Mat-2.4177 Seminar on Case Studies in Operations Research

Helsinki University of Technology

Optimization of Multiplier and Rebalancing in Variable Proportion Portfolio Insurance Strategy

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1 Introduction

This is the project report for the case study Optimization of Variable Proportion Portfolio Insurance (VPPI) strategy. The paper presents the use of VPPI with different constraints to optimize the returns of a portfolio with a given risk limit. The study is conducted for Sampo Bank as a part of the project seminar in operations research in Helsinki University of Technology.

Throughout history, investors have sought after strategies to hedge their portfolios against downturns. One of the simplest methods to avoid risk is to divide the portfolio into two parts, risk-free and risky asset. When the time evolves and market conditions change, the proportions of risky and riskless assets in the portfolio are adjusted according to a chosen strategy. The adjustment is done to maximize the returns of the asset portfolio. The choice for a strategy is mostly based on the investor's risk attitude and objectives of the investment.

In chapter one, we introduce the case company as well as the research problems and project objectives. In chapter two, a thorough literature review is conducted. The literature review consists of some of the most common asset allocation strategies and the advantages and deficiencies of each strategy. We start chapter three by representing the setup of the model, that is, how the market data can be simulated with the help of geometric Brownian motion and jump diffusion. After that, three different VPPI strategies are suggested and also different kinds of risk control methods are represented and the implementation of each one is regarded. In chapter four we represent the results of the project. The performances of different VPPI strategies with different parameters are tested, using both real and simulated market data. Finally, we represent the most important conclusions of the project.

1.1 Case company: Sampo Bank

Sampo Bank is the third largest bank in Finland and a part of Danish bank conglomerate Danske Bank. The 'Sampo Bank' name is used as a trading and brand name for the parent company's Finnish and Baltic operations. Previous to 2007, Sampo Bank was part of the Finnish banking and insurance group Sampo.

Danske Bank's home markets consist of Denmark, Finland, Sweden, Norway, Estonia, Latvia, Lithuania, Ireland and Northern-Ireland. However, the bank has offices also in the UK, Germany, Poland, Luxembourg and Russia.

The financial result for the Finnish branch was 274 million euros in 2007 and it had 1,1 million banking customers.

1.2 Research problems and objectives

From a wide set of different asset allocation strategies our focus is on Variable Proportion Portfolio Insurance (VPPI). The logic behind VPPI is to invest certain percentage of the portfolio to risky assets. This follows a formula:

$$E = mC, \tag{1}$$

where E is the exposure to the risky asset, m is a time-variant multiplier and C is the cushion, the difference between the portfolio value V and the floor F.

The objective of the study is to come up with an applicable rule for determining the multiplier m against time. All the essential variables considering financial market fluctuations must be included for the model to best optimize gains in different situations. The rule must consider both typical fluctuations under normal market conditions and extreme events less frequent, but typical to stock markets.

With these words, the following research questions have been identified:

- What method should be used in choosing the multiplier to maximize expected value of the portfolio?
- What methods are used in choosing the multiplier?
- What are the main benefits and downsides of VPPI compared to other portfolio insurance strategies?

1.3 Methodology

In the beginning, a set of project phases was determined. The critical action points in the project were following:

1. Conducting literature review
2. Deciding on the optimal allocation mechanism
3. Model implementation with Matlab
4. Testing the model with real market data

First, a literature review concerning different asset allocation strategies is conducted. Special considerations are put on dynamic strategies and especially on VPPI. University libraries and databases as well as student research projects will be utilized to find relevant literature and articles. In order to thoroughly understand the optimization problem, each member of the project team will carefully study the relevant material.

The objective of the project is to find an optimum rule for choosing multiplier and rebalancing frequency with regards to VPPI in such way that the expected return is optimized. When all the relevant aspects have been identified, the optimization problem will be developed. There are some constraints that must be taken into account. For example, risky asset exposure must be in range 0-100% and multiplier can only change 1 unit at a time step due to limited liquidity. Also the risk mustn't exceed a certain level.

It is possible to give the multiplier a certain fixed, optimized value. In this case, a variable multiplier is chosen and an optimum strategy for changing it will be developed. The multiplier can be chosen by using different criteria.

For example, (historical or implied) volatility can be applied. Other possible criteria include: current risky asset value, market trend (bear/bull market), risky asset change and trend of the risky asset value. An optimal rule for multiplier selection is developed by carefully considering each of these possibilities.

The rebalancing frequency can be chosen to be fixed e.g. weekly. Or alternatively, a rebalancing might occur after an adequate change in the risky asset value. Also other strategies will be considered. In essence, the rebalancing strategy must react to market changes quickly enough. On the other hand, the shorter the rebalancing period, the higher the transaction costs will be.

When the optimal allocation mechanisms have been found, they will be programmed using Matlab. The constructed models will be tested using real market data delivered by project contact persons at Sampo Bank. Sufficient amount of data is promised to be available. If necessary, additional data can be produced by simulating Brownian motion with occasional jumps. The models will be tested with different kind of data to ensure success in different kind of situations.

This kind of optimization problem has several possible solutions. Different solutions perform very differently depending on the market behavior. VPPI typically performs poor in an oscillating market. The aim is to find a mechanism that could perform well during all kind of market fluctuation. The results of the project will be viewed critically and possible flaws of the constructed model will be identified.

2 Literature review

Commonly investment decisions are made on the basis of efficient markets hypothesis (EMF) [6]. This hypothesis states that it is impossible to “beat the market” in the long run as current market prices reflect all relevant information. Therefore, it is unattainable to do better than the market through maneuvers of market timing or expert stock selection. Although EMF is a foundation stone for investment theory, it has raised a degree of contradictions. [7]

There have been studies (e.g. Lo and MacKinlay 1988) that contain implications on risky assets (shares) not following a random walk on which EMF bases on. In their research Lo and MacKinlay (1988) drew a conclusion that even a reserved ability to forecast the stock market is highly advantageous.

Research on finance has traditionally used mathematical and statistical tools to develop optimal investment strategies. The techniques of genetic algorithms and programming have been increasingly employed in this field. The objective is, and has been to come as close to perfect foresight as possible [6]. In reaching this objective, there has emerged several asset allocation strategies that determine the proportions of the portfolio invested in risky and risk-free assets.

2.1 Asset Allocation Strategies

When considering asset allocation strategies the two most referred terms are strategic and tactical asset allocation. Strategic asset allocation takes an overlook on a portfolio's robustness to achieve long-term objectives. Tactical asset allocation, on the other hand, concentrates on the short-term and aims to respond to market borne deviations from the objectives set in strategic asset allocation. There are several ways to control this deviation in aiming to maximize investor's gains. That is, the investor must decide on a set of rules which determine the reaction on market fluctuations that affect the portfolio value. These rules are called asset allocation strategies. [1]

A strategy is chosen on the basis of investor's risk tolerance. In a fluctuating market different strategies lead to different outcomes. There are no "good" or "bad" strategies for asset allocation. The chosen strategy should always be evaluated against current market conditions. One strategy may perform better in certain conditions as the other one could be stronger in different situation. Furthermore, some investors are willing to take larger risks than others with their choices of asset allocation strategies. In the end, the feasible investment strategy is determined by those bearing the risk and enjoying the gains from investments. [2]

There are certain terms used to describe different market conditions. A bull market is linked with increasing investor confidence and raising stock market prices. A bear market is a prolonged period during which stock prices fall. Prices fluctuate constantly on the open market; a bear market is not a simple decline, but a substantial drop in the prices of a range of issues over a defined period of time. Bear market is usually accompanied with economy recession as on the other hand, bull market is experienced during economic boom. [4]

Strategies that sell and buy risky-assets on the basis of market fluctuations are called portfolio insurances (PI). Constant proportion portfolio insurance (later referred to as the CPPI), popularized by Black and Jones [3], is one of the most commonly used PI-method. It consists of two features: the ability to guarantee initial investment and the ability to provide participation to the market performance. Portfolio insurance strategies have better down-side protection and better upside potential than buy-and-hold strategies [2].

In this paper, we next take a closer look on three common asset allocation strategies and then go on to viewing the most interesting one: variable proportion portfolio insurance. The three strategies presented next are: buy-and-hold; constant-mix and constant-proportion portfolio insurance.

