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Pricing of Junior Mezzanine Tranches of Collateralized Loan Obligations

FINAL REPORT

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Abbreviations

CDF	Cumulative Distribution Function
CDR	Cumulative Default Rate
CDS	Credit Default Swap
CDO	Collateralized Debt Obligation
CLO	Collateralized Loan Obligation
CVaR	Conditional Value at Risk
DM	Discount Margin
ID	Interest Diversion
IRR	Internal Rate of Return
LIBOR	London Interbank Offered Rate
OC	Over-collateralization
RR	Recovery Rate
SPV	Special Purpose Vehicle

1 Introduction

Collateralized loan obligations (CLO) are structured asset backed securities that compound a pool of corporate loans (collateral portfolio) with notes of varying cash-flow priority (obligations). A CLO is constructed as a special purpose vehicle (SPV) that finances the CLO by issuing the collection of notes, namely, tranches, and purchases the collateral portfolio of corporate loans by searching for appropriate, typically B-rated, loans for the portfolio.

The cash flow structure of a CLO allocates the interest and principal incomes of the portfolio of loans to the notes according to a pre-specified prioritization scheme, termed the payment waterfall of the CLO. This study considers CLOs with standard prioritization scheme, termed simple subordination, in which the coupon (and principal) payments of senior notes are paid before those of mezzanine and subordinate notes. The cash flow stream of a standard CLO is illustrated in Figure 1.

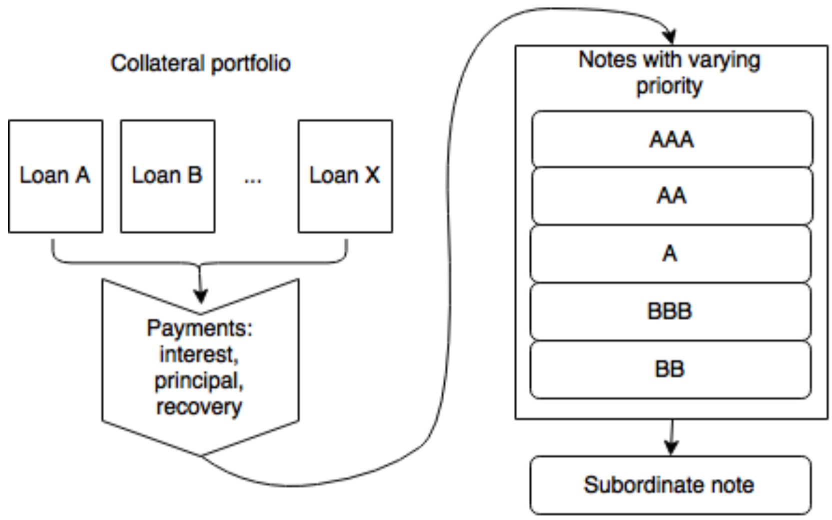

FIGURE 1: STANDARD CLO.

The prioritization scheme of a CLO includes checks for sufficient collateralizations for senior notes, which allow for the higher credit quality of those notes ensuring that the possibility of losses is minimal. These checks may cause premature principal payments, which enable passing the checks in the future. Because the senior tranches are less risky, they pay smaller coupon payments than the riskier junior tranches. The riskiest note is the subordinate note, which receives payments only if there is cash remaining after all the coupon and principal payments to the other notes have been made.

Currently, because of the zero reference rates of many central banks, there are few attractive fixed income securities available in the market for an investor seeking high returns. This makes the junior notes of CLOs an attractive investment opportunity nowadays. However, because of the complex payment structure of these securities, it is not straightforward to assess the risk of these securities, and consequently, finding a fair price may be troublesome. Hence developing a method for analyzing the cash flows of a CLO can provide valuable information on the risks of the notes of different priority, and consequently alleviate finding a fair price for these securities.

This study presents a comprehensive analysis on different factors affecting the cash flows of the most junior mezzanine tranche of a CLO, termed also as BB tranche. The analysis is made by constructing a model which generates different default schemes for the loans of the collateral portfolio and distributes the received payments to the notes of a CLO under these schemes. The analysis results in fair prices in terms of expected returns and assesses the risk of the note by presenting distributions of the possible returns under different scenarios. We study the return distributions in a case study by applying the model for an example CLO.

The rest of this study is organized as follows. Section 2 provides a review of some relevant literature on CLO modelling and presents the references used for this study. Section 3 then presents the methods and explains the selected approaches for the developed model. In Section 4 the used data is clarified and Section 5 presents the results of junior mezzanine tranche returns under different scenarios. Section 6 explains model validation and discusses the most important limitations of our model. Section 7 concludes the report with a summary of our methods and findings.

2 Literature review

There is extensive literature on different pricing models of portfolio derivatives (Amsdorf and Halperin, 2008). A compact and comprehensive overview of the field of CLO modelling has been made by, e.g., Sepci et al. (2009). They made a rough division of the CLO models in academic literature into (i) static and (ii) stochastic models. This division is made by the different approaches to modelling defaults of the portfolio loans.

2.1 Static models

In static models, the loans of the underlying portfolio are assumed to default with a constant rate. These kinds of models can also include assumptions of constant prepayments of the loans and constant recovery rates from the defaulted loans. These specify the expected amounts of the defaulted loans, the cash recovered from the defaults and the cash obtained from loans where no defaults occur. The expected cash flows obtained from the loan portfolio is then distributed to the liabilities of CLO, the tranches and the subordinate equity holders, by the specified payment rules.

2.2 Stochastic models

The stochastic models use same assumptions of default, prepayment and recovery rates and similarly distribute the portfolio proceeds according to the payment rules of the CLO. The main feature that distinguishes the stochastic models from static models is the treatment of loan defaults. Defaulting a portion of loans in static models is in contradiction with real behavior, where loans either default or do not default. Moreover, the stochastic models allow for varying amounts of loans to default in each period, which is also in keeping with real world behavior. These features are obtained by simulating different default scenarios using random number generation. This simulation can also include simulation of different prepayment and recovery rates. The distributions from which the default behavior is generated are defined so that they are in harmony with historical or market data. Unlike static models, stochastic models allow for examination of the tail of the distribution.

Giesecke and Kim (2011) modelled CLO's cash flows to analyze the risk of different CLO tranches. They pointed out that if realistic estimates of the default probabilities are desired, historical default data should be preferred to market derivative pricing data. If default probabilities are estimated from the market pricing data, they will not be real world probabilities but risk-neutral probabilities instead, which are often greater than real world default probabilities, due to the fact that most investors are risk-averse.

2.3 Default simulation

Li (2000) introduced a method to model defaults of financial instruments using a random variable called "time-until-default", or simply survival time, which denotes the length of time that a security survives. He derived some properties for the distribution of the random variable, such as exponential distribution over certain periods under mild assumptions. Perhaps the most notable outcome of Li's study was the notorious default correlation model using copula functions. This model was widely used before the subprime crisis and some have even blamed this model as the cause of the financial crisis (Salmon, 2009). Whether or not the model played a major role in the calculations of market players involved in the crisis, it is evident that lack of comprehension of the nature of default correlation was one great cause for the crisis, which Li worried already in Whitehouse (2005).

While default time modelling has been widely studied in academic literature, the recovery rates of loans are rarely given as much consideration. For example, Sepci et al. (2009) simply assumed a constant recovery rate in their model. The only stochastic model for recovery rates recognized from academic literature was made by Duffie and Gârleanu (2001). Nevertheless, while they modelled the times of defaults with stochastic differential equations, they simply assumed that the cash amounts recovered from defaulted loans follow a uniform $U(0,1)$ distribution, relative to the par amount of the loans.

2.4 Historical data

For realistic estimates of the default time distributions and recovery rates, the most important references are the annual default studies by Moody's. These studies are published in the first quarter of each year, and we use the studies made in years 2011 and 2016 (the former study contains some additional information on Moody's default rates). Moody's rates different corporate loans by how prone they are to defaults. Moreover, Moody's monitors the default events that occur to these and some un-rated loans (such as loans of formerly rated issuers).

It is important to note here the definition of default by Moody's. Moody's defines default as four different credit events. These include (i) a missed or delayed interest or principal payment, (ii) a bankruptcy of the debt issuer, (iii) a distressed exchange (e.g., the obligor offers the creditors a restructured debt with diminished financial obligation as compared to the original obligation), and (iv) a forced change in the terms of the credit agreement (such as forced currency re-denomination imposed by the debtors sovereign). This definition is provided in, for example, the default study of year 2011. Hence the default events monitored by Moody’s include also less severe credit events than the bankruptcies of the obligors.

Using historical data of the defaults, Moody's calculates statistics of the defaults, such as the average cumulative default rates (CDR) and recovery rates (RR). The average CDR represents an estimate of the expected cumulative probability function of the time until default for loans. Moody's methodology to calculate the cumulative default rates is described by Hamilton and Cantor (2006). They explain that the Moody’s CDRs estimate unbiased default probabilities as the average CDRs presented by Moody's take into account, for example, the censoring bias due to rating withdrawals. The data on recovery rates is available both as post-default trading prices and as ultimate recovery rates, that is, the actual cash recovered from the issuers after an event of default.

2.5 Pricing

After the default scenarios have been generated and all the calculations of the tranche cash flows have been made, a fair value for a tranche can be calculated by discounting the expected cash flow yielded by the model (Sepci et al., 2009). The selection of appropriate discount factor can, however, be ambiguous, because it should take into account features such as credit risk and liquidity premium, defining of which requires subjective knowledge. Another approach to valuate a CLO bond is to generate the scenarios under a risk-neutral probability measure, and to calculate the discounted cash flows using the risk-free rate as the discount factor (Kim, 2010).

Nevertheless, it is possible to avoid the assumption of an appropriate discount rate and still perform modelling with real world probabilities instead of the risk-free probabilities by using internal rates of return (IRR) of par priced notes as the value. This is usually done to valuate equity tranches, because the subordinate note does not have a fixed coupon payment and hence the only way to assess the periodical payments of the equity tranche is to calculate its IRR. Such analysis for par priced equity tranche is presented, for example, in Tavakoli (2008).

3 Methodology

The CLO model is presented in this section. The first subsection describes the estimation of future reference interest rates. Then we move on to discussing the default scenario generation by simulation of (i) default times and (ii) recovery cash flow received from defaulted loans. Next, we explain how the cash flows of the collateral portfolio of a CLO are derived under different default scenarios, followed by describing how these cash flows are distributed to the notes of a CLO. The last two parts of this section discuss what the simulation scheme is and how different pricing measures are calculated from simulation outcomes.

3.1 LIBOR forward curve

The coupons of the notes and the loans in the portfolio of the CLOs considered in this study depend on the 3-month London Interbank Offered Rate (LIBOR). Hence to obtain realistic cash flows in our CLO model, we estimate the future 3-month LIBOR rates based on market information. One of the most traded interest rate derivatives are swap agreements on the 3-month LIBOR, and we estimate the future LIBOR rates based on the market information of these agreements. In an interest rate swap, one party pays periodical payments according to a floating reference rate, and another party pays a fixed periodical payment. Hence, at the initialization of a swap contract it is assumed that

$$\sum_{i=1}^{n} d(T_i) \cdot (\Delta c) = \sum_{j=0}^{m} d(t_j)(\delta_j \mathrm{E}[r_j]),$$

where c is the constant payment, Δ and δ are fractions of year that correspond the periods of the payments, $d(\cdot)$ is the discount factor and $\mathrm{E}[r_j]$ is the market expectation of the 3-month LIBOR rate (Fujii et al., 2010).

Assuming that expectations dynamics hold, we denote $\mathrm{E}[r_i] = l_i$, where l_i is the forward rate of the 3-month LIBOR to period i. The day count of typical swap agreements (and for those from which the market data is obtained) is semi-annual and the day count of 3-month LIBOR is quarterly. Hence

$$\sum_{i=1}^{2m} d(T_i) \cdot (c_m/2) = \sum_{j=0}^{4m} d(t_j)(l_j/4),$$

where m is the tenor (maturity) of the swap and c_m the constant payment corresponding to this tenor. We make a simplifying assumption that the LIBOR changes only at even years, from which follows that $l_j = l_{j+1} = l_{j+2} = l_{j+3} \; \forall \, j = 1 + 4k, k = 0,1,2, \ldots.$ Discounting with 3-month LIBOR and approximating $1 + l_k^{6m}/2 = (1 + l_k/4)^2$ (where l_k is the 3-month forward LIBOR to period k) yields

$$\sum_{i=1}^{2m} \left[\prod_{k=1}^{i} \left(1 + \frac{l_k}{4}\right)^2 \right]^{-1} \left(\frac{c_m}{2}\right) = \sum_{j=0}^{4m} \left[\prod_{k=1}^{i} \left(1 + \frac{l_k}{4}\right) \right]^{-1} \left(\frac{l_j}{4}\right),$$

$$\sum_{i=1}^{m} \left[\prod_{k=1}^{i-1} \left(1 + \frac{L_k}{4}\right)^{-4} \right] \left[\left(1 + \frac{L_i}{4}\right)^{-2} + \left(1 + \frac{L_i}{4}\right)^{-4} \right] \left(\frac{c_m}{2}\right) = \sum_{j=0}^{m} \left[\prod_{k=1}^{i-1} \left(1 + \frac{L_k}{4}\right)^{-4} \right] \left[\sum_{h=1}^{4} \left(1 + \frac{L_j}{4}\right)^{-h} \right] \left(\frac{L_j}{4}\right),$$

where the annual 3-month forward LIBORs are denoted as $L_k = l_K, K = 4(k-1) + i, i = 1,2,3,4$ (because we assume only yearly change in the 3-month LIBOR). From the above equation we can solve the rates by bootstrapping, when the constant payments can be obtained from market data, for example online on the webpages of ICE Benchmark Administration (IBA).

