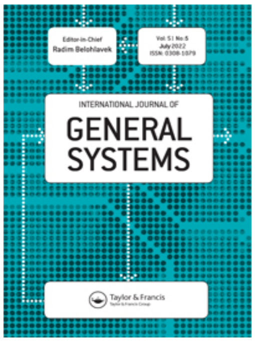

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Advancing construction of conditional probability tables of Bayesian networks with ranked nodes method

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ABSTRACT

System models based on Bayesian networks (BNs) are widely applied in different areas. This paper facilitates the use of such models by advancing the ranked nodes method (RNM) for constructing conditional probability tables (CPTs) of BNs by expert elicitation. In RNM, the CPT of a child node is generated using a function known as the weight expression and weights of parent nodes that are elicited from the expert. However, there is a lack of exact guidelines for eliciting these parameters which complicates the use of RNM. To mitigate this issue, this paper introduces a novel framework for supporting the RNM parameter elicitation. First, the expert assesses the two most probable states of the child node in scenarios that correspond to extreme states of the parent nodes. Then, a feasible weight expression and a feasible weight set are computationally determined. Finally, the expert selects weight values from this set.

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1. Introduction

Bayesian networks (BNs) (e.g. Fenton and Neil 2013) are utilized in numerous system models to represent uncertain knowledge and conduct reasoning under uncertainty. Their application areas include, e.g. medical decision making (Hill et al. 2021; Constantinou et al. 2015), risk and safety management (Xia et al. 2017, 2018; Baraldi et al. 2015; Dahll and Gran 2000), project management (Freire et al. 2018; Yet et al. 2016; Perkusich et al. 2015), maintenance, policy and military planning (Mancuso et al. 2021; Barons, Wright, and Smith 2017; Barreto and Costa 2019), software defect prediction (Fenton et al. 2008), and adaptive student testing (Plajner and Vomlel 2020). A BN describes a system as a directed acyclic graph with nodes representing random variables and arcs indicating their direct dependencies. The dependencies are quantified as conditional probabilities. With sufficient data available, both the graph structure and the conditional probabilities of a BN can be determined with data-fitting approaches (Scanagatta, Salmerón, and Stella 2019; Alsuwat et al. 2020; Neapolitan 2004). Alternatively, the whole BN or some parts of it can be constructed by eliciting information from domain experts (Kjærulff and Madsen 2013; Fenton and Neil 2013), which is a common practice in the construction of various types

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of system models (Pai and Gaonkar 2020; Zhu et al. 2020; Jin et al. 2019; Brunelli 2018). The graph and the probabilities of the BN encode together the joint probability distribution of the nodes. This property enables one to make fast probabilistic queries about the nodes. The results of the queries are communicated well by the visual graph. These features of BNs make them useful system models in many probabilistic domains. In addition, influence diagrams (e.g. Howard and Matheson 2005), that are the decision theoretical extension of BNs, offer further means to support decision making under uncertainty in systemic environments (e.g. Virtanen, Raivio, and Hämäläinen 1999, 2004).

In many practical applications, BNs contain specific types of nodes called ranked nodes (Fenton, Neil, and Caballero 2007). They represent through discrete ordinal scales quantities that can be considered continuous but may lack well-established interval or ratio scales. Like with any nodes having discrete scales, the dependencies between ranked nodes are quantified in conditional probability tables (CPTs). A CPT defines the probability distributions of the descendant, the child node, for all combinations of states of its direct predecessors, the parent nodes. Because the measurement scales of ranked nodes are often subjective, e.g. {low, medium, high}, the construction of their CPTs is usually based on expert elicitation. However, as a single CPT may consist of dozens or even hundreds of elements, it is often impossible to have the expert assess all the required probabilities due to, e.g. cognitive strain or scarcity of time (Druzdel and van der Gaag 2000; Monti and Carenini 2000). To deal with this challenge, Fenton, Neil, and Caballero (2007) have developed the ranked nodes method (RNM) in which the CPT is generated based on a small number of parameters elicited from the expert. These parameters are (1) a function called the weight expression that defines the basic rule by which the parent nodes affect the child node, (2) weights of the parent nodes that represent their relative strengths of influence on the child node, and (3) a variance parameter that describes how precisely the state of the child node is known if the states of the parent nodes are known. There are four alternatives for the weight expression in RNM. The whole CPT can be constructed by using one of them together with single values of the weights and the variance parameter. It is also possible to divide the CPT into parts based on the states of selected parent nodes and generate each part with a part-specific weight expression and part-specific values of the weights and the variance parameter. Constructing the CPT in parts is referred to as the use of a partitioned weight expression.

RNM is implemented in AgenaRisk software (Agena Ltd 2021) through which the method has been utilized in many BN system models (e.g. Freire et al. 2018; Fenton et al. 2008; Kaya and Yet 2019). Research on the methodological properties of RNM has also been conducted. Fenton, Neil, and Caballero (2007) report on the benefits of RNM over the manual construction of a CPT, i.e. assessing each CPT element individually, in a case study concerning reliability evaluation of electronic components. While the manual construction was laborious and led to the CPT being probabilistically inconsistent, RNM enabled the construction of a consistent CPT with much smaller elicitation effort from the experts. Mkrtchyan, Podofillini, and Dang (2016) investigate the ability of CPTs constructed with RNM to portray probabilistic relationships typical in human reliability analysis. Laitila and Virtanen (2020) elaborate the theoretical principle of RNM and explore how well CPTs in various real-life BN-based system models can be reproduced with RNM. They have also established elicitation practices concerning the application of RNM to nodes with continuous scales (Laitila and Virtanen 2016, 2022). Noguchi, Fenton, and Neil (2019) study the capability of RNM to represent the explaining away property of binary variables.

Despite the easy deployment with AgenaRisk and the methodological insight provided by the aforementioned studies, the effective use of RNM suffers from the lack of exact guidelines for the elicitation of the parameters from a domain expert (Mkrtchyan, Podofillini, and Dang 2016; Laitila and Virtanen 2016). This shortcoming can result in a cumbersome and time-consuming elicitation process in which suitable values for the parameters are sought through trial and error (Laitila and Virtanen 2016). For a subject-matter expert ignorant of the technicalities of RNM, it may even, incorrectly, appear impossible to construct a desired type of CPT by using the method. The practices presented by Laitila and Virtanen (2016, 2022) ease the elicitation of RNM parameters, but they are applicable only for ranked nodes defined through discretized continuous scales.

This paper brings further relief to the elicitation challenges of RNM by presenting a new elicitation framework designed for ranked nodes whose ordinal scales are defined by subjective labels. The framework consists of two elicitation procedures called "initial" and "supplementary", and a computational procedure. In the initial elicitation procedure, a domain expert makes mode pair assessments, i.e. assesses two most probable states of the child node, in scenarios that correspond to extreme combinations of states of the parent nodes. This is followed by the computational procedure that diagnoses whether any of the four alternative weight expressions of RNM is feasible with regard to the mode pair assessments. A weight expression is feasible if, with some weights, it yields for the extreme scenarios such probability distributions of the child node that the two most probable states match the mode pair statements of the expert. If a weight expression is feasible, the procedure produces also the set of feasible weights, i.e. all the values of weights with which the probability distributions of the child node in the extreme scenarios become compatible with the mode pair assessments. If a weight expression is not feasible, the outcome of the procedure is information on how the mode pair assessment should change for the infeasibility to become resolved. With a feasible weight expression, point values for the weights are selected from the feasible weight set with the supplementary elicitation procedure. Here, the expert decides the point value of a single weight by reviewing the probability distribution of the child node in a single extreme scenario. A value for the variance parameter is chosen in the same procedure. The framework covers two ways of using RNM. The primary way is that the whole CPT of the child node is generated with a single weight expression as well as with single values of the weights and the variance parameter. The secondary way is constructing the CPT in parts with a partitioned weight expression. A need to apply a partitioned weight expression is indicated by the outcomes of the procedures of the framework.

The elicitation framework facilitates the construction of BN system models with RNM in multiple ways. By utilizing the initial elicitation procedure and the computational procedure, a feasible weight expression and the related set of feasible weights are found with low elicitation effort and without deep technical understanding of RNM. With an available MATLAB (The MathWorks, Inc 2019) implementation of the computational procedure (Laitila 2021), the feasible weight expression and the feasible weight set are solved in a typical RNM application in a matter of seconds given the mode pair assessments concerning the extreme scenarios. Thus, with the expert providing only ordinal probabilistic assessments to well-structured elicitation questions, the feasible weight expression and the set of feasible weights are readily discovered. The feasible weight set eases the selection of point values for the weights by considerably narrowing down their ranges. In addition, with the supplementary elicitation procedure, the weights receive transparent interpretations as the point value of each weight is decided based on a single extreme scenario. These interpretations help to justify the selected point values. The execution of the supplementary elicitation procedure is facilitated by an accessible MATLAB implementation (Laitila 2021). The framework encompasses the use of a partitioned weight expression through extended versions of the aforementioned procedures and implementations. They address the application of a partitioned weight expression with a level of detail lacking from the existing literature on RNM.

The paper is organized as follows. Section 2 provides an overview and a comparison of CPT construction methods that are complementary to RNM. Section 3 briefly presents the concepts of ranked nodes and RNM. Furthermore, the challenges related to the parameter eliciation of RNM are discussed. Section 4 presents the steps and the procedures of the new elicitation framework when using a single weight expression. In turn, Section 5 discusses the use of the framework with a partitioned weight expression. The application of the framework is demonstrated throughout these sections with an example BN. Finally, Section 6 provides concluding remarks and discusses topics for further research. Additional results are presented in the supplementary material, with references to them marked in the paper by "S.#".

2. Overview of parametric methods for eliciting conditional probability tables

Defining CPTs by expert elicitation is typically the most challenging part of the construction of a BN (Renooij 2001; Druzdel and van der Gaag 2000). The size of a CPT grows exponentially with the number of parent nodes. Therefore, the number of probabilities to be specified even for a single CPT may rise to tens or hundreds. Assessing so many probabilities coherently and without biases can be virtually impossible for a domain expert due to mental fatigue or lack of time. Conventional probability elicitation techniques, such as probability wheel or reference lottery, may be used to mitigate cognitive biases when the number of required assessments is small. However, they are generally considered to be too time-consuming for the construction of CPTs (Renooij 2001; Druzdel and van der Gaag 2000).

To deal with the elicitation challenge of CPTs, their construction is often carried out through methods referred to as parametric probability distributions (Druzdel and van der Gaag 2000), canonical distributions (Russell and Norvig 2003), canonical models (Pearl 1988), and filling-up methods (Mkrtchyan, Podofillini, and Dang 2016; Rohmer 2020). These methods enable the construction of a CPT through parameters that are assessed by the expert and whose number is significantly smaller than the number of elements in the CPT. Table 1 lists well-known parametric methods along with information on their features. These methods are next briefly discussed and compared to RNM.

Røed et al. (2009) have developed a method that shares some key characteristics with RNM. Also in their method, the construction of a CPT is based on a functional relationship between the parents and the child node. Similarly to RNM, the relationship is encoded by weights assigned for the parents and a parameter that defines the level of dispersion of

Table 1. Features of parametric methods for construction of CPTs concerning non-binary nodes.

