

Prediction of remaining lifetime

(Jäljelläolevan eliniän ennustaminen)

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1 Introduction

Predicting the remaining lifetime of a component is an important problem e.g. in optimising system maintenance. The component lifetime is often modelled using only a static probability distribution that does not take into account any component monitoring data. Moreover, many wear and crack growth models are purely deterministic even when it is known that the phenomena are stochastic by nature.

In this work a small literature survey of different methods of lifetime prediction is done. A probability model for predicting the remaining lifetime of a component under some degradation mechanism is developed. The lifetime predictions of such model should be updateable with degradation monitoring data. Fatigue crack growth (FCG) is used as the case degradation mechanism.

The probability model is realised as a Matlab program. Simulations with the program are used to demonstrate the uses and benefits of this kind of model. The Matlab program is handed over to the customer to be used after the end of the project.

This work is part of VTT research project SYSTELI - Systems analysis in life management.

2 Summary of literary survey

2.1 Gibbs sampler

Gibbs sampler is also called alternating conditional sampling.

Let $\theta = (\theta_1, \dots, \theta_d)$ be the parameter vector of interest and let y denote the observed data. Each iteration of the Gibbs sampler cycles through the d components of θ , drawing each component conditional on the value of all the others. There are thus d steps in iteration t . At each iteration t , each θ_j^t is sampled from the conditional distribution given all other components of θ : $p(\theta_j | \theta_{-j}^{t-1}, y)$. Thus, each θ_j is updated conditional on the latest value of θ for the other components, which are the iteration t values for the components already updated and the iteration $t-1$ values for the others.

The sampling converges to a stationary distribution that is the joint posterior distribution $p(\theta|y)$. (Gelman et al. 1995)

2.2 One dimensional linear Kalman filter

The purpose of a Kalman filter is to estimate the state of a system from measurements that contain random errors.

Let x_t denote the state variable. We suppose x_t satisfy the linear equation $x_{t+1} = Ax_t + w_t$, where w_t is random process noise with mean $E(w_t)=0$ and variance $\text{Var } w_t = Q$. We make a measurement y_t of x_t , which is related to x_t by equation $y_t = Bx_t + v_t$, where v_t is random measurement noise with mean $E(v_t)=0$ and variance $\text{Var } v_t = R$. The measurement noise is not correlated with process noise. From these data and an initial estimate x_e of x_0 with the variance of error in this estimate, denoted P , the Kalman filter will give the best estimate of x . (Joseph)

2.3 Acceptance – rejection sampling

Acceptance-rejection sampling is used for generating a sample value of a random variable X with probability distribution π_x . The steps are:

1. Generate a candidate x of X assuming X is uniformly distributed.
2. Take from π_x the probability $P(X=x)$.
3. Generate a random number u from the uniform distribution on $(0,1)$.
4. If $P(X=x) > u$, x is accepted as the sample value. If not, start again from step 1.

The steps are repeated until the desired number of samples is achieved. (See e.g. Chib and Greenberg 1995).

2.4 Bayesian analysis in fatigue crack growth models

Stochastic fatigue crack growth in steel structures subject to random loading

Uncertainty in fatigue crack growth under service load conditions arises from the statistical characteristics of crack growth under constant amplitude loading and from random variable amplitude loading. This study generalises previous stochastic fatigue crack growth models by incorporating a time-dependent noise term described by arbitrary marginal distributions and autocorrelations to model the uncertainty in the crack growth under constant amplitude loading. A computationally efficient approach for handling wide-band random loadings based on the rainflow method of stress cycle identification is also developed. The method is illustrated with a fatigue reliability analysis. (Zheng & Ellingwood 1998)

Model uncertainty and Bayesian updating in reliability-based inspection

In this paper, a Bayesian procedure is proposed to quantify the modelling uncertainty, including the uncertainty in mechanical and statistical model selection and the uncertainty in distribution parameters. The procedure is developed first using a simple example and then is applied to a fatigue reliability problem, with the combination of two competing crack growth models and considering the uncertainty in the statistical distribution parameters for each model. This Bayesian failure probability analysis can be incorporated with information from non-destructive inspections performed on the structure to derive more realistic reliability estimates. The procedure for updating the mechanical model, probabilistic model, distribution parameter

statistics and reliability is illustrated for the fatigue reliability problem. (Zhang & Mahadevan 2000)

Bayesian analysis for the results of fatigue test using full-scale models to obtain the accurate failure probabilities of the Shinkansen vehicle axle

Bayesian analysis was performed to estimate an appropriate value of the uncertain propagation rate of cracks that can be initiated at the wheelseat of a Shinkansen vehicle axle. In the analysis, fatigue life distribution obtained by numerical simulation that employed the crack propagation rate obtained from small specimens was used as the prior distribution. Then it was modified by the results of the fatigue test of full-scale models as additional information to obtain the posterior distribution. It was indicated that the variances of fatigue life distribution reduced through the analysis. By using the crack propagation rate obtained from the posterior fatigue life distribution, the failure probabilities of the Shinkansen vehicle axle in operation, that were calculated previously by using the crack propagation rate due to the experiment of small specimens were recalculated. The resulting probabilities of failure were almost the same as those that were not modified, but were slightly lower. Although the difference was not so significant, it was thought that more confident values of the failure probability were obtained. (Akama 2002)

Reliability of aircraft structures under non-periodic inspection: a Bayesian approach

This article proposes a Bayesian analysis methodology to determine appropriate non-periodic inspection intervals of fatigue-sensitive aircraft structures, so that their reliability remains above a prespecified minimum level throughout their service life. The methodology is based on assumptions about the probability distribution function of the time to crack initiation, the law of crack propagation, the probability of crack detection and the failure rate before and after crack initiation. The Bayesian approach proposed in this paper is allows one to utilise judiciously the results of earlier inspections for the purpose of determining the time of the next inspection and estimating the values of several parameters involved in the problem that can be treated as

uncertain. Numerical simulations verify the above-mentioned capabilities of the Bayesian method. (Deodatis et al. 1996)

2.5 Models of fatigue crack growth (FCG)

Modelling FCG with the Paris-Erdogan equation

One of the most used expressions of crack growth rate is the Paris-Erdogan equation

$$\frac{dx}{dN} = C(\Delta K)^n,$$

where x is the crack length, N is the time, $\Delta K = \Delta K(x)$ is the stress intensity amplitude, and C and n are constants. (See e.g. Kozin and Bogdanoff 1989, Provan 1987.)

The parameters of the Paris-Erdogan equation are sometimes estimated from existing data by using linear regression (Bigerelle & lost 1999). Taking the logarithm of both sides of the Paris-Erdogan equation, we get

$$\ln \frac{dx}{dN} = \ln C + n \ln \Delta K,$$

which is linear in terms of $\ln C$ and n . If we now denote $\mathbf{x} = (1 \ \ln \Delta K)^T$ as the vector of independent variables, $\mathbf{b} = (\ln C \ n)^T$ as the parameter vector and $y = \ln dx/dN$ as the dependent variable, we can apply the familiar model $y = \mathbf{x}^T \mathbf{b} + \varepsilon$, for which the parameters can be estimated with the least-squares method.

Applying the inverse transform, we end up with the model

$$\frac{dx}{dN} = e^e C(\Delta K)^n,$$

which fulfils the physically obvious demand of dx/dN being non-negative.

The problem with this model is that the standard assumptions of the linear regression model demand that $E(\varepsilon)=0$ and $\text{Var } \varepsilon$ is constant. This may not be the case, which makes the least-squares estimates of the parameters questionable. Furthermore, it is commonly known that ΔK depends on x , which in turn makes its role as an independent variable ambiguous.

Modelling FCG with Markov chains

Markov chain models have been developed for FCG by Bogdanoff and Kozin (1985). This kind of model is based on the following assumptions:

1. The component is subject to repetitive 'damage cycles' with constant severity.
2. The damage levels are discrete (or may be discretized): $1, 2, \dots, j, \dots, b$. Level b corresponds to a 'failure state', i.e. reaching that level marks the end of the component's lifetime.
3. The progress from state j is independent of the way in which state j has been reached (the Markov property)

In addition, one more assumption is made in the case of a so-called one-step model:

4. The damage level in one damage cycle either remains unchanged or increases by one, and thereby cannot skip any levels.

If, in this context, the random variable "time (number of damage cycles) needed to reach state b " - i.e. the component's remaining lifetime - is denoted by W_b , then its cumulative distribution can easily be calculated starting from any distribution for the initial damage state, because

$$P(W_b < t) = P(\text{The damage level at time } t \text{ has reached } b),$$

which in turn is $\pi_b(t) = \pi_0 \Pi^t$, where π_0 is the initial distribution and Π is the transition probability matrix of the model (for details see Bogdanoff and Kozin 1985).

Elementary model

FCG has also been modelled in the following, very simple way (Sheikh & Younas 1995): the damage (crack length) D at duty cycle N is assumed to be

$$D(N) = D_0 N^\beta,$$

where $D_0 = D(1)$ is the initial crack length (constant), and $\beta \sim N(u, s^2)$. Taking the logarithm of this, we get

$$\ln D(N) = \ln D_0 + \beta \ln N,$$

which implies that $\ln D(N)$ is normally distributed with mean

$$E(\ln D(N)) = \ln D_0 + \ln N E(\beta)$$

and variance

$$\text{Var}(\ln D(N)) = (\ln N)^2 \text{Var } \beta.$$

Now the distribution for the cycle **N** at which the damage reaches a critical level D_c can be derived as follows:

$$\begin{aligned} P(\mathbf{N} > N) &= P[D(N) < D_c] = P[\ln D(N) < \ln D_c] \\ &= \Phi[(\ln D_c - E(\ln D(N))) / (\text{Var}(\ln D(N)))^{1/2}]. \end{aligned}$$

The parameters of this model, namely, $E(\beta)$ and $\text{Var } \beta$, can be estimated from a measured data set $D_i = D_0 N^{\beta_i}$, or from a set of observed component lifetimes **N_i**.

The most striking weakness of this model is that it allows $dD(N) / dN = \beta D(1) N^{\beta-1}$ to be negative, conflicting with reality.

3 The models used in the computer program

3.1 Counting and updating the probability distributions of cumulative damage

Let X_t denote the degree of damage, like the crack length, at time t . X_t takes values from a discrete set $\{0, 1, \dots, b\}$, where $X=b$ means failure of the component. Let $P(X_0 = i) = \pi_i(0)$ be the probability that $X=i$ at $t=0$, and $\pi(0) = (\pi_0(0), \dots, \pi_b(0))$ is the initial distribution of X . We may assume that $\pi(0) = (\pi_0(0), \dots, \pi_b(0)) = (1, 0, \dots, 0)$, for example.

The transition probabilities of X from state i to state j form the transition probability matrix Π :

$$\Pi = (\Pi_{ij}) = P(X_{t+1} = j \mid X_t = i).$$

The transition probability matrix is assumed to be stationary, that is, same for all t . Assuming that the damage is non-decreasing, the matrix Π is an upper triangular matrix. Moreover, it is a band matrix, since the damage rarely grows several units in one time step. For example in fatigue crack growth the band is quite narrow in the left upper corner (small crack) and slowly widens to the right lower corner (large crack).

We have measurements Y of the damage X . Y takes values from the same set as X : $y_t \in \{0, 1, \dots, b\}$. The measurement is assumed to be inaccurate, and dependent on the value of X according to the conditional probabilities $P(Y = y \mid X = x)$, $x, y \in \{0, 1, \dots, b\}$. X is not measured at $t = 0$.