3.2 Default time generation

Similar to the approach of Li (2000), we model defaults of corporate loans as a point events, occurring after a length of time. We model the survival time of a security, that is, the time-until-default, as a random variable T that has a cumulative distribution function (CDF) $F(t) = \mathrm{P}(T \leq t)$. T is a continuous random variable that measures the length of time, measured from the present time, until a default occurs. For example, for a non-defaulted security, $F(0) = 0, F(\infty) = 1$, and $F(t_1) = p_1$, where p_1 is the probability that the security defaults after t_1 length of time at the latest.

3.2.1 Default probability distribution

3.2.1.1 Marginal distributions

The distributions of probabilities of defaults of individual loans can be obtained, for example, from the historical default rates presented in Moody's Investors Service (2016). These rates present the actual realized relative amounts of defaults occurring for corporate loans of different credit quality, as measured by alphanumeric Moody's ratings. Hence, assuming that the default times of loans have come from the same distributions for loans of similar credit rating, these historical rates present a large sample estimate of the true probability distribution of default times.

Another approach to obtain default probabilities would be, for example, to estimate them from credit-default-swap (CDS) rates (Chan-Lau, 2006). In CDS one party pays a quarterly fee in exchange for the other party paying the experienced losses in case the reference obligor defaults during the life of the contract. Estimation of default probabilities can be illustrated using the following one-period example.

The protection seller is exposed to an expected loss L equal to $L = p(1 - RR)$, where p is the one-period default probability and RR the expected recovery rate at default. Assuming risk-neutrality (and fair pricing arguments and frictionless markets) the CDS spread S, that is, the quarterly fee paid for by the protection buyer, is then $S = p(1 - RR)/(1 + r)$, where r is the risk-free rate. By direct observation of the market CDS spreads and using historical recovery rates, it would then be possible to estimate the default probabilities p. A similar approach could be worked to estimate default probabilities from the market prices of the loans, if reliable data of these would be available.

While the CDS estimation can provide more recent data on the default probabilities of individual loans, it does not allow for the estimation of real world probabilities, because the risk-neutrality assumption is not realistic for real world investors. Hence the probabilities estimated this way would be the risk-neutral probabilities, instead. The same applies for the loan market price estimation, and the loan market prices do not satisfy the frictionless markets assumption either, because the market of B-rated loans is illiquid. For these reasons, we prefer the Moody's historical default rates as the most reliable estimates of the real world probabilities that a loan of a certain credit quality defaults, under normal economic circumstances.

3.2.1.2 Joint distribution

The default rates provide the marginal probability distributions of defaults of individual loans. Li (2000) presented a copula function approach that links univariate marginal to their full multivariate distribution. Specifically, form m uniform random variables $U_1, U_2, \ldots, U_m$, the joint distribution C, termed a copula function is

$$C(u_1, u_2, \ldots, u_m, \rho) = \Pr(U_1 \le u_1, U_2 \le u_2, \ldots, U_m \le u_m),$$

where ρ is the correlation parameter of the multivariate distribution. Li (2000) showed that for a given set of marginal distributions $F_1(x_1), F_2(x_2), \ldots, F_m(x_m)$ the function

$$C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m), \rho) = F(x_1, x_2, \ldots, x_m)$$

results in a multivariate distribution function with marginal distributions $F_1(x_1), F_2(x_2), \ldots, F_m(x_m)$. The most popular copula function used to link marginal default time distributions is the multivariate copula

$$C(u_1, u_2, \ldots, u_m) = \Phi_m(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_m), \mathbf{\Sigma}),$$

where Φ_m is the m-variate and Φ the univariate normal distribution and $\mathbf{\Sigma}$ is the correlation matrix of the m-variate normal distribution.

It is, however, difficult to assign realistic correlations between the default times of loans in the collateral portfolio. Suppose, for example, two loans that belong to companies in the same industry. If the size of the industry shrinks, the default probability of both loans may increases. On the other hand, if one of the companies succeeds in increasing its market share, the other company's loans may become more prone to default. Hence by only considering the correlations of two loans by the industries of the companies (which is the only data available for correlation estimation provided in a typical CLO portfolio sheet), it is difficult to assess any realistic correlations for the loans.

Modelling correlation could be useful when modelling of economic crises, because during these crises the defaults of loans appear to be more correlated than in normal circumstances (the timings of defaults of corporate loans tend to cluster into economic crises). However, this behavior could also be explained by temporal increase in the default probabilities of individual loans during economic crises, which can be modelled by scaling the default probabilities of different periods.

For the aforementioned reasons, we do not include default correlation modelling in our model, and instead generate the default times from their marginal distributions directly. Note that this corresponds to modelling zero correlations between the default times of individual loans.

3.2.2 Inverse CDF sampling

Generation from an arbitrary empirical probability distribution function is possible with the inverse CDF method. Suppose data X from a distribution p with a CDF $F(x)$. That is,

$$F(x) = \Pr(X \le x) = \int_{-\infty}^{x} p(u)\mathrm{d}u.$$

To generate from this distribution, we first draw a random value U from a uniform distribution on $[0,1]$. Then we let $X = F^{-1}(U)$. Then the random variable X will follow exactly the distribution p. Hence to simulate from p, we can use the following procedure:

1. Draw u from $U(0,1)$.
2. Calculate $x = F^{-1}(u)$, i.e., solve $F(x) = u$ for x.

Now that our variable is time until default of a loan T, and the CDF is assumed to match the historical default rates, the cumulative distribution function is

$$F(t) = \begin{cases} p_1, & if\ t = t_1, \\ p_2, & if\ t = t_2, \\ & \cdots \\ p_n, & if\ t = t_n, \\ 1, & if\ t > t_n, \end{cases}$$

where $p_i, i = 1, \dots, n$ are the average cumulative default rates to years $t_i = 1,2, \dots, n$ and where the last state corresponds a time out of the boundaries of our consideration.

Because our cash flow model is discrete but generated U are continuous, the generated default times $t = F^{-1}(u)$ have to be approximated to match the closest period i of the model, that is, we approximate $t_i \approx F^{-1}(u)$. Hence a rounding rule for the generated default times has to be defined.

The defaults of loans will have two effects: the recovery rate of the loan is obtained instead of the interest stream and principal payment, and the principal of the loan portfolio is reduced by the size of the defaulted loans. The recovered cash from a defaulted loan typically occurs with a delay after the default time. Based on this, an upward rounding rule on the default times can be justified, and hence we use this rounding rule.

Using this rounding rule, we define

$$F^{-1}(u) = \begin{cases} t_1, & if\ 0 \le u \le p_1, \\ t_2, & if\ p_1 < u \le p_2, \\ \cdots & \cdots \\ t_n, & if\ p_{n-1} < u \le p_n, \\ \infty, & if\ u > p_n, \end{cases}$$

where again the last state is out of consideration boundaries. This method is illustrated in Figure 2: Default time generation. The presented distribution shows the probability distribution for B3-rated loans.

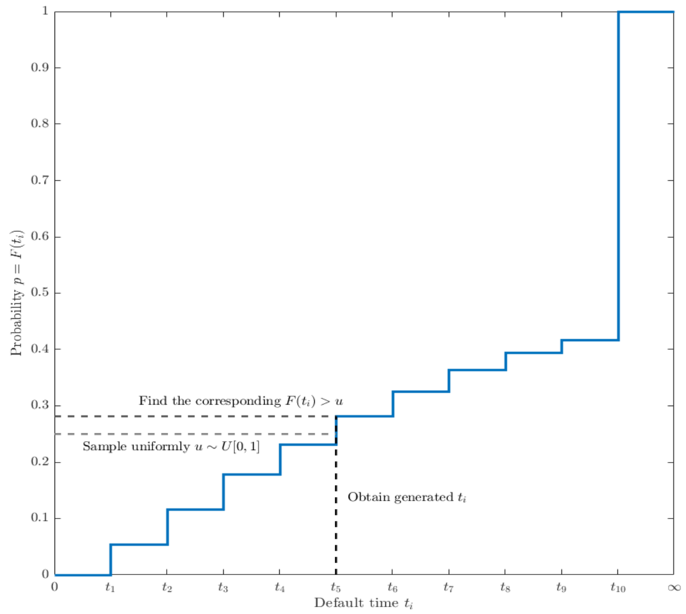

FIGURE 2: DEFAULT TIME GENERATION.

3.2.3 Quarterly intervals

Our model has quarterly intervals, but the default data in Moody's Investors Service (2016) is presented on annual basis. Hence to generate default times that can occur during any period of our model, we interpolate the empirical CDF of the default times. We fitted a piecewise linear function to the default time data, that is,

$$F(t) = \begin{cases} p_1 \cdot \left(\dfrac{t}{t_1}\right), & if\ t \leq t_1, \\ \dfrac{p_2 - p_1}{t_2 - t_1} \cdot (t - t_1) + p_1, & if\ t \leq t_2, \\ \dots & \\ \dfrac{p_n - p_{n-1}}{t_n - t_{n-1}} \cdot (t - t_{n-1}) + p_{n-1}, & if\ t \leq t_n, \\ 1, & if\ t > t_n. \end{cases}$$

This function is then used to generate default times using the inverse CDF method.

3.2.4 Economic crisis modelling

We test some characteristics that will affect the proceeds of CLO notes. One of these characteristics is the default rates for the loans, that is, if the cumulative probability distribution of default times differs from the historical averages presented by Moody's Investors Service (2016).

We test different default rates as if the default rates would increase from the historical averages for some period during the lifetime of the CLO. This corresponds to modelling an economic crisis, a period during which defaults of corporate loans are much more common than during an ordinary economic state (in addition to other exceptional circumstances). Suppose that the probability of the loan defaulting before times t_1 or t_2 are $\Pr(T \leq t_1)$ and

$\Pr(T \leq t_2)$, respectively. We model an economic crisis happening between time t_1 and t_2 by increasing the probability of defaults of loans happening between these times, that is, by scaling up the conditional probability $\Pr(T \leq t_2 | T > t_1)$. First we calculate this probability from (known) $\Pr(T \leq t_1) = a$ and $\Pr(T \leq t_2) = b$ using the Kolmogorov definition of conditional probability as

$$\Pr(T \leq t_2 | T > t_1) = \frac{\Pr(T \leq t_2 \ \cap \ T > t_1)}{\Pr(T > t_1)} = \frac{b-a}{1-a}.$$

Suppose now that a crisis occurs at time interval $I^c = [t_1^c, t_2^c]$ and suppose $t_2 \in I^c$. Then, the conditional probability of a loan defaulting in this time interval is scaled by a crisis factor γ to get

$$\Pr(T \leq t_2 | T > t_1, t_2 \in I^c) = \gamma \frac{\Pr(T \leq t_2 \ \cap \ T > t_1)}{\Pr(T > t_1)} = \gamma \frac{b-a}{1-a}.$$

Note that if the crisis does not occur in a considered time interval, the conditional probability is $\Pr(T \leq t_2 | T > t_1, t_2 \notin I^c) = (b-a)/(1-a)$.

Finally, we can calculate a new cumulative probability function recursively $F(t) = \mathrm{P}(T \leq t)$ by integrating over the conditional probabilities as

$$F(t) = \Pr(T \leq t) = \sum_{t_i \leq t} \Pr(T \leq t_i \ \cap \ T > t_{i-1}) = \sum_{t_i \leq t} \Pr(T \leq t_i | T > t_{i-1}) \Pr\,(T > t_{i-1})$$

$$F(t) = \sum_{t_i \leq t, t_i \in I^c} \Pr(T \leq t_i | T > t_{i-1}, t_i \in I^c) \Pr(T > t_{i-1}) + \sum_{t_i \leq t, t_i \notin I^c} \Pr(T \leq t_i | T > t_{i-1}) \Pr(T > t_{i-1})$$

$$F(t) = \sum_{t \geq t_i \in I^c} \Pr(T \leq t_i | T > t_{i-1}, t_i \in I^c)[1 - \Pr(T \leq t_{i-1})] + \sum_{t \geq t_i \notin I^c} \Pr(T \leq t | T > t_i)[1 - \Pr(T \leq t_{i-1})],$$

where $i \geq 1$ and $t_0 = 0$. We estimate a reasonable value for the crisis factor by studying the default data during different crises in Moody's Investors Service (2016). In 2009, the default rate of all Moody's rated loans was 5.02% and in 2008 this rate was 2.51%. On average, the Moody's rated loans' default rate has been 1.57%, from year 1983 to year 2015. Hence during the financial crisis in 2008-2009, the default rates were $(5.02\% + 2.51\%)/(2 \cdot 1.57\%) = 2.4$ times greater than on average. Hence to model a similar change in the default rates as in 2008-2009 crisis we set the crisis factor to $\gamma = 2.5$ for two years during some two-year period in the CLOs lifetime.

3.3 Recovery rate generation

We model recovery rates as ultimate recovery rates, meaning the portion of principal that is ultimately recovered from the defaulting company, often with a delay of approximately one year. In contrast with market recovery rates, the ultimate recovery rates are typically higher and receiving an ultimate recovery does not require activity from the CLO manager. An alternative modeling possibility would be to use market recovery rates and assume that the CLO manager immediately trades the defaulted loan away.