2.1.1 Buy-and-hold

Buy-and-hold strategy is a ‘do-nothing’ strategy. An initial mix of risky and risk-free asset is bought and then held. In buy-and-hold strategy no rebalancing of the portfolio is required which neglects the need for constant monitoring. As a consequence, buy-and-hold strategy has low management costs.

Reproducing Perold and Sharpe (1988), let’s consider an example where you have \$60 in shares (risky) and \$40 in cash (risk-free). The difference \$60 –

$40 = $20 represents a 'cushion' and the risk-free position $40 stands for a 'floor'. Portfolio value will never drop below this floor. On the other hand, the upside potential in buy-and-hold strategy is unlimited. That is, the value of a portfolio can rise indefinitely along with the escalating stock market.

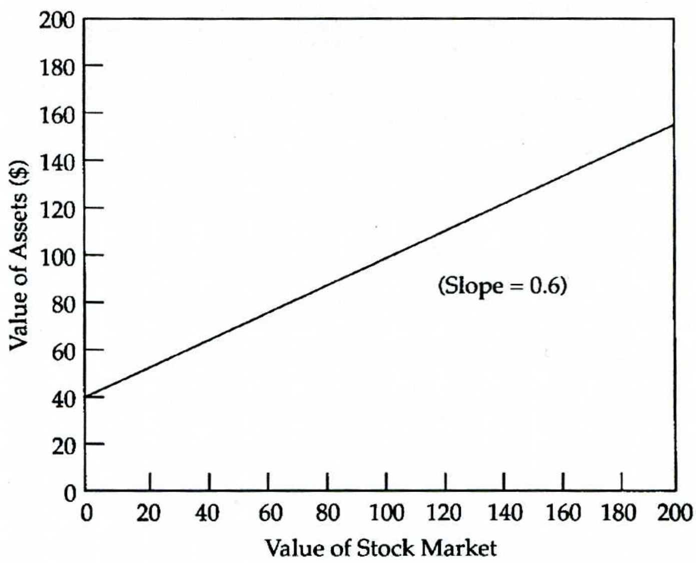

Figure 1: Payoff diagram for 60/40 buy-and-hold strategy. Adapted from Perold and Sharpe (1988).

Some characteristics of the strategy are:

- Risky and risk-free asset weightings alter according to market fluctuations.
- The portfolio value is linearly dependent on stock market value. That is, buy-and-hold portfolio increases in relative value by the share of shares (risky assets) in the mix.
- The portfolio will do at least as well as the floor. That is, its value will never go below the initial investment in risk-free asset.
- Upside potential is unlimited.

To sum, buy-and-hold strategy poses largest potential for reward and loss. The larger the initial percentage invested in shares the larger the reward from buy-and-hold strategy during bullish market. The opposite goes for bearish market.

2.1.2 Constant-mix

In constant-mix strategies a constant proportion of portfolio is invested in risky assets (shares). When the values of assets in the portfolio change, the investor buys and sells a mix determined by some constant ratio. That is, there is a rule that forces the investor to buy and sell stocks along with changing market conditions in order to keep the portfolio on the constant mix.

In following Perold and Sharpe's (1988) illustration, let's take an overlook on an investor who is managing a 60/40 constant mix portfolio. That is, $60 of the portfolio value is in risky assets (shares) and $40 in risk-free assets. Now the investor faces a decline of 10% in the value of shares. As a consequence, the shares drop to $54 in value and the whole portfolio to $94. The proportion of shares in the portfolio is now 54/94=57.4% which is below the chosen mix proportions. To get back to 60/40 mix, the investor must now purchase new shares. If the stock market had gone up, the investor would have sold some portion of the shares.

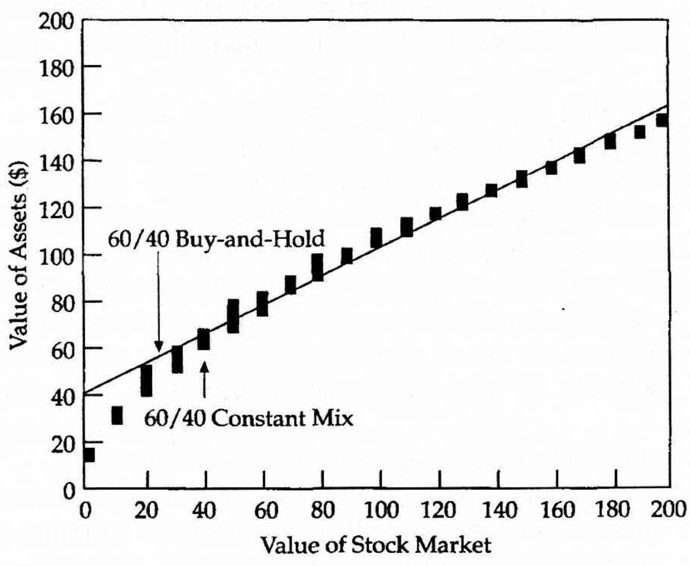

Figure 2: Payoff diagram for constant-mix and buy-and-hold strategies. Adapted from Perold and Sharpe (1988).

Figure 2 illustrates constant-mix strategy versus buy-and-hold. Each black box corresponds to one of 2000 possible results for an investor who rebalances his portfolio after any 10-point move in the stock market [2].

The timing for rebalancing differs by investors. Some portfolios are rebalanced at certain time intervals but more often the action will be undertaken when there has been a change of a specified percentage in the value of the portfolio. In concrete terms, the rebalancing implies buying of shares as their value declines and selling shares when they are rising.

In sum, the main characteristic of constant-mix strategy is its underperformance during upward going share market. Similarly, the strategy is winning when the market is down sloping.

2.1.3 Constant Proportion Portfolio Insurance (CPPI)

As we saw constant-mix selling shares when they rose, CPPI strategy does the opposite. In applying CPPI, a floor must be selected. As noted earlier, this floor represents the minimum value under which the portfolio cannot go. As we denote the difference between the floor and the portfolio value as the cushion, CPPI strategy is simply a rule that keeps the weight of risky assets in our basket as a constant multiple of the cushion [9]. Let's look this little bit closer.

Again, we have a good illustration of CPPI in action in Perold and Sharpe (1988). We have a portfolio of $100 value with the floor set on $75 and the multiplier fixed at 2. As a consequence, the initial investment in shares is $50 and the cushion equals $100-$75=$25. Now let's consider the stock market declining by 10%. We see risky assets decreasing from $50 to $45 resulting in $95 for the total portfolio value. Further on, the floor was set on $75 so our new cushion is now $95-$75=$20. According to CPPI strategy our exposure to risky assets (shares) should be multiplier times the cushion which is $40. As a consequence, we have to sell $45-$40=$5 worth of shares to satisfy our investment rule. In sum, with CPPI shares are sold when they fall in value and bought when their value is rising.

Figure 3 (adapted from Perold and Shapiro 1988) illustrates CPPI strategy versus buy-and-hold. The black boxes each stand for an outcome of 2000 possible results in the case where the investor is rebalancing his portfolio after any 10-point move in the market. [2]

As noted earlier, genetic programming has been used increasingly in deciding on the multiplier [3]. Genetic algorithm, on which the programming builds, is a computing technique to find or approximate solutions to optimization and search problems [8]. The approach is composed of several genetic operators such as mutation, selection of the fittest and crossover [3].

CPPI strategy's deficiency is evident. The allocation multiplier is fixed and doesn't adjust according to fluctuations in market. It contains only data from historical volatility and is therefore vulnerable to sharp fluctuations in the market volatility [5]. As a consequence, there is demand for strategy that takes also this side into consideration.