			Parameters assessed				
Method	$N \mid n, m$*	$N \mid n = 3, m = 5$	Probabilities	Weights of parent nodes	Dispersion parameter	Other	Optional number of parameters to use
Hassall et al.	n	3		X			
RNM**	$n + 1$	4		X	X		X
InterBeta**	$2(m - 1) + n$	11	X	X		X	X
Røed et al.	$(m - 1)n$	12	X		X		
WSA	$m^2 - m + n$	23	X	X			
EBBN	$m^2 - m + 2n$	26	X				
Likelihood	$(n + 2)m + 1$	26	X			X	
Func. interpol.	$2^n(m - 1)$	32	X				
Noisy-MAX	$n(m - 1)^2$	48	X				
Cain***	$n(m - 1)^2$	48	X				
Chin et al.	$n(m^2 - m)$	60				X	

*$N \mid n, m$ is the number of parameters elicited when a child node and its n parents have m states each.
**The numbers of parameters correspond to default forms of use of the methods.
***The method does not provide a computational routine for the construction of a CPT when the child node has more than three states, i.e. $m > 3$.

the probability distributions. However, whereas in RNM there are four alternative weight expressions to choose from, the method of Røed et al. is limited to a single function. This function is similar to one weight expression of RNM that involves taking weighted averages of the states of the parent nodes. Hassall et al. (2019) have also proposed a method in which the conditional probability distributions of the child node are calculated utilizing weighted averages of the parent states. However, the method omits the expert evaluating the dispersion of the distributions. Instead, the dispersion levels are dictated by the numbers of states of the nodes and the weights assigned for the parents. Related to this feature, if a child node has an odd number of states m, the middle state necessarily obtains the probability $1/m$ for any combination of the parent states. In this regard, the method is likely to produce CPTs that require more manual editing than CPTs constructed with RNM.

The EBBN (Elicitation for Bayesian Belief Networks) method (Wisse et al. 2008), the weighted sum algorithm (WSA) (Das 2004) and the Cain calculator (Cain 2001) are based on the interpolation of conditional probability distributions. These methods begin with the expert assessing the conditional probability distributions of the child node for so-called anchor combinations of states of the parent nodes. The remaining conditional probability distributions of the CPT are then derived by interpolating between the anchor distributions. Both the anchor state combinations and the interpolation techniques vary between the methods. Like RNM, these methods involve the parent nodes obtaining weights reflecting their strength of influence on the child node. However, contrary to RNM, the dispersion of the derived distributions is not user-controlled, but emulates those of the anchor distributions.

The functional interpolation method (Podofillini, Mkrtchyan, and Dang 2014) and the InterBeta method (Barons, Mascaro, and Hanea 2021) are also based on utilizing interpolation to derive missing probability distributions of a CPT from method-specific anchor distributions assessed by the expert. However, the probabilities of the anchor distributions are not interpolated directly. In the functional interpolation method, each anchor distribution is approximated by a normal distribution so that best-fit estimates of the mean and variance parameters are determined. The missing probability distributions of the CPT are calculated through normal distributions whose mean and variance parameters are interpolations of the estimates concerning the anchor distributions. The InterBeta method involves a similar principle except that Beta distributions are used instead of normal distributions. The InterBeta method also provides the expert an option to assign weights to parent nodes, their states, or their state combinations. Therefore, the method has mark X in the last column of Table 1. By increasing the details of weighting, a greater range of probabilistic relationships of the nodes can be portrayed. The alternative weighting options of InterBeta are in that regard similar to the use of partitioned weight expressions in RNM.

The noisy-MAX method (Diez 1993; Srinivas 1993) is designed for settings in which parent nodes represent individual causes for a common effect portrayed by the child node. The expert must specify CPT entries that express the individual ability of each cause to bring about the effect. The rest of the CPT is calculated using the assumption that, in the presence of several causes, each one affects the child node independently of the others. Noisy-MAX handles nodes with multiple ordinal states (i.e. multiple states on an ordinal scale) whereby it is a methodological extension of the noisy-OR method (Pearl 1988) designed for binary nodes. Fenton, Neil, and Caballero (2007) note that RNM allows to portray a greater range of probabilistic relationships than noisy-MAX. In addition, Noguchi, Fenton, and Neil (2019) show that with RNM, the explaining away property of binary nodes can be represented more extensively than with noisy-OR.

Kemp-Benedict (2008) has developed the likelihood method where different state combinations of the parent nodes are seen as moving the probability distribution of the child node away from a "typical distribution" in a systematic way. The typical distribution is assessed by the expert and represents the probability distribution of the child node in the absence of information about the parents. The conditional probability distributions in the CPT are obtained by multiplying the typical distribution by likelihood terms. These terms are composed of weighting factors that the expert has selected for the states of the child node and the parents. No detailed guideline for the elicitation of the weighting factors is presented by Kemp-Benedict (2008). Hansson and Sjökvist (2013) provide some instruction and note that the method becomes very complex if the child node has more than three states. With RNM, exact guidelines for the parameter elicitation exist for ranked nodes formed by discretzing continuous scales (Laitila and Virtanen 2016, 2022). Furthermore, this paper presents new guidance concerning ranked nodes with subjective labeled states. Also, contrary to the likelihood method, the number of parameters to be elicited in RNM does not generally increase with the number of states of the nodes.

A method by Chin et al. (2009) utilizes the methodology of the analytic hierarchy process (AHP) (Saaty 2000) in the CPT construction. The method begins with the expert performing pairwise comparisons of the probabilities of the states of the child node given the state of an individual parent node. These comparisons are then used to calculate conditional probability distributions of the child node regarding single parent nodes. By taking products of these distributions, the final probability distributions of the CPT are obtained. As opposed to this method, the existing elicitation guidelines for RNM allow the expert to evaluate the probabilistic behavior of the child node for specified state combinations of all the parent nodes. In this regard, the elicitation concerning RNM provides a clear way for the expert to consider the joint effect of the parent nodes on the child node.

To help compare the required elicitation effort of the methods discussed above, Table 1 presents how many quantitative expert assessments they require when a child node has n parent nodes and all the nodes have m states. Formulas applicable to any values of n and m are displayed along with the numerical values specific for the case $n = 3$ and $m = 5$. In this case, the CPT of the child node consists of $m^{n+1} = 625$ elements whereby its direct assessment would require the expert to specify $m^n(m-1) = 500$ probabilities. Compared to this number, all the discussed methods drastically reduce the number of quantitative assessments required from the expert. Moreover, Table 1 indicates that RNM is among the best ones with regard to this reduction capability. Concerning the Cain calculator, it should be noted that the method does not provide a computational routine for the CPT construction when the child has more than three states (Cain 2001). Furthermore, regarding RNM and the InterBeta method, the numbers in Table 1 correspond to the default ways of using the methods. As discussed above, they both provide the option of specifying more parameters to have CPTs portraying a greater range of probabilistic relationships.

Based on the above considerations, RNM enables quick quantification of CPTs for verification and their systematic refinement without excessive manual editing of individual CPT elements. The ability to generate CPTs quickly fits well also with the utilization of sensitivity analysis in the construction of a BN. With sensitivity analysis, see, e.g. [66], one can identify the CPT elements to which the behavior of the BN shows highest sensitivity. Attention can then be focused on refining these probabilities. Besides the small number of parameters to be elicited, a favorable feature of RNM is that the alternative weight expressions can help experts to understand and describe the probabilistic relationships between nodes (Fenton, Neil, and Caballero 2007). Furthermore, RNM is implemented in AgenaRisk software (Agena Ltd 2021), which supports its easy deployment. Of the other methods discussed, only noisy-OR and noisy-MAX have implementations in existing well-known BN software like GeNIe (BayesFusion, LLC 2021), Netica (Norsys Software Corp. 2021), and Hugin (Hugin Expert A/S 2021). For the likelihood method and the method of Hassall et al., online implementations are available (Luedeling and Goehring 2022; Hassall 2019). However, these implementations are not associated with a wide range of functionalities of BN analysis, contrary to the aforementioned software.

To recapitulate, the methodological principle of RNM is complementary to those of other parametric methods. The alternative weight expressions used in RNM give the expert both flexibility and cognitive support for describing probabilistic relationships of nodes. In addition, the small default number of parameters to be elicited, the option to specify more parameters for depicting complex probabilistic relationships, and the existing software implementation are favorable features that promote the use of RNM for constructing BN-based system models in various application fields.

3. Ranked nodes method (RNM)

3.1. Ranked nodes

A ranked node is a discrete random variable whose states are expressed with an ordinal scale such that each state can be considered to represent a range of values of a continuous quantity. If there is no well-established continuous scale to measure the quantity, the ordinal scale of the ranked node consists of descriptive labels that may be subjective. If a well-established continuous scale exists, the states of the ranked node can be defined by discretizing the continuous scale.

Figure 1 displays an example BN representing how the success level of performing a given work assignment depends on the skill and concentration levels of an employee as well as on the challenge level of the assignment. All the nodes in the BN are ranked nodes. Furthermore, as all the quantities represented by the nodes lack well-established continuous scales, the ordinal scales of the nodes consist of descriptive labels.

3.2. Functioning of RNM

Let there be the parent nodes $X_1, \ldots, X_n$ and the child node X_C defined on discrete ordinal scales. Furthermore, let there be random variables $\chi_1, \ldots, \chi_n$ defined on a unit scale $[0, 1]$ and a random variable χ_C that depends on them according to a regression model

$$\chi_C = f(\chi_1, \ldots, \chi_n, \boldsymbol{w}) + e, \quad e \sim N(0, \sigma^2), \tag{1}$$

where the regression function $f(\cdot)$ is called a weight expression, the related regression coefficients $\boldsymbol{w} = (w_1, \ldots, w_n)$ are called weights, and e is an error term that follows a normal distribution with a zero mean and a variance σ^2. The functioning of RNM is based on associating $X_1, \ldots, X_n, X_C$ with $\chi_1, \ldots, \chi_n, \chi_C$ so that knowing X_i to be in a given state x_i on the

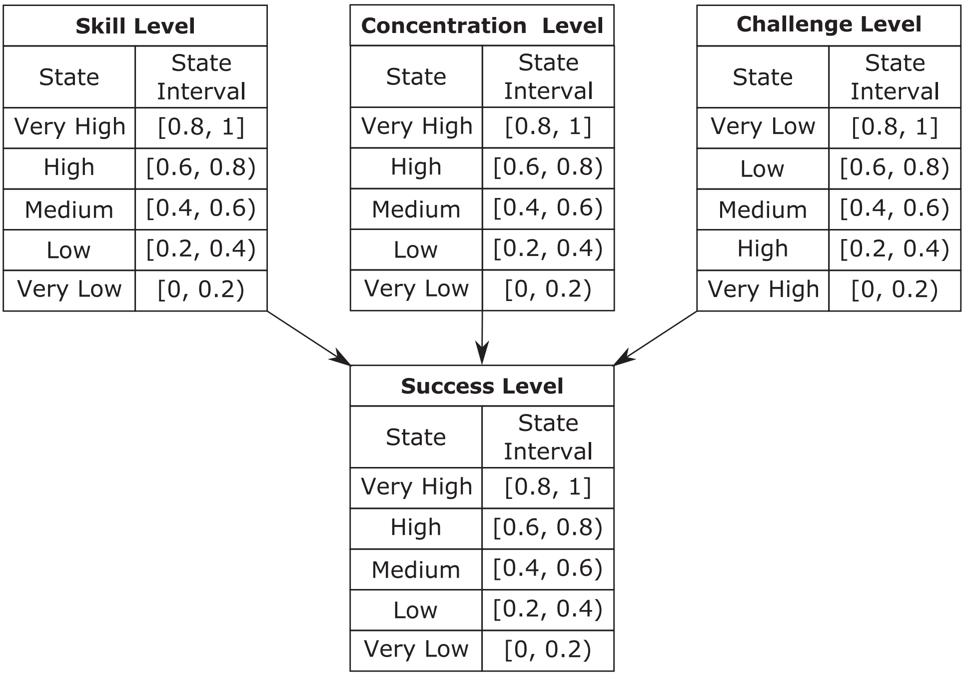


Figure 1. Example BN.

ordinal scale corresponds to knowing χ_i to lie within a specific subinterval $[a_i, b_i]$, called a state interval, on the unit scale. The state intervals associated with consecutive states of a node are also consecutive, of equal width, and cover up the whole unit scale. Moreover, the order of the state intervals with each node must be defined so that the direction of influence of the parent nodes on the child node becomes correctly portrayed. As an example, Figure 1 displays the state intervals associated with the states of the nodes of the example BN. The state intervals indicate that high levels of skill and concentration and low levels of challenge all promote high levels of success.