The historical average ultimate recovery rate for loans has been 80.4% in the 1987-2015 timeframe as stated by Moody's Investors Service (2016). In effect, the recovery rates can be as low as zero and on the other hand occasionally even surpass 100%. In our model we limit recovery rates to the range between zero and one.

As in Duffie and Gârleanu (2001), we model the recovery rate as a uniformly distributed random variable. The mean of the distribution is easily changeable in the model and is currently set at 60% based on a conservative expert expectation. The range of the random variable is limited to be from 20% to 100%.

3.3.1 Recovery rate adjustment

To incorporate market information into our model, we adjust the random recovery rate based on the market price of the loan in relation to the other loans in the portfolio. We calculate the expected value of the loan's cash flow $CF_i(j), j = 1, \ldots, m_i$ (where m_i is the maturity of the loan) of each loan i at the assumed 60% recovery rate and calculate the ratio between the net present value of this expected cash flow (discounting with 3-month LIBOR) and the price of the loan. We then calculate the average of these ratios. The recovery rates of the cheap loans are adjusted down and the recovery rates of the expensive loans up.

Ultimately the ultimate recovery rates are generated by:

$$RR_i = \begin{cases} 0, & if\ rr_i \leq 0 \\ rr_i, & if\ 0 < rr_i < 1, \\ 1, & if\ rr_i \geq 1 \end{cases}$$

where

$$rr_i = u_{a,b} + \left(\frac{\mathrm{E}\left[\sum_{j=1}^{m_i} d(j) CF_i(j)\right]}{P_i} - \frac{1}{n} \sum_{i=1}^{n} \frac{\mathrm{E}\left[\sum_{j=1}^{m_i} d(j) CF_i(j)\right]}{P_i} \right),$$

where $CF_i(j)$ is the (random) cash flow of loan i at period j (consisting of coupon, principal, and recovery payments), $d(j)$ is the discount factor corresponding to period j and the 3-month LIBOR rate of that period, P_i is the price of the loan, and n is the number of loans in the portfolio. $u_{a,b}$ is a realization of a uniform random variable $U_{a,b} \sim U(a,b) = U(0,1)(b-a) + a$. Parameters a and b define the mean and variance for the random variable $U_{a,b}$. With mean 60%, for example, the maximum variance is received with $a = 0.2$ and $b = 1$, and we use these values for RR generation.

3.4 Portfolio cash flow model

The notes of the CLO pay coupons quarterly, and for this reason we model the cash flows of the CLO quarterly. Under normal circumstances, each corporate loan i in the CLO's portfolio pays coupon payments C_i in every period k until the maturity of the loan and the loan principal F_i at the maturity t_i^m of the loan. The coupons of the loans are floating rate coupons, that is,

$$C_i(k) = (L_k + s_i)F_i,$$

where L_k is the 3-month London Interbank Offered Rate (LIBOR) in period k and s_i is the spread of the loan which is the additional interest paid by the loan, in addition to the reference rate. For most of the loans a LIBOR rate floor f_i is also defined, which protects the loan buyers from interest rate decreases by setting a minimum to the reference rate. Under these circumstances, the periodical coupon payment becomes

$$C_i(k) = [\min(L_k, f_i) + s_i]F_i.$$

We do not model prepayments of the loans, and hence the loans in the portfolio act as if they were floating rate bonds.

Each loan has a certain probability $\Pr(T \leq t_k | T > t_{k-1})$ of default in each period k. If a loan defaults, it will no longer pay any coupons and instead it will make a payment equal to RR_iF_i once, where RR_i is the recovery rate of the loan. The recovery cash flows received in the case of default are often delayed due to the circumstances of the corporation that lead to default. Hence we model the recovery rates to be paid after a delay of one year.

We denote the n-vector of realized default times as $\mathbf{T} = (\tau_1, \tau_2, \dots, \tau_n)$, where τ_i's are the realizations of the default times of each loan i and n is the number of loans in the loan portfolio. We then denote the maturities of each loan as an n-vector $\mathbf{t} = (t_1^m, t_2^m, \dots, t_n^m)$. The cash flow received from the loan portfolio in each period k is then

$$CF(k) = \sum_{i=1}^{n} \mathbf{1}(t_k \leq t_i^m)\mathbf{1}(t_k < \tau_i)[C_i(k) + \mathbf{1}(t_k = t_i^m)F_i] + \mathbf{1}(t_k = \tau_i + 1)RR_iF_i,$$

where $\mathbf{1}(A) = 1$, if statement A is true and 0, otherwise and $C_i(k)$ is the coupon payment of loan i as defined above. The coupon payments of the collateral portfolio are termed the interest payments and the rest of the payments are the principal payments of the portfolio. These are treated somewhat differently in the payment waterfall of the CLO. Hence we write the cash flows of the collateral portfolio also as

$$CF(k) = CF_{interest}(k) + CF_{principal}(k),$$

where $CF_{interest}(k) = \sum_{i=1}^{n} \mathbf{1}(t_k \leq t_i^m)\mathbf{1}(t_k < \tau_i)C_i(k)$ and $CF_{principal}(k) = \sum_{i=1}^{n}[\mathbf{1}(t_k = t_i^m)\mathbf{1}(t_k < \tau_i) + \mathbf{1}(t_k = \tau_i + 1)RR_i]F_i$.

In addition to the cash flows of the portfolio, the operation of CLO requires monitoring the par value of the portfolio in each period. The par value of the portfolio is

$$F_{portfolio}(k) = \sum_{i=1}^{n} \mathbf{1}(t_k < \tau_i)\, F_i.$$

3.4.1 Portfolio adjustments

At the time when the notes of a CLO are priced, the whole collateral portfolio has not usually been purchased. In keeping with this, we model a CLO where most of the loans of the collateral portfolio have already been purchased, but where approximately fifth of the loans have not yet been purchased. These loans are used as adjustable loans for the sensitivity analysis of the CLO. We complete the collateral portfolio with 20 identical artificial loans, so that the adjustment of the average characteristics of these loans is straightforward (they are identical) and that they behave as the rest of the portfolio in terms of default scenarios (separate loans). Suppose that the current par amount of the portfolio is $F_{portfolio}(-1)$ (before the simulation is initiated) and the target par amount is $F_{portfolio}(0)$ (the initial par amount of the portfolio). The target par amount is selected according to the sensitivity analysis scenario. Then the par amount of each of these loans is

$$F_{artificial} = \frac{F_{portfolio}(0) - F_{portfolio}(-1)}{20}$$

These artificial loans can be used to modify, for example, the (i) weighted average spread (ii) the weighted average rating factor and (iii) the initial par amount of the portfolio. The first two are the averages of characteristics of individual loans, weighted by the par amount of the loans, that is,

$$s_{WA} = \frac{\sum_{i=1}^{n} F_i s_i}{F_{portfolio}(0)}, \qquad rf_{WA} = \frac{\sum_{i=1}^{n} F_i rf_i}{F_{portfolio}(0)},$$

where rf_i is a numerical value of the credit quality of loan i (defined from the alphanumeric rating factors and the conversions provided by Moody's), termed the rating factor of a loan.

3.5 Liability model

The liability model chiefly implements the payment waterfall, allocating given portfolio interest and principal payments to interest and principal payments of the tranches. Along with scheduled interest payments, changes in the portfolio structure might trigger early principal payments to senior tranches in order to reduce their risk. To facilitate this, par subordination and overcollateralization percentages are tracked before and after each batch of payments. A summary of each key concept is given below, after which a typical payment waterfall is described.

3.5.1 Par subordination

Let $t = 0, \dots, T$ denote a tranche in descending order of seniority. Par subordination of a tranche τ is defined as

$$ParSub_\tau(k) = F_{portfolio}(k) - \sum_{t=0}^{\tau} F_t(k),$$

during period k where F_t is the par of tranche t, i.e., as the portfolio par less par of all senior tranches divided by the portfolio par.

Par subordination serves to describe the amount of "buffer" par that needs to default before that tranche incurs losses. The more senior the tranche, the more par subordination it has. During the years of operation of the CLO, some of its assets are expected to default, reducing the par subordination of each tranche (given that there is any still left). The price of each tranche then reflects whether the defaults and their effect on par subordination has been lower of higher than expected.

3.5.2 Overcollateralization tests

Overcollateralization (OC) for each period k and for each tranche τ in turn is defined as

$$OC_\tau(k) = \frac{F_{portfolio}(k)}{\sum_{t=0}^{\tau} F_t(k)},$$

i,e, as the portfolio par divided by the sum of par of all non-junior tranches. Similar to the par subordination figures, OC serves to indicate the amount of buffer and risk the tranche is subject to. The CLO bylaws often define so called trigger levels for the OC of each tranche, that when met, trigger early payments to senior tranches in order to recover OC levels to acceptable levels. The principal payment needed to re-establish a required tranche OC level OC_τ^T is therefore given by

$$OC_\tau(k) = OC_\tau^T \Leftrightarrow \frac{F_{portfolio}(k)}{\sum_{t=0}^{\tau} F_t(k)} = \frac{F_{portfolio}(k)}{\sum_{t=0}^{\tau} F_t{}'(k)}$$

$$\Rightarrow F_0(k) - F_0'(k) \equiv \Delta F_0(k) = \frac{\left(OC_\tau^T - OC_\tau(k)\right) \cdot \left(\sum_{t=0}^{t} F_t(k)\right)^2}{\left(OC_\tau^T - OC_\tau(k)\right) \cdot \left(\sum_{t=0}^{t} F_t(k)\right) + F_{portfolio}(k)},$$

where $F_t'(k)$ is the tranche par after payment. The early payments are made from interest proceeds, reducing the funds available to make payments to junior tranches.

3.5.3 Interest coverage tests

Interest coverage tests test the CLO's interest coverage, i.e., the ratio of interest proceeds generated by the CLO's assets compared to the interest cost of the CLO's debt. Upon failure, interest proceeds are directed to early amortization until the given coverage ratios are met, similarly as with OC tests. The model discussed here currently only supports CLOs with no interest coverage tests.

3.5.4 Interest diversion tests

Interest diversion tests are similar to OC tests in that their triggers are defined in terms of overcollateralization. However, when triggered, a fraction of the remaining interest income is spent on acquiring new assets, hence diversifying the loan portfolio.

After the reinvestment period (see below), interest diversion test failure results in principal payments according to the priority of payments. However, interest diversion test failure can cause only 50% of the remaining proceeds to be directed to early amortization at most.

3.5.5 Reinvestment period

The bylaws of a CLO often also define a reinvestment period, during which early principal payments from the loan portfolio are reinvested to new loans instead of making payments to the senior tranches. Modeling the reinvestment period requires one to take assumptions as to the qualities of the reinvested loans. By introducing new loans following the median qualities of the current loan portfolio, the portfolio would start to converge towards an unrealistically homogeneous structure, and sampling new loans according to the current portfolio distribution would significantly increase the number of samples required to reach accurate results. Realistically modelling the reinvestment period was deemed to be complex and it was excluded from the model's scope. For this reason, modelling interest diversion test was also made according to post-reinvestment period means. Depending on the CLO manager's incentivization, she might be encouraged to take aggressive bets during the reinvestment period, exposing its creditors to additional risk. Therefore, excluding reinvestment period from the model potentially underestimates risk.

3.5.6 Payment waterfall

Interest proceeds and principal payments from the loan portfolio are distributed to the creditors of the CLO. The securities issued by the CLO belong to a tranche indicating its level of subordination w.r.t to other of the CLO. The payment waterfall describes the subordination of payments as well as other fees and costs relevant to the CLO. The payment waterfall is additionally subject to a series of overcollateralization (OC) and interest diversion (ID) tests.

In its current form, the model allocates interest payments as follows (namely, as per the example CLO provided by the client). Currently, only overcollateralization tests are assumed for the initial CLO model. OC tests and principal payments resulting from test triggers are modelled.

1. Senior management fee to the CLO manager (typically 0.20% of par p.a.)
2. Interest of AAAs
3. If an event of default has occurred, then to pay the principal of AAAs
4. Interest of AAs
5. If AAA/AA OC test is breached, then to pay the principal of AAAs until AAA/AA OC test is satisfied.
6. Interest of As
7. If A OC test is breached, then to pay the principal of AAAs until A OC test is satisfied.
8. Interest of BBBs
9. If BBB OC test is breached, then to pay the principal of AAAs until BBB OC test is satisfied.
10. Interest of BBs
11. If BB OC test is breached, then to pay the principal of AAAs until BB OC test is satisfied.
12. During the reinvestment period only (4 first years): If the interest diversion test is breached, then to use 50% of the remaining proceeds to buy new loans into the portfolio until the interest diversion test is satisfied.
13. Junior management fee to the CLO manager (typically 0.30% of par p.a.)

14. Rest to the subordinated note

Principal payments aside from those used to satisfy outstanding interest payments and those borne from OC test failures are similarly distributed on tranche seniority, i.e., AAA is paid all principal payments until all par is repaid. After this, AA is paid principal and so on, until all principal payments from the loan portfolio are repaid. Due to the possibility of defaults in the loan portfolio, it is entirely possible for the principal cash flow from the portfolio to be insufficient to repay all liabilities. This leads to losses in the more junior tranches of the CLO, in accordance to their higher riskiness and price.