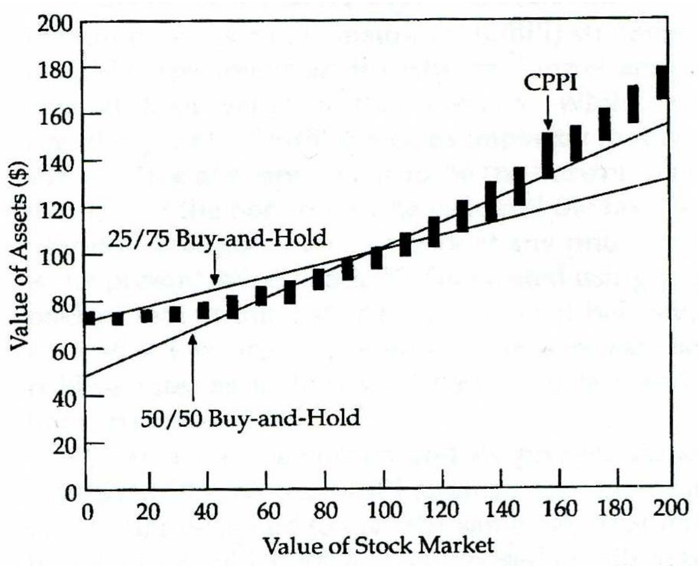

Figure 3: Payoff diagram for CPPI and buy-and-hold strategies. Adapted from Perold and Sharpe (1988)

2.2 Modifications of CPPI

We now introduce some extensions that can by applied to CPPI. The strategy described above is so called *constant floor CPPI*, the floor grows at constant rate through investment period. J-F Boulier and A Kanniganti [15] present two *variable floor strategies*.

The problem with a fixed growth rate floor is that the cushion can become too small to provide exposure when market is performing very well, especially when t approaches T. And there is also a problem of portfolio to become too exposed throughout a rising period. A highly exposed portfolio is very vulnerable in case of a downturn. Boulier and Kanniganti introduced so called ratchet strategy and margin strategy to solve these problems.

Ratchet strategy The idea is to increase level of insurance by putting "excess" cushion into the floor. The floor is increased if

$$mC_t > pV_t, \qquad (2)$$

where constant $p > 0$, V_t is the value of the portfolio, cushion $C_t = V_t - F_t$ and $R_t = V_t - E_t$. Now we have a new floor and a new exposure

$$F_{t+1} := \frac{m-p}{m} V_t \qquad (3)$$

$$E_{t+1} := m(V_t - F_{t+1}) = pV_t. \qquad (4)$$

This strategy is meant to be used in discrete time.

Margin strategy Now we use an extra floor to limit exposure. The idea is to set the initial floor at a value higher than the minimum. Later this "margin" floor can be used to augment the exposure if it falls too low. Let us denote the margin by M_0. We now define the floor as follows

$$F_0 = V_0 e^{-rT} + M_0 \qquad (5)$$

Various methods can be used with the margin but usually it is reduced portion by portion when exposure goes under predetermined level.

In sum, CPPI is a strategy where the multiplier m is constant. What happens if we let the multiplier to change in relation to fluctuating market? This would be VPPI strategy.

2.2.1 Variable Proportion Portfolio Insurance (VPPI)

Variable proportion portfolio insurance, often referred as dynamic portfolio insurance (DPI), is an extended version of CPPI. How it differs from CPPI is its varying multiplier on deciding the proportions of risky and risk-free asset in the portfolio. Since the market is constantly changing there is no reasoning to keep this multiplier fixed like in CPPI strategy (Chen and Chang 2005).

In VPPI, the allocation between shares and risk-free assets is operated dynamically in aiming to maximize the portfolio return also taking into consideration the constraints set by the investor [5]. For example, the investor might want to bind the multiplier to reach values only from a certain interval to match his or her risk attitude.

There are both qualitative and quantitative methods for deciding on the multiplier. Qualitative methods could be such as the investor's forecast profiles on risks and return. Quantitative approaches usually incorporate market volatility measures. These could be historical, current and/or implicit volatilities. Some of the other factors used are the expected return of the risky asset and effective interest rate levels. [5] As proposed already in discussion on CPPI, a high multiplier is beneficial during market growth and the adverse during falling stock market.

Case example of VPPI Now let us take a closer look to VPPI in the case with zero interest rate for asset without risk. Rebalancing of the portfolio is done either periodically or when the value of the risky asset changes a certain predefined percentage. For example, let assume a portfolio of $V=100$, a floor value of $F=75$ and a multiplier of $m=2$. Now using equation 1 gets the proportion, which goes to risky assets is in this case $E=50$.

Thus, the initial mix is 50/50 risky asset/cash. Suppose the risky asset depreciates 10%, so the investor's shares will fall from 50 to 45. The total value of the portfolio then $V=95$, and the cushion is $C=20$. According to the VPPI rule $m=2$, the new stock position is $E=40$ assuming the multiplier stays the same. This requires the sale of 5 of shares and investment of the proceeds in cash.

2.3 Theoretical consideration

In this section, some factors that should be considered when drafting asset allocation strategy are discussed. First, we will review volatility factors as they have been treated in the relevant literature. From there we move on to discuss jump-diffusion process which holds an importance when generating realistic data to be analyzed.

Black-Scholes model for stock market returns embraces certain shortcomings which lessen its use in real life situations. One of these is its deficiency in taking into account abrupt crashes and upsurges that are well expected in the stock market. [11]

An alternative for Black-Scholes model is provided by jump-diffusion processes, which are widely used to simulate price development different assets (e.g. options [12]). Jump-diffusion is a combination of geometric Brownian motion and Poisson process controlled jumps [11]. The former stands for the general diffusion or fluctuation experienced in stock market returns, and the latter aims to model above mentioned changes which are more sudden by nature.

Thus, the model for the asset price is following:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right) \tag{6}$$

Here *W(t)* is a standard Brownian motion, *N(t)* is a Poisson process with rate λ, μ is the drift, σ is the volatility and V_i is a sequence of independent identically distributed nonnegative random variables, such that $Y = \log(V)$ has an asymmetric double exponential distribution.

The stochastic differential equation has an analytical solution:

$$S(t) = S(0) exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\} \prod_{i=1}^{N(t)} V_i \tag{7}$$

To conclude, different asset allocation strategies were presented as discussed in literature. Based on this, we move on to set up our model for determining an optimal rule for determining the multiplier m in VPPI strategy.

3 Model

The selection of the multiplier defines the degree of the performance preferred by the investor. Let us denote the multiplier by m and the floor by F_0. The investment is allocated between risk-free assets R_0 and risky assets $E_0=mC_0$. Under continuous trading we have the following equation:

$$V_t = mC_t + R_t, \qquad (8)$$

where V_t is the value of the portfolio, cushion $C_t = V_t\text{-}F_t$ and $R_t = V_t\text{-}E_t$. Risk-free assets (bonds) B_t are assumed to have constant interest rate r and therefore the floor is $F_t = F_0e^{rt}$. For B_t we have an equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \qquad (10)$$

where μ is the expected rate of growth of stocks, σ is the standard deviation and W_t is standard Brownian motion.

Consider time period [0,T] for the investment. For a capital guarantee derivate security that guarantees the initial amount at the end of maturity T we clearly have $F_0 = V_0e^{-rT}$. The investor can also choose his risk profile by altering the floor as well as the multiplier.

3.1 Measuring portfolio performance

We now introduce a common portfolio performance measure Sharpe ratio that we are about to use to determine relative performance between various strategies. The Sharpe's measure is defined as follows:

$$S_p = \frac{(r_p - r_f)}{\sigma_p}, \tag{11}$$

where r_p is the expected return of the portfolio, r_f is the risk-free interest rate and σ_p is the standard deviation of the portfolio. The Sharpe ratio measures the reward to volatility trade-off.

3.2 Model constraints

The rule for asset re-allocation dynamics that maximizes the expected return of the portfolio must respect the following constraints:

- Multiplier cannot change more than 1 unit at any one time step due to limited liquidity
- Multiplier must be in the range 1-5
- Risky asset exposure must be in range 0% - 100%
- Maximum risk limit: 95% CVaR must be less than 2% from the notional amount
- Rebalancing costs (fixed amount 0.01% of notional amount + 0.05% of rebalancing notional)

3.3 Volatility-based strategy

The multiplier can be defined by calculating the 20-day annual historical volatility of the stock market. The basic principle is to decrease m when market is very volatile and increase m when there is less volatility. Thus the possibility of making heavy losses when the stock market is going down rapidly is eliminated. On the other hand, we cannot exploit any quick increases in the market value.