A given conditional probability $P(X_C = x_C \mid X_1 = x_1, \ldots, X_n = x_n)$, i.e. an element of the CPT of X_C, is computed in RNM according to

$$\begin{aligned} &P(X_C = x_C \mid X_1 = x_1, \ldots, X_n = x_n) \\ &\quad = P(\chi_C \in [a_C, b_C] \mid \chi_1 \in [a_1, b_1], \ldots, \chi_n \in [a_n, b_n], \chi_C \in [0, 1]), \end{aligned} \tag{2}$$

where the right-hand side of the equation is calculated based on Equation (1). Note that while $\chi_C \in (-\infty, \infty)$ according to the regression model (1), the condition $\chi_C \in [0, 1]$ is included in Equation (2). In practice, the computation of Equation (2) involves taking equidistant sample points from the state intervals $[a_i, b_i]$, $i = 1, \ldots, n$, and integrating normal distributions truncated to $[0, 1]$ over the state interval $[a_C, b_C]$. The computational process is explained in detail by Laitila and Virtanen (2020).

The parameters that are elicited from the expert are the weight expression $f(\cdot)$, the related weights $\boldsymbol{w}$, and the variance parameter σ^2. Being the regression function in

Equation (1), the weight expression f can be interpreted as the basic rule by which the parent nodes affect the child node. There are four alternative weight expressions: WMEAN, WMIN, WMAX, and MIXMINMAX. With given sample points of $\chi_1, \ldots, \chi_n$, WMEAN yields their weighted average. MIXMINMAX produces a weighted average of their minimum and maximum. WMIN and WMAX result in values that are below and above the average of the sample points, respectively. The functional forms of the weight expressions are presented in Section S.1 (supplementary material). The basic conditions that the weights of the different weight expressions must fulfill are defined by so-called preliminary sets of weights W^f as follows:

$$W^{\mathrm{WMEAN}} = \left\{(w_1, \ldots, w_n) \in \mathbb{R}^n \mid w_1, \ldots, w_n \in [0,1], \sum_{i=1}^{n} w_i = 1\right\}, \tag{3a}$$

$$W^{\mathrm{WMIN}} = \{(w_1, \ldots, w_n) \in \mathbb{R}^n \mid w_1, \ldots, w_n \geq 1\}, \tag{3b}$$

$$W^{\mathrm{WMAX}} = \{(w_1, \ldots, w_n) \in \mathbb{R}^n \mid w_1, \ldots, w_n \geq 1\}, \tag{3c}$$

$$W^{\mathrm{MIXMINMAX}} = \{(w_{MIN}, w_{MAX}) \in \mathbb{R}^2 \mid w_{MAX} \in [0,1], w_{MIN} = 1 - w_{MAX}\}. \tag{3d}$$

The variance parameter σ^2 defines how dispersed the conditional probability distributions of the child node are. It reflects how precisely the state of the child node is known given the states of the parent nodes.

Instead of generating the whole CPT with a single weight expression and single values of the weights and the variance parameter, the CPT may also be generated in parts based on the states of selected parent nodes. Then, each part of the CPT is generated with fixed part-specific values of the RNM parameters that are elicited from the expert separately for each part. This manner of using RNM is called a partitioned weight expression.

3.3. Challenges with elicitation of parameters of RNM

Concerning ranked nodes defined through subjective labeled states, there are no detailed guidelines for the elicitation of the RNM parameters in the literature. Therefore, the elicitation is likely to be performed through trial and error. That is, the CPT of the child node is repeatedly generated with alternative parameter values until it is considered to represent the views of the expert with sufficient accuracy. However, without proper technical understanding of RNM, the trial-and-error process easily reduces to a random search of the parameter values which makes the elicitation cumbersome.

For the elicitation of the weight expression, some instruction is presented by Fenton, Neil, and Caballero (2007). According to it, the elicitation can be supported by having the expert assess the mode of the child node for different combinations of extreme states of the parent nodes. Table 2 presents these combinations for the example BN along with examples of assessments of the mode of *Success Level* marked with the number 1 in each row. The idea is now that a suitable weight expression is deduced based on the mode assessments of the expert. However, no exact guideline for the deduction is presented in Fenton, Neil, and Caballero (2007). Therefore, technical insight on RNM is required in order to conclude whether some weight expression is suitable or whether a partitioned weight expression is needed. For example, the probabilistic behavior of *Success Level* indicated by the mode statements in Table 2 is representable with both WMIN and MIXMINMAX. However, this

Table 2. Expert assessments of the mode of *Success Level* for the combinations of the extreme states of the parent nodes.

Row	Scenario	Skill Level	Concentration Level	Challenge Level	Success Level				
					Very High	High	Medium	Low	Very Low
1	–	Very High	Very High	Very Low	1				
2	$x^{D,1}$	Very Low	Very High	Very Low		2	1		
3	$x^{D,2}$	Very High	Very Low	Very Low		2	1		
4	$x^{D,3}$	Very High	Very High	Very High			1	2	
5	$x^{R,1}$	Very High	Very Low	Very High				1	2
6	$x^{R,2}$	Very Low	Very High	Very High				1	2
7	$x^{R,3}$	Very Low	Very Low	Very Low			2	1	
8	–	Very Low	Very Low	Very High					1

Note: Green and red highlight the states of the parent nodes associated with the state intervals (0.8, 1] and [0, 0.2], respectively. In each row, the mode assessment of the expert is indicated by the number 1. In Rows 2–7, the assessment of the expert on the second most probable state of Success Level is indicated by the number 2.

is not straightforward to realize if one, e.g. only knows the verbal descriptions of the weight expressions but lacks deeper understanding of the technical functioning of RNM.

As there are no elicitation guidelines for the weights and the variance parameter, their values are decided through trial and error. Then, the main challenge is that without proper instruction, it may not be easy to distinguish and comprehend the effect of individual weights on the CPT (Laitila and Virtanen 2016). Moreover, especially with $f = \text{WMIN}$ and $f = \text{WMAX}$, the preliminary weight set W^f (Equations (3b) and (3c)) defines a vast range for the search of suitable weight values.

Besides the above problems regarding the parameter elicitation, the use of RNM through a partitioned weight expression is hampered by the lack of instructions for deciding whether a CPT should be generated in parts and how such a partition should be established. This shortcoming is noted by Mkrtchyan, Podofillini, and Dang (2016) when they explore the applicability of RNM for portraying probabilistic relationships typical in human reliability analysis. The lack of guidance for the use of a partitioned weight expression can increase the burden of the trial and error practice in the CPT construction. As a result, RNM may be discarded as a too cumbersome way to construct required CPTs.

Another elicitation challenge is specific to a typical application setting of RNM in which all the parent nodes and the child node have the same number of states. In this setting, the probabilistic relationship between the nodes can be represented with RNM only if the nodes are elementarily RNM-compatible (Laitila and Virtanen 2020). A child node and its parent nodes are called elementarily RNM-compatible if (1) the nodes have the same number of states and (2) the states of each node can be sorted so that when the parent nodes are in states of equal rank, the mode of the child node is the state with the same rank. If one fails to realize that the nodes are not elementarily RNM-compatible, time may be spent in vain for the elicitation of suitable parameters even though there actually exists none. For example, suppose that during the elicitation of the weights for the example BN, the expert evaluates a scenario in which all the parent nodes are in the state *Medium*. If she deems that the mode of *Success Level* is anything else than *Medium*, there are no values of the parameters that can yield a CPT portraying her opinion correctly.

The challenges discussed above are alleviated by Laitila and Virtanen (2016, 2022) with approaches concerning the application of RNM to ranked nodes formed by discretizing continuous scales into ordinal scales. However, if the ranked nodes lack such continuous scales, the approaches cannot be used because their elicitation means are based on associating points on the continuous scales of the nodes with points on the unit scale used in RNM.

4. Elicitation framework for parameters of RNM

In order to mitigate the challenges discussed in the previous section, a new framework for the elicitation of the RNM parameters is presented. It is designed for situations in which a child node and its parent nodes all have the same number of states. This is often the case in real-world applications of RNM (e.g. Xia et al. 2017, 2018; Baraldi et al. 2015; Freire et al. 2018; Perkusich et al. 2015; Barreto and Costa 2019; Fenton et al. 2008; Kaya and Yet 2019; Xue et al. 2016). The framework can be applied to nodes with unequal number of states with means described by Laitila and Virtanen (2016). For example, one may initially define an equal number of states for all the nodes and construct a CPT with the framework. After that, states of selected nodes are either divided or merged and the CPT is updated in accordance to the views of the expert.

The steps of the framework are displayed in Figure 2. The starting point is that there is a BN fragment, i.e. a piece of a BN, consisting of a child node X_C and its parent nodes $X_1, \ldots, X_n$ which all are ranked nodes and have m of states. The ending point is that suitable parameters have been determined, and a CPT is generated with them for further review. Steps 1, 2, and 4 require probabilistic assessments of the expert and they are highlighted in Figure 2 with grey boxes. Step 3 is a computational procedure that does not require involvement of the expert. The primary way of using RNM in the framework is that the entire CPT is generated with a single weight expression as well as with single values of the weights and the variance parameter. If no single values of the parameters are suitable, a partitioned weight expression is applied.

The steps of the framework with regard to the use of a single weight expression are next presented. Throughout the presentation, the example BN displayed in Figure 1 is used to demonstrate the execution of the steps in practice. The application of the framework with a partitioned weight expression is discussed in Section 5 in the same way. While the BN in Figure 1 is used as an illustrative example, the steps would be applied in a similar manner to any other instance of a child node and its parents being ranked nodes with an equal number of ordinal states.

4.1. Step 1: check of elementary RNM-compatibility of nodes

In Step 1, the elementary RNM-compatibility of the nodes is checked. The expert assesses the mode of the child node in the m scenarios in which all the parent nodes are in the states of equal rank on their ordinal scales. For example, with the BN in Figure 1, the mode assessment is required for $m = 5$ scenarios of which two correspond to Rows 1 and 8 of Table 2. The corresponding mode assessments presented in the table now indicate the elementary RNM-compatibility of the nodes.

If the nodes are not elementarily RNM-compatible, one should try to update them accordingly. The first means is to try to define for the nodes new ordinal scales with which the compatibility is achieved. In addition, it can be considered whether some of the nodes

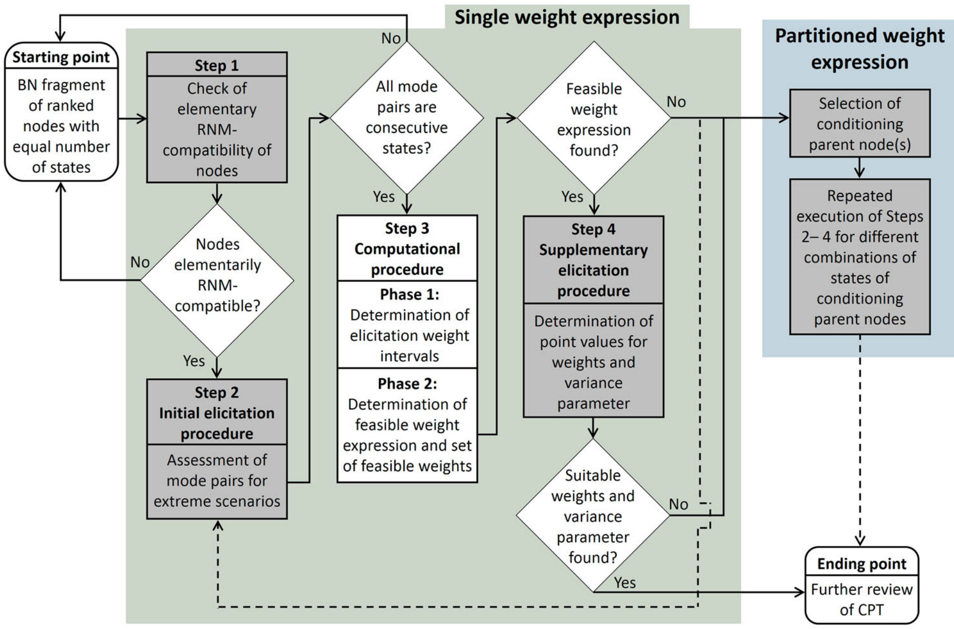


Figure 2. Steps of the elicitation framework for RNM. The steps requiring involvement of the expert are shaded with grey.

could be replaced by new nodes representing slightly different quantities for which the construction of suitable ordinal scales is easier. If performing such measures is not possible or does not help, the probabilistic interaction between the nodes cannot be portrayed with a CPT generated with RNM.