3.6 Monte Carlo methods

Monte Carlo methods, as defined by Mooney (1997) among others, are a class of algorithms characterized by the use of repeated random sampling to yield results about a probabilistic system that would otherwise be difficult or impossible to derive analytically. For instance, by the law of large numbers, the sample mean of a random variable should converge towards its expected value. In the case of CLO cash flow modeling, the method is particularly helpful in that is allows estimating the distribution of tranche cash flows by simulating the performance of their underlying loan portfolio, given that one is able to sample from the random distributions describing the loans. The payment waterfalls fundamental to each CLO allocate asset cash flows to each tranche in a non-trivial way, making it difficult to derive the exact trance flows exactly.

In our Excel model, deterministic calculations are created to determine tranche cash flows given default times and recovery rates for each loan. Default times and recovery rates are then repeatedly sampled and their respective cash flows are recorded via data tables. The resulting estimated cash flow distributions are then summarized to facilitate pricing assessments. The expected computation time is proportional to the number of samples and furthermore carrying out sensitivity analysis on any of the models inputs multiplies the number of calculated samples by the number of different input values used. It is therefore important to weigh the importance of both accuracy and computation time to make a balanced trade-off between them. Additionally, maximizing the performance of the deterministic calculations in turn reduces needed computation time per sample and allows for improved accuracy and total computation time. In this instance, a run with 1000 samples was found to yield sufficient accuracy while still consuming a reasonable time to calculate.

3.7 Pricing and risk analysis

3.7.1 Expected internal rate of return

The internal rate of return r for a cash flow $x_0, x_1, \dots, x_n$, where $x_0 < 0$ and $x_k \geq 0, \forall k > 0$, is defined as the solution to the equation

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} = \sum_{k=0}^{n} \frac{x_k}{(1+r)^n}$$

If the cash flow stream is known, it is possible to solve the above equation by, for example, some numerical solver. However, when the possible events of defaults are taken into account, the cash flow stream received varies according to the experienced losses in the collateral portfolio.

Nevertheless, it is possible to obtain a distribution for the internal rate of return by simulation. Suppose we can generate observations $\boldsymbol{x}^i = (x_1^i, \dots, x_{n^i}^i)$ of a random cash flow stream $\boldsymbol{X} = (X_1, \dots, X_N)$ (where n^i is a realization of random time of last payment N). We can then calculate the internal rate of return r^i for this realized cash flow by solving the above equation for r. By repeating this procedure numerous times, we obtain realizations of the random variable R, which is the internal rate of return of the random cash flow stream $\boldsymbol{X}$. From these realizations we can calculate different descriptive statistics.

After generating m realizations of the internal rates of return $r^i, i = 1,2, \dots, m$, it is possible to estimate the expected internal rate of return by the sample average, that is,

$$\widehat{\mathrm{E}[R]} = \frac{1}{m}\sum_{i=i}^{m} r^i$$

We can also estimate the conditional value at risk (CVaR) on 5% level, that is, the expected value of the 5% of the worst outcomes. The population 5%-CVaR can be estimated from m realizations of a random variable by the sample average of the 5% of the smallest samples, that is,

$$\widehat{\mathrm{CVaR}_{5\%}}[R] = \frac{1}{0.05m}\sum_{i=1}^{0.05m} r^{*i},$$

where $0.05m$ was assumed to be an even number and r^{*i}s are the realized IRRs ordered ascending.

3.7.2 Discount margin of the tranches

The CLO notes are valued by the discount margin (DM) of a par priced note. The DM of a floating-rate security is the return over the return of a reference security (we calculate DM so that the price of the return of the security corresponds a par-priced security). Let us denote the random return of a CLO tranche t as R_t and the internal rate of return of a security of which payments depend on the 3-month LIBOR as r_{LIBOR}. Then the DM of the tranche t is $DM_t = R_t - r_{LIBOR}$. We consider the expected value of the return over the reference security, that is, the expected DM $\mathrm{E}[DM_t] = \mathrm{E}[R_t] - r_{LIBOR}$, which can be calculated from sample averages of the realizations of R_t. We also consider the worst case scenarios explicitly, that is, we compare the average of returns of the 5% worst outcomes to the reference rate by calculating $\mathrm{CVaR}_{5\%}[DM_t] = \mathrm{CVaR}_{5\%}[R_t] - r_{LIBOR}$ for different CLO notes of interest.

4 Data used

The data used consists mainly of the four CLOs and seven loan portfolios provided by the client. The properties of the CLOs and portfolios were inspected thoroughly to familiarize ourselves with the data. This was to achieve better understanding of the common contents and properties of available data when investing in a CLO. As the format of especially the loan portfolios varied greatly, we first extracted the necessary data from the portfolios and CLOs for ease of examination. With this, we noticed that some portfolios were lacking information essential for our model, like Moody's ratings, and hence could not be used to full extent.

The range of values for each property was examined as well as the connections between values of different properties. This way, we were able to assess which properties and values would have the most impact on the portfolio's output cash flows and therefore the CLO. Those impacts were further investigated in sensitivity analysis. As the data consisted only of one complete pair of a CLO and its underlying portfolio, those were chosen for use in the sensitivity analysis to ensure better analysis on the results.

In addition to the data provided by the client, some external data was needed for the model. As mentioned in Section 2.4, Moody's historical data for defaults and recovery rates were used. Not only were the values used in forming the model, the historical data of financial crises played a notable role in sensitivity analysis. Additionally, the LIBOR forward curve was used in the model calculations as mentioned in Section 3.1. For validation, that LIBOR forward curve was also compared to the estimate given by the client.

5 Results

5.1 Base case

The parameters of the base case are in Table 1. In the base case loans follow the Moody’s rating based marginal default probabilities. For example, the average annual default probability for B2 rated loans is 4.2% in the the base case (WARF of the portfolio is approximately that of B2 rated loans).

TABLE 1: PARAMETERS OF THE BASE CASE.

Prices of notes	*Average RR*	*New loan spread*	*New loan rating*	*Portfolio WA spread*	*Portfolio WARF*	*WA spread of notes*	*Initial OC*
100	60%	400bps	B2	429bps	2739	2.38	109.9%

All Monte Carlo simulations were done using 1000 repetitions. The resulting discount margins of the lower tranches are presented in Figure 3. As can be seen in the figure, all tranches excluding the subordinate note have the same discount margin in 100% of the runs (two runs of BB had DMs 6.4% and 0.4%, however). This means that they do not suffer practically any losses in the 1000 runs and therefore the BB tranche clearly dominates the other tranches with a higher discount margin. The subordinate tranche absorbs all the losses and therefore has a wide distribution of discount margins.

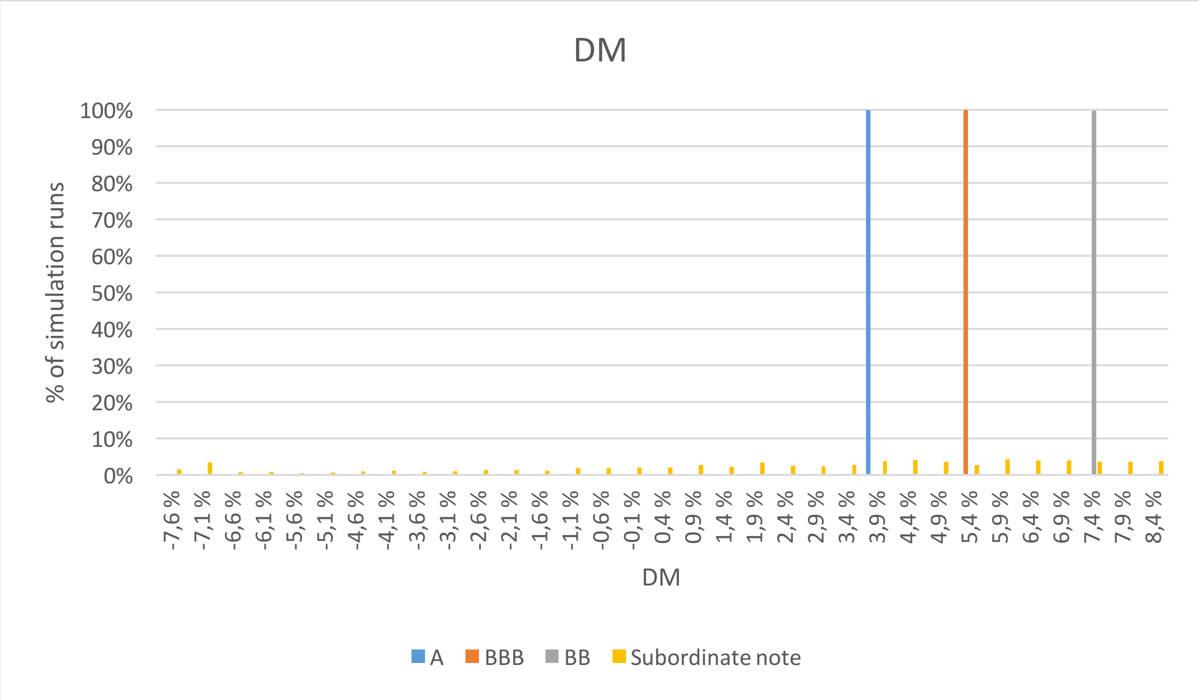

Figure 3: Discount margins of lower tranches in the base case.

Figure 4 illustrates the discount margin distribution of the underlying loan portfolio. The average discount margin of the portfolio is significantly lower than the discount margin of the BB tranche. This is because the first tranche is distinctly largest in terms of par amount and has a low IRR and very low risk. The portfolio DM distribution is not very wide, e.g., only in two cases the DM of the portfolio falls below 1.4%, and in this base case the subordinate tranche absorbs just about all the risk.

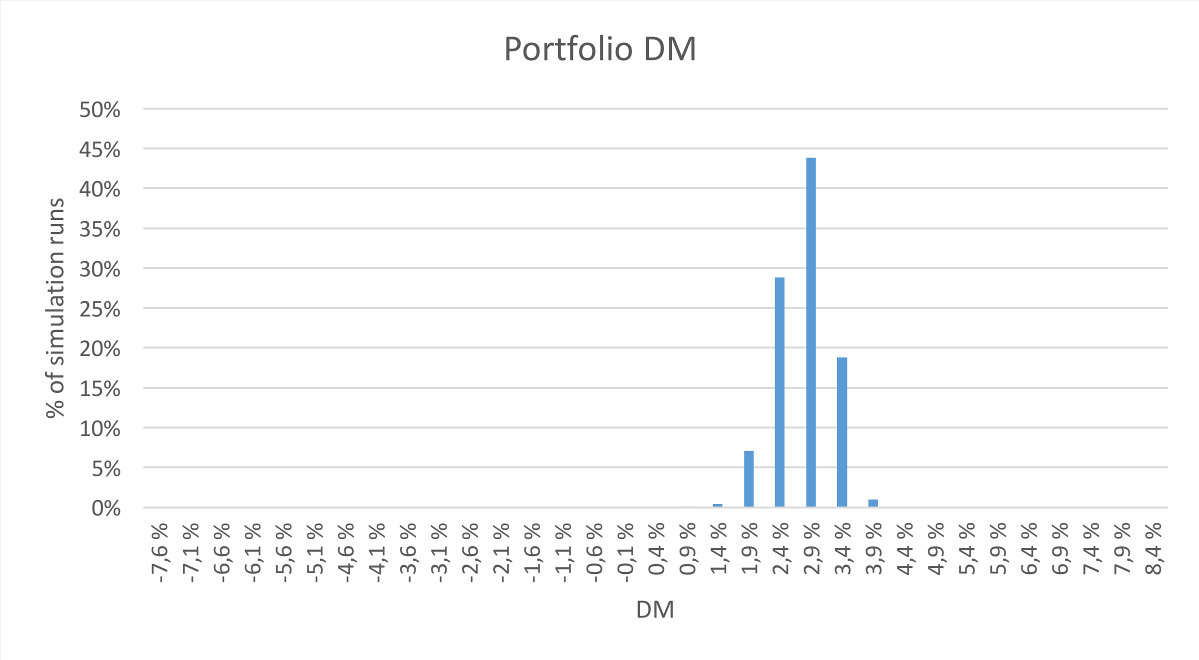

FIGURE 4: DISCOUNT MARGINS OF THE LOAN PORTFOLIO IN THE BASE CASE

The last figure for the base case, Figure 5, presents the relation between the BB tranche discount margin and the relative number of defaults occurred out of the 164 loans in the collateral portfolio. The shading of the points illustrates the numbers of simulation rounds having the corresponding result (more realizations correspond darker shading). Like observed before, in most of the cases the BB tranche DM is very high. This figure illustrates that even with an increase in the amount of defaults the discount margin stays unchanged. Only in a few cases with many defaults, the discount margin of the BB tranche has dropped, and these cases had approximately 50 defaults (30% of the loans in the portfolio) each. The decrease in discount margin was larger with more occurred defaults, so there is some correlation between the number of defaults and the BB tranche discount margin.