When using real market data, implied (instead of historical) volatility is used. The multiplier (m) will be determined in the following way (Table 1). Here vol_{t-1} denotes the 20-day historical annual volatility of the risky asset on the previous trading day.

Table 1: Defining the multiplier with the help of volatility.

$vol_{t-1}(\%)$	≤ 10	$10 < vol \leq 15$	$15 < vol \leq 20$	$20 < vol \leq 25$	> 25
m_t	5	4	3	2	1

Additionally, the multiplier will not be increased unless it has been the same for at least 5 business days. Thus, the multiplier won't be increased too quickly if there is only a short upward movement in the stock market value.

When calculating the volatility we have used a Matlab m-file called "Moving variance" from Aslak Grinsted (2005).

3.4 Trend-based strategy

In trend based strategy the multiplier is changed according to market performance. The multiplier is increased by 0.5 when market is in uptrend and reduced by 0.5 when market is in downtrend. The trend is defined with moving averages. We used Exponentially Weighted Moving Averages (EWMA) with leading 9 days average and lagging 26 days average. EWMA uses exponential weights giving much more importance to recent observations. We use 10 consecutive observations after lagging and leading averages crosses to determine the trend change. Consecutive observations are needed to decrease sensitivity to the market changes and to avoid unnecessary multiplier changes.

3.5 Strategy based on volatility and trend

We have considered strategies that define the multiplier with the help of volatility or current trend of the stock market. Now we develop a strategy that combines these two methods, that is, we use both volatility and current trend to determine the multiplier.

If the market is in downtrend, the multiplier is decreased by 0.5 regardless of the current volatility. If the market is in uptrend, the multiplier is changed according to the current volatility:

- volatility > 30%, m remains the same
- 25% < volatility < 30%, m is increased by 0.2
- 15% < volatility < 25%, m is increased by 0.4
- volatility <15%, m is increased by 0.6

The risk of the portfolio can be measured in several ways. In this report we concentrate on using Value at Risk (VaR). However, we also consider often used measurement Conditional Value at Risk (CVaR) from the theoretical point of view even though it was left out of the model due to convenience reasons. Next, we briefly provide the theory of these risk measures and after that we move on to considering how these risk measures should be applied in this project.

3.6 Value at Risk

Value at Risk (VaR) is a common method that is broadly used by security firms and investment banks to measure the market risk of their asset portfolios. VaR measures the worst expected loss over a given time horizon under normal market conditions at a given confidence level. Thus, a bank might state that the daily VaR of its trading portfolio is USD 10 million at the 95% confident level. This would indicate that there is a 5% probability that the daily loss would exceed USD 10 million.

Let us assume that we use a confidence level c and the probability distribution of the future portfolio value is *f(w)*. To determine VaR, we have to find the worst possible realization W* such that the probability of exceeding this value is *c*:

$$c = \int_{W^*}^{\infty} f(w)dw \tag{12}$$

On the other hand:

$$1 - c = \int_{-\infty}^{W^*} f(w)dw = P(w \leq W^*) = p \tag{13}$$

Thus we have a probability of p that the portfolio value is less than W* which equals to Value at Risk at a confidence level *c*.

The problem with VaR is that it gives no information on the possible loss if we hit the lower tail p (p=1-c, where c is the confidence level). Conditional Value at Risk (CVaR) is a risk measure that repairs this deficiency.

3.7 Conditional Value at Risk

CVaR at a confidence level c is the expected return on the portfolio in the worst p% of cases. Thus, CVaR is always worse than (or equal to) VaR. CVaR at a confidence level c can be defined in the following way (here W* is VaR at a confidence level *c*):

$$CVaR = E(W|W < W^*) = \frac{\int_{-\infty}^{W^*} wf(w)dw}{\int_{-\infty}^{W^*} f(w)dw}, \quad \int_{-\infty}^{W^*} f(w)dw = p \tag{14}$$

When determining the multiplier and rebalancing frequency, we must ensure that the risk does not exceed a certain limit. At this project certain limits have been set regarding the risk limit and rebalancing frequency:

- CVaR at a 95% confidence level must be less than 2% from the notional amount
- Maximum rebalancing frequency daily

In essence, we have to set a rebalancing period that is more than 1 day but is short enough to meet the risk limit condition. In other words, the volatility of the portfolio value must not grow too big. We also have to remember that there is a lag of 1 or 2 days before a new allocation takes place.

Let us assume that the initial portfolio value equals 100. Let us now take a certain lower tail of the probability distribution of the future portfolio return (*f(w)*). The expected value of this tail must be -2 and there must be only a 5% probability that we hit this lower tail. When these conditions are met, the CVaR of the portfolio return at a confidence level 95% is 2% from the notional amount.

We will make an assumption that the future return of the portfolio, f(w), is normal distributed with an expected return μ=0 and volatility σ. By using equation 14 we can determine a certain σ that satisfies the risk limit condition. By trial and error, we find a value of σ=1 that satisfies the condition of the risk limit:

$$\int_{-\infty}^{-1,64} f(w)dw = 1 - c = p = 0,05$$

$$CVaR_{0,95} = \frac{\int_{-\infty}^{-1,64} wf(w)dw}{\int_{-\infty}^{-1,64} f(w)dw} = -2 \qquad (15)$$

So *f(w)* is actually a standardized normal distribution (μ =0, σ =1). However, the result is that the volatility of the portfolio mustn't exceed 1% during a rebalancing period. However, the portfolio consists of a riskless investment (bonds) and a risky part (stocks). Let R(t) be the share of bonds at time t and E(t) the share of stocks in the portfolio (thus R(t)+E(t)=V(t)). The portfolio variance can then be determined as (here σ_B denotes the volatility of bonds and σ_S the volatility of stocks):

$$\sigma_p^2 = R(t)^2\sigma_B^2 + E^2\sigma_S^2 \qquad (16)$$

We can assume that σ_B =0. Thus, the maximum variance of the risky part (stocks) will be:

$$\sigma_S = \frac{\sigma_p}{E(t)}, \tag{17}$$

where σ_P =1.

Let us assume that there are 250 trading days a year and the stock market annual volatility is σ. Then we can calculate the maximum rebalancing period T (days), if we already have determined the risky asset exposure E(t):

$$\frac{\sigma_p}{E(t)} = \frac{1}{E(t)} = \sigma_S \cdot \sqrt{\frac{T}{250}} \Leftrightarrow T = \frac{250}{E(t)^2 \cdot \sigma_S^2} \tag{18}$$

For an example we could assume the annual volatility of the stock market to be 20% and the current portfolio risky asset exposure E(t)=0,3 (30%). By using equation 18 we get a maximum rebalancing period of T=6,94 days. With this rebalancing frequency the CVaR of the portfolio is exactly 2% from the notional amount.

Another strategy is to remain a fixed rebalancing period T_F. Then the exposure of the risky asset has to be set small enough, so that the risk limit isn't exceeded. From equation 18 we can solve the maximum exposure.

$$E(t) = \frac{1}{\sigma_S}\sqrt{\frac{250}{T_F}} \tag{19}$$

If we assume once again that the annual volatility of the stock market σ_S=20%, we can calculate the maximum exposure with different values of the fixed rebalancing period (table 2).

Table 2: Maximum risky asset exposure with different rebalancing periods.

T_F	1	2	3	4	5	6	7	10	15	20	30	50
$E_{max}(\%)$	79	56	46	40	35	32	30	25	20	18	14	11

3.8 Implementation

As described in the earlier chapter, we could use CVaR to control the portfolio value change during one rebalancing period. The disadvantage of this strategy is that the exposure will remain at a low level even if the value of the portfolio well exceeds the floor level at a certain time t. This results in low returns even during bull market. The performance of the volatility-based strategy when using CVaR is illustrated in figure 5. In this certain case the value of the stock index has risen to over 500% of the initial value. However, the portfolio final value is under 200% of the initial value. The multiplier remains low (ranging from 1 to 3), and the exposure changes very rapidly at some points. The exposure might be very different even on consecutive time steps because of the nature of the risk control method.

