- *Synthesizing statistical analysis with subjective perceptions of uncertainties*: In many problems, there are significant uncertainties about the future consequences of alternative portfolios. Although much research has been done to improve the characterization of uncertainties, further efforts are needed to build models that accommodate not only data-driven estimates of uncertainties based on statistical techniques but also subjective views on uncertainties, obtained through structured processes of expert judgement elicitation. Incorporating such views (or beliefs, see Baker et al., 2020) may be especially useful when there is not much relevant data from the past. In pursuing such an integration, probabilistic approaches are appealing in that they make it possible to synthesize inputs from diverse information sources within the sound framework of standard probability theory.
- *Optimizing decision processes for portfolio selection*: It is often possible to develop an improved portfolio model through efforts that reduce uncertainties by providing more accurate estimates about the consequences of the alternative assets being considered. If these efforts are insufficient, the remaining uncertainties may be so significant that there is a high probability of choosing a sub-optimal portfolio *ex-post*. Still, if such efforts towards reducing uncertainties consume plenty of resources, the benefits of being able to choose a better portfolio may not compensate for the resources that are consumed by these efforts. Thus, these kinds of inherent tradeoffs can be studied by employing the concept of Value of Information to design the selection process optimally in view of all stages that span from the formulation of the portfolio model to the implementation of the recommended portfolio. Specifically, the *optimal design of processes for portfolio selection* is thus an important topic that can be approached, for example, through simulation–optimization studies to derive well-founded triage decision rules (Keisler, 2004) that can be combined with portfolio optimization. For instance, if exceptionally excellent investment opportunities are encountered, it may be optimal to seize them promptly. In contrast, decisions on less attractive opportunities can be postponed and resolved later through portfolio optimization.
- *Designing robust and new assets based on portfolio optimization*: Portfolio problems often involve significant uncertainties whilst many decision-makers are risk averse. Therefore, instead of focusing on the solution that is optimal for a single choice of parameter values that proves erroneous *ex-post*, it may be advantageous to identify *robust* portfolios that perform reasonably well across a wide range of parameter values. Here, the computation of non-dominated portfolios based on the concept of stochastic dominance, for instance, is a powerful approach as one can derive conclusions about which alternative assets are included in all non-dominated portfolios and are therefore robust in view of the full range of parameter values. Although much work has been carried out in this area, these analyses can be extended to address fresh research questions in portfolio optimization. For example, rather than taking the set of alternative assets as a given, one may ask how new assets should be *designed* so that they will be included in many or most non-dominated portfolios.
- *Ensuring the validity of models*: Because portfolio optimization models rely on the information available at the time of the investment decision, the recommended portfolio is guaranteed to be optimal with respect to this information only. Thus, when the investment environment evolves, it is crucial to validate the underlying modelling hypotheses and carefully assess the selected portfolio's realized (future) performance. This kind of *ex-post* validation complements *ex-ante* approaches such as robustness analysis. It also helps build an integrated framework for establishing empirically grounded conclusions about the effectiveness of optimization models in practice. Validation can extend beyond examining a single portfolio to consider all non-dominated portfolios (Kao & Steuer, 2016; Pavlou et al., 2019). Ideally, validation tests should not be limited to simple measures of profitability/return; instead, they should span a broader range of criteria in light of possible changes in the decision-makers' preferences over time.
- *Enhancing human interactions*: The interactions through which information is elicited from decision-makers and stakeholders, as well as the ways in which results of portfolio optimization are communicated to them are key determinants of the quality and eventual impact of decision support. For example, if the elicitation process is inadequate, the model will not be valid; and if the results are not clearly communicated, the decision recommendations may not be understood or trusted. Thus, there is a need to enhance these human interactions, for instance through new

elicitation protocols and visualization techniques. These advances need not be geared to developing very large portfolio models, which might require excessive elicitation effort. It may be more fruitful to build more parsimonious models whose results can be appreciated interactively in workshops, for instance.

While many portfolio optimization models are sophisticated for a good reason, we caution against developing models whose features are unnecessarily complex for their intended use. One reason is that the simultaneous compounding of multiple approaches in representing preferences and uncertainties, for example, may give rise to models that lack transparency. This can make it hard to explain the recommendations and, at worst, may contribute more to confusion than clarity. Yet if models fail to capture key concerns that actually matter (e.g., significant uncertainties or interdependencies between assets), there is a risk of losing validity due to an inadequate problem representation. Against this backdrop, there is a continuing need for reflective case studies with real data and real decision-makers, as only such case studies permit warranted conclusions about what kinds of portfolio models work well in practice. Such conclusions cannot be reached by speculating how imaginary decision-makers would interact with overly sophisticated or simplistic models of hypothetical problems.

7. Conclusions

For a long time, portfolio optimization has been one of the most prominent fields of operational research, marked by significant theoretical, methodological, and algorithmic advances as well as compelling achievements in supporting the allocation of resources across a stunningly diverse range of problems. This celebrated history notwithstanding, much work is needed to expand its capabilities further. For example, more expressive preference models are needed to fully capture concerns such as sustainability; the impacts of unforeseen disruptions need to be accounted for in adapting risk measures for better risk management; and ever-better optimization techniques are needed to solve large portfolio models that are presently intractable. Furthermore, progress in synergistic areas such as text mining, deep learning, and human–computer interaction can be exploited to create capabilities for building, solving, and using portfolio optimization models beyond the state of play.

Overall, building on this paper's extensive scope, which covers divisible and indivisible assets, we see exciting opportunities that arise from the cross-fertilization between these complementary and synergistic areas of portfolio optimization. For example, the need to quantify and mitigate financial risks has spawned risk measures that can now be deployed to support the management of real assets. Conversely, advances in building preference models for multiple attributes can be used to address concerns such as the UN Sustainable Development Goals that transcend the scope of conventional financial evaluation criteria.

Efforts are also needed to understand even better in what kind of problems portfolio optimization has most to offer. Many of the benefits of portfolio models come from the possibility of allocating resources to alternative assets based on a more systematic and comprehensive evaluation than what would otherwise be the case. Still, if the costs of formulating such models outweigh the benefits, more straightforward approaches, such as taking decisions on individual assets sequentially one by one based on triage decision rules, may be considered. Thus, one should not take it for granted that the construction and solution of a full-blown portfolio optimization model is optimal.

Although portfolio optimization is a comparatively mature field, there are promising avenues for advancing its frontiers. These frontiers include but are not limited to incremental enhancements to its mathematical and methodological algorithmic core, because there are attractive opportunities that stem from the challenges encountered in problems that have not yet been satisfactorily mastered. In developing this emerging agenda, there is a need for a sufficiently close interplay between theory and practice to ensure pertinent aspects of validity, ranging from well-founded modelling activities to effective interaction with decision-makers. This combination of methodological rigour and practical relevance will ensure that the future of portfolio optimization will be bright indeed.

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