The marginal probability distribution $\pi(1)$ of X for $t = 1$, where $\pi_i(1) = P(X_1 = i)$, is obtained directly from

$$p(1) = p(0)\Pi.$$

At $t = 1$ we make a measurement Y_1 of X_1 , and want to update $\pi(1)$ with the result of the measurement, that is, we want the conditional probability distribution $\pi(1 \mid Y_1)$. By the Bayes' rule

$$p_i(1 | Y_1) = P(X_1 = i | Y_1 = y_1) = \frac{P(Y_1 = y_1 | X_1 = i)P(X_1 = i)}{P(Y_1 = y_1)},$$

where

$$P(Y_1 = y_1) = \sum_{i=0}^b P(X_1 = i)P(Y_1 = y_1 | X_1 = i).$$

The probability distributions for following time steps are obtained by recursion from the preceding distributions. Suppose we have the conditional distribution $\pi(T | \mathbf{Y}_{t \leq T})$, where $\mathbf{Y}_{t \leq T}$ denotes all the measurements $Y_t = y_t$ with $t \leq T$. If we do not make a measurement at $t=T+1$, the conditional distribution of X for $t=T+1$ is

$$p(T+1 | \mathbf{Y}_{t \leq T}) = p(T | \mathbf{Y}_{t \leq T})\Pi.$$

If we do make the measurement at $t=T+1$ and thus have $Y_{T+1} = y_{T+1}$, we get

$$\begin{aligned} p_i(T+1 | \mathbf{Y}_{t \leq T}, Y_{T+1}) &= P(X_{T+1} = i | \mathbf{Y}_{t \leq T}, Y_{T+1} = y_{T+1}) \\ &= \frac{P(Y_{T+1} = y_{T+1} | X_{T+1} = i)P(X_{T+1} = i | \mathbf{Y}_{t \leq T})}{P(Y_{T+1} = y_{T+1})} \end{aligned}$$

where $P(X_{T+1} = i | \mathbf{Y}_{t \leq T})$ is the i th component of $\pi(T+1 | \mathbf{Y}_{t \leq T})$ and

$$P(Y_{T+1} = y_{T+1}) = \sum_{i=0}^b P(X_{T+1} = i)P(Y_{T+1} = y_{T+1} | X_{T+1} = i).$$

The remaining lifetime is predicted by repeatedly multiplying the current conditional distribution with the transition probability matrix Π . From these distributions we can see the probability that the cumulative damage has exceeded the given limit (e.g. thickness of the pipe wall) at different time steps.

3.2 Parameters used in the model

We denote by X_t the fatigue crack length at time t . X_t takes values from a discrete set $\{0, 1, \dots, b\}$, where $b = 100$, and $X=100$ means the crack has penetrated through the component. We chose $b = 100$ because it can be interpreted as the percentage of the component thickness. A real crack length is assigned to a discrete crack length as indicated by its integer part.

The crack length at the beginning is assumed to be zero or undetectable or very small (less than 1% of the thickness of the component). The initial distribution of X is thus $\pi(0) = (1, 0, \dots, 0)$.

The assumptions and forming of the transition matrix Π and the probability model of flaw sizing (the probabilities $P(Y = y \mid X = x)$) are presented below.

3.3 The transition probability matrix

Our transition probability matrix is based on simulated crack growth data. The data is generated using the Paris-Erdogan equation.

The Paris-Erdogan equation for fatigue crack growth and a randomisation of the equation were presented in chapter 2.5. For a single time step ($dN=1$) the randomised equation becomes

$$dx = e^\varepsilon C(\Delta K)^n.$$

Provan (1987) gives that ΔK is roughly proportional to square root of x :

$\Delta K = \beta \sqrt{x}$, where β is constant. The amount of crack growth in one time step is thus

$$dx = e^\varepsilon C(\beta \sqrt{x})^n,$$

where C , n and β are constants and ε is normally distributed random variable with mean $E(\varepsilon) = 0$ and variance $\text{Var } \varepsilon = \sigma_\varepsilon^2$. The expected amount of crack growth is thus dependent on the current crack length.

To generate data we used parameter values $C = 0.005$, $n = 1.3$, $\beta = 1$, and $\sigma_\varepsilon = 1.7$. The simulated data consisted of one hundred realisations.

For each state i , the numbers of single step transitions from state i to state j ($j=0,1,\dots, 100$) in the data are calculated. These numbers are divided by the total number of single step transitions from the state i , giving single step transition probabilities from state i to states j . The transition probabilities are assumed stationary, dependent only on the current crack length (i.e. the state i).

In real life applications the transition probability matrix can be formed using actual crack growth data as we used simulated data. The matrix can also be updated as new data is collected.

3.4 Probability model of flaw sizing

Simola and Pulkkinen (1998) have discussed probability models of flaw sizing. We use their logit model of the relationship between the actual and measured flaw sizes. The logit model assumes the relation

$$\ln \frac{y}{d-y} = \mathbf{b}_0 + \mathbf{b}_1 \ln \frac{x}{d-x} + \mathbf{x}$$

between the measured flaw depth y and the actual flaw depth x . d is the component thickness ($x, y \leq d$), $\mathbf{b}_0 \in (-\infty, \infty)$ and $\mathbf{b}_1 > 0$ are regression parameters, and ξ is a normally distributed random error, $\xi \sim N(0, \sigma_\xi^2)$. If we denote $z = \ln \frac{y}{d-y}$ and $\mathbf{m} = \mathbf{b}_0 + \mathbf{b}_1 \ln \frac{x}{d-x}$, then $z \sim N(\mathbf{m}, \sigma_\xi^2)$. Z has the cumulative distribution function $F_Z(a) = P(z < a) = \Phi((a - \mathbf{m})/\sigma_\xi)$, where Φ is the cumulative distribution function of the normal distribution $N(0, 1)$.

For probabilities $P(Y = y | X = x)$ we need the cumulative distribution function of measured flaw depth Y . From $F_Z(a)$ we get

$$F_Z(a) = P\left(\ln \frac{y}{d-y} < a\right) = P\left(\frac{y}{d-y} < e^a\right) = P\left(y < \frac{de^a}{1+e^a}\right) = F_Y\left(\frac{de^a}{1+e^a}\right).$$

Denoting $\hat{a} = \frac{de^a}{1+e^a}$ we get the cumulative distribution function of Y :

$$F_Y(\hat{a}) = F_Z\left(\ln \frac{\hat{a}}{d-\hat{a}}\right) = \Phi\left(\frac{1}{\sigma_\xi} \left(\ln \frac{\hat{a}}{d-\hat{a}} - \mathbf{m}\right)\right).$$

In the following y and x denote elements from the discrete set $\{0, 1, \dots, b\}$ such that y represents the real interval $[i, i+1)$ and x the interval $[j, j+1)$, where i and j are integers. We have

$$\begin{aligned} P(Y = y | X = x) &= P(Y < i+1 | X = x) - P(Y < i | X = x) \\ &= F_Y(i+1 | X = x) - F_Y(i | X = x). \end{aligned}$$

Here the condition $X=x$ is approximated using the mean of the F_Y values at the end points of the interval $[j, j+1)$:

$$F_Y(i | X = x) \approx \frac{1}{2} [F_Y(i | X = j) + F_Y(i | X = j + 1)].$$

So we have

$$\begin{aligned} P(Y = y | X = x) &\approx \frac{1}{2} [F_Y(i+1 | X = j) + F_Y(i+1 | X = j + 1)] \\ &\quad - \frac{1}{2} [F_Y(i | X = j) + F_Y(i | X = j + 1)] \\ &= \frac{1}{2} \Phi \left(\frac{1}{s_x} \left(\ln \frac{i+1}{d-(i+1)} - b_0 - b_1 \ln \frac{j}{d-j} \right) \right) \\ &\quad + \frac{1}{2} \Phi \left(\frac{1}{s_x} \left(\ln \frac{i+1}{d-(i+1)} - b_0 - b_1 \ln \frac{j+1}{d-(j+1)} \right) \right) \\ &\quad - \frac{1}{2} \Phi \left(\frac{1}{s_x} \left(\ln \frac{i}{d-i} - b_0 - b_1 \ln \frac{j}{d-j} \right) \right) \\ &\quad - \frac{1}{2} \Phi \left(\frac{1}{s_x} \left(\ln \frac{i}{d-i} - b_0 - b_1 \ln \frac{j+1}{d-(j+1)} \right) \right) \end{aligned}$$

The parameters b_0 , b_1 and σ_ξ^2 can be modified to fit the measurements. We have used as default values $b_0=0.06$, $b_1=1.25$ and $\sigma_\xi^2=0.22$. These values are the parameter estimates of an actual set of measurements (see Simola and Pulkkinen 1998).

4 *About the program*

Here is a short description of what the program does. The complete program manual is in Appendix A.

The program asks you to give the following information.

- The name of a previously created transition probability matrix (an excel file) or the parameters for creating one as explained in chapter 3.3.
- The name of a previously created measurement accuracy matrix (an excel file) or the parameters for creating one as explained in chapter 3.4. A measurement accuracy matrix contains the probabilities $P(Y = y \mid X = x)$ with rows corresponding to the x values and columns to the y values.
- The measurement data (an excel file).
- The probability distribution of the initial state.

The program plots three kinds of figures. These three figures are plotted using no measurement data, using the first measurement data, using the first and second measurement data, and so on. If you have given m measurements, the program will produce $3 \cdot (m+1)$ figures. Examples of these figures are in the manual in Appendix A (Figure 19, Figure 20 and Figure 21).

The first kind of figure plots the individual measurements, and a few predictive realisations starting from the time of the last measurement T_y . The crack length where a realisation starts is chosen from the probability distribution $\pi(T_y)$ by acceptance-rejection sampling (see chapter 2.3). The amount of crack growth in each time step is chosen randomly from the corresponding row of the transition matrix, weighting with the probabilities of the different transitions. The program asks the number of realisations to be plotted and the time span to be plotted.

The second kind of figure plots the predictive cumulative distribution function and the predictive probability density function at the final state b . The predictive cdf was discussed in chapter 'Modelling FCG with Markov chains'

(page 8). The predictive pdf is obtained from the predictive cdf by numerical differentiation.

The third kind of figure plots a histogram of the state probability vector $\pi(T_1)$. The program asks for the time T_1 .

5 Results from sample simulations

Two sets of measurement data were used in simulations, each consisting of five measurements taken at instants 100, 200, 300, 400 and 500. The measurements are shown in Table 1.

Table 1 Measurement data used in simulations

Time	Measurement set 1	Measurement set 2
100	0,7	0,2
200	3,4	4
300	12,5	20,9
400	22,1	43,2
500	48,2	40,1

For the simulations a transition probability matrix was generated using 100 realisations (see chapter 3.3) and this matrix was then used throughout all simulations. In chapter 5.1 are presented the results from simulations using the default variance $\sigma_{\xi}^2=0.22$ in the flaw sizing model. For comparison the results of the same form for cases of reduced and increased variance σ_{ξ}^2 are shown in chapters 5.2 and 5.3, respectively.

5.1 Default variance in the flaw sizing model

5.1.1 Measurement set 1

In Figure 1 different subsets (the individual measurements are marked by star-symbols) of the measurement set have been used in determining the probability distribution of the damage state at the time of the last measurement in the subset. From this time and distribution 50 realisations of crack growth have been simulated in each case.