4.2. Step 2: initial elicitation procedure

Step 2 is an initial elicitation procedure in which the expert assesses a so-called mode pair, i.e. the most probable state and the second most probable state of the child node, in $2n$ specific scenarios. The first half of the scenarios corresponds to the type represented by Rows 2–4 of Table 2. These scenarios are called "D-scenarios" referring to the idea that a single parent node has "dropped" to its lowest state on the ordinal scale (the state associated with the subinterval $[0, 1/m]$ on the unit scale) while the others remain in their highest states (the states associated with the subinterval $[(m-1)/m, 1]$ on the unit scale). The second half of the scenarios corresponds to the type displayed in Rows 5–7 of Table 2. These scenarios are called "R-scenarios" as one parent node can be thought to have "risen" to its highest state while the others remain in their lowest states.

Rows 2–7 of Table 2 display examples of mode pair statements concerning the D- and R-scenarios. In the rows, numbers 1 and 2 indicate the expert views of the most and second most probable states of *Success Level*.

A conditional probability distribution generated with RNM always has the two most probable states of the child node being consecutive on the ordinal scale (Laitila and Virtanen 2021). Therefore, in order for RNM to be suitable for representing the views of the expert, each of the mode pair statements concerning the D- and R-scenarios must consist of two consecutive states of the child node. The mode pair statements depicted in Table 2 are suitable from this point of view.

If there are mode pair statements that do not consist of two consecutive states of the child node, one should try to redefine the nodes by using similar means as described in the end of Section 4.1. However, in such a case, the execution of the framework has to be started again from Step 1.

4.3. Step 3: computational procedure for determining feasible weight expression and set of feasible weights

In Step 3, a computational procedure is used to check the feasibility of weight expressions with regard to the mode pair statements. If the procedure finds a weight expression to be feasible, it also determines the related set of feasible weights for it. The feasibility of the weight expressions and the weights are defined as follows.

Definition 4.1: Let there be the mode pair statements of the expert for the D- and R-scenarios. If there exist a weight expression f and weights $\boldsymbol{w} \in W^f$ with which the probability distributions generated for the D- and R-scenarios become compatible with the mode pair statements of the expert, then f and $\boldsymbol{w}$ are said to be feasible with regard to the mode pair statements.

The computational procedure consists of two phases. In Phase 1, the mode pair statements of the expert are used to form so-called "elicitation weight intervals" for the weights of each weight expression. In Phase 2, the feasibility of the weight expressions and the related sets of feasible weights are determined by studying the elicitation weight intervals with regard to feasibility conditions specific to each weight expression. The phases are explained below more specifically. Technical details of Phases 1 and 2 are presented in Sections S.2 and S.3 (supplementary material). A MATLAB implementation of the computational procedure is also available (Laitila 2021).

4.3.1. Phase 1: determination of elicitation weight intervals based on mode pair statements

Let $x^{D,i}$ denote the ith D-scenario, i.e. a scenario in which the parent node X_i is in its lowest state and the other parent nodes are in their highest states. Correspondingly, let $x^{R,i}$ denote the ith R-scenario with $p^{D,i}$ and $p^{R,i}$ designating the mode pair statements concerning $x^{D,i}$ and $x^{R,i}$. Referring to the example BN, if *Skill Level*, *Concentration Level*, and *Challenge Level* are represented by X_1, X_2, and X_3, Row 2 of Table 2 depicts the scenario $x^{D,1} =$ (Very Low, Very High, Very Low) and the mode pair statement $p^{D,1} = (\mathit{Medium}, \mathit{High})$. The other D- and R-scenarios are also presented in the table.

Depending on the weight expression, either the mode pair statement $p^{D,i}$ or $p^{R,i}$, or both, is used to form an elicitation weight interval for the weight w_i. In the case of WMEAN, both $p^{D,i}$ and $p^{R,i}$ yield the intervals $[\underline{w}_i^D, \overline{w}_i^D]$ and $[\underline{w}_i^R, \overline{w}_i^R]$ for w_i, respectively. With WMIN, $p^{D,i}$ produces the interval $[\underline{w}_i^D, \overline{w}_i^D]$ for w_i but the mode pair statements concerning the R-scenarios do not provide any intervals. On the other hand, with WMAX, the interval

Table 3. Elicitation weight intervals determined based on the mode pair statements in Table 1.

Mode pair statement	WMEAN	WMIN	WMAX	MIXMINMAX
$p^{D,1}$	$[\underline{w}_1^D, \overline{w}_1^D] = [0.38, 0.50]$	$[\underline{w}_1^D, \overline{w}_1^D] = [1.2, 2.0]$	–	$[\underline{w}_{MAX}^{D,1}, \overline{w}_{MAX}^{D,1}] = [0.48, 0.60]$
$p^{R,1}$	$[\underline{w}_1^R, \overline{w}_1^R] = [0.14, 0.25]$	–	$[\underline{w}_1^R, \overline{w}_1^R] = [0.29, 0.67]$	$[\underline{w}_{MAX}^{R,1}, \overline{w}_{MAX}^{R,1}] = [0.17, 0.29]$
$p^{D,2}$	$[\underline{w}_2^D, \overline{w}_2^D] = [0.38, 0.50]$	$[\underline{w}_2^D, \overline{w}_2^D] = [1.2, 2.0]$	–	$[\underline{w}_{MAX}^{D,2}, \overline{w}_{MAX}^{D,2}] = [0.48, 0.60]$
$p^{R,2}$	$[\underline{w}_2^R, \overline{w}_2^R] = [0.14, 0.25]$	–	$[\underline{w}_2^R, \overline{w}_2^R] = [0.29, 0.67]$	$[\underline{w}_{MAX}^{R,2}, \overline{w}_{MAX}^{R,2}] = [0.17, 0.29]$
$p^{D,3}$	$[\underline{w}_3^D, \overline{w}_3^D] = [0.50, 0.63]$	$[\underline{w}_3^D, \overline{w}_3^D] = [2.0, 3.33]$	–	$[\underline{w}_{MAX}^{D,3}, \overline{w}_{MAX}^{D,3}] = [0.36, 0.48]$
$p^{R,3}$	$[\underline{w}_3^R, \overline{w}_3^R] = [0.25, 0.38]$	–	$[\underline{w}_3^R, \overline{w}_3^R] = [0.67, 1.2]$	$[\underline{w}_{MAX}^{R,3}, \overline{w}_{MAX}^{R,3}] = [0.29, 0.41]$

$[\underline{w}_i^R, \overline{w}_i^R]$ for w_i is formed through $p^{R,i}$ but the mode pair statements concerning the D-scenarios do not result in any intervals. In the case of MIXMINMAX, both $p^{D,i}$ and $p^{R,i}$ yield the intervals $[\underline{w}_{MAX}^{D,i}, \overline{w}_{MAX}^{D,i}]$ and $[\underline{w}_{MAX}^{R,i}, \overline{w}_{MAX}^{R,i}]$ for the weight w_{MAX}, respectively. As an illustration, Table 3 presents the elicitation weight intervals implied by the mode pair statements displayed in Table 2.

The construction of the elicitation weight interval $[\underline{w}_i^*, \overline{w}_i^*]$ ($* = D$ or $* = R$) based on the mode pair statement $p^{*,i}$ originates from a feature of RNM that in the scenario $x^{*,i}$, w_i is generally the only weight that defines the compatibility of the conditional probability distribution $P(X_C \mid x^{*,i})$ with the mode pair statement $p^{*,i}$. The technical explanation of this feature is presented in Section S.2 (supplementary material). Concerning the feature, the interval $[\underline{w}_i^*, \overline{w}_i^*]$ contains all the values of w_i by which $P(X_C \mid x^{*,i})$ becomes compatible with $p^{*,i}$. Especially, when $w_i = \underline{w}_i^*$ or $w_i = \overline{w}_i^*$, either (1) the states of X_C specified in $p^{*,i}$ are together its most probable states with equal probabilities or (2) the mode of X_C specified in $p^{*,i}$ is unique and the second most probable state specified in $p^{*,i}$ has an equal probability with the third most probable state of X_C. This property of $[\underline{w}_i^*, \overline{w}_i^*]$ generally holds independent of the individual values of the other weights w_j – as long as all the weights as a whole belong to the set of preliminary weights W^f defined in Equations (3a)–(3d). Thereby, it provides for the weight w_i a clear interpretation regarding the probabilistic behavior of the child node.

To exemplify the above property of the elicitation weight intervals, consider the interval $[\underline{w}_1^D, \overline{w}_1^D] = [1.2, 2.0]$ related to WMIN in Table 3. Now, Figure 3(a) displays the probability distribution of the scenario $x^{D,1}$ that WMIN yields with any $\boldsymbol{w} = (w_1, w_2, w_3) \in W^{\text{WMIN}}$ such that $w_1 = \underline{w}_1^D = 1.2$, together with the variance parameter $\sigma^2 = 0.01$. The states *Medium* and *High* of *Success Level* have equal probabilities and they are together the most probable states. The probability distribution in the figure represents an extreme example of a distribution still compatible with the mode pair statement $p^{D,1} = (Medium, High)$ displayed in Row 2 of Table 2. In turn, Figure 3(b) illustrates the probability distribution of the scenario $x^{D,1}$ obtained by using any $\boldsymbol{w} = (w_1, w_2, w_3) \in W^{\text{WMIN}}$ such that $w_1 = \overline{w}_1^D = 2.0$. The state *Medium* is now the unique mode while the states *High* and *Low* are together the second most probable states. Thereby, this probability distribution represents another type of extreme case still compatible with the mode pair statement $p^{D,1} = (Medium, High)$.

It should be noted that with WMIN and WMAX, exceptions to the demonstrated property of the interval $[\underline{w}_i^*, \overline{w}_i^*]$ may occur when $n \geq 2(m-1)$, i.e. when the number of parent nodes n is almost twice the number of states of the nodes m or larger, and $p^{D,i}$ consists of

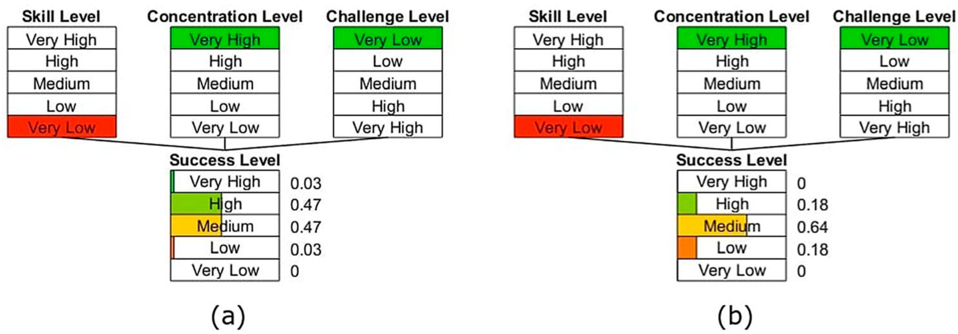


Figure 3. Probability distribution of *Success Level* obtained for the first *D*-scenario $x^{D,1}$ using WMIN with the variance parameter $\sigma^2 = 0.01$ and any values of weights $(w_1, w_2, w_3) \in W^{\mathrm{WMIN}}$ having (a) $w_1 = \underline{w}_1^D = 1.2$ or (b) $w_1 = \overline{w}_1^D = 2.0$.

the two highest states of the child node (a condition specific for WMIN) or $p^{R,i}$ consists of the two lowest states (a condition specific for WMAX). In these specific cases, the compatibility of $P(X_C \mid x^{*,i})$ with $p^{*,i}$ ($* = D$ with WMIN and $* = R$ with WMAX) may depend also on weights other than w_i, see Section S.3 (supplementary material) for more information. This feature of WMIN and WMAX is taken into account while determining their feasibility in Phase 2 of the computational procedure.