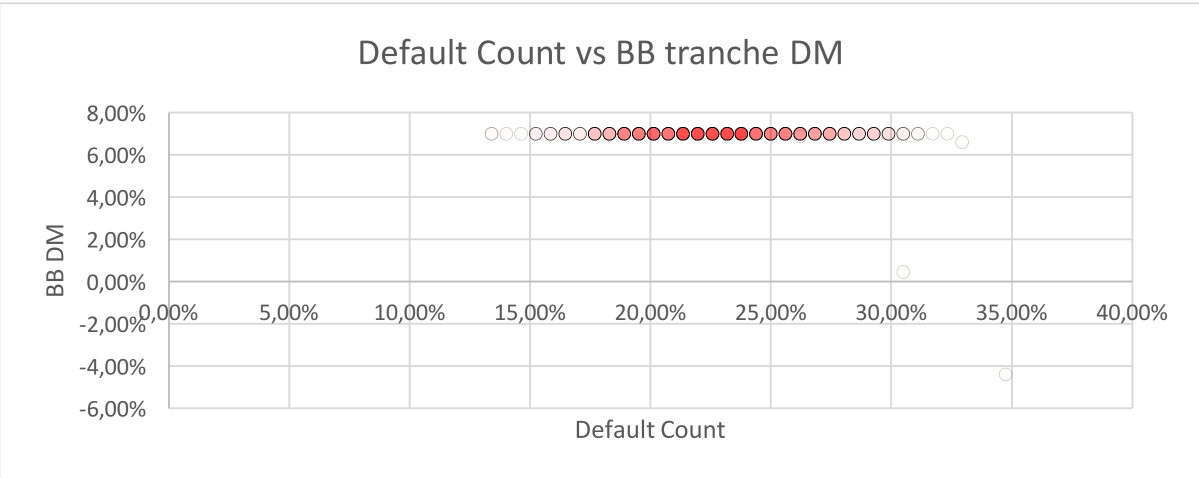

FIGURE 5: BB TRANCHE DISCOUNT MARGIN IN RELATION TO DEFAULT COUNT IN THE BASE CASE.

5.2 Default scaling

The default rates of the loans have a large impact on the CLO losses. Our model includes an option to increase the default probability per year, as described in section 3.2.4. For example, using scaling by two means that the marginal default probability is doubled for that year. First, the effect of general scaling, i.e. increasing the default rate for each year, was studied. Then, different years were scaled with different values.

5.2.1 General scaling

First, a value of 1.25 was selected for scaling the default probabilities. Other parameters were kept constant (as in Table 1). As seen from Figure 6, the effect on the BB tranche was minimal. In over 95% of the simulation runs the discount margin is as high as in the base case, but the tail of the distribution extends to the negative side (up to -1.1%). As seen from the subordinate note discount margin distribution, the increased amount of loan defaults affects the CLO. Fortunately for the BB tranche, the subordinate note is still able to absorb most of the losses.

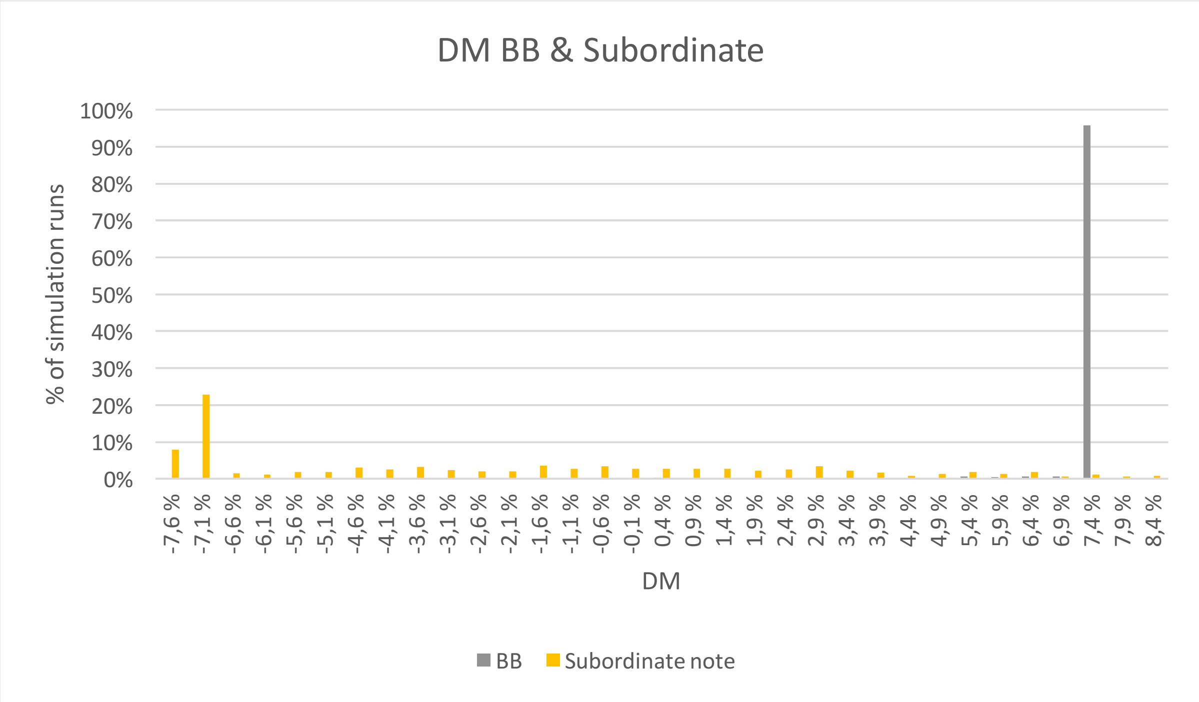

Figure 6: Discount margin of BB tranche and subordinate note in case of default scaling (1,25).

Then, the scaling was increased to 1.4. The resulting discount margins of the BB trance can be seen in Figure 7. With this scaling, some impact on the returns of BB tranche were experienced, but still in 90% of the runs the discount margin is as high as in the base case. The effect of the scaling on the realized defaults of loans is illustrated in Figure 8. Starting from the beginning, the number of defaults increases steadily, but clearly faster than in the base case.

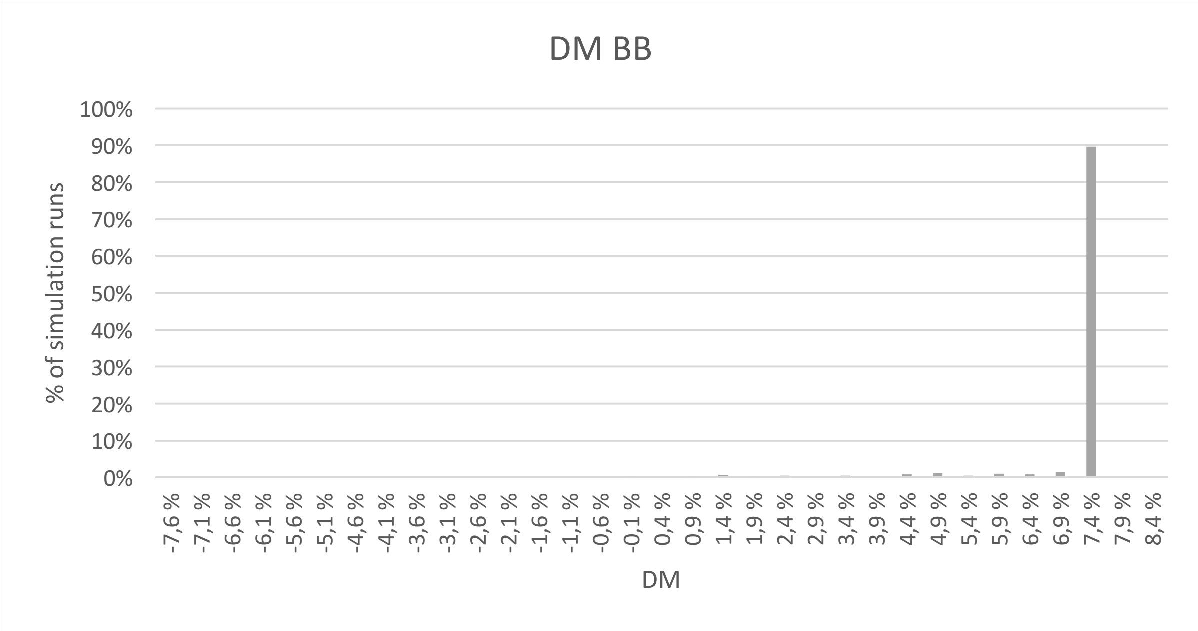

FIGURE 7: DISCOUNT MARGIN OF BB TRANCHE IN CASE OF DEFAULT SCALING (1.4).

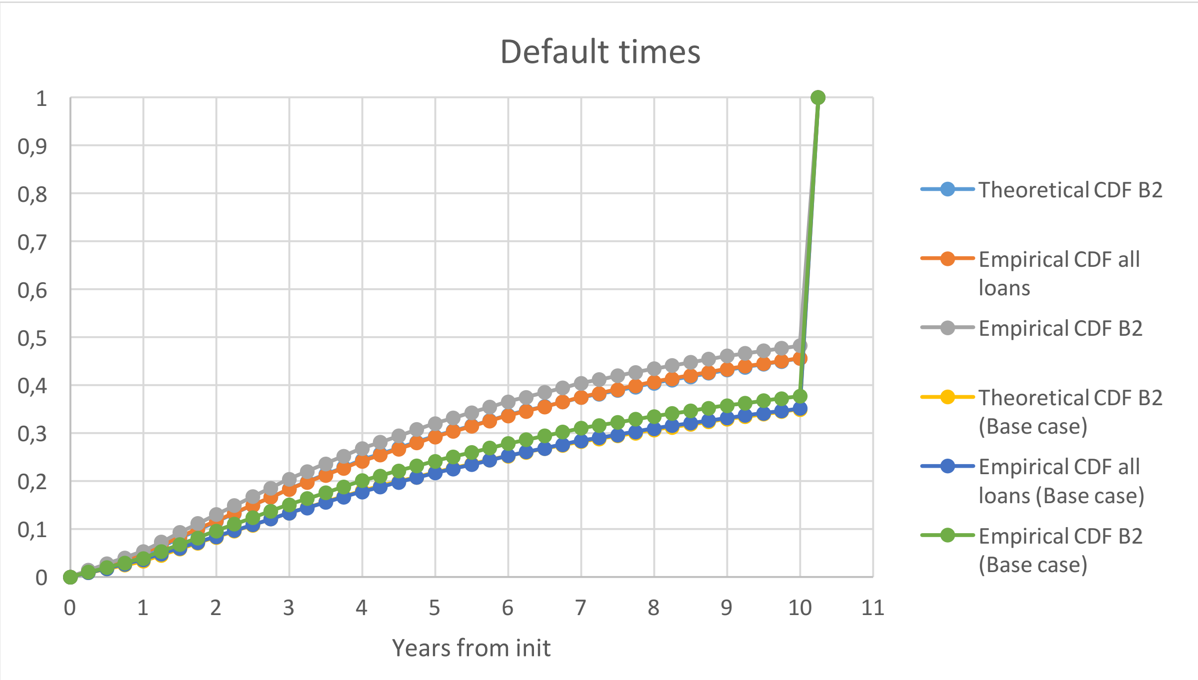

FIGURE 8: DEFAULT TIMES IN BASE CASE VERSUS CASE DEFAULT SCALING (1.4).

5.2.2 Scaling per year

Instead of scaling all years, only one or two years were then scaled to simulate the effects of a financial crisis. Additionally, the effects of the year on which this crisis would hit were investigated. Starting from the year of initialization of the CLO, a bigger scale of 2 was set on each year one by one. The results were such that scaling on year two had the biggest effect, year three and one some effect and the rest a smaller effect, ever decreasing towards year 10.

Then the effect of scaling two consecutive years was compared to the effect of setting a bigger scale on one year. Scaling years two and three with 2 had a greater effect on the losses of the BB tranche than scaling the year two with a value of 2.5, but if the scale of year two was increased up to 3, then the losses exceeded those. Therefore, a short financial crisis has to be very significant to cause more losses than a longer financial crisis.

5.3 Crisis modelling (high default probability few years)

During the financial crisis of 2008-2009, there were a lot more defaults and the recovery rates were poor (Moody's Investors Service, 2011). To simulate this kind of a crisis as a worst-case-scenario, the defaults of years two and three were scaled with 2.5 and the average recovery rate was lowered to 50%.

The effect of scaled defaults for years two and three can be clearly seen in Figure 9. During the first year the amount of defaults is the same as in the base case. After the first year, the amount of defaults starts increasing rapidly until the end of the third year after which the amount of defaults increases at approximately the same pace as in the base case. At the end of the third year 30% of the loans had defaulted on average, whereas in the base case at that point only 15% had defaulted.