We can see that as time grows, uncertainty of the crack length grows. Comparing the situation at time 500 using no measurements and using four measurements shows clearly how updating with measurements narrows the

distribution. It can be noted in particular that using only the last measurement provides about as vague a picture of the situation at time 500 as using only the first measurement. It appears therefore that it is the combined effect of all measurements that makes the information accurate.

In Figure 2 the probability of the final state has been plotted against time, starting in each case from the time of the last measurement used. This is in effect the cumulative distribution function of the lifetime of the component (see chapter 'Modelling FCG with Markov chains' on page 8 for explanation). In addition, the difference between each time step in these cdf's has been plotted for a numerical approximation of the corresponding probability density functions. This shows how using more measurements gives a more centred distribution for the component lifetime. The reason for the last case giving an early high peak can be seen by comparing the last two plots of Figure 1: using only the last measurement gives a more pessimistic picture of the situation.

Finally, Figure 3 shows the probability distribution of the damage state calculated at time 600 in each case (the state probability vectors $\pi(600)$). Again, this illustrates the more accurate information obtained by using more measurements.

5.1.2 Measurement set 2

Figure 4, Figure 5 and Figure 6 show the same results for measurement set 2. The main phenomenon here is the fourth measurement resulting in a very pessimistic picture and the fifth measurement giving a corrective direction. The notable difference from measurement set 1 is in which measurement subsets give the most centred distributions. This seems to depend on how well the measurement data is in accordance with the "mean behaviour" determined by the transition probability matrix (depicted by the cases where no measurements have been used), i.e. earlier recorded data.

Figure 1 and Figure 4 show how updating can also move the distribution.

5.2 Reduced variance in the flaw sizing model

In this section a variance $\sigma_{\xi}^2=0.05$ instead of the default value $\sigma_{\xi}^2=0.22$ has been used in the flaw sizing model. The simulation results are shown in Figure 7, Figure 8, Figure 9, Figure 10, Figure 11 and in Figure 12 below. As can be expected, the distributions are now more centred since measurements are considered more reliable. However, the simulated realisations still show that using all measurements narrows the distribution. Furthermore, the most recent measurement has now a greater effect.

5.3 Increased variance in the flaw sizing model

Here, a variance of $\sigma_{\xi}^2=1.00$ instead of the default value $\sigma_{\xi}^2=0.22$ has been used. Consequently, all results are now characterised by wide distributions due to the uncertainty behind the measurements. Results are displayed in Figure 13, Figure 14, Figure 15, Figure 16, Figure 17 and in Figure 18.

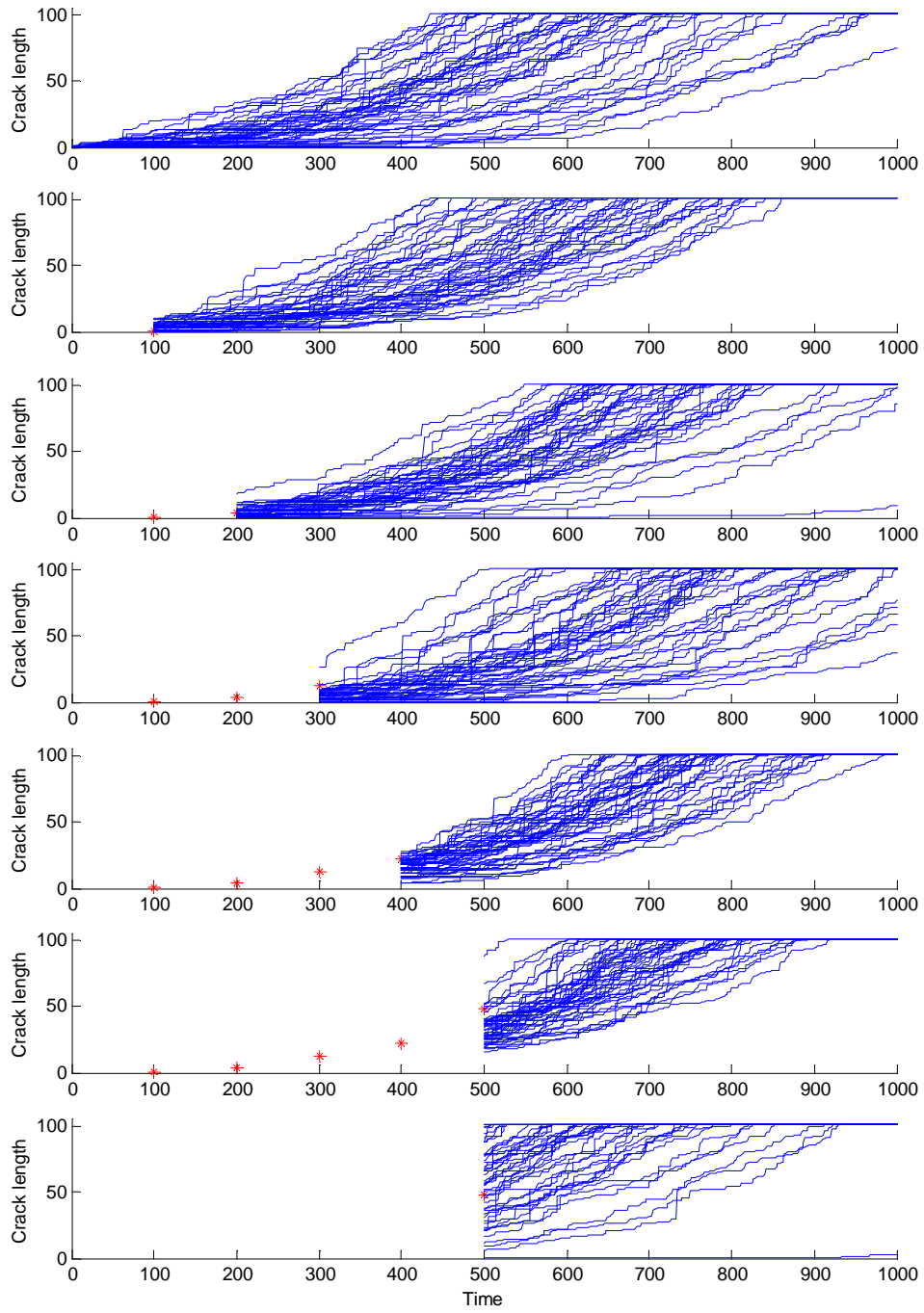


Figure 1: Crack growth realisations with variance $s_x^2=0.22$, measurement set 1.

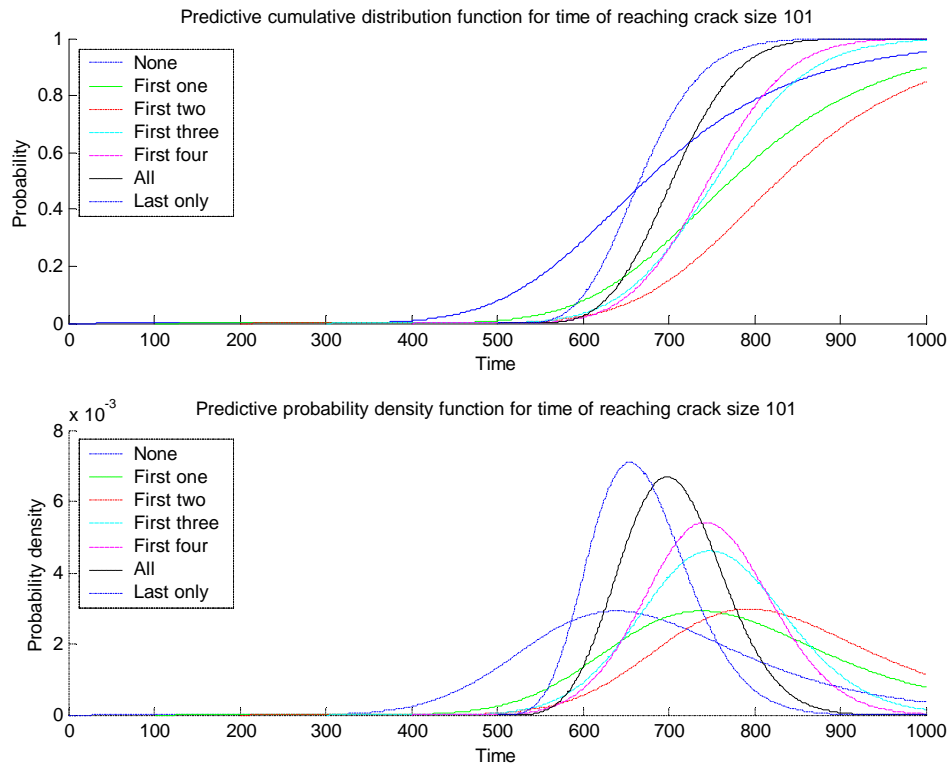


Figure 2: The predictive cdf's and pdf's, variance $S_x^2=0.22$, measurement set 1. The legend indicates the measurements used.

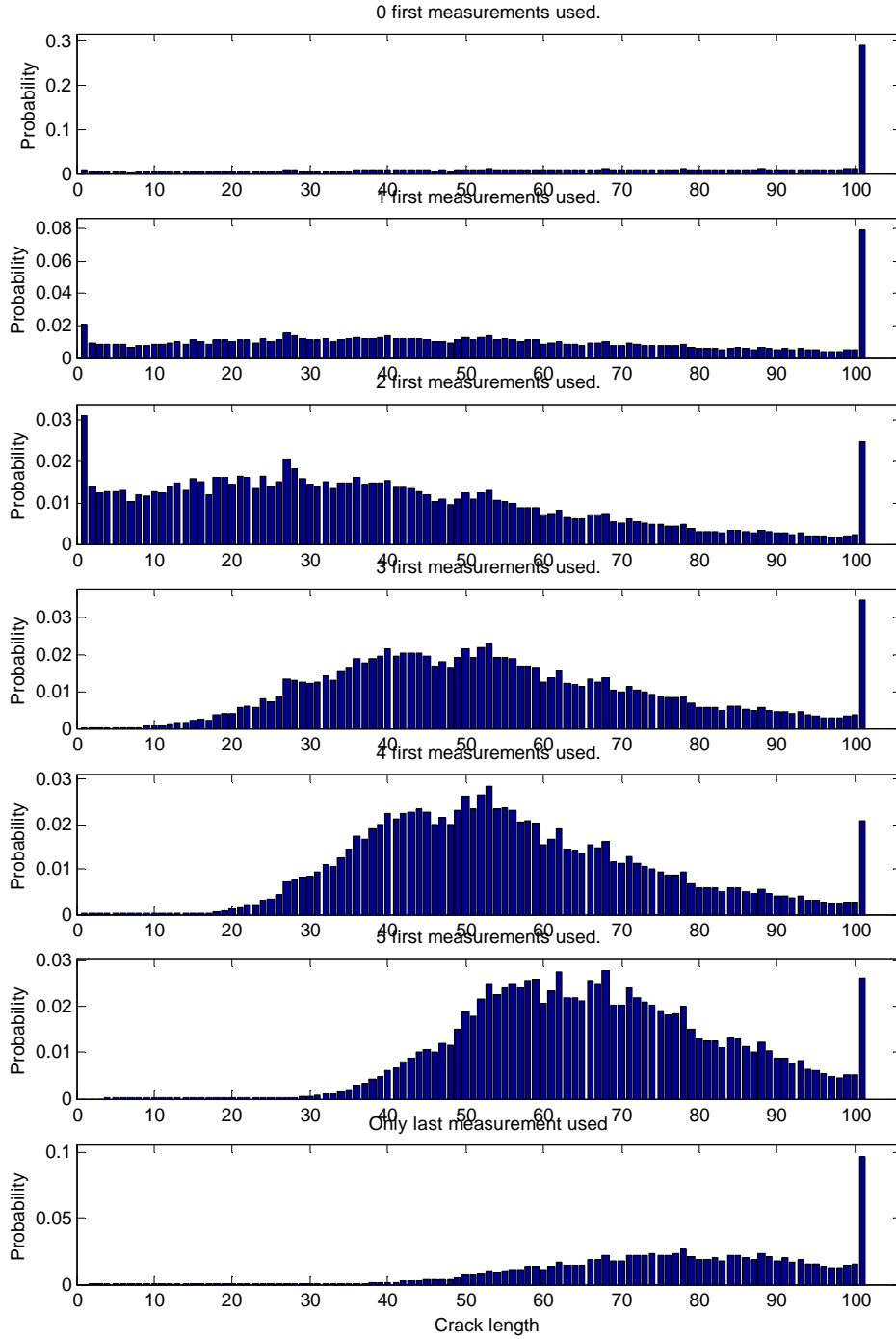


Figure 3: Histograms of the state probability vector $p(600)$. Variance $s_x^2=0.22$, measurement set 1.