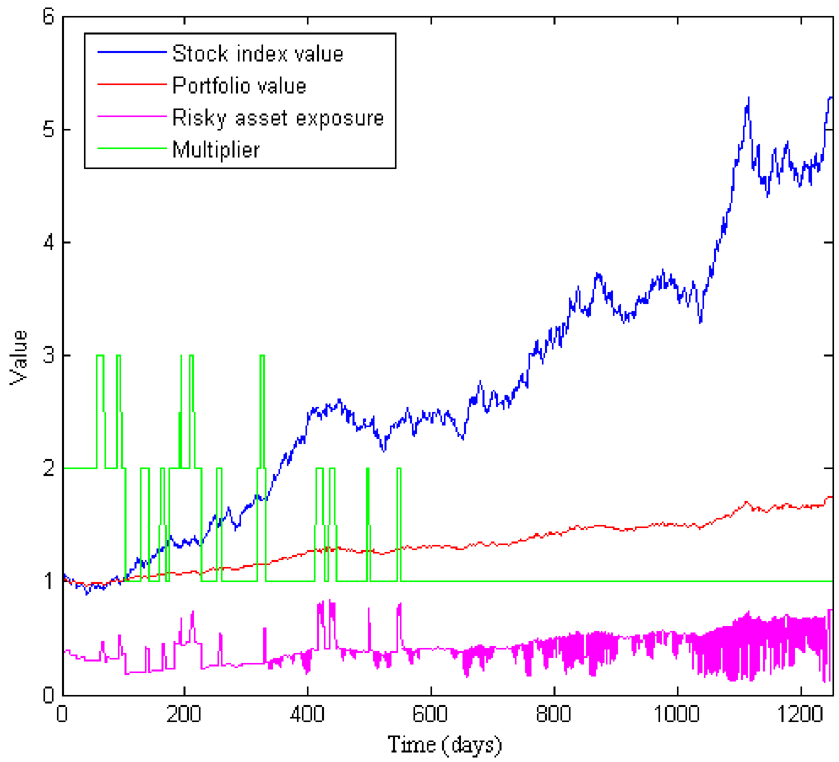

Figure 4: Stock index, portfolio value (volatility based strategy with CVaR), multiplier and risky asset exposure.

Obviously, we should not use this kind of risk measure because it drastically weakens the performance of the VPPI. A more reasonable solution would be to compare the portfolio value V(t) to the floor value F(t). When V(t) is well above F(t), we should put no restrictions to the risky asset exposure because there is only a minimal risk that V(t) would drop to F(t). However, if V(t) is near F(t), so that V(t)>F(t), we should take actions to ensure that the floor level is not reached.

We could use CVaR to determine the maximum allowed exposure if the portfolio value is close to the floor level. In this case, we would have to calculate CVaR at each time step and the calculation time would increase significantly. This kind of risk limit would also be quite difficult to implement and would require iterative measures instead of straight-forward calculation.

Consequently, we control the risk by calculating how much the portfolio value exceeds the floor value at each time step. If the portfolio value (V(t)) is under 102% the value of the floor (F(t)), we set the risky asset exposure to 0% and permanently move all assets to the riskless investment asset. Thus, the portfolio value must meet the following condition at each time step:

$$\frac{V(t) - F(t)}{F(t)} > 0.02 \tag{20}$$

This safety measure makes it very improbable that the portfolio value would drop through the floor. The result would eventually be quite the same even if we used more sophisticated, CVaR based risk limits.

4 Results

In this chapter different simulations of VPPI asset allocation rules are discussed. First, we consider whether to use generated or real market data in simulations. Then we move on to reviewing performances of chosen strategies and end up with conclusions.

4.1 Data types

It can be argued whether to use generated or real historical market data when running the simulations. Below (Figures 5-8) are presented typical examples of both types in the form of stock market returns and volatilities.

4.1.1 Real market data

The VPPI strategies have been tested by using real market data. The data comes from two sources. The first one is S&P 500 which is a stock market index containing the stocks of 500 Large-Cap corporations, most of which are American. The observation time is from January 1, 1990 to April 14, 2008 (4608 trading days). The other source is Dow Jones EURO Stoxx 50 which is stock index designed by Stoxx Ltd and contains 50 sector leaders in the Eurozone. The observation time is from January 4, 1999 to April 14, 2008 (2361 trading days). Thus the total observation time is 6969 trading days (roughly 28 years), including daily stock index and implied volatility.

The value of stock index (the value at first day is set to one) is shown in figure 5. One can see on this picture some rapid increases and decreases as is typical to stock markets. The amount of these stock index fluctuations can be measured by implied volatility, shown to same time period as was used in Fig. 5 in Fig.6. Histogram of five day period of daily returns, the timeframe we are mostly using in this study, is shown in Fig. 7. From Fig. 7 one can see that the return is not normally distributed, but contains tails, which broadens the distribution and such increase the amount of risk one have to take, when investing to the stock index.