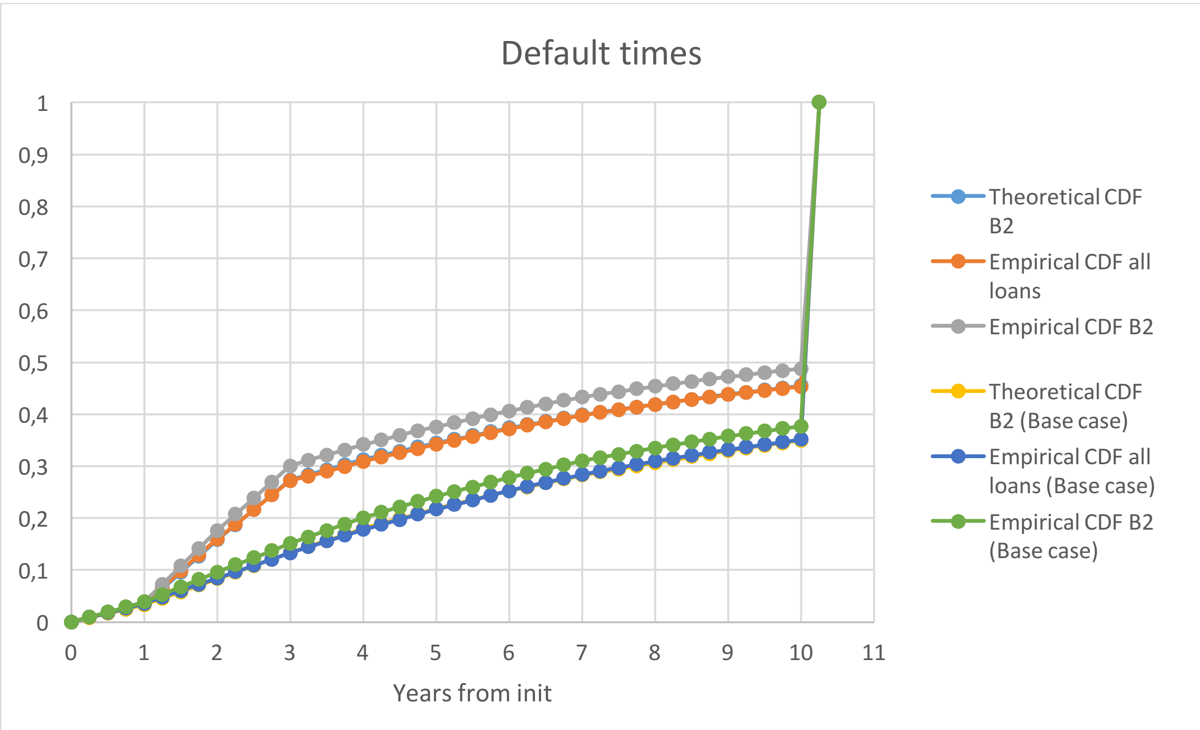

FIGURE 9: DEFAULT TIMES IN BASE CASE VERSUS CRISIS CASE.

The large amount of defaults combined with a lower recovery rate lead to vast losses in tranches of the CLO. The effect of these losses on the BB tranche are illustrated in Figure 10. Now, only in 10% of the runs, the discount margin was as high as in the base case. Most of the simulation runs resulted in great losses -- more than 65% of the runs resulted in a negative discount margin and more than 45% resulted in a discount margin equal to or lower than -7,6%.

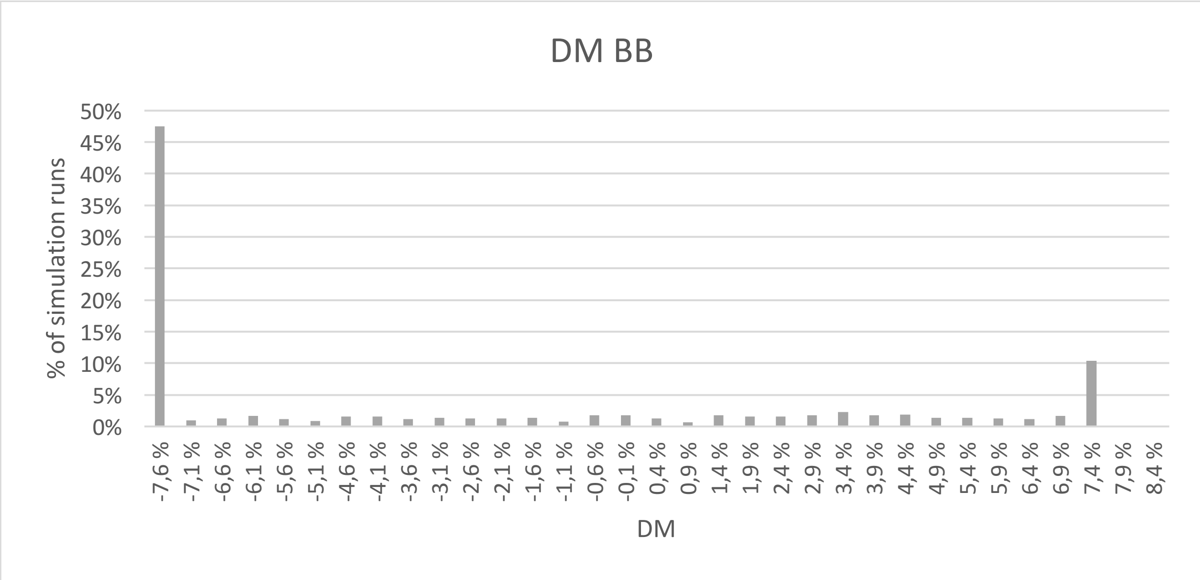

FIGURE 10: DISCOUNT MARGIN OF BB TRANCE IN CRISIS CASE.

The relationship between the DM of the BB tranche and the underlying portfolio is nicely shown in Figure 11. The figure presents the correlation between the portfolio discount margin and the BB tranche discount margin in this crisis case. As the BB tranche suffers great losses in this case, there is clear correlation between the discount margins of the tranche and the loan portfolio. One can observe, for example, that in this case the severe losses of the BB tranche require the DM of the portfolio to fall below 1.4%, approximately. The range of BB tranches discount margin is from -14% to 7.5%, whereas the underlying portfolio's ranges from -1.8% to 2.4%.

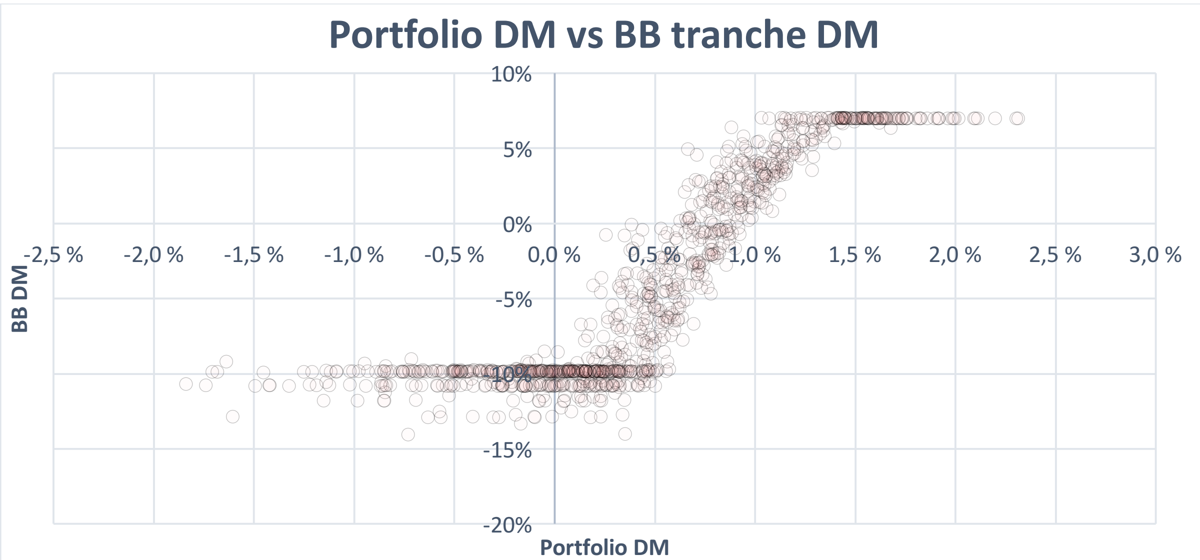

FIGURE 11: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN IN CRISIS CASE.

Figure 12 presents the same comparison of default counts and the BB tranche discount margin as Figure 5 in the base case. This time, the average amount of defaults is clearly bigger. The worse distribution of BB tranche discount margin is also visible in the figure. With more defaults, the BB tranche suffers more losses on average. The figure illustrates how the losses increase when the number of default starts to approach 50.

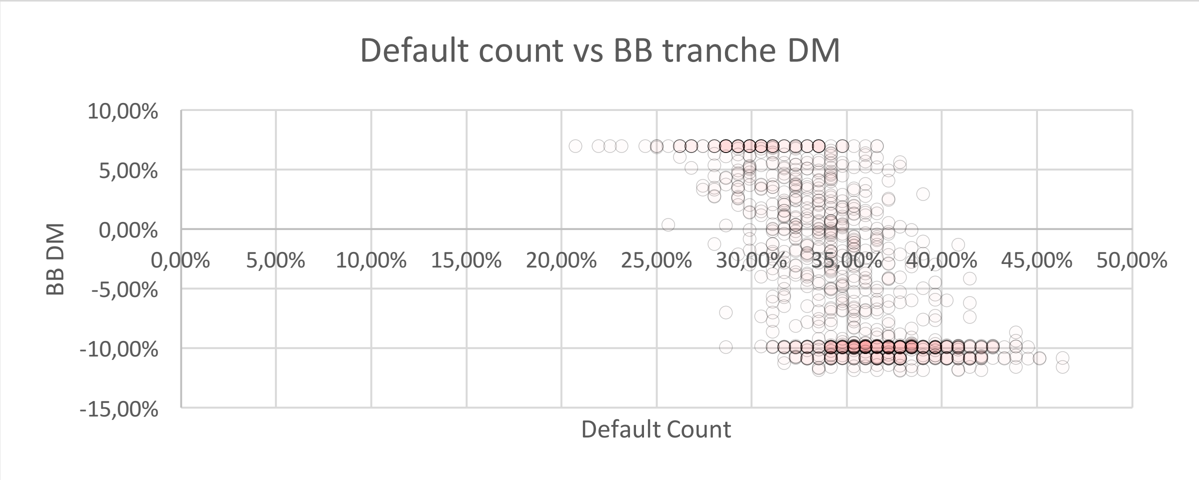

FIGURE 12: BB TRANCHE DISCOUNT MARGIN IN RELATION TO DEFAULT COUNT IN CRISIS CASE.

5.4 Over collateralization tests

We simulate the CLO cash flows with decreasing portfolio par amounts. This can be thought to simulate, for instance, already realized defaults. Figure 13 shows the relation between expected DM and its CVaR compared to the lowered portfolio par amounts. We see that the BB tranche is well protected for the first 15 lost in portfolio par, but the CVaR is greatly reduced after this. The expectation starts to fall significantly after a loss of 25. With a portfolio par value loss of 40 we seem to be nearing the limit at approximately 3% expected DM and −10% as the CVaR 5%. The selected range of initial par amounts corresponds to par subordination of BB tranche in the range of 0-9%. The corresponding par amounts and par subordination levels are presented in Table 2.

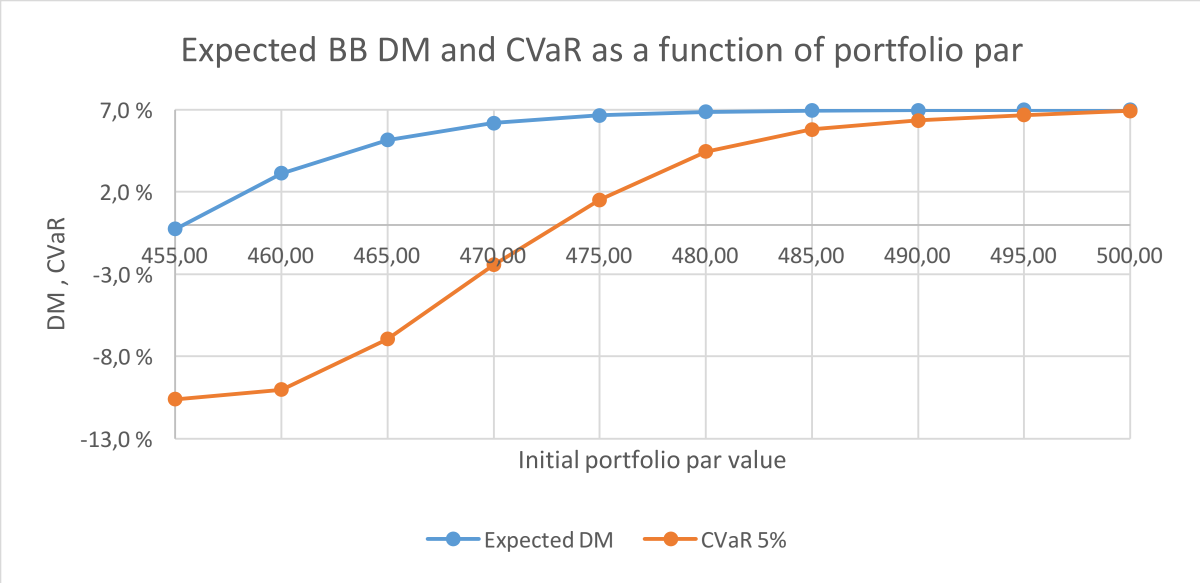

FIGURE 13: EXPECTED DISCOUNT MARGIN OF BB TRANCHE AND CVAR IN RELATION TO PORTFOLIO PAR.

TABLE 2: INITIAL PORTFOLIO PAR AND BB TRANCHE PAR SUBORDINATION.

Initial portfolio par	500	495	490	485	480	475	470	465	460	455
BB tranche par sub. (%)	9.00	8.08	7.14	6.19	5.21	4.21	3.19	2.15	1.09	0.00

5.5 Bad new loans

The quality of the portfolio manager affects in the quality of the new loans in the portfolio. Because we do not model the reinvestment period, the quality of the portfolio manager only affects in the quality of the 20% of loans in the portfolio, that have not yet been purchased and instead will be assumed to have a certain quality (see Section 3.4.1).

5.5.1 New loans spread 200

To investigate the effect of lower quality loans we first lower the coupon spread from 400 bps to 200 bps for the newly issued loans, which corresponds a drop of the weighted average spread of the portfolio from to 427 bps to 389 bps. The effect is minimal. The portion of BB tranche having 0% losses moves from 99% to 98% and the DM 5%-CVaR from 6.92% to 5.97%. However, this clearly shows that changing the spread of about 10% of the loans has an immediate effect on the profits of the portfolio and BB tranche. Figure 14 shows that as the lower spread loans hurt the portfolio DM, we start to see BB DM drop in the low end. The losses begin again after the DM of the portfolio falls below approximately 1.4%.