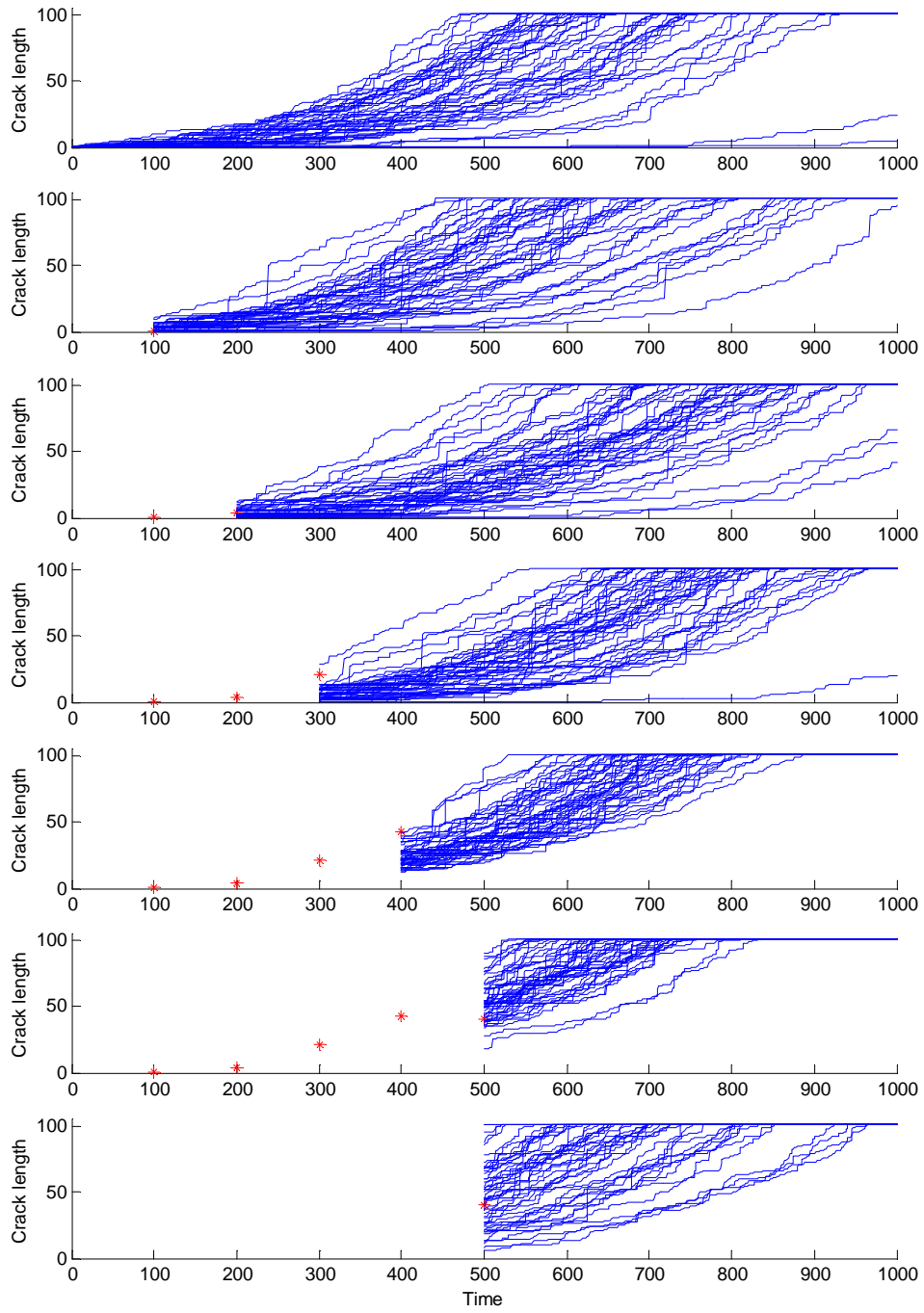


Figure 4: Crack growth realisations with variance $s_x^2=0.22$, measurement set 2.

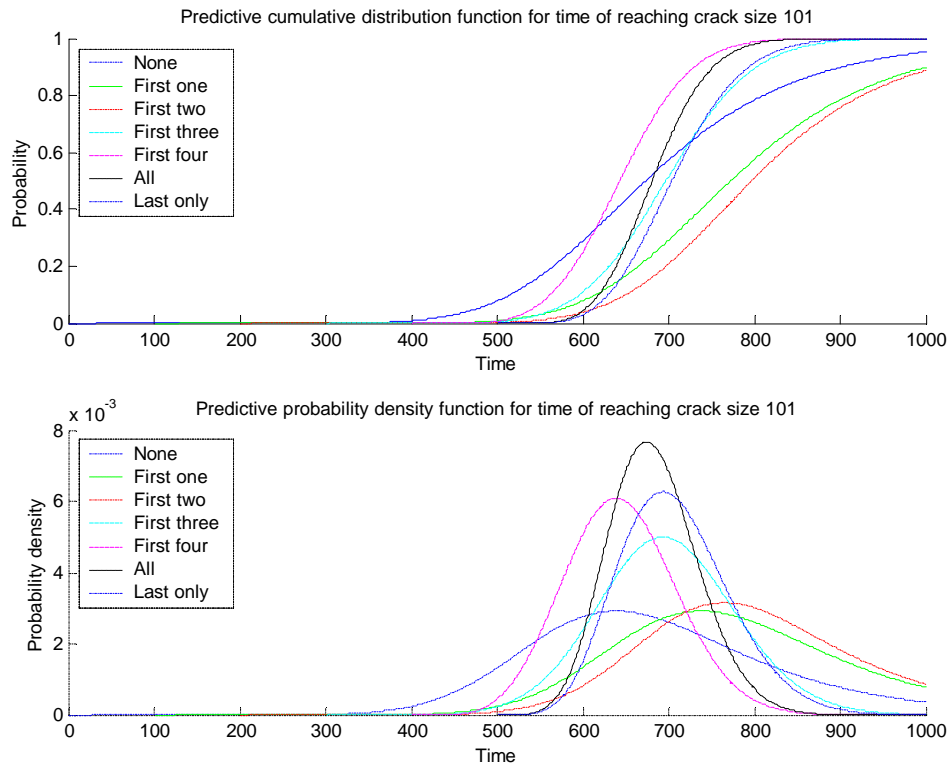


Figure 5: The predictive cdf's and pdf's, variance $S_x^2=0.22$, measurement set 2. The legend indicates the measurements used.

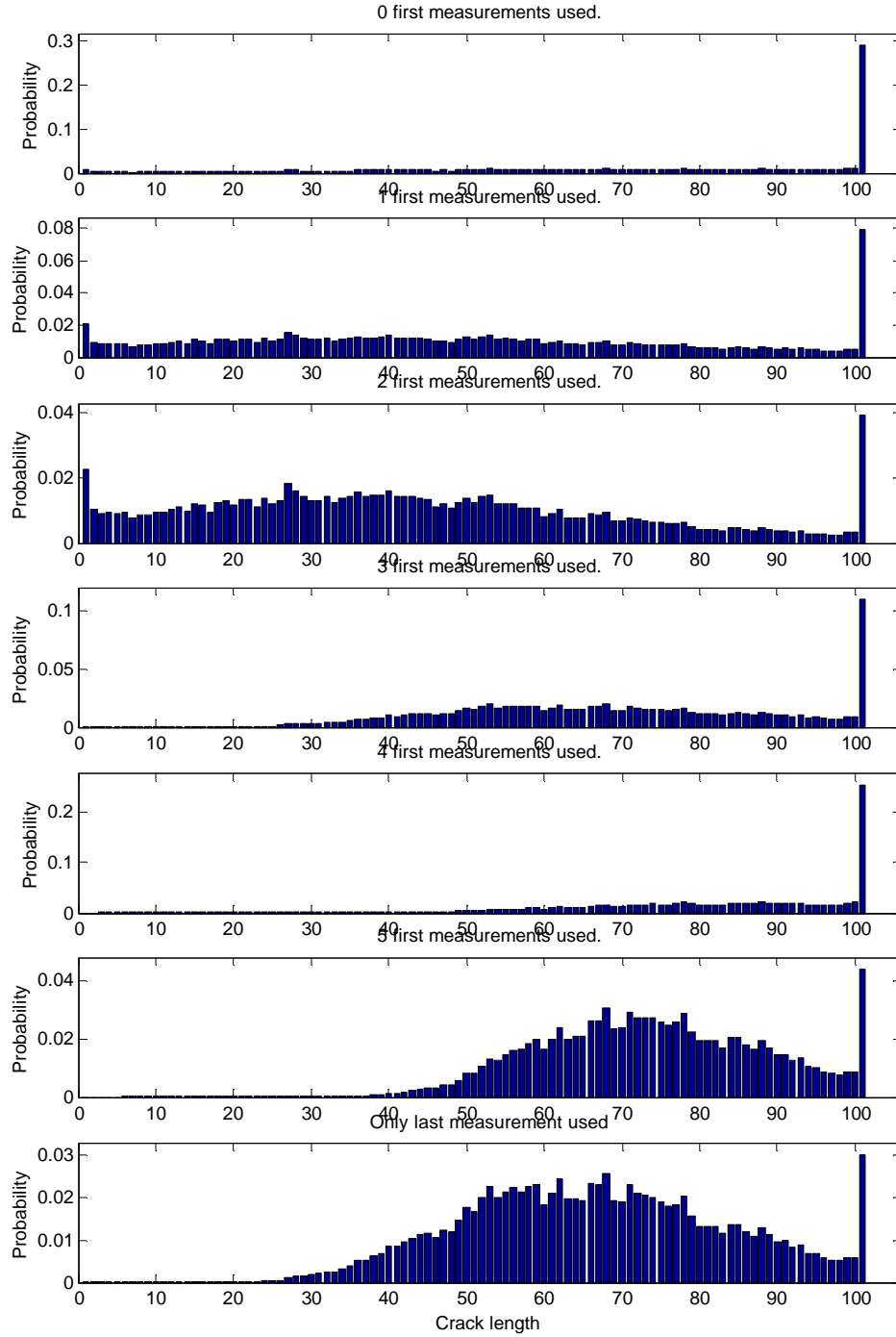


Figure 6: Histograms of the state probability vector $\mathbf{p}(600)$. Variance $s_x^2=0.22$, measurement set 2.