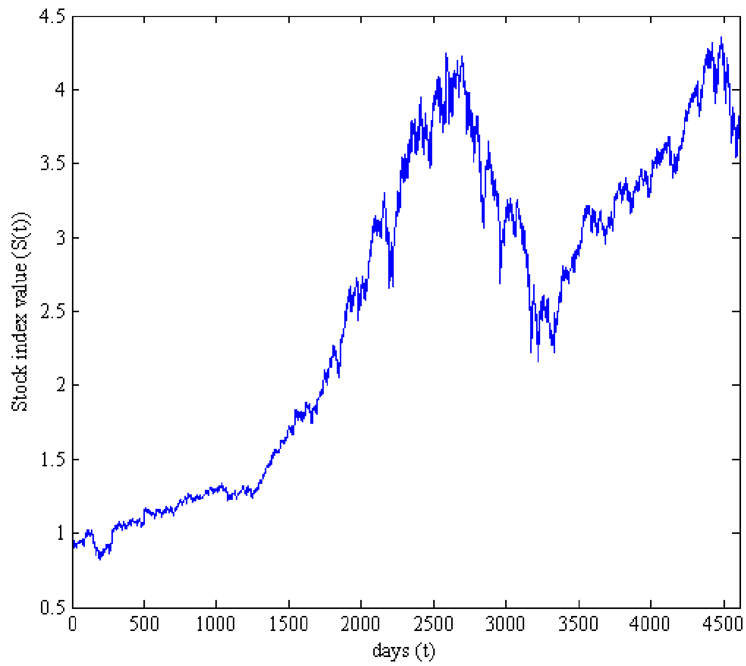

Figure 5: The stock index value of S&P 500

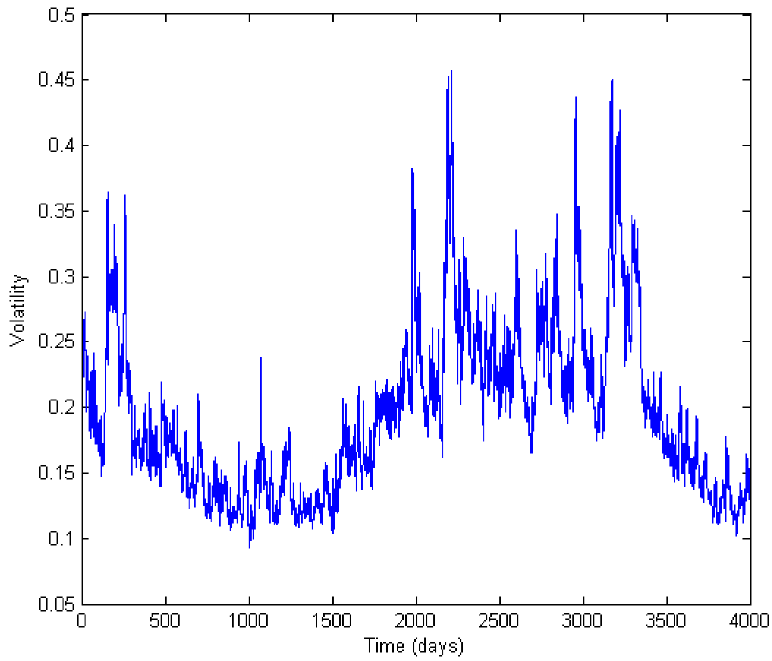

Figure 6: The implied volatility of S&P 500

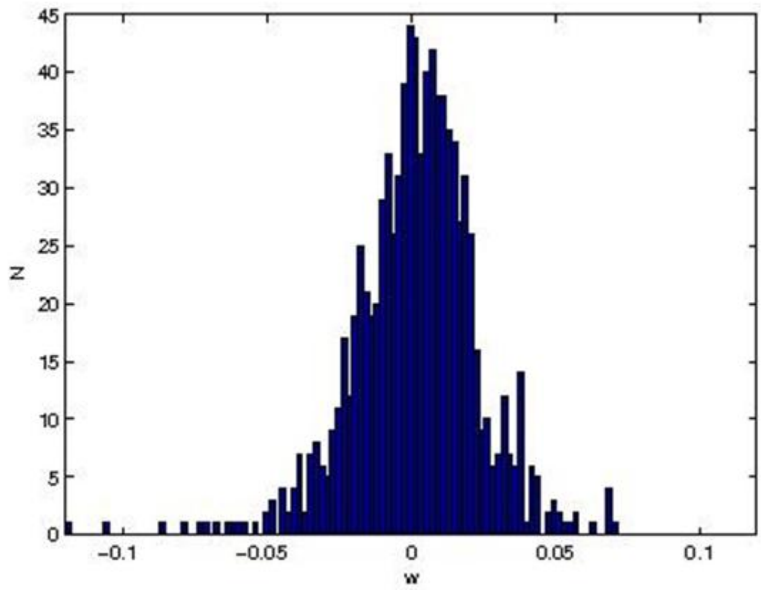

Figure 7: Histogram of five day returns of S&P 500

4.1.2 Simulated market data

To simulate data we used parameters μ=0.11, risk-free interest rate 0.04, and volatility σ=0.2 [3]. Data was simulated to include 5 years of trading days, 50 weeks a year and 5 trading days a week. The simulated stock index value is illustrated in figure 7 and the corresponding volatility in figure 8. As can be clearly seen the data generated by geometric Brownian motion discussed in chapter 3 generates the stock index data, which differs in the case of real data especially with respect to volatility, which are much more noisy in our case, when compared to real market data. This is something what one expects to see, when we are dealing with the Brownian motion without any long-range memory. Such things as high implied volatility fluctuations with respect to long time scales are not seen with our simulated data. More sophisticated methods [16], which better take care of this problem have been created, but are not used here, where we are mainly interested in VPPI performance.

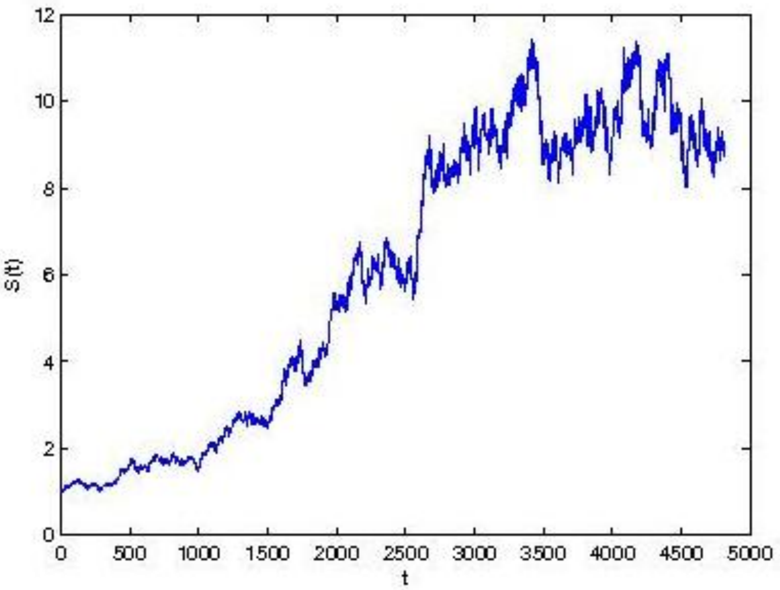

Figure 8: Simulated stock index value

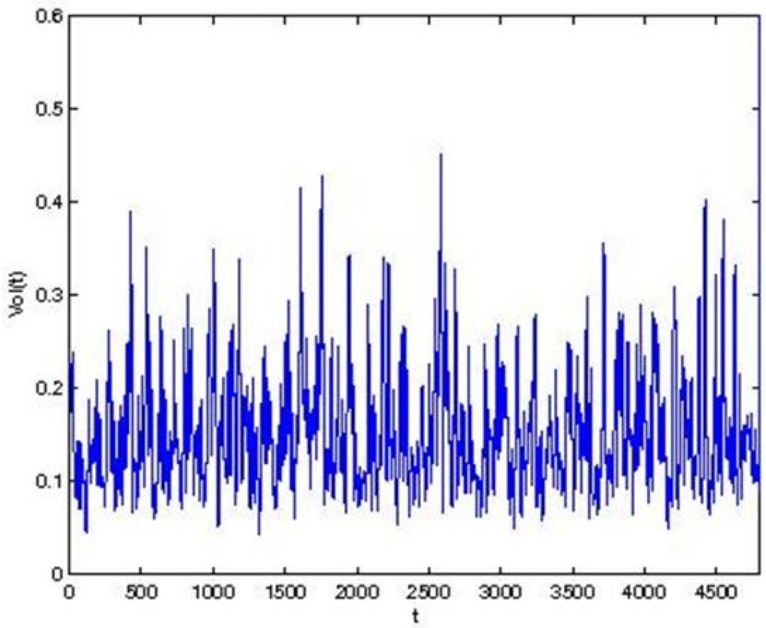

Figure 9: Volatility of generated data

As we can see, simulation generated data gives congruent results with the actual data. A definite advantage in using generated data is its infinite quantity as it can contain a lot longer time horizon than real historical data.

From here on we move to comparing CPPI and different VPPI strategies. We consider the performance of CPPI strategy and use this as a benchmark for our VPPI simulations. We employ both data types in determining the optimal multiplier in the case of VPPI.

4.2 CPPI as benchmark

In order to be able to benchmark the different VPPI strategies we ran simulations with different CPPI values. As assumed the expected return and volatility rise as the multiplier is increased. These expected value results are shown in figure 10 and the volatility results in figure 11. Intuitively, well designed VPPI strategy should perform at least equally well as CPPI with the same volatility. Hence the objective is to evaluate the performance between CPPI and VPPI strategies that we have devised.

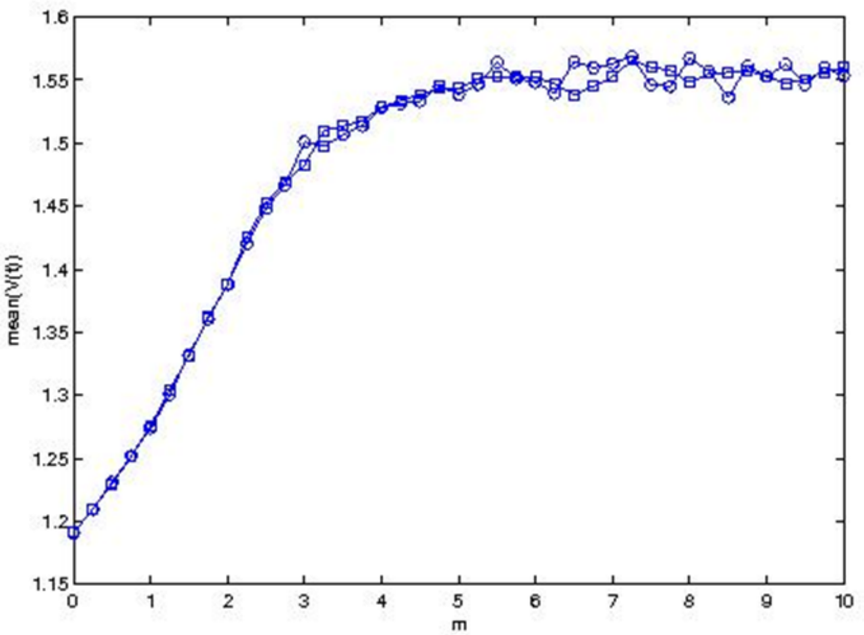

Figure 10: Expected portfolio value in CPPI strategy. Expected portfolio value in linearly decreasing VPPI. Squares 10000 simulation run, circles 20000 simulation run.