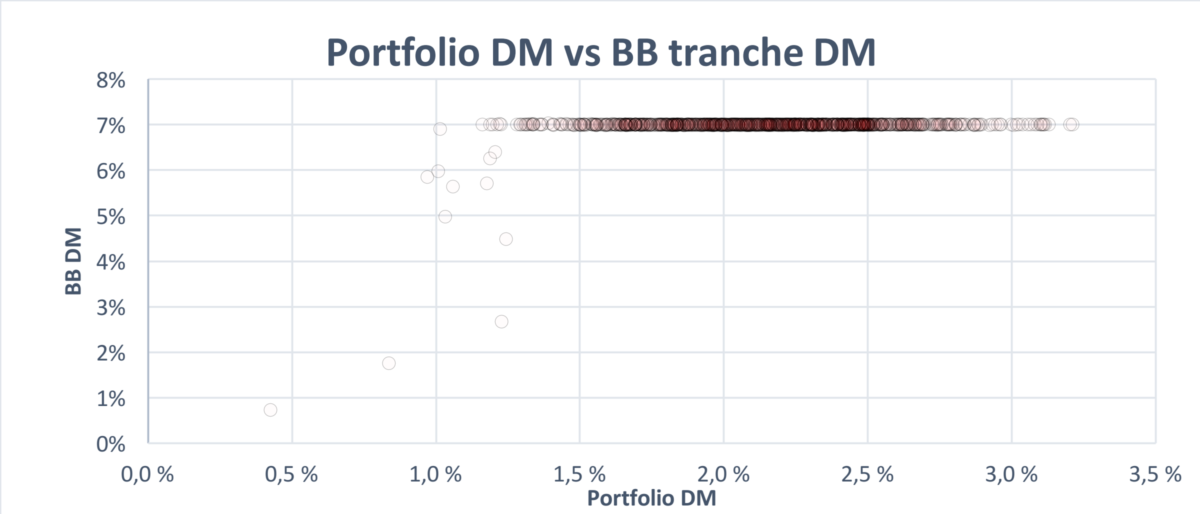

FIGURE 14: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN WITH POOR SPREAD FOR NEW LOANS. WA SPREAD OF COLLATERAL PORTFOLIO 389 BPS.

5.5.2 New loans rating Caa3

We lower the rating of newly issued loans from B2 to Caa3 to see how the rating of the newly issued loans affects the BB tranche lowering the portion of 0% loss to 96% and the DM CVaR to 4.36% which is worse than the CVaR for the BBB tranche. As expected and shown in Figure 15, the increased default probability lowers the portfolio DM and the BB tranche DM incurs a larger hit in the low end. This drop of loans rating corresponds to the increase of weighted average rating factor (WARF) from 2737 to 3753.

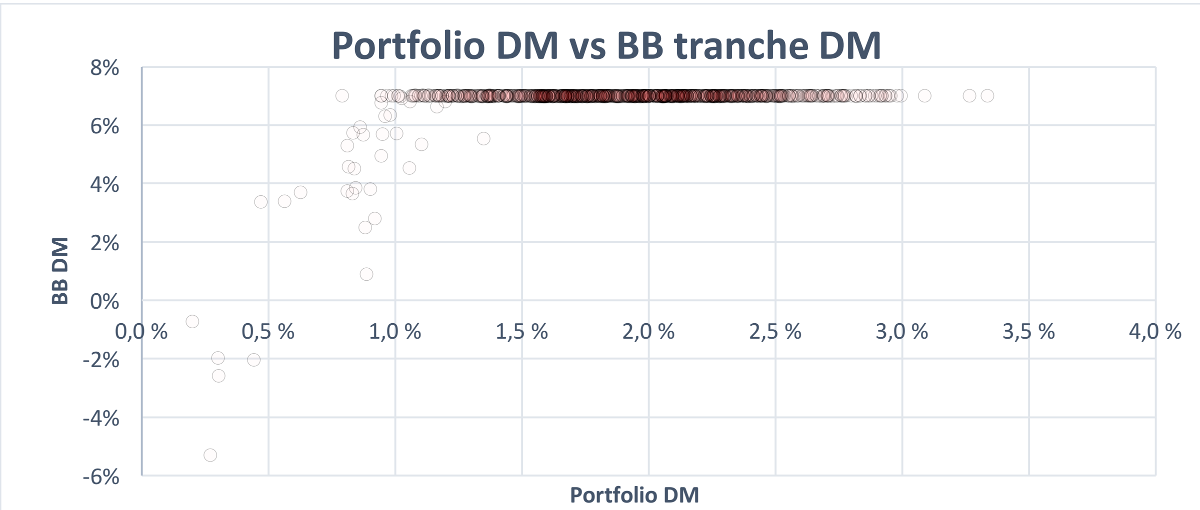

FIGURE 15: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN WITH POOR RATINGS FOR NEW LOANS. WARF = 3753 AND 20% OF THE LOANS ARE CAA3 RATED.

5.5.3 Bad rating and low spread of artificial loans

To simulate a completely dreadful portfolio manager we combine the above cases of picking bad quality loans in both aspects and thus lower the spread to 200 and the rating to Caa3. As shown by Figure 16, the effect is slightly larger than that of the ratings alone. The portion of 0% loss has fallen to 93% and the DM CVaR 5% is at 3.96%.

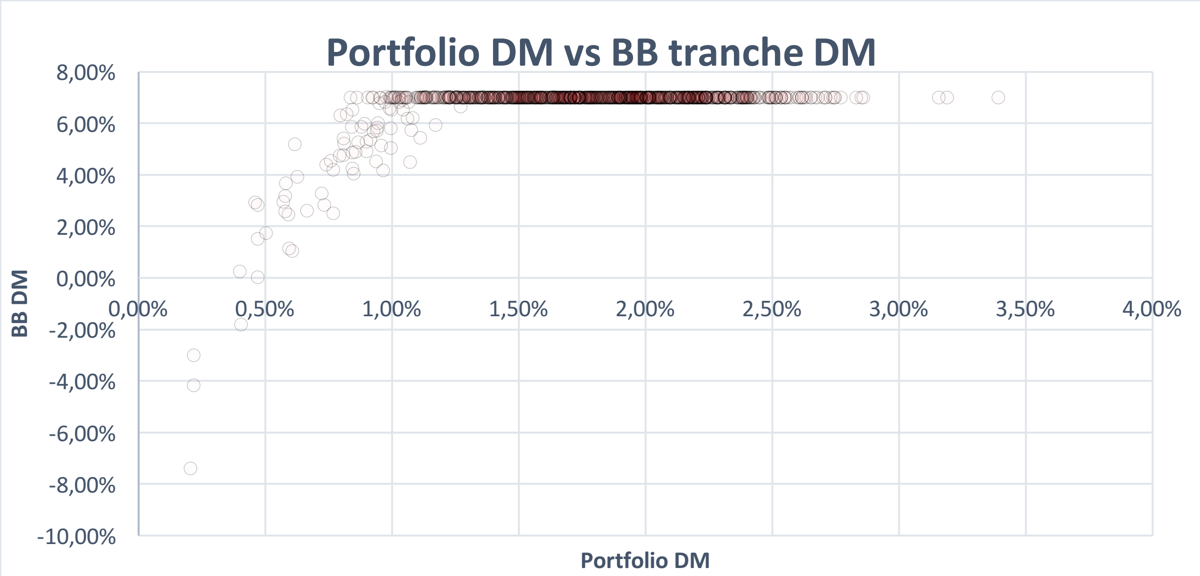

FIGURE 16: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN WITH POOR RATINGS AND POOR SPREAD FOR NEW LOANS. PORTFOLIO WA SPREAD 389 BPS AND WARF 3753.

5.6 Bad liability structure

5.6.1 AAA spread too large

To investigate how a change in the liability structure affects the BB tranche, we adjust the spread of the most senior tranche up. In practice this will increase the profitability of the AAA tranche and leave less cash flow to be shared between the junior tranches. Adjusting the spread from 1.65% up to 1.95%. This has very little effect lowering the BB DM CVaR to 6.70%. Increasing the spread further to 2.5% results in the CVaR decreasing to 5.35%. This increase corresponds to increase of weighted average spread of the notes increasing from 2.39% to 2.97%. Figure 17 clearly demonstrates that the point to witch the portfolio DM may decrease before hurting BB DM has moved up meaning that the BB tranche will be hit earlier (at approximately 1.7% instead of 1.4%) in terms of the portfolio DM.

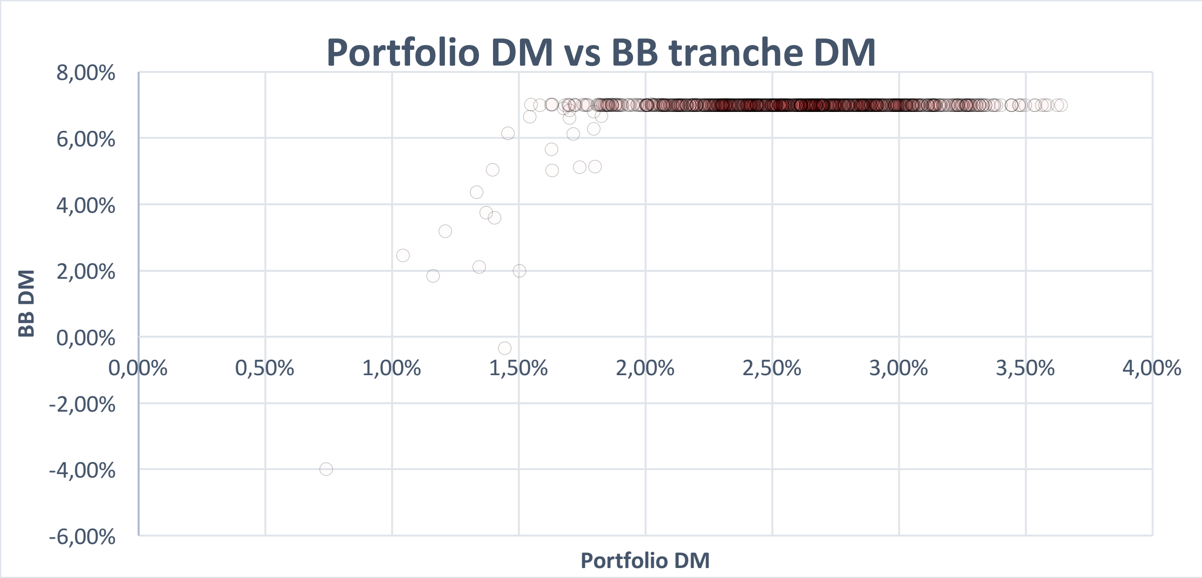

FIGURE 17: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN WITH INCREASED AAA SPREAD. WA SPREAD OF NOTES 2.97%.

5.6.2 Narrow subordinate note

We lower the par amount of the subordinate note from 50 to 25 and increase the par amount of AAA tranche by the same amount. This effectively increases the risk of the BB tranche significantly as the downside protection provided by the subordinate note is significantly smaller (by decreasing the par subordination of BB tranche). This lowers the portion of 0% loss in the BB tranche to 88% and the DM CVaR to 0.99%. Figure 18 shows that the BB tranche gets hit earlier (at 2% as opposed to 1.4%) than in the previous example in terms of portfolio DM.

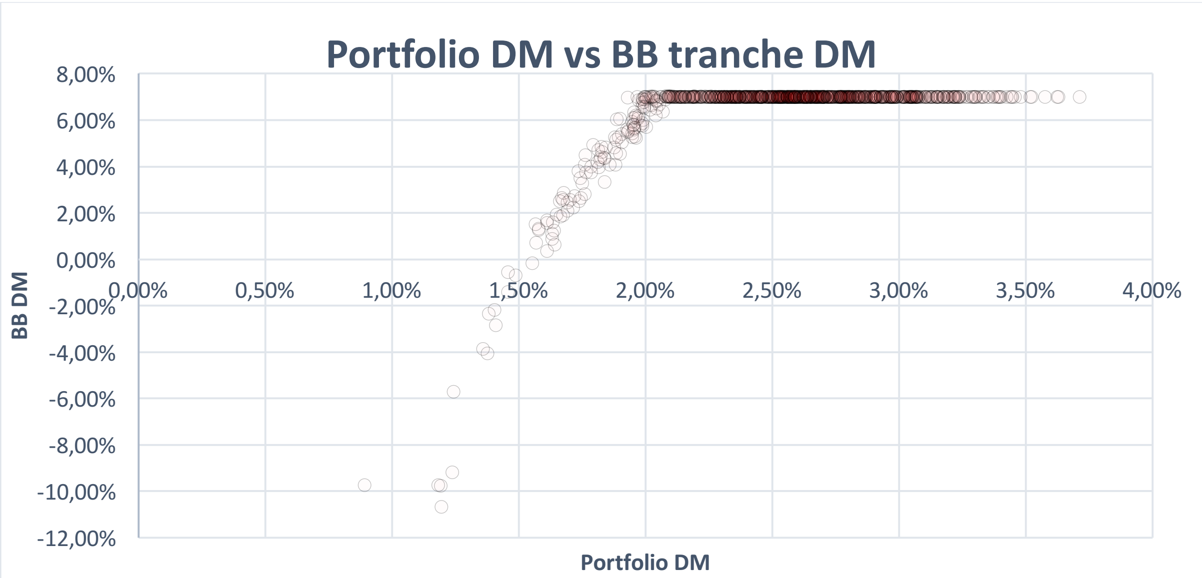

FIGURE 18: BB TRANCHE'S DISCOUNT MARGIN IN RELATION TO PORTFOLIO'S DISCOUNT MARGIN WITH NARROWER SUBORDINATE NOTE AND THICKER AAA. INITIAL PAR SUB 104.2%.