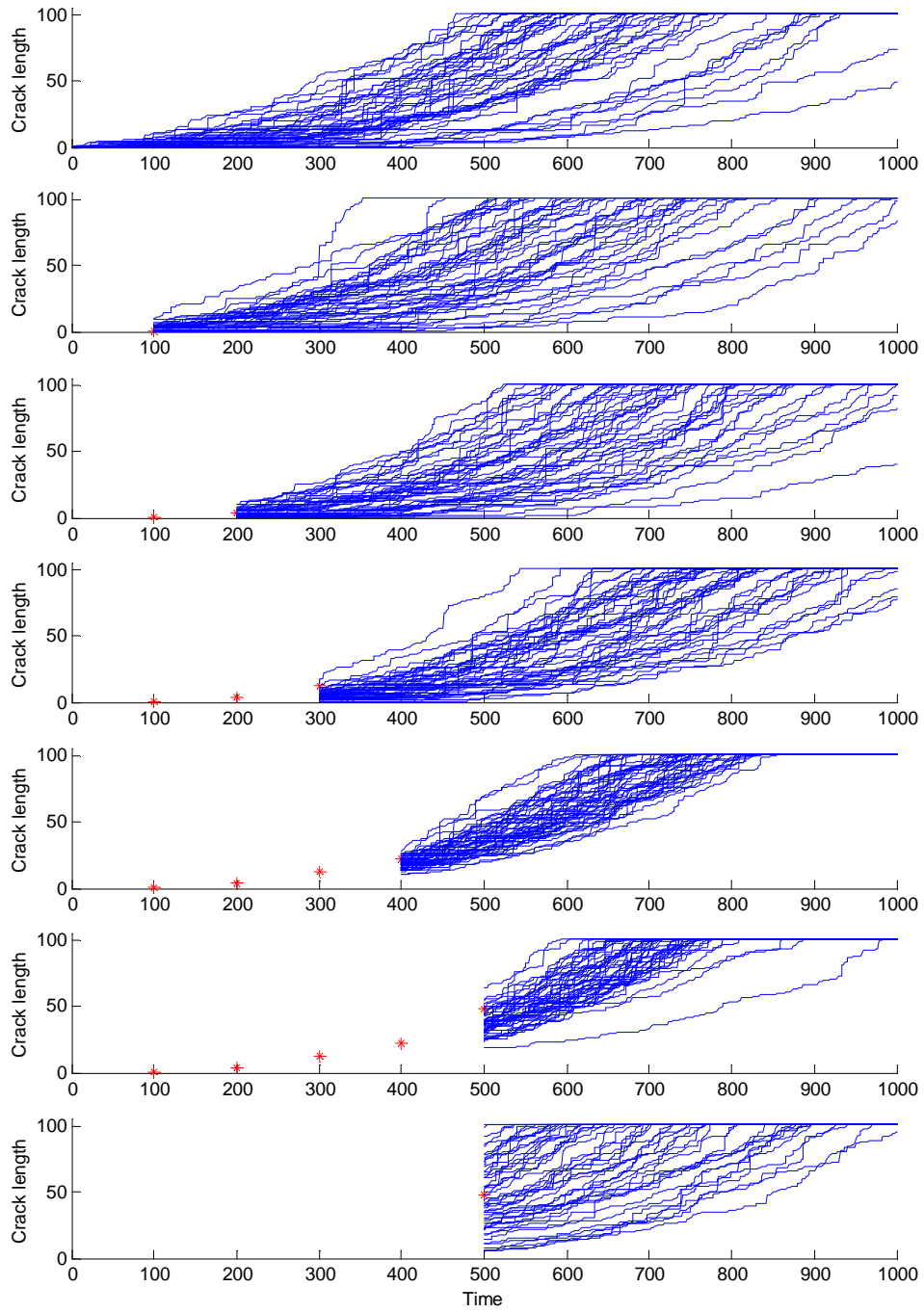


Figure 7: Crack growth realisations with variance $s_x^2=0.05$, measurement set 1.

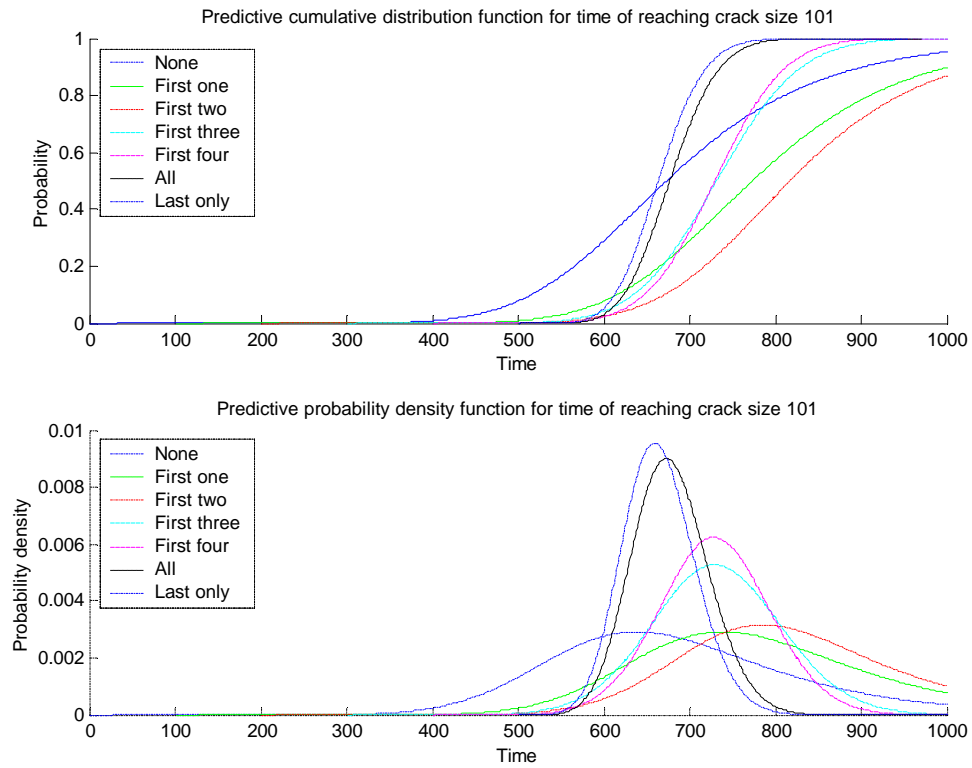


Figure 8: The predictive cdf's and pdf's, variance $S_x^2=0.05$, measurement set 1. The legend indicates the measurements used.

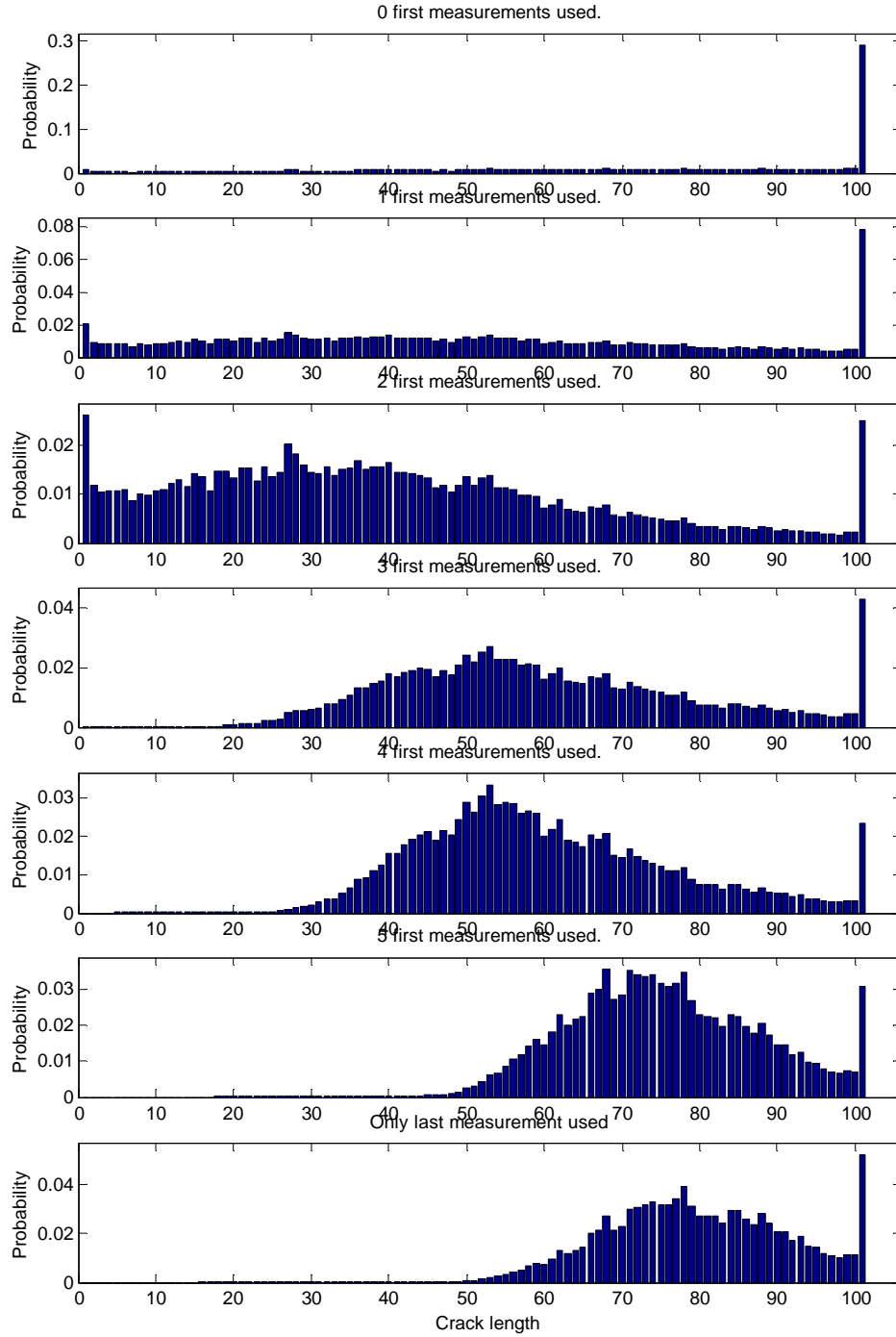


Figure 9: Histograms of the state probability vector $p(600)$. Variance $s_x^2=0.05$, measurement set 1.