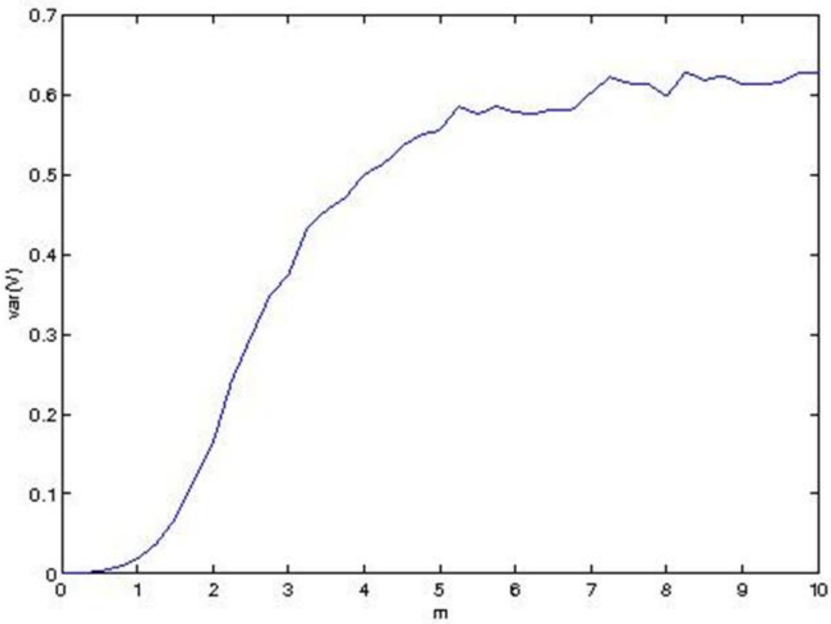

Figure 11: Portfolio variance in CPPI strategy

4.3 Simulation with real market data

We have taken eight different 5-year periods (1250 trading days) from the market data for the simulation. For example, from the S&P 500 data we have taken periods starting from day 1, 626, 1251, 1876, 2501 and 3126. Obviously, most days appear in two different periods, but on the other hand, the number of simulations is increased.

We apply all the three strategies (volatility, trend, volatility & trend) to the market data. In the following simulations a fixed rebalance period of 5 days has been used. However, whenever the multiplier is changed, a rebalance occurs immediately. The risk limit of equation 20 has been used. The results of the simulations and the performance of the stock index are presented in table 3. The Sharpe ratios have been calculated assuming a 4% annual risk-free interest rate.

Table 3: The simulations of different VPPI strategies with real market data.

Strategy	Average end value	STD	Average annual return	Average total cost	Sharpe ratio
Volatility	1.2756	0.3402	1.0499	0.0750	0.173
Trend	1.3986	0.6586	1.0694	0.1218	0.276
Volatility&Trend	1.4247	0.6663	1.0734	0.0780	0.312
Stock index	1.4539	0.8096	1.0777	-	0.293

The histograms of the returns with different strategies are illustrated in figures 13, 14 and 15. The returns of the stock index are presented in figure 12.