As another technical detail concerning each weight expression, the bounds of the elicitation weight intervals are actually dependent on the variance parameter σ^2 and the number of sample points s taken from the state intervals of parent nodes in CPT generation. However, the differences between bounds with different values of σ^2 and s are insignificant from the practical point of view (Laitila and Virtanen 2021). In the computational procedure, the default values are $\sigma^2 = 1/(4m^2)$ and $s = 5$ that are applied also in the illustrations of Figure 3. When a CPT is generated with any weight expression using $\sigma^2 = 1/(4m^2)$, the majority of the probability mass of the child node is always shared either by its two or three most probable states (Laitila and Virtanen 2021). Thus, if the weights are taken from the elicitation weight intervals and $\sigma^2 = 1/(4m^2)$, the resulting probability distributions of the child node in the *D*- and *R*-scenarios represent well the mode pair statements of the expert. On the other hand, when the CPT is generated with $s = 5$, the conditional probability distributions reflect well the underlying regression model of RNM (Laitila and Virtanen 2020). This sampe size is also the default in AgenaRisk implementation of RNM.

4.3.2. Phase 2: determination of feasible weight expression and set of feasible weights

In Phase 2, the elicitation weight intervals formed in Phase 1 are studied with regard to feasibility conditions that are specific to each weight expression. If the feasibility conditions of a weight expression f are fulfilled, f is feasible and the procedure provides the related set of feasible weights F^f. Here, $F^f \subset W^f$, i.e. the set of feasible weights is a subset of the preliminary weight set W^f defined in Equations (3a)–(3d). Furthermore, any single feasible weight w_i necessarily belongs to the elicitation weight interval $[\underline{w}_i^*, \overline{w}_i^*]$ ($* = D$ or $* = R$), see Section S.3 (supplementary material) for further discussion. For a feasible weight expression f, the procedure also produces "central weights" $\boldsymbol{w}^0 \in F^f$ that represent an average type of element of F^f. On the other hand, if a weight expression is not feasible, the procedure indicates which feasibility condition is being violated and gives information on how the mode pair statements should change for the violation to become resolved. This feature is indicated in Figure 2 with a dashed line running to the box of Step 2.

4.3.2.1. WMEAN. With WMEAN, the feasibility conditions concern weight intervals $[\underline{w}_i, \overline{w}_i]$ that are formed based on the elicitation weight intervals $[\underline{w}_i^D, \overline{w}_i^D]$ and $[\underline{w}_i^R, \overline{w}_i^R]$ as the intersection $[\underline{w}_i, \overline{w}_i] = [\underline{w}_i^D, \overline{w}_i^D] \cap [\underline{w}_i^R, \overline{w}_i^R]$ for each $i = 1, \dots, n$. The feasibility conditions derived in Section S.3 (supplementary material) are

(I) The weight intervals $[\underline{w}_i, \overline{w}_i]$, $i = 1, \dots, n$ must all be non-empty sets. If $[\underline{w}_i, \overline{w}_i]$ is an empty set with a given i, the mode pair statements $p^{D,i}$ and $p^{R,i}$ indicate contradictory values for the weight w_i. In other words, $p^{D,i}$ and $p^{R,i}$ suggest that the strength of influence of the parent node X_i on the child node is too different in the scenarios $x^{D,i}$ and $x^{R,i}$.
(II) The sum of the lower bounds $\underline{w}_i$, $i = 1, \dots, n$, must not exceed 1. If this condition is violated, too many parent nodes are considered to have too large strength of influence on the child node.
(III) The sum of the upper bounds $\overline{w}_i$, $i = 1, \dots, n$, must not be smaller than 1. A sum less than 1 indicates that too many parent nodes are considered to have too small strength of influence on the child node.

If the feasibility conditions I, II, and III are satisfied, WMEAN is feasible with regard to the mode pair statements of the expert. The set of feasible weights F^{WMEAN} is

$$F^{\text{WMEAN}} = \Bigg\{ \boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n \mid w_i \in [\underline{w}_i, \overline{w}_i] \; \forall \, i = 1, \dots, n, \; \sum_{i=1}^{n} w_i = 1 \Bigg\}. \quad (4)$$

4.3.2.2. WMIN and WMAX. With WMIN and WMAX, the elicitation weight intervals $[\underline{w}_i^*, \overline{w}_i^*]$ ($* = D$ with WMIN and $* = R$ with WMAX) are examined with respect to feasibility conditions that are analogical between the weight expressions. Therefore, only the conditions regarding WMIN are now discussed. The feasibility conditions of both weight expressions are derived in Section S.3 (supplementary material).

With WMIN, there are generally two feasibility conditions:

(I) All the upper bounds $\overline{w}_i^D$, $i = 1, \dots, n$ must be at least equal to 1. If $\overline{w}_i^D$ is less than 1 with some i, the parent node X_i exhibits too small strength of influence on the child node in the scenario $x^{D,i}$.
(II) The intervals $[\max\{\underline{w}_i^D, 1\}, \overline{w}_i^D]$, $i = 1, \dots, n$, must include weights $\boldsymbol{w} = (w_1, \dots, w_n)$ by which the probability distributions $P(X_C \mid x^{R,i})$, $i = 1, \dots, n$, become compatible with the mode pair statements $p^{R,i}$. The violation of this condition means that the strengths of influence of the parent nodes on the child node indicated by the D-scenarios are not valid in the R-scenarios.

If $n \geq 2(m-1)$ and there is a mode pair statement of the form $p^{D,i} = (x_C^m, x_C^{m-1})$, an additional, third, feasibility condition is also checked:

(III) When $n \geq 2(m-1)$ and $p^{D,i} = (x_C^m, x_C^{m-1})$, the intervals $[\max\{\underline{w}_j^D, 1\}, \overline{w}_j^D]$, $j = 1, \ldots, n$, must include weights $\boldsymbol{w} = (w_1, \ldots, w_n)$ by which the probability distribution $P(X_C \mid x^{D,i})$ becomes compatible with the mode pair statement $p^{D,i}$. This condition can be violated under the given circumstances if weights other than w_i are large enough.

If the feasibility condition I is satisfied, the fulfillment of the condition II is analyzed numerically. Different weight combinations $\boldsymbol{w} = (w_1, \ldots, w_n)$ are constructed by taking equidistant sample points from the intervals $[\max\{\underline{w}_i^D, 1\}, \overline{w}_i^D]$, $i = 1, \ldots, n$. With each $\boldsymbol{w}$, the probability distributions of the R-scenarios are generated, and the fulfillment of the feasibility condition II is checked. If there is a need, probability distributions of specific D-scenarios are also generated for checking the feasibility condition III. The weights $\boldsymbol{w}$ through which all all conditions I, II, and III are satisfied form the set of feasible weights F^{WMIN}.

4.3.2.3. MIXMINMAX. In the case of MIXMINMAX, the feasibility conditions concern weight intervals $[\underline{w}_{MAX}^D, \overline{w}_{MAX}^D]$, $[\underline{w}_{MAX}^R, \overline{w}_{MAX}^R]$, and $[\underline{w}_{MAX}, \overline{w}_{MAX}]$ that are formed from the elicitation weight intervals $[\underline{w}_{MAX}^{D,i}, \overline{w}_{MAX}^{D,i}]$ and $[\underline{w}_{MAX}^{R,i}, \overline{w}_{MAX}^{R,i}]$, $i = 1, \ldots, n$, as the intersections

$$\begin{aligned}
[\underline{w}_{MAX}^D, \overline{w}_{MAX}^D] &= [\underline{w}_{MAX}^{D,1}, \overline{w}_{MAX}^{D,1}] \cap \cdots \cap [\underline{w}_{MAX}^{D,n}, \overline{w}_{MAX}^{D,n}], \\
[\underline{w}_{MAX}^R, \overline{w}_{MAX}^R] &= [\underline{w}_{MAX}^{R,1}, \overline{w}_{MAX}^{R,1}] \cap \cdots \cap [\underline{w}_{MAX}^{R,n}, \overline{w}_{MAX}^{R,n}], \\
[\underline{w}_{MAX}, \overline{w}_{MAX}] &= [\underline{w}_{MAX}^D, \overline{w}_{MAX}^D] \cap [\underline{w}_{MAX}^R, \overline{w}_{MAX}^R].
\end{aligned} \tag{5}$$

The feasibility conditions derived in Section S.3 (supplementary material) are

(I) The interval $[\underline{w}_{MAX}^D, \overline{w}_{MAX}^D]$ must not be an empty set. The violation of this condition means that the strength by which the parent node in the lowest state affects the child node in a given D-scenario varies too much between different D-scenarios.

(II) The interval $[\underline{w}_{MAX}^R, \overline{w}_{MAX}^R]$ must not be an empty set. The violation of this condition indicates that the strength by which the parent node in the highest state in a given R-scenario influences the child node varies too much between different R-scenarios.

(III) The interval $[\underline{w}_{MAX}, \overline{w}_{MAX}]$ must not be an empty set. This condition is violated when the effect that the parent nodes pose on the child node in the D-scenarios is not consistent with that observed in the R-scenarios.

If all the above feasibility conditions are satisfied, MIXMINMAX is feasible. The set of feasible weights $F^{\text{MIXMINMAX}}$ is

$$\begin{aligned}
F^{\text{MIXMINMAX}} = \{\boldsymbol{w} = (w_{MIN}, w_{MAX}) \in \mathbb{R}^2 \mid \\
w_{MAX} \in [\underline{w}_{MAX}, \overline{w}_{MAX}], w_{MIN} = 1 - w_{MAX}\}.
\end{aligned} \tag{6}$$

4.3.3. Illustration of Phase 2 with example BN

When Phase 2 of the computational procedure is applied to the weight intervals in Table 3, WMEAN, WMAX, and MIXMINMAX are all found to be infeasible weight expressions. With WMEAN, the feasibility condition I is violated because the intervals $[\underline{w}_i^{D,i}, \overline{w}_i^{D,i}]$ and $[\underline{w}_i^{R,i}, \overline{w}_i^{R,i}]$ do not intersect each other with any value of i. Regarding this violation, the computational procedure indicates that the mode pairs of *Success Level* should be raised in the D-scenarios and/or lowered in the R-scenarios for WMEAN to become a feasible weight expression. Here, "raising" ("lowering") the mode pair means changing the value of one or both of its elements so that the new value is higher (lower) on the ordinal scale of the child node than the initial value. For example, if $p^{D,1}$ is raised from $p^{D,1} = (Medium, High)$, see Row 2 of Table 2, to $p^{D,1} = (High, \text{Very High})$, the elicitation weight interval $[\underline{w}_1^{D,1}, \overline{w}_1^{D,1}]$ changes from $[0.38, 0.50]$ to $[0.14, 0.25]$ which coincides with $[\underline{w}_1^{R,1}, \overline{w}_1^{R,1}] = [0.14, 0.25]$, see Table 3. In order to resolve the violation of the feasibility condition I completely, also the mode pairs of the other D- and/or R-scenarios would need to undergo similar types of adjustments.

In the case of WMAX, the feasibility condition I is not satisfied because the upper bounds $\overline{w}_1^R$ and $\overline{w}_2^R$ in Table 3 are smaller than 1. To fix this matter, the computational procedure indicates that the mode pairs $p^{R,1}$ and $p^{R,2}$ of *Success Level* displayed in Rows 5 and 6 of Table 2 should be raised upwards.

With MIXMINMAX, the elicitation weight intervals $[\underline{w}_{MAX}^{D,i}, \overline{w}_{MAX}^{D,i}]$, $i = 1, 2, 3$, in Table 3 intersect only at $w_{MAX} = 0.48$. Thereby, the feasibility condition I is satisfied with $w_{MAX} = 0.48$ being the only feasible value of w_{MAX}. The single intersection point is the result of the fact that the mode pair $p^{D,3} = (Medium, Low)$ in Row 4 of Table 2 is close to but not equal with the values $p^{D,1} = p^{D,2} = (Medium, High)$ in Rows 2 and 3. Correspondingly, the condition II is satisfied with $w_{MAX} = 0.29$ being the only feasible value of w_{MAX}, i.e. the only point in the intersection of the elicitation weight intervals $[\underline{w}_{MAX}^{R,i}, \overline{w}_{MAX}^{R,i}]$, $i = 1, 2, 3$. In this case, the origin of the single intersection point is that $p^{R,3} = (Low, Medium)$ in Row 7 of Table 2 is close to but not equal to $p^{R,1} = p^{R,2} = (Low, \text{Very Low})$ in Rows 5 and 6. However, as the values 0.48 and 0.29 are not equal, the condition III is violated making MIXMINMAX infeasible. To overcome this violation, the computational procedure indicates that one should lower mode pairs of the D-scenarios and/or raise mode pairs of the R-scenarios.