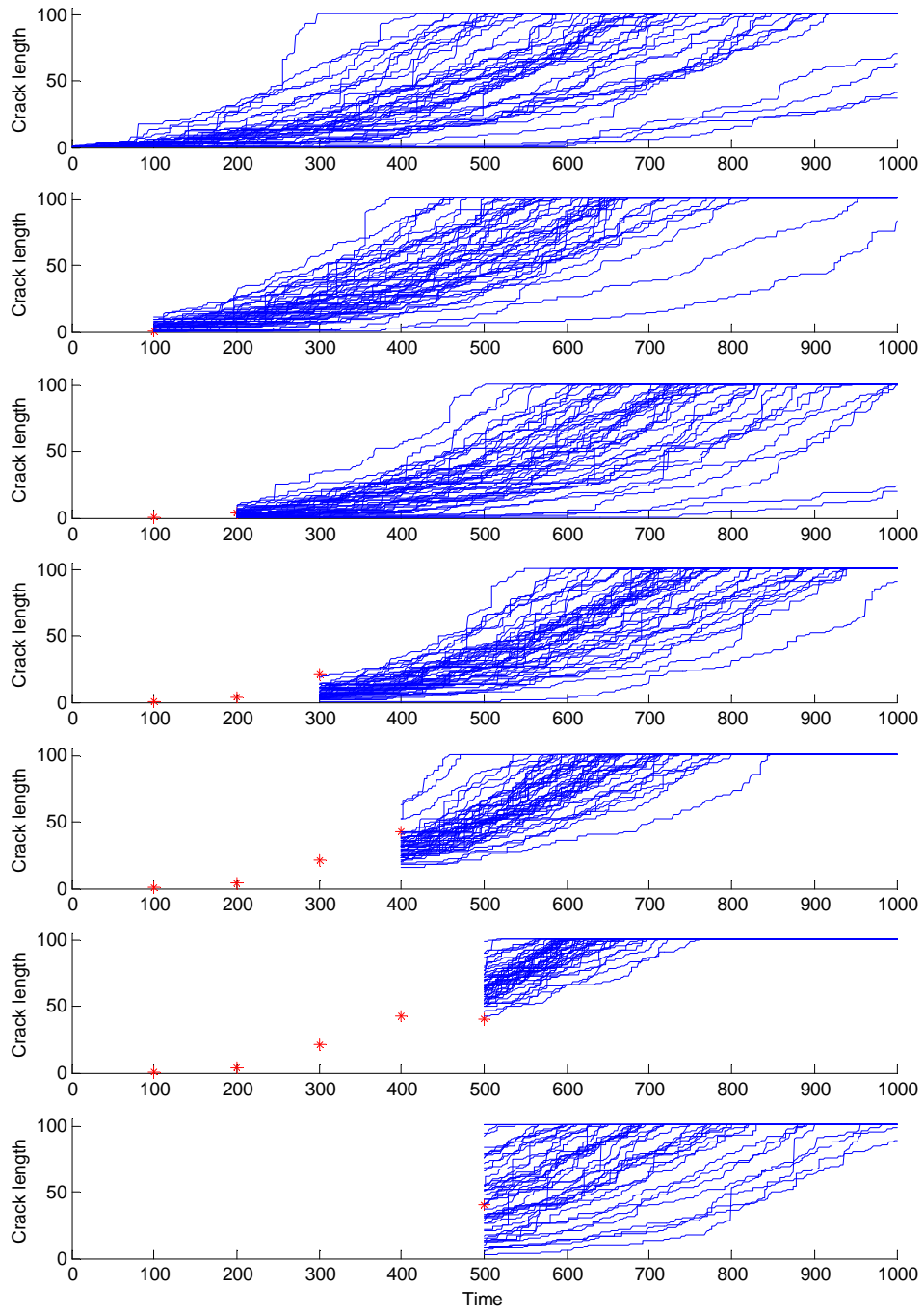


Figure 10: Crack growth realisations with variance $s_x^2=0.05$, measurement set 2.

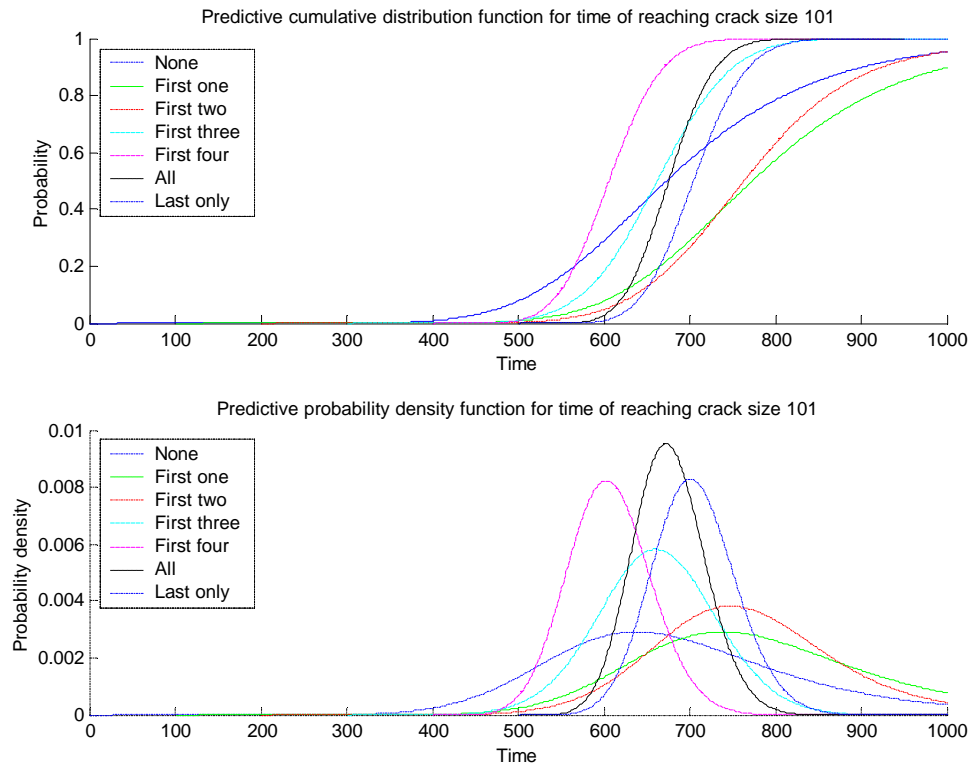


Figure 11: The predictive cdf's and pdf's, variance $S_x^2=0.05$, measurement set 2. The legend indicates the measurements used.

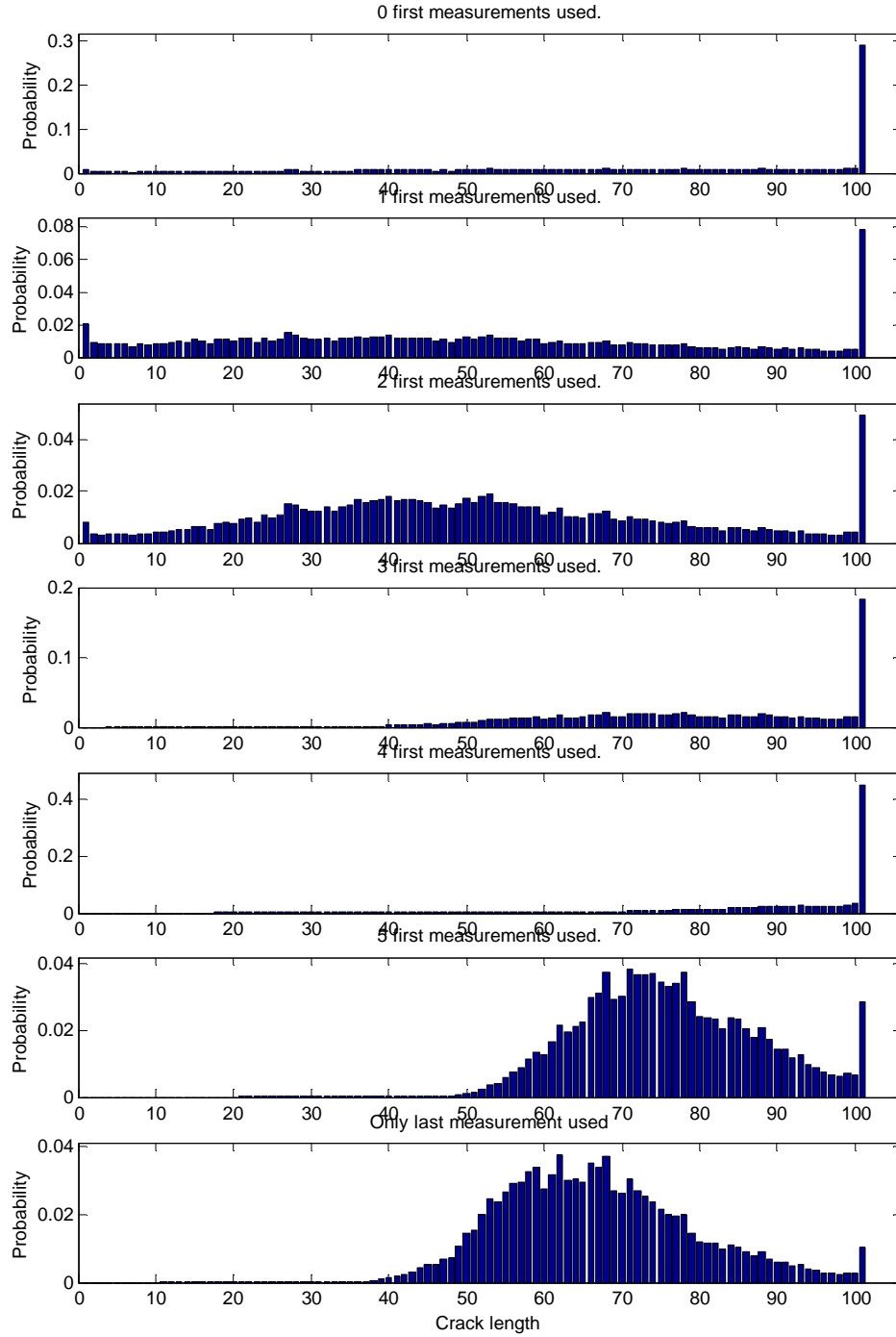


Figure 12: Histograms of the state probability vector $p(600)$. Variance $s_x^2=0.05$, measurement set 2.