5.7 Conclusive table of results

We conclude the results of the analysis on effects of different factors affecting the returns of the junior mezzanine tranche in the following three tables. Table 3 presents different statistics affecting the return of BB tranche and the expected discount margin. The effect of different changes to the return of the collateral portfolio of the expected DM of the BB tranche are presented in Table 4, and Table 5 presents effects on the return of BB tranche by modifications which have no effect on the discount margin of the collateral portfolio. Table 6 explains the used abbreviations.

TABLE 3: VALUES FOR THE BASE CASE.

$E[DM_{BB}]$	$CVaR_{5\%}[DM_{BB}]$	$Avg[N_D]$	$Avg[N_D^{2,3}]$	$E[DM_p]$	$CVaR_{5\%}[DM_p]$	$WAvg[s_i]$	$WAvg[s_t]$	$WARF$	$ParSub_{BB}(0)$
7.00%	6.92%	58.4	15.7	2.56%	1.67%	4.27%	2.38%	2739	9%

TABLE 4: VALUES FOR CASES WITH MODIFICATIONS AFFECTING PORTFOLIO DM.

Modification	*Portfolio DM change*	*Other effect*	$E[DM_{BB}]$	$CVaR_{5\%}[DM_{BB}]$
Default probabilities scaled by 1.25, all years	$\mathrm{E}[DM_p] = 2.02\%$ $\mathrm{CVaR}_{5\%}[DM_p] = 1.00\%$	$\mathrm{Avg}[N_D] = 69.2$	6.87%	4.49%
Default probabilities scaled by 1.4, all years	$\mathrm{E}[DM_p] = 1.78\%$ $\mathrm{CVaR}_{5\%}[DM_p] = 0.71\%$	$\mathrm{Avg}[N_D] = 75.2$	6.63%	1.06%
Default probabilities scaled by 2.5, years 2&3, and $Avg[RR] = 50\%$	$\mathrm{E}[DM_p] = 0.46\%$ $\mathrm{CVaR}_{5\%}[DM_p] = -1.00\%$	$\mathrm{Avg}[N_D^{2,3}] = 38.6$	-3.99%	-11.66%
New loans spread from 4.00% to 2.00%	$\mathrm{E}[DM_p] = 2.15\%$ $\mathrm{CVaR}_{5\%}[DM_p] = 1.20\%$	$\mathrm{WAvg}[s_i] = 3.89\%$	6.95%	5.97%
New loans rating changed from B2 to Caa3	$\mathrm{E}[DM_p] = 1.92\%$ $\mathrm{CVaR}_{5\%}[DM_p] = 0.79\%$	$\mathrm{WARF} = 3753$ $\mathrm{Avg}[N_D^{2,3}] = 19.2$	6.86%	4.36%
New loans spread 2.00% and rating Caa3	$\mathrm{E}[DM_p] = 1.74\%$ $\mathrm{CVaR}_{5\%}[DM_p] = 0.74\%$	$\mathrm{WAvg}[s_i] = 3.89\%$ $\mathrm{WARF} = 3753$	6.84%	3.96%

TABLE 5: VALUES FOR CASES WITH MODIFICATIONS AFFECTING FACTORS OTHER THAN PORTFOLIO DM.

Modification	*Effect*	$E[DM_{BB}]$	$CVaR_{5\%}[DM_{BB}]$
Initial portfolio par -10	$ParSub_{BB}(0) = 7.14\%$	6.96%	6.34%
Initial portfolio par -25	$ParSub_{BB}(0) = 4.21\%$	6.65%	1.51%
Initial portfolio par -45	$ParSub_{BB}(0) = 0.00\%$	0.27%	-10.62%
Subordinate note par decreased to $25M (from $50M)	$ParSub_{BB}(0) = 4.00\%$	6.64%	0.99%
AAA spread increased from 1.65% to 2.50%	$\mathrm{WAvg}[s_t] = 2.59\%$	6.91%	5.35%

TABLE 6: MEANINGS OF THE ABBREVIATIONS USED FROM TABLE 3 TO

Abbreviation	*Meaning*
$\mathrm{Avg}[N_D]$	Average default count during whole CLO lifetime
$\mathrm{Avg}[N_D^{2,3}]$	Average default count during years 2 and 3
$\mathrm{E}[DM_p]$	Portfolio expected DM
$\mathrm{CVaR}_{5\%}[DM_p]$	Portfolio DM CVaR at 5% level
$\mathrm{WAvg}[s_i]$	Weighted average spread of portfolio
$\mathrm{WAvg}[s_t]$	Weighted average spread of the tranches
WARF	Weighted average rating factor of portfolio
$ParSub_{BB}(0)$	Initial par subordination of BB tranche

6 Discussion

6.1 Pricing ideas

To derive a viable price for the BB tranche based on our model, we propose using a combination of the expected DM and the DM 5%-CVaR to derive a price fitting the purchasers risk preference. The probability of the different economic circumstances could also be considered to combine the prices implied by different market scenarios. Moreover, should the initial state, such as initial par subordination, change, our model gives useful information in assessing the changes into the distribution of the return of the junior mezzanine tranche.

For example, suppose weights w_{E} and $w_{\mathrm{CVaR}_{5\%}} = 1 - w_{\mathrm{E}}$ given by the risk preferences of the purchaser. Then the risk-adjusted expectation of the DM of a par priced junior mezzanine note would be $\mathrm{E}_{ra}[DM_{BB}] = w_{\mathrm{E}}\mathrm{E}[DM_{BB}] + (1 - w_{\mathrm{E}})\mathrm{CVaR}_{5\%}[DM_{BB}]$. On the other hand, suppose that a pessimistic investor is expecting an economic crisis with probability p_{crisis}.

Then the crisis-adjusted expectation of DM could be calculated by the linearity of expected value as $\mathrm{E}_{ca}[DM_{BB}] = p_{crisis}\mathrm{E}\left[DM_{BB}^{crisis}\right] + (1 - p_{crisis})\mathrm{E}\left[DM_{BB}^{basecase}\right]$.

6.2 Validation

The behavior of our model was constantly validated during the building process. Both the asset and the liability side models were validated together and separately. The validation was done with the help of our client, by reflecting on earlier results from the literature, and with sanity checks.

Sanity checks were done most frequently. They included, for example, making sure that the generated default and recovery rates were positive and that the cash flows were of sensible size. Default and recovery rates were also compared against historical distributions. On the liability side sanity checks included making sure that the losses hit the subordinate note first and then the tranches from bottom up. The tranche behavior was validated by changing the price of each tranche on their turn and then verifying that the change in expected IRR was of reasonable size and to the right direction, i.e. when tranche price is lowered, the expected IRR increases.

6.3 Limitations

6.3.1 Lack of modelling reinvestment period

Modelling the reinvestment period was excluded from the scope of this project. Not modelling the reinvestment has two dimensional effects in the lifecycle of the CLO. Firstly, the maturity of the CLO notes is shorter than if reinvestment period would be implemented, because the principal proceeds received during the reinvestment period would be used to purchase new loans instead of paying back the principal to the most senior tranche and because the interest diversion test would be satisfied by purchasing new loans instead of paying back the principal. Secondly, the possible unavailability of suitable loans will not be modelled. Shorter lifecycle of the CLO naturally somewhat reduces the risk of the BB tranche (and other tranches), since payments are received at a faster pace.

Modelling reinvestment period would leave more impact on the quality of the CLO manager. Our analysis only considered the quality of the portfolio manager by the quality of the new issues in the portfolio, and this effect was small with bad quality, but still reasonable, new loans.

6.3.2 Default correlation

We did not include default correlation modelling for reasons discussed in Section 3. Positively correlated default times would mean that more loans would default simultaneously, and hence the possibility of collateral portfolio losses to hit mezzanine tranches could increase. While we model the economic crises separately, our model does not allow for analyzing the probability of such crises occurring.

If default correlation were to be included in the model, already defaulted loans would increase the probability of more loans to default. Hence default correlation modelling would make conducting analysis on the effect of earlier defaulted loans possible. Note that we did analyze the effect of par losses in the collateral portfolio (in Section 5.4), but this analysis does not take into account the increased default probability through correlation of defaults.

6.3.3 Effect of loan prices small

Our model includes the prices of the loans of collateral portfolio only by minor adjustment to the recovery rates of the loans. If prices of loans would be used when calculating the default probabilities, they would have greater effect on the returns of BB tranche. This kind of adjustment for our model would enable analyses on the effect of price of portfolio declining. Moreover, the distribution of loan prices (relative price of some loans smaller than rest of the loans, and hence these loans are riskier than other loans) could be taken into consideration.

6.3.4 Other risk measures

For this project, no suggestions for the most suitable risk measure for BB tranche was found from the academic literature. This motivated the use of a typical metric used in risk analysis, the conditional value at risk. However, whether this measure is the most suitable in the context of pricing remains debatable.

7 Conclusion

In this report we have reviewed relevant literature on CLO pricing and discussed our methodology to the pricing exercise. Using actual loan and CLO data, we have presented outputs from a base case as well as several alternative scenarios, where tranche cash flow performance sees varying levels of impact. In addition, we discussed the validation of the modelling process and evaluated main limitations of the approach.

In the literature review, both a static and a stochastic approach were evaluated and the stochastic approach selected as the appropriate one. Existing default modeling and pricing work were additionally reviewed as a basis for our approach. Recovery rate modelling was specifically reviewed in order to accurately represent the loan portfolio's performance.

The themes of the literature review are reflected in the methodology section that gives a detailed characterization of the model's behavior. We derived the estimations for the default time and recovery rate forming the core of the portfolio model and calibrated them to Moody's default data. We described the liability model and related concepts as well as motivated necessary simplifications. We implemented a Monte Carlo simulator to sample from the stochastic portfolio model and evaluate the distributions of cash flows throughout the wider CLO model. We also gave a definition of the data format used by the model and described the process to standardize given data to the input format.

The base case results for the studied CLO exhibited fluctuating returns only on the subordinate note, implying the BB tranche to possess a very attractive risk-to-return ratio. Several alternative scenarios were evaluated to assess what economic shifts alter the performance of the BB tranche. Simply scaling up default probabilities or introducing inferior loans to the asset portfolio gave negligible effects on the BB tranche's performance while worsening the performance of the subordinate note. Simulating a financial crisis similar to the one endured in 2008-2009 by increasing default intensities and reducing recovery rates however impacted severe losses on the BB and senior tranches. This analysis implies that the BB tranche remains attractive so long as there are no major upsets in the economy, while sufficient trouble in the asset portfolio performance could cause a narrow junior tranche to have a major share of its proceeds and capital wiped out.

We have validated our model both on a per-component basis and from end to end by using sanity checks, the client's consideration and reflection on previous results in the literature. We identified two leading limitations in the model; disregarding correlation among the loan portfolio and the CLO's reinvestment period both potentially underestimate the riskiness of the tranche cash flows, and their incorporation to the model would unquestionably improve the model's accuracy.

Appendix: Self-assessment

CLO are complex derivatives where the complexity is borne firstly of a large loan portfolio and secondly of a long list of allocation rules used to ultimately determine the issued notes' cash flows. The project team had no prior experience in structured finance and pricing of asset backed securities. Study of the problem domain was therefore needed in order to deliver results, and the team has learned a lot about financial modelling and structured finance in the course of the project.

The strategy taken by the team was to first develop a simple model in the form of an initial hypothesis that was to be incrementally developed based on internal validation and client feedback. The approach possessed a number of pros and cons. Namely, delivering working prototypes to the client already at the early stages helped gather feedback and direct next steps for the team, mitigating the risk of failing to provide a valid deliverable. On the other hand, the approach dictated that the extent of the initial literature review was limited, leading to oversight in some steps of the model's implementation and excess work done in diagnosing and fixing resulting bugs. In addition, the task allocation done in the initial stages was substantially altered in the course of the project, which raises the question if attempting such early allocation was indeed necessary.

In setting the project objective, the client articulated a vast number of possible factors that affect the price of a CLO tranche. Of these factors, a subset was selected. Initial findings where the junior mezzanine tranche's pricing was found to be very attractive led the team to focus on risk analysis, assessing what which factors and to what extent need to change in order for the tranche's performance to deteriorate. This shift of focus takes a risk-return view of the pricing problem, meaning that it still addresses the pricing objective of the project.

Microsoft Excel was selected as the modelling tool in the beginning of the project, since it was seen to be flexible and familiar to the client, while being adequate in terms of performance. By the end of the project, implementing Monte Carlo simulation and sensitivity analysis in the model were found to bring Excel to its limits. Additionally, the iterative development approach resulted in a model that could be clearer. If developed further, it might be beneficial to rewrite the model in a language such as R or Python.

In summary, the project has exposed the team to a new and exciting field, where previously learned operations research expertise was applied. The team learned a lot of the field and project work in the course of the project, both from the literature and from interactions with the client. The team would like to thank Professor Salo and Dr. Gustafsson for their guidance and feedback, and furthermore wish Dr. Gustafsson the best of luck in managing his portfolio.

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