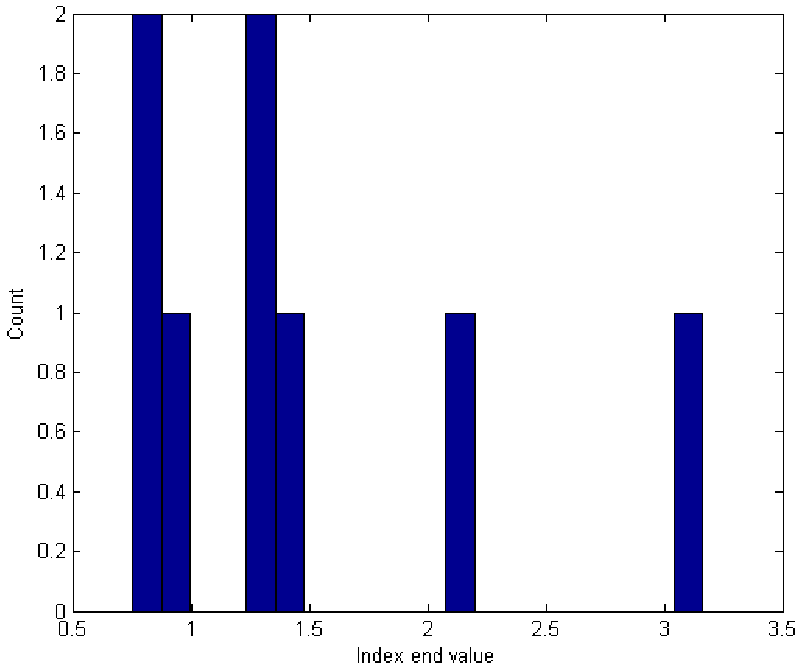

Figure 12: Index end value S&P 500

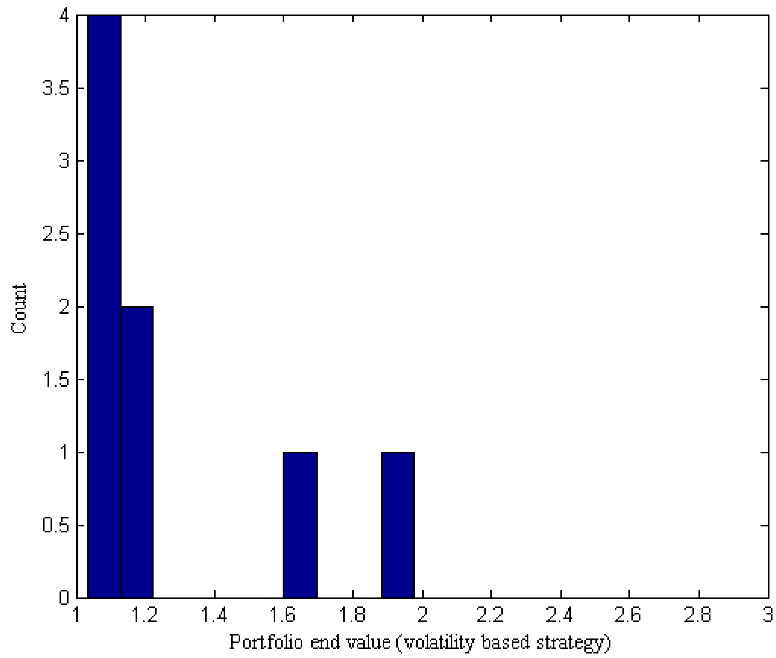

Figure 13: Portfolio value with volatility based strategy

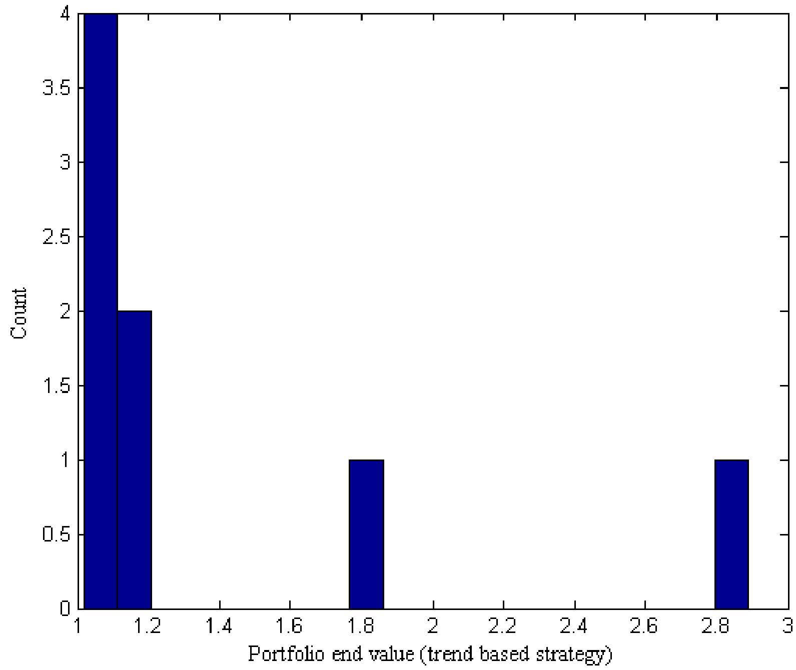

Figure 14: Portfolio end value with trend based strategy

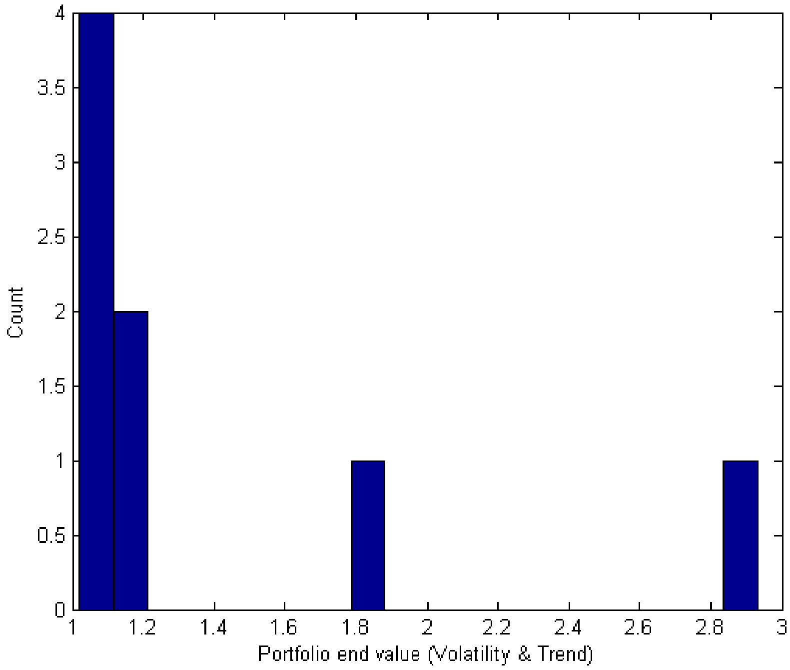

Figure 15: Portfolio end value with volatility & trend based strategy

4.4 Simulation with generated data

We tested our algorithms with simulated data using trend based strategy and volatility & trend strategy. The capital invested in risk-free assets has a growth of three days during weekends, after every 5 days period. With simulated data and real stock market data we used one day time lag in rebalancing, so that when rebalance takes place, it is done at the price of the next day's closing price.

During our simulations we decided that an appropriate simulation run length is 5000 simulations. It is also a length our computers are able to execute in a decent time, several hours. According to our experience of the simulations, the mean has a quite low standard error, about less than 2%. But a problem arises with the variance. It does not seem to converge even if the simulation length is increased to 20000 or more.

The simulation results are concurrent with the results obtained using real market data. The volatility & trend based strategy performs only slightly better than the strategy relying solely on trend. Interesting notion is that neither of the above mentioned strategies is able to outperform CPPI in terms of volatility versus risk.

Strategy	Average end value	STD	Average annual return	Average total cost
Trend	1.4162	0.6391	1.0721	0.0581
Volatility & Trend	1.4347	0.5737	1.0749	0.0611

4.4.1 Case example of linearly decreasing multiplier

We also considered much more simple strategies to allocate assets. One of these strategies called the linearly decreasing multiplier was found to perform quite well. The idea is to start with some multiplier value and decrease it linearly to 0 within the given five year time frame. Starting with high exposure will enable the portfolio value to grow in the beginning of the hence creating possibility to compounding profits. These expected value results are shown in figure 16 and the volatility results in figure 17. The results are very similar to CPPI in terms of volatility versus expected return.

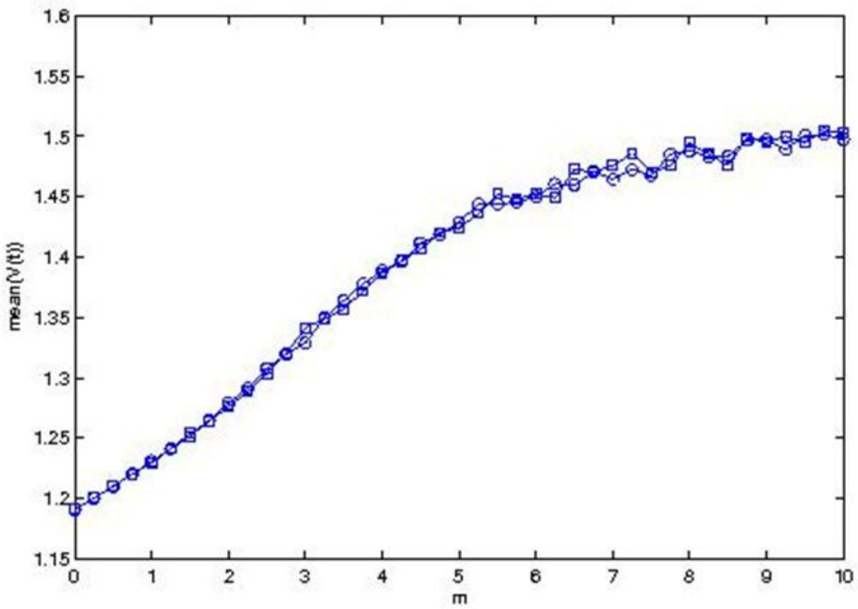

Figure 16: Expected portfolio value in linearly decreasing VPPI. Squares 10000 simulation run, circles 20000 simulation run.

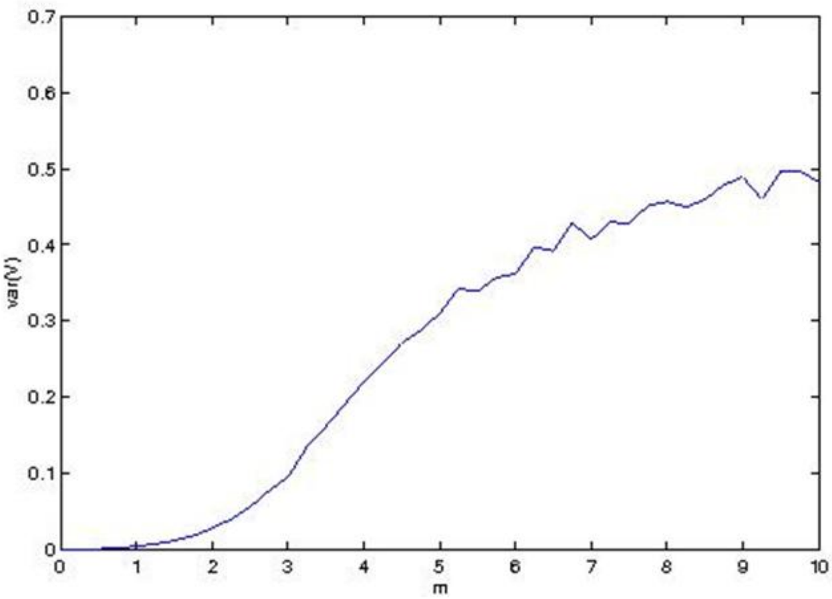

Figure 17: Portfolio variance in linearly decreasing VPPI

4.5 Rebalancing costs

Effects of various rebalancing periods were tested with all strategies, but no confidently measurable differences were discovered. This could indicate that with our strategies the costs caused by rebalancing have a relatively greater impact on expected portfolio terminal value than the actual portfolio performance in terms of capturing the market fluctuations.

For volatility & trend -based strategy we received following results with rebalancing periods 5, 10 and 15 days: average portfolio end values 1.3766, 1.4262 and 1.4347; average standard errors 0.5469, 0.5729 and 0.5766; average total costs 0.110, 0.063 and 0.055 respectively. It is clear that performance differences with different rebalancing periods are very much a result of decreased costs.

4.6 Conclusions

The results of the simulations with real market data include great amount of uncertainty. The amount of simulations (8) is very small so the errors are quite large. With simulated data we use samples of several thousands. So the average returns and standard deviations as well as Sharpe ratios might be misleading.

However, we can make some qualitative conclusions. The volatility based strategy seems to perform worse than the other two strategies. The strategy based on both volatility and trend yields the highest average returns. All in all, VPPI is working as it should be. The possibility of negative returns is eliminated, but in the same time the average return decreases when compared to the stock market index. And according to the Sharpe ratios, the strategy based on both volatility and trend outperforms the pure stock index.

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