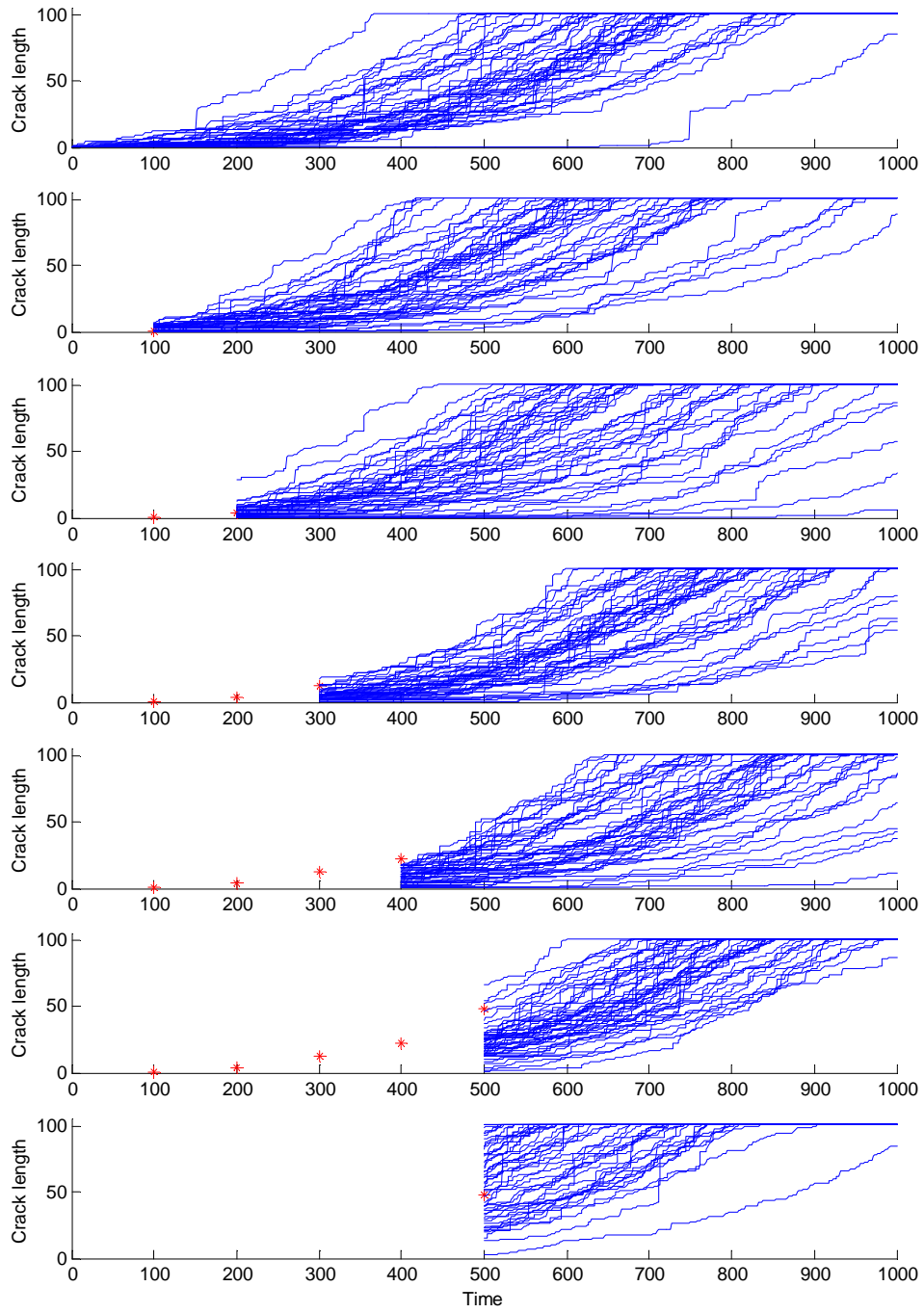


Figure 13: Crack growth realisations with variance $s_x^2=1.00$, measurement set 1.

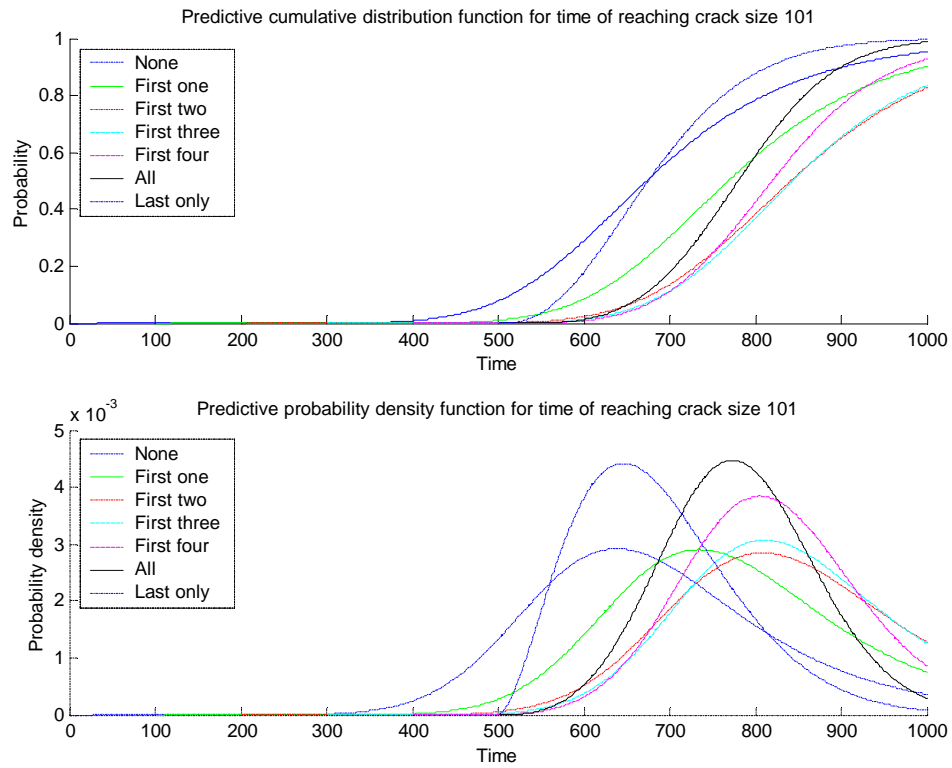


Figure 14: The predictive cdf's and pdf's, variance $S_x^2=1.00$, measurement set 1. The legend indicates the measurements used.

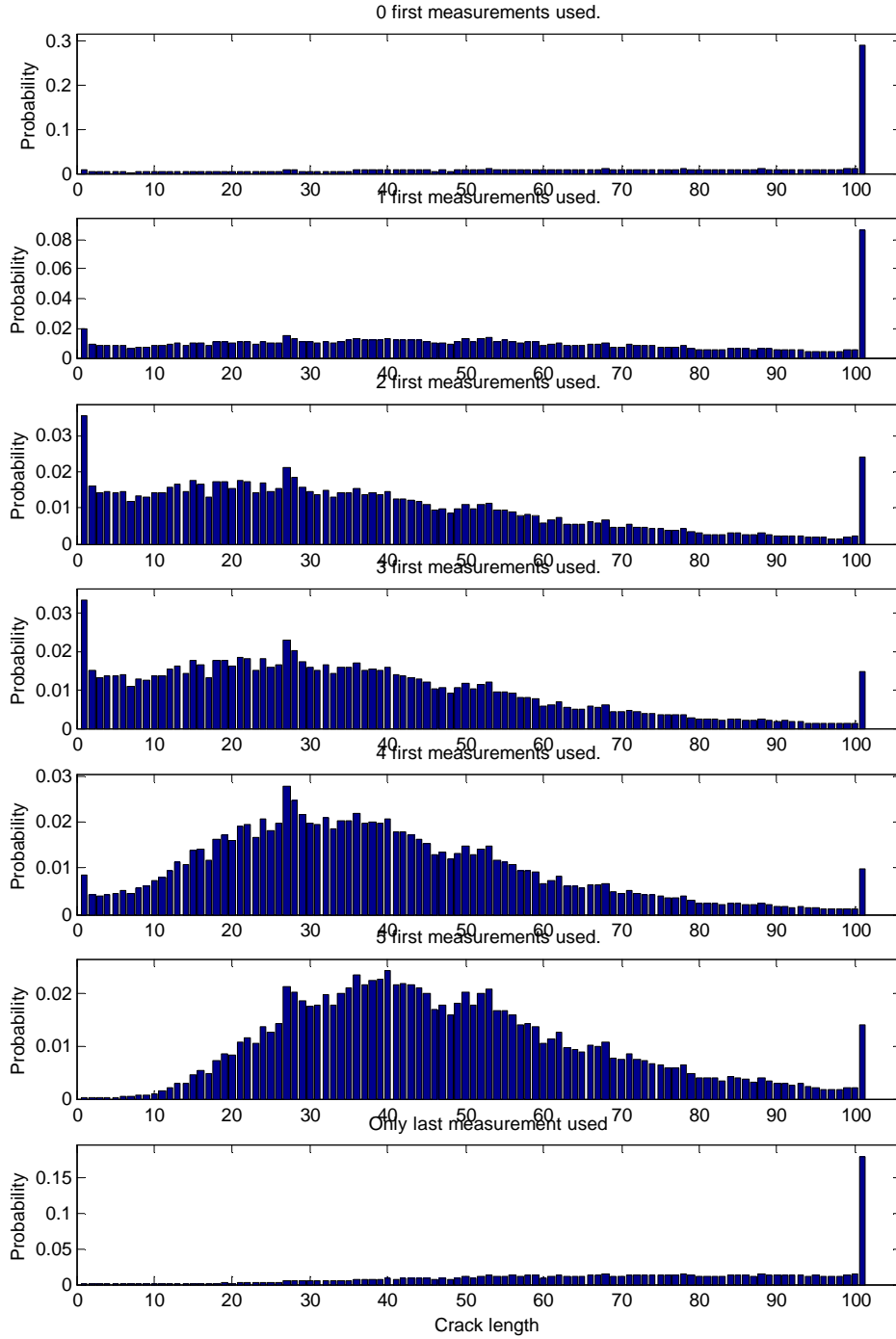


Figure 15: Histograms of the state probability vector $\mathbf{p}(600)$. Variance $s_x^2=1.00$, measurement set 1.

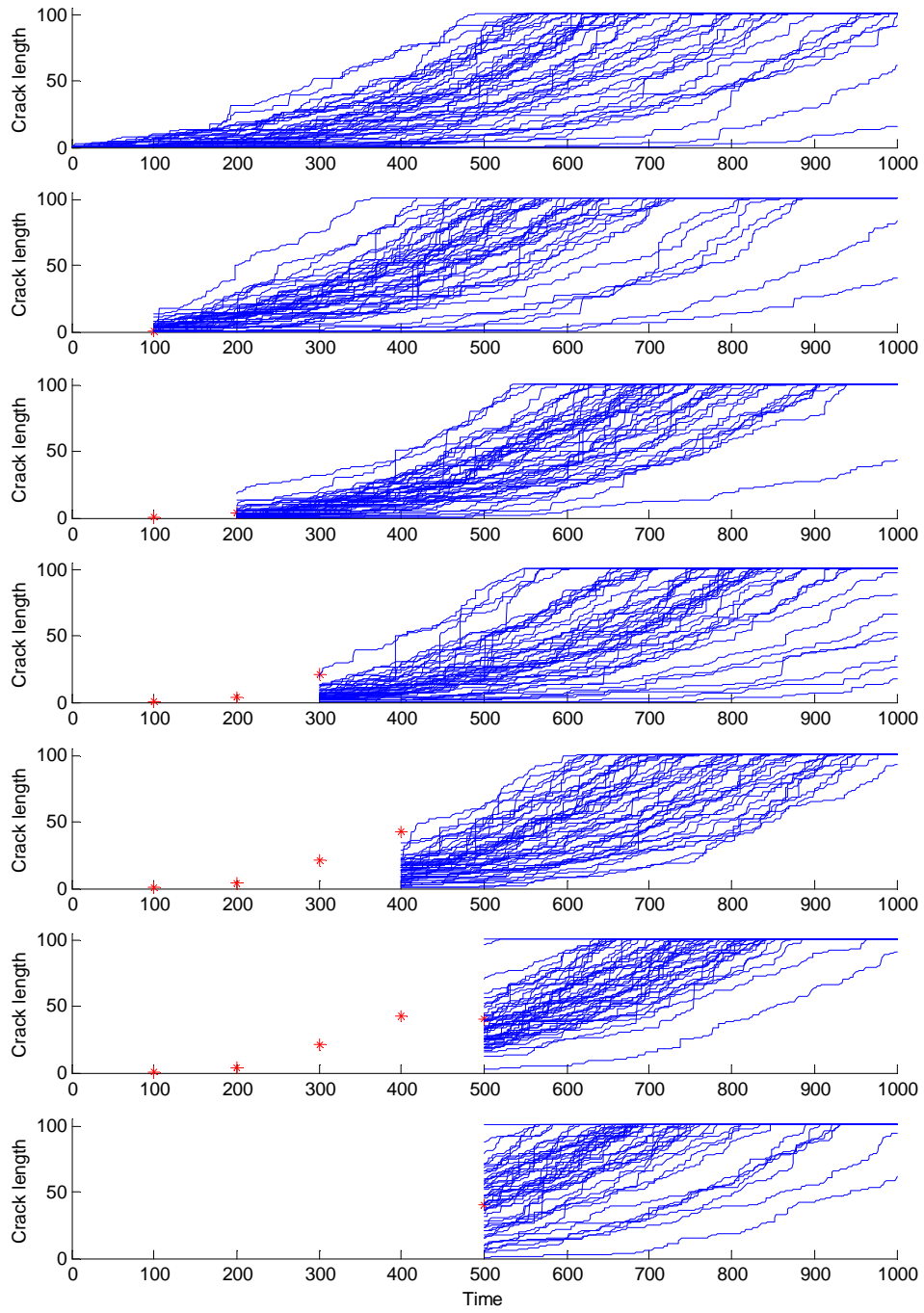


Figure 16: Crack growth realisations with variance $s_x^2=1.00$, measurement set 2.