Unlike the other weight expressions, WMIN is determined to be feasible with a specifc set of feasible weights. The upper bounds of the weight intervals of WMIN $\overline{w}_i^{D,i}$, $i = 1, 2, 3$, in Table 3 fulfill the feasibility condition I. Furthermore, the numerical analysis of the elicitation weight intervals carried out in the computational procedure reveals that also the feasibility condition II is satisfied. There is no need to check the feasibility condition III because the special circumstances concerning it do not apply now. Figure 4 illustrates all the weight values analyzed. The coordinate axes correspond to the elicitation weight intervals $[\underline{w}_i^{D,i}, \overline{w}_i^{D,i}]$, $i = 1, 2, 3$ of WMIN in Table 3. The weights through which the feasibility conditions I and II are fulfilled, i.e. the weights forming the feasible weight set F^{WMIN}, are highlighted in green. Most of the weights within the elicitation intervals are feasible. Yet, if in a weight combination $\boldsymbol{w} = (w_1, w_2, w_3)$ both w_1 and w_2 are near the upper bounds of their elicitation weight intervals ($\overline{w}_1^{D,1} = \overline{w}_2^{D,2} = 2.0$), $\boldsymbol{w}$ is infeasible.

The central weights of WMIN provided by the procedure are $\boldsymbol{w}^0 = (1.54, 1.54, 2.63)$. Figure 5 displays the probability distributions obtained with $\boldsymbol{w}^0$ and $\sigma^2 = 0.01$ for all the

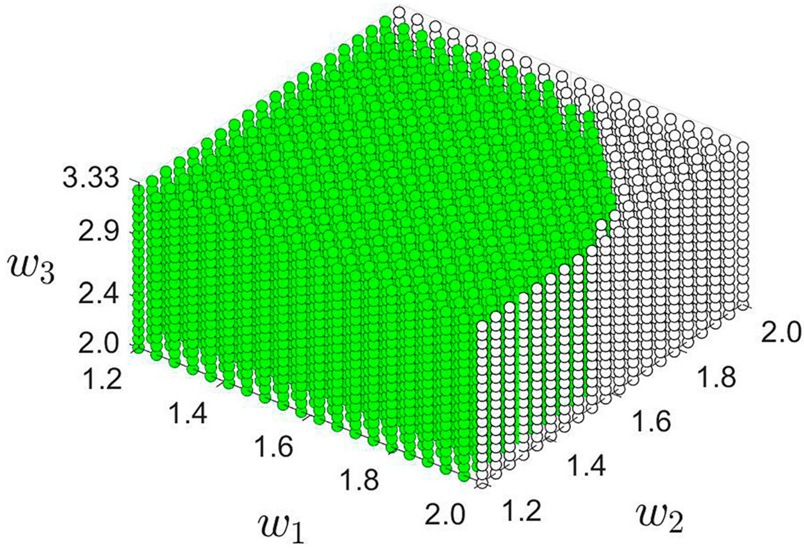


Figure 4. Set of feasible weights F^{WMIN} determined for WMIN based on the mode pair statements displayed in Table 1. The weights belonging to F^{WMIN} are highlighted in green. The coordinate axes correspond to the elicitation weight intervals presented in Table 2. The weights w_1, w_2, and w_3 refer to *Skill Level*, *Concentration Level*, and *Challenge Level*, respectively.

D- and *R*-scenarios. The distributions are compatible with the mode pair statements in Rows 2–7 of Table 2. In all the probability distributions displayed, virtually all of the probability mass of *Success Level* is shared between its three most probable states. Thus, the distributions are in line with the discussion in Section 4.3.1 concerning the default value of the variance parameter $\sigma^2 = 0.25/m^2 = 0.25/5^2 = 0.01$.

4.4. Step 4: supplementary elicitation procedure for selection of point values for weights and variance parameter

Step 4 of the approach is carried out if a feasible weight expression f and the related set of feasible weights F^f are discovered in Step 3. In Step 4, suitable point values for the weights $\boldsymbol{w}$ are selected from F^f along with a suitable value for the variance parameter σ^2. If there are more than one feasible weight expressions, any of them can be selected for the execution of Step 4.

Compared to the preliminary weight set W^f defined in Equations (3a)–(3d), the set F^f considerably narrows down the range of possible weight values. Therefore, with the help of F^f, it may be easy to select point values for $\boldsymbol{w}$ and σ^2 by freely applying trial and error as discussed in Section 3. Alternatively, the values of $\boldsymbol{w}$ and σ^2 can be decided in a more structured manner with a supplementary elicitation procedure in which the expert further considers the probabilistic behavior of the child node in either the *D*- or *R*-scenarios. The basic idea in the procedure is the same with all the weight expressions and it is next presented. After that, features specific to different weight expressions are discussed. Finally, the procedure is briefly demonstrated with the example BN. A MATLAB implementation to

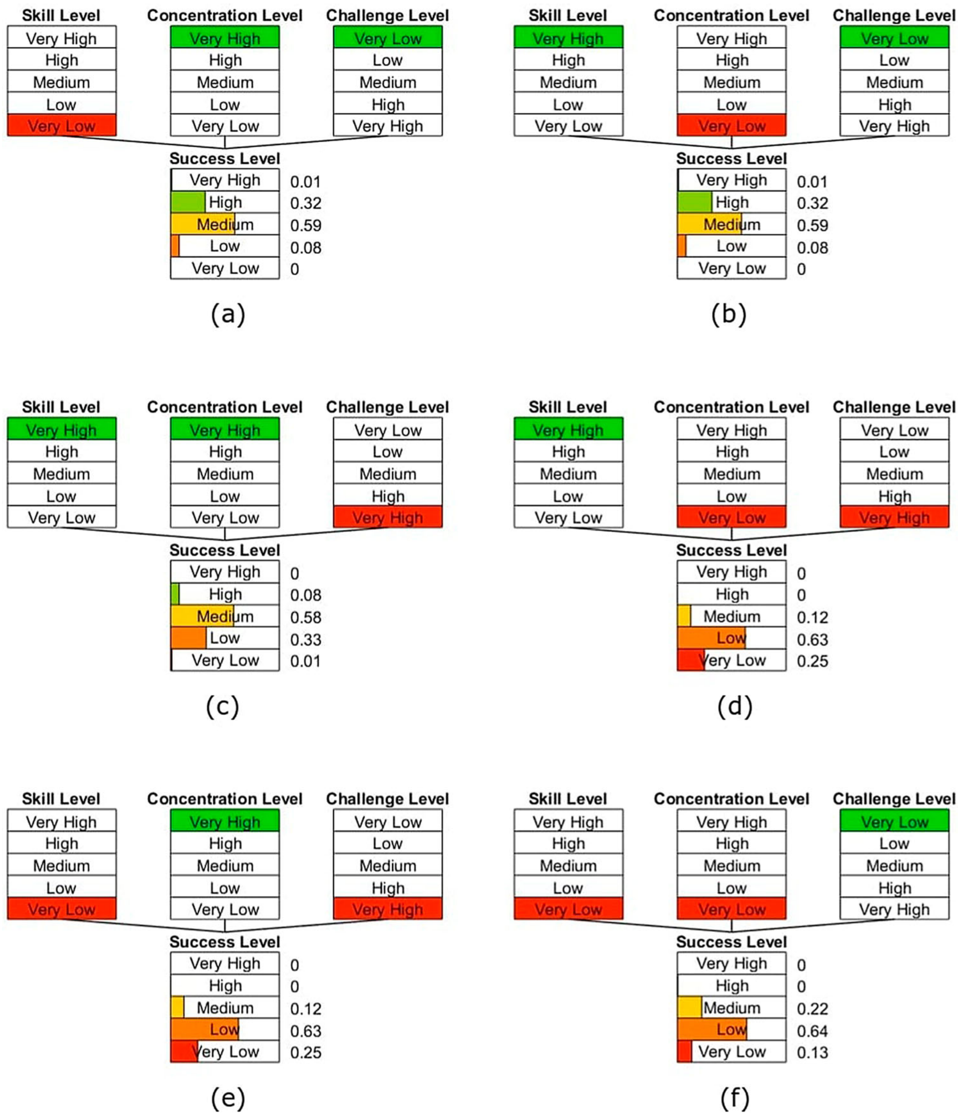


Figure 5. Probability distributions obtained for the *D*- and *R*-scenarios of the example BN using WMIN with the central weights $\boldsymbol{w}^0 = (1.54, 1.54, 2.63)$ and the variance parameter $\sigma^2 = 0.01$.

support the execution of the procedure is available (Laitila 2021). Using the RNM parameters defined by the expert, it generates and visualizes the probability distributions for the *D*- and *R*-scenarios. The illustrations in Figures 3, 5, and 6 have been produced with the implementation.

4.4.1. Description of elicitation procedure

The elicitation procedure begins with designating either the *D*-scenarios or the *R*-scenarios as "primary scenarios" denoted by $x^{P,i}$. Which way the scenarios are designated depends on the weight expression. Generally, the only weight that affects the conditional probability

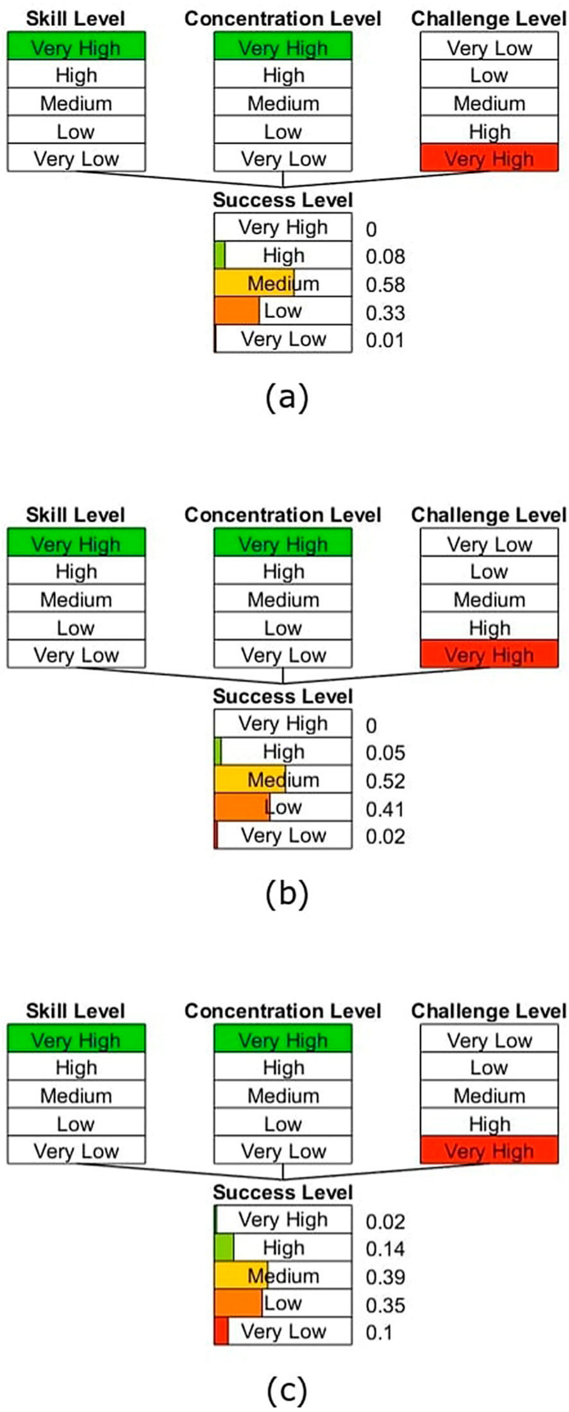


Figure 6. Probability distribution of the scenario $x^{D,3}$ obtained using WMIN with (a) $\boldsymbol{w} = (1.54, 1.54, 2.63)$, $\sigma^2 = 0.01$, (b) $\boldsymbol{w} = (1.54, 1.54, 3.0)$, $\sigma^2 = 0.01$, and (c) $\boldsymbol{w} = (1.54, 1.54, 3.0)$, $\sigma^2 = 0.03$.

distribution $P(X_C \mid x^{P,i})$ is w_i. This feature enables the elicitation of a point value for w_i by having the expert review $P(X_C \mid x^{P,i})$ generated with different values of w_i. At the same time, a value for σ^2 is also assessed.

The elicitation concerning $P(X_C \mid x^{P,i})$ proceeds as follows. First, $P(X_C \mid x^{P,i})$ is generated with some feasible values of the weights $\boldsymbol{w}$ and the variance parameter σ^2. Here, the central weights $\boldsymbol{w}^0$ obtained in Step 3 can be used as the initial weights when the first primary scenario is taken under review. Furthermore, the initial value of the variance parameter can be $\sigma^2 = 0.25/m^2$ used in the computational procedure in Step 3. The expert then reviews and comments on the generated distribution. Based on the comments, the values of w_i and σ^2 are adjusted and $P(X_C \mid x^{P,i})$ is regenerated for a new review. If the expert thinks that probability mass in $P(X_C \mid x^{P,i})$ should be moved from one side of the mode to the other, w_i is to be changed. The limits for the adjustment of w_i are dictated by the feasible weight set F^f. On the other hand, if the expert desires the probability mass to be more (less) dispersed altogether, the value of σ^2 should be increased (decreased). The practice of regenerating $P(X_C \mid x^{P,i})$ and adjusting the values of w_i and σ^2 is repeated until the distribution is deemed suitable by the expert.

After a value for w_i is selected based on the scenario $x^{P,i}$, it is kept fixed when eliciting values for other weights with the other primary scenarios. Furthermore, the value of σ^2 selected with a given primary scenario is used as the initial value when another primary scenario is taken into review.

It may happen that alternative values of σ^2 are considered appropriate in different primary scenarios. Then, one should first check whether there is any single value of σ^2 that could be applied in all the scenarios. If no single value is considered suitable, the use of a partitioned weight expression is necessary. Instead, if suitable values for the weights and the variance parameter are found, the ending point of the framework is reached, see Figure 2.

4.4.2. Specific features with WMIN and WMAX

With WMIN, the primary scenarios are the D-scenarios whereas with WMAX they are the R-scenarios. With both WMIN and WMAX, before the expert starts to review the probability distributions of the primary scenarios, the parent nodes X_i as well as the scenarios $x^{D,i}$ and $x^{R,i}$ are reordered according to increasing values of the central weights $\boldsymbol{w}^0 = (w_1^0, \dots, w_n^0)$. After the reordering, $\boldsymbol{w}^0$ fulfills $w_1^0 \leq \cdots \leq w_n^0$. The scenarios are addressed by starting from $x^{P,n}$ and descending towards $x^{P,1}$. Thereby, the values of the weights that affect the CPT elements the most become fixed first. Here, changing w_i does not require any changes to be made to the other weights.

4.4.3. Specific features with WMEAN

In the case of WMEAN, it can be freely decided whether the primary scenarios are the D-scenarios or the R-scenarios. The review of the probability distributions is carried out in a similar way as with WMIN and WMAX. However, unlike with either WMIN or WMAX, when the weight w_i is changed, also the weights $w_1, \dots, w_{i-1}$ must be updated to new values in line with the conditions defining F^{WMEAN} in Equation (4). In contrast, the weights $w_{i+1}, \dots, w_n$ decided before w_i stay in their fixed values. At some point of the review process, this update scheme of the weights may reveal that there are no weights in F^{WMEAN}

by which the expert's views on all the primary scenarios could be portrayed accurately enough. In this type of situation, it is necessary to use a partitioned weight expression.

If Step 4 is successfully executed, one may discover that the probability distributions $P(X_C \mid x^{P,i})$ obtained with the final weights slightly differ from those initially reviewed and accepted by the expert. This is due to technical properties of WMEAN discussed more in Section S.4 (supplementary material). However, as the changes in single probabilities are generally less than 0.01, the effects are insignificant from the practical point of view.

4.4.4. Specific features with MIXMINMAX

With MIXMINMAX either the *D*-scenarios or the *R*-scenarios can be designated as the primary scenarios. Then, with any primary scenario $x^{P,i}$, point values for the weight w_{MAX} and the variance parameter σ^2 are selected by reviewing the probability distribution of $x^{P,i}$. After this, the expert evaluates whether the distribution of $x^{P,i}$ is applicable also in all the other primary scenarios. If different probability distributions are considered to be necessary in various primary scenarios, the use of a partitioned weight expression is needed.

4.4.5. Demonstration with example BN

Recall from Section 4.3.3 that WMIN is feasible with regard to the mode pair statements presented in Table 2. The central weights belonging to the feasible weight set F^{WMIN} are $\boldsymbol{w}^0 = (1.54, 1.54, 2.63)$. By following the elicitation procedure for WMIN, the *D*-scenarios are designated as the primary scenarios with the correspondence $x^{P,i} = x^{D,i}$, $i = 1, 2, 3$. As $w_1^0 \leq w_2^0 \leq w_3^0$, the parent nodes and the scenarios are already in the desired order.

Because w_3^0 is the largest central weight, the expert first reviews the probability distribution of the scenario $x^{D,3}$. The initial distribution is generated with the weights $\boldsymbol{w}^0$ and the variance parameter $\sigma^2 = 0.25/m^2 = 0.01$, see Figure 6(a). The main remark of the expert is that the distribution is too strongly focused to the state *Medium* and that the probabilities of the states *Medium* and *Low* should be more alike. In order to shift the probability distribution more towards *Low*, the weight w_3 should be increased. The feasible weight set F^{WMIN} depicted in Figure 4 indicates that the upper bound of w_3 is $\overline{w}_3 = 3.33$. The value of w_3 is then raised from 2.63 to 3.0 which produces the probability distribution displayed in Figure 6(b). While the probabilities of *Medium* and *Low* of *Success Level* have now become more alike, the expert says that the distribution is now too heavily focused on these two states. This indicates that σ^2 should be increased. When σ^2 is raised from 0.01 to 0.03, the more dispersed distribution depicted in Figure 6(c) is obtained.

Let the expert be satisfied with the probability distribution displayed in Figure 6(c). Then, the elicitation procedure would continue with the review of the probability distribution of the scenario $x^{D,2}$ in order to assess a value for w_2, and possibly a new value for σ^2. The distribution of $x^{D,2}$ would initially be generated with the current weights $\boldsymbol{w} = (1.54, 1.54, 3.0)$ and the variance parameter $\sigma^2 = 0.03$. After addressing the scenario $x^{D,2}$, the review of the scenario $x^{D,1}$ would be carried out in the same manner. As each primary scenario is used to define a value for a single weight, one should be able to find point values for all the weights by addressing all the primary scenarios. However, the elicitation procedure could indicate that no single value of the variance parameter is adequate in all the primary scenarios. In such a case, a partitioned weight expression should be taken into use.

5. Application of elicitation framework with partitioned weight expression

A partitioned weight expression is used if no feasible weight expression is found in Step 3 or no values of the weights and the variance parameter are deemed suitable in Step 4. Then, the CPT of the child node is divided into parts based on the states of one or more parent nodes, referred to as "conditioning parent nodes". Each part is separately generated with a fixed part-specific weight expression along with fixed part-specific values of the weights and the variance parameter that are elicited from the expert.

The application of a partitioned weight expression begins with the selection of conditioning parent nodes. This is followed by the elicitation of the part-specific RNM parameters for the accordingly divided CPT. These actions are explained below in more detail, see also Figure 2.

5.1. Selection of conditioning parent nodes

In order to restrain the elicitation effort of the expert, it is desirable to have as few conditioning parent nodes as possible. Moreover, a need to generate a CPT extensively in parts contradicts the basic idea of RNM, i.e. that the probabilistic behavior of the child node corresponds to a simple general rule. If the use of several conditioning nodes appears necessary, it is worth considering whether the group of parent nodes could be modified. For example, aggregating the effect of some of the nodes into a new auxiliary node can help to create a setting in which a partitioned weight expression is not needed. However, if new parent nodes are declared, the execution of the elicitation framework should be restarted from Step 1.

A good candidate for a conditioning parent node is such that the expert finds it natural to describe how changes in the states of the other parent nodes affect the child node for a given state of the candidate. To illustrate this idea, consider the example BN in Figure 1. Suppose that the expert can readily describe how the challenge level of an assignment defines the effect that a certain drop in the skill or concentration level of an employee has to the expected success level of performing the assignment. This indicates that *Challenge Level* is a good candidate for a conditioning node.

It is also possible to use conditioning parent nodes that are not ranked nodes. For example, any kind of labeled or boolean node suffices as well. For any combination of states of such conditioning nodes, the part-specific RNM parameters can be elicited by applying Steps 1–4 to the non-conditioning parent nodes exactly in the same way as described in Section 4. The given states of the conditioning non-ranked nodes only define conditions under which the expert should consider the effect of the non-conditioning ranked nodes to the child node. In the description that follows, all parent nodes, including the conditioning ones, are ranked nodes with the same number of states as the child node.

5.2. Elicitation of part-specific parameters of RNM for divided CPT

Once the conditioning parent nodes are selected, the CPT of the child node is divided into parts based on combinations of their states. For example, if *Challenge Level* is the conditioning parent node in the example BN, the CPT of *Success Level* is divided into five parts according to the *Challenge Level* states. For each part of the divided CPT, part-specific parameters are elicited from the expert by applying slightly modified versions of Steps 2–4 of the framework. If there are n_c conditioning parent nodes with m states each, the modified versions of Steps 2–4 explained below have to be executed m^{n_c} times in total.

5.2.1. Modified Step 2

Let n denote the number of all parent nodes and n_c denote the number of conditioning parent nodes. In Step 2, the expert assesses the mode pair of the child node in $n - n_c + 1$ D-scenarios and equally many R-scenarios that are specific for the part of the CPT being addressed. These part-specific D- and R-scenarios are analogical to those defined in Section 4.2. Now, the conditioning parent nodes are in their fixed states in all of the scenarios. In one D-scenario, the non-conditioning nodes are all simultaneously in their highest states whereas in one R-scenario they all are simultaneously in their lowest states. In the rest of the D- and R-scenarios, the non-conditioning nodes alternate in being alone in their lowest and highest states, as explained in Section 4.2.

As an example, Table 3 presents the D- and R-scenarios related to the state *High* of *Challenge Level* when it is the conditioning parent node in the example BN. Here, $n = 3$, $n_c = 1$, and the number of D- and R-scenarios each is $n - n_c + 1 = 3 - 1 + 1 = 3$. Some of the D- and R-scenarios coincide because there are three parent nodes. The table also displays example mode pair statements for *Success Level*.

5.2.2. Modified Step 3

In Step 3, a computational procedure is executed to determine feasibility of weight expressions and feasible weight sets based on the mode pair statements assessed in Step 2. This procedure is an extended version of the procedure presented in Section 4.3. It examines the feasibility of the weight expressions with two alternate settings concerning the conditioning parent nodes. The settings are elaborated below and the details of the procedure are explained in Section S.5 (supplementary material). A MATLAB implementation of the procedure is also made accessible (Laitila 2021).

In the first setting, all the conditioning parent nodes are "computationally inactive" in the examined CPT part. This means that their states are not associated with subintervals of the unit scale and they are not given any weights. Thereby, their states only represent conditions under which the effects of the non-conditioning parent nodes to the child node are to be considered and the related RNM parameters are to be determined. The computational procedure determines elicitation weight intervals, feasible weight expressions, sets of feasible weights, and central weights only for the non-conditioning parent nodes.

In the other setting, the feasibility of the weight expressions is studied so that exactly one of the conditioning parent nodes is "computationally active" in the examined CPT part. This node, denoted by X_q, is assigned with a weight and its states are associated with subintervals of the unit scale. It is involved in the generation of the CPT part in the same manner as the non-conditioning parent nodes. On the other hand, the rest of the conditioning parent nodes are computationally inactive. Each conditioning parent is set in turn to be X_q. With a given X_q, elicitation weight intervals are first formed for X_q based on the mode pair statements of the D- and R-scenarios in which all the non-conditioning parents are in the same ordinal state (scenarios $x^{D,3}$ and $x^{R,3}$ in Table 4). Using these intervals and the other mode pair statements, elicitation weight intervals are formed for the non-conditioning parents analogously to Phase 1 of Step 3 in Section 4.3.1. Then, the elicitation

Table 4. *D*- and *R*-scenarios related to the state *High* of the conditioning parent node *Challenge Level* along with mode pair statements for *Success Level*.

				Success Level				
Scenario	Skill Level	Concentration Level	Challenge Level	Very High	High	Medium	Low	Very Low
$x^{D,1}$	Very Low	Very High	High			2	1	
$x^{D,2}$	Very High	Very Low	High			2	1	
$x^{D,3}$	Very High	Very High	High			1	2	
$x^{R,1}$	Very High	Very Low	High			2	1	
$x^{R,2}$	Very Low	Very High	High			2	1	
$x^{R,3}$	Very Low	Very Low	High				2	1

Note: The notation $x^{D,i}$ and $x^{R,i}$ refers now to the part-specific *D*- and *R*-scenarios. With each scenario, the statement of the expert on the mode and the second most probable state of Success Level are indicated by the numbers 1 and 2, respectively.

weight intervals are examined with respect to feasibility conditions analogously to Phase 2 of Step 3 in Section 4.3.2. For each feasible weight expression f, the procedure produces a feasible weight set F^f and central weights $\boldsymbol{w}^0$.

When the computational procedure is executed in a given CPT part, there are several alternative outcomes. A feasible weight expression may be found only with all conditioning parent nodes being computationally inactive. On the other hand, it could be found only with specific conditioning parent nodes being computationally active. It is also possible that in the CPT part, multiple feasible weight expressions are discovered when all conditioning nodes are computationally inactive and/or when specific conditioning nodes are computationally active. Because the results of Step 3 are part-specific, the feasible weight expressions and weights, and the possible computationally active parent node, can vary in different parts of the CPT.

Concerning the mode pair statements in Table 4, the computational procedure discovers that both WMIN and MIXMINMAX are feasible provided that the conditioning parent node *Challenge Level* is computationally active. The feasible set $f^{\text{MIXMINMAX}}$ is characterized by the interval $[\underline{w}_{MAX}, \overline{w}_{MAX}] = [0.25, 0.31]$ of the weight w_{MAX}, cf. Equation (3d). The central weights determined are $\boldsymbol{w}^0 = (w^0_{MIN}, w^0_{MAX}) = (0.72, 0.28)$. The feasible weight set f^{WMIN} of WMIN is displayed in Figure 7. In this case, the central weights are $\boldsymbol{w}^0 = (w^0_1, w^0_2, w^0_3) = (1.9, 1.9, 7.0)$.

5.2.3. *Modified Step 4*

In Step 4, point values are selected for the weights of the parent nodes and the variance parameter, provided that a feasible weight expression is discovered in Step 3. The point values are specific to the CPT part that is being analyzed.

If a feasible part-specific weight expression is found in Step 3 with all conditioning parent nodes being computationally inactive, Step 4 should primarily be executed using that one. This helps to minimize the elicitation effort of the expert. The part-specific point values for the weights and the variance parameter are in this case selected by carrying out Step 4 according to the description in Section 4.4.

When the feasible part-specific weight expression involves a computationally active conditioning parent node X_q, the selection of weights in Step 4 becomes slightly altered compared to the description in Section 4.4. If the feasible weight expression is WMIN, WMAX or WMEAN, the first weight to be decided is w_q of X_q. The point value of w_q

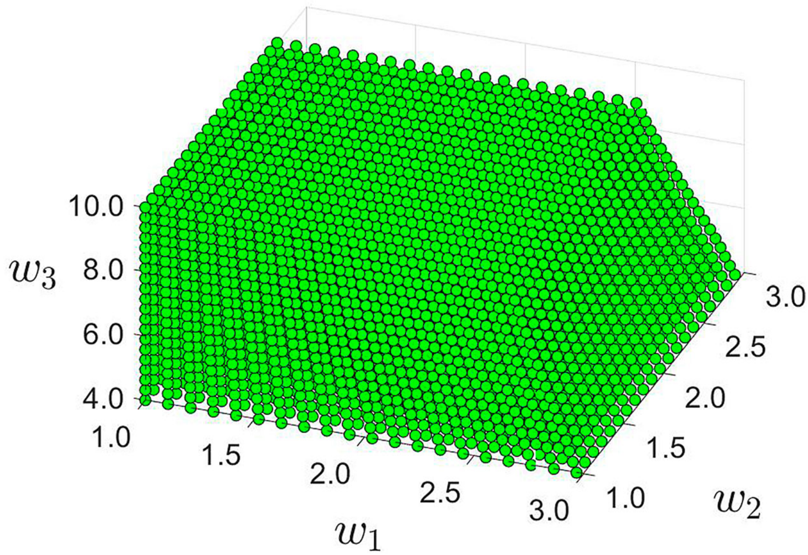


Figure 7. Set of feasible weights F^{WMIN} determined for WMIN based on the mode pair statements displayed in Table 3. The coordinate axes correspond to the elicitation weight intervals of individual weights. The weights w_1, w_2, and w_3 refer to *Skill Level*, *Concentration Level*, and *Challenge Level*, respectively.

is assessed by the expert based on reviewing the probability distribution of the primary scenario in which all the non-conditioning parent nodes are in the same ordinal state. The scenario $x^{D,3}$ ($x^{R,3}$) in Table 4 is an example of such a scenario when D-scenarios (R-scenarios) are the primary scenarios. After the value of w_q is chosen, the selection of the weights of the non-conditioning parent nodes w_i proceeds in the order of decreasing values of the central weights w_i^0, as described in Section 4.4. If the feasible weight expression is MIXMINMAX, the point value of w_{MAX} is selected by reviewing first the probability distribution of the primary scenario in which all the non-conditioning nodes are in the same state. By using this point value, a probability distribution for the other primary scenario is generated for a review. The expert then has to evaluate whether this other probability distribution is applicable in all those scenarios.

Like in the non-partitioned case, the part-specific value of the variance parameter σ^2 is selected with any weight expression while reviewing the probability distributions of the primary scenarios. If no single value of σ^2 is considered adequate in all of them, the CPT can be further partitioned to represent the views of the expert more comprehensively.

Regarding the example of Table 4, one could try executing Step 4 either with WMIN or MIXMINMAX. With WMIN, the primary scenarios are the D-scenarios and $x^{D,3}$ would be the first scenario to be reviewed. In the case of MIXMINMAX, the primary scenarios could be either the D- or the R-scenarios. Depending on the choice, the first scenario to be reviewed would be either $x^{D,3}$ or $x^{R,3}$.

6. Conclusion

This paper discussed how the lack of exact guidelines complicates the elicitation of parameters of RNM and thereby hampers the application of the method in constructing BN system models. In order to alleviate this challenge, the paper presented a novel framework for applying RNM. It consists of two elicitation procedures and a computational procedure which enable systematic elicitation of the parameters from the expert by easily understandable questions about the probabilistic behavior of the child node. The framework is primarily designed for ranked nodes whose ordinal scales are defined by subjective labeled states. In this respect, the framework complements existing elicitation approaches that are applicable only to ranked nodes formed by discretizing continuous scales. The new framework can also be applied to those kinds of nodes by regarding their discretization intervals as the subjective labeled states.

The framework eases the use of RNM through various ways which further encourages the deployment of RNM in practical applications. To begin with, a feasible weight expression and a set of feasible weights are determined based on expert assessments of the two most probable states of the child node in scenarios in which the parent nodes are in their extreme states. Therefore, a suitable weight expression can be found with low elicitation effort and without deep technical insight on RNM. At the same time, also the range of possible values for each weight is narrowed down considerably compared to the initial range defined by the preliminary weight set. This narrowing helps the selection of appropriate point values for the weights. The feasible weight expression and the feasible weight set are solved with a computational procedure for which a MATLAB implementation is available. With this implementation, the execution of the computational procedure takes typically only some seconds of time. Thus, the feasible weight expression and the feasible weight set are discovered virtually immediately after obtaining ordinal probabilistic assessments from the expert. The structured elicitation questions and the swift utilization of the answers computationally are features of the framework that streamline the elicitation of RNM parameters compared to the current practice, i.e. the ungoverned search through trial and error.

To bring additional support to the selection of the point values of the weights, the framework includes a procedure in which they are elicited one by one by having the expert describe in more detail the probabilistic behavior of the child node in the extreme scenarios. Simultaneously, a suitable value for the variance parameter is chosen. To facilitate the execution of the procedure, an existing MATLAB implementation generates and visualizes the probability distributions of the extreme scenarios using the RNM parameters defined by the expert. The fact that the value of a single weight is specified based on a single scenario makes the origins of the weights easy to track. This transparent interpretation is helpful if there is a need to explain or justify the weights that are being used, e.g. to different stakeholders related to the BN application in which RNM is utilized.

The framework addresses the use of RNM with both a single weight expression and a partitioned weight expression. While a single weight expression is the default way to use RNM, the need to shift to a partitioned weight expression is indicated by the outcomes of the procedures discussed above. Covering the two forms of using RNM and indicating the need to change from one to the other are properties of the framework that further ease the use of RNM. These properties are also unique in the sense that the earlier literature on RNM does not present any guidelines for deciding whether to use a single or a partitioned weight expression.

The framework also allows for the elicitation of the RNM parameters based on statements of multiple experts or data. When there are multiple experts, they can together define the labeled states of nodes and decide the probabilistic assessments required in the elicitation procedures. Alternatively, once the states of the nodes are defined, the framework can initially be used separately with each individual expert. If the same weight expression is feasible with each of them, a single set of feasible weights can be established as the intersection or the union of the individual feasible weight sets. The experts can then select together point values for the weights from this common set.

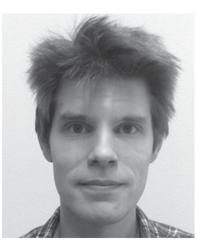
If there is data available for estimating probability distributions of the child node in the extreme scenarios, the estimated distributions can be utilized in the elicitation procedures. The data could be, e.g. any measurements or records enabling to calculate relative frequencies of the child's states for the combinations of the parents' extreme states. Then, in the initial elicitation procedure, mode pairs can be selected according to these distributions. In the supplementary elicitation procedure, the expert can use the estimated distributions as a reference when selecting point values for the weights and the variance parameter. Alternatively, the point values of the parameters can be determined by data fitting so that the estimated distributions are approximated with distributions generated with RNM. Overall, the new framework enables a new way to combine expert knowledge and data in the construction of CPTs included in a BN system model.

A tentative version of the framework was applied in a case study involving the construction of a BN-based system model of an air surveillance network. In the construction of the BN with RNM, the domain experts found the use of the framework preferable to the practice of trial and error when eliciting the RNM parameters. A theme for future research are more structured experiments in which the framework is tested with humans in various setups to understand better its strengths and weaknesses. With such experiments, further knowledge could be acquired on, e.g. how fast, easily and accurately CPTs can be constructed by applying the framework. Comparisons could be made to RNM used without the framework, other parametric methods for CPT construction, and elementwise assessment of CPTs with and without conventional probability elicitation techniques. The comparisons could also address biases concerning expert elicitation that might be faced or mitigated when using the different means. Another future research theme concerns the design of the framework. At present, the framework is meant to be used when a child node and its parents all have the same number of states, which is a common setting in RNM applications. While there are means for using the framework with nodes having non-equal numbers of states, its further development with regard to this capability would bring more relief to the use of RNM.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributors



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