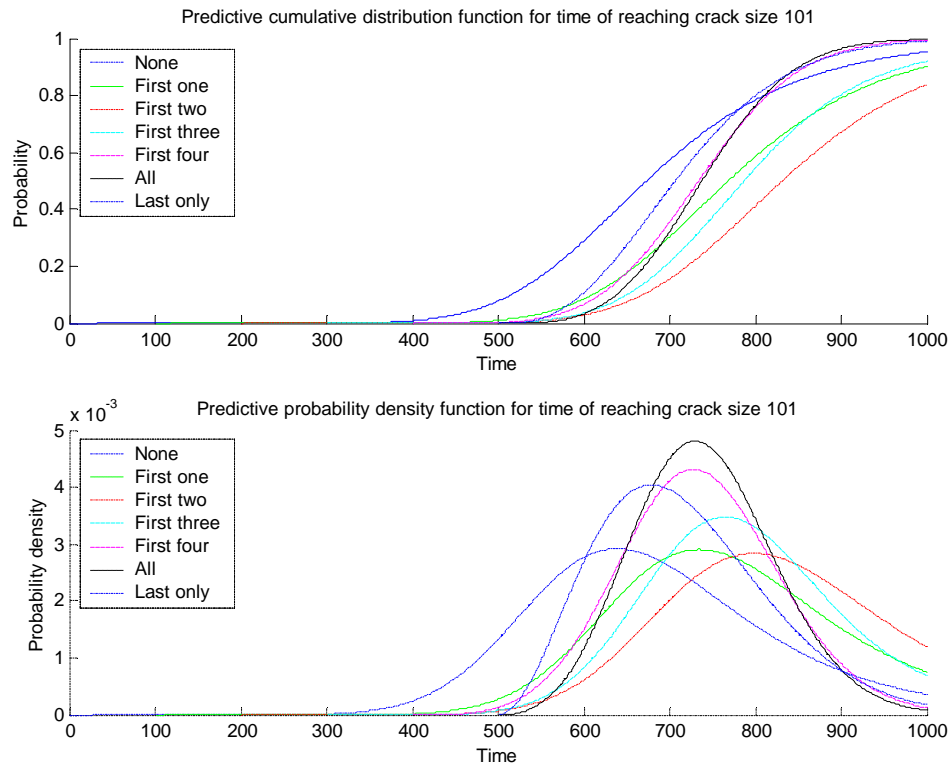


Figure 17: The predictive cdf's and pdf's, variance $S_x^2=1.00$, measurement set 2. The legend indicates the measurements used.

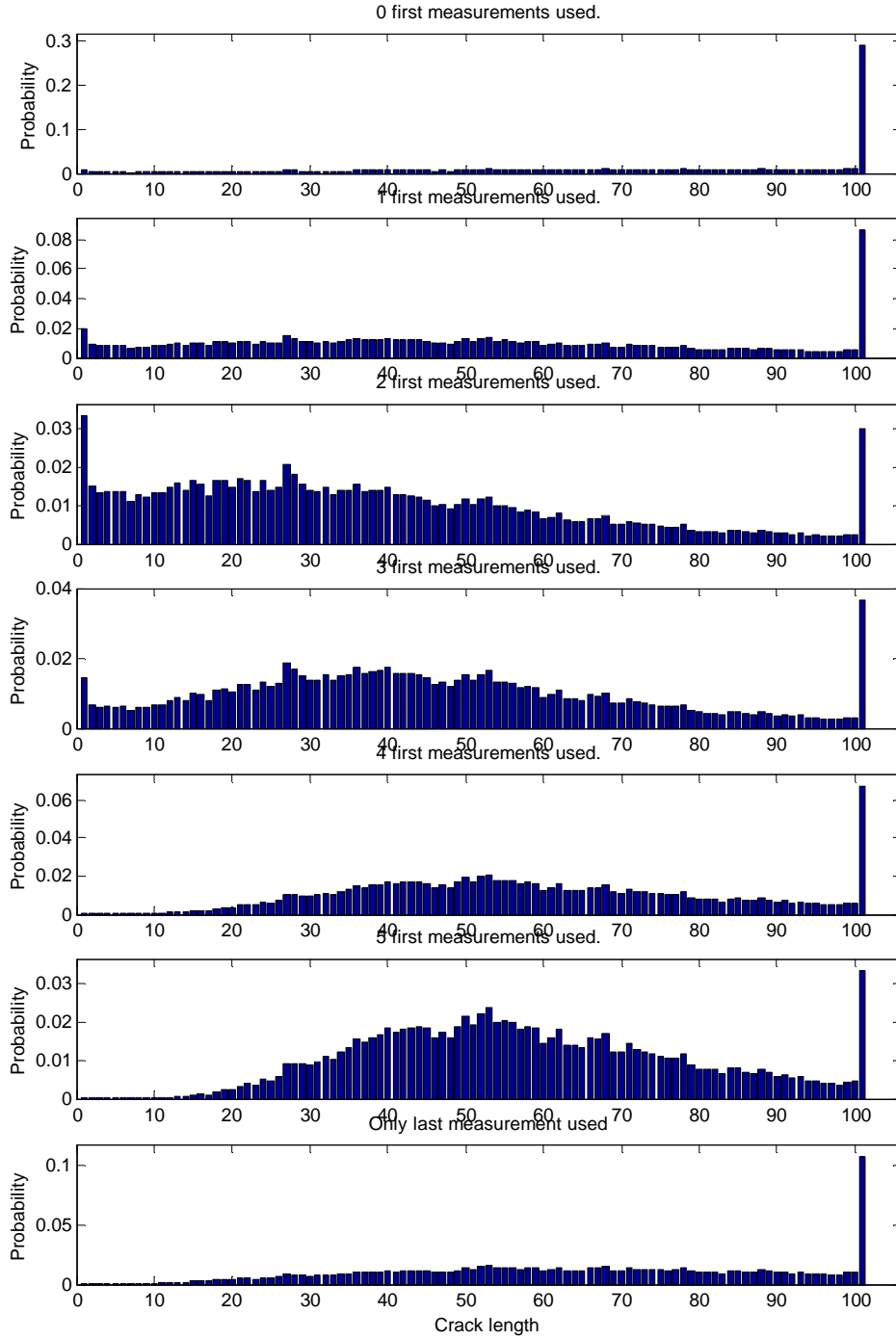


Figure 18: Histograms of the state probability vector $p(600)$. Variance $s_x^2=1.00$, measurement set 2.

6 Conclusions

We have created a computer program that calculates and updates predictive probability distributions of the remaining lifetime of a component.

An updateable model can reduce the uncertainty of predictions of the remaining lifetime. We showed that updating with measurements narrows the probability distributions, and can even move the distribution.

It seems that using more than one measurement for updating will give sharper distributions than using only one (the latest) measurement. The measurements appear to have a combined effect that is more than just the latest information.

We used fatigue crack growth as the test case. Other degradation mechanisms could be modelled similarly, for example bacterial growth in food industry. The main requirement is to have some monitoring data for the generation of the transition probability matrix. Our computer model can easily be scaled for different situations and environments.

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Appendix

A The FCG program manual

A.1 Preface

This is the manual for Matlab program `fcg.m`, the program for predicting fatigue crack growth. The growth is first predicted without any measurement data and then with measurement data. The program asks for some parameters and then plots figures of the predicted crack growth, the predicted cumulative distribution function, the predicted probability density function, and histograms of the probabilities of crack sizes at a user defined time.

Before you start to use the program `fcg.m`, please read through this manual, or at least chapters A.2 and A.4.

This program was developed as a student case study for VTT Industrial Systems in the spring 2002. The case study was part of the course “Project work seminar in operations research”, organised by the Laboratory of Systems Analysis in Helsinki University of Technology. The author of this program is Ms. Tiina Komulainen. Send your questions and report problems by e-mail to Tiina.Komulainen@hut.fi.

A.2 Requirements

You need to have Matlab version 6.1, Statistics Toolbox (for Matlab) and Microsoft Excel installed on your computer.

For smooth operation move the following files to your Matlab work-folder:

`accmeas.m`

`accmeas2.m`

`createfpaths.m`

`createpaths.m`

`distributions.m`

`fcg.m`

plotrealizations.m
plotstate.m
plottime.m
transmatrix.m

You will also need your measurement data, so move it also to the work-folder
for example as
measurements1.xls

A.3 The program files

The program consists of ten .m-files:

accmeas.m
accmeas2.m
createfpaths.m
createpaths.m
distributions.m
fcg.m
plotrealizations.m
plotstate.m
plottime.m
transmatrix.m

The main program file is `fcg.m`.

A.4 FCG program in brief

Here is a short description of the program and how it works.

1. The program asks if you want to create a new transition probability matrix. If you do, the program asks for some parameters and whether you would like to save the new transition probability matrix. If you chose not to create a new transition probability matrix, you will have to give the name of an existing transition probability matrix in excel-form, e.g. `tfmatrix1.xls`.

2. The program asks if you want to create a new measurement accuracy matrix. If you do, the program asks for some parameters and whether you would like to save the new measurement accuracy matrix. If you chose not to create a new measurement accuracy matrix, you will have to give the name of an existing measurement accuracy matrix in excel-form, e.g. `accmatrix1.xls`.
3. The program asks you for the measurement data, which should be in an excel-file. The data should be in two columns so that the measurement times are in the first column and the measurement values are in the second column.
4. The program asks you to give the probability distribution of the initial state. You can use the default vector in which the initial crack length is zero with probability $p=1$.
5. Finally the program asks you for the plotting parameters. It takes a while and three figures for each measurement are plotted. See Figure 19, Figure 20 and Figure 21.

A.5 FCG program: the long version

A.5.1 Starting

Type

```
>fcg(1);
```

in the command window of Matlab. The program prints:

```
>This is a function for predicting fatigue crack growth
>
>If you press enter (ctrl), default value will be used!
```

A.5.2 Creating transition probability matrix

After the first two lines the program prints:

```
>1. Creating transition probability matrix
>
```

```
>Do you want to create a new transition probability matrix?  
Y/N [Y]:
```

If you choose 'n', the program asks you to give the name of the transition probability matrix. The program does not accept false names, or files that do not exist.

```
>Do you want to create a new transition probability matrix?  
Y/N [Y]: n  
>Give the name of the excel-file that contains  
> the transition probability matrix: tfmatrix.xls
```

If you choose to create a new matrix, the program prints the following questions. If you want to use the default parameters displayed in brackets, press enter (ctrl).

```
>Do you want to create a new transition probability matrix?  
Y/N [Y]: y  
>  
>Parameters for transition probability matrix.  
>Give the number of realizations (20):
```

The number of realizations should be around 100 for accurate estimates, but it may take a while for the computer to count so many realizations.

```
>The length of the realizations (1000):
```

The length of the realizations should be around 1000 if you are going to use the default parameters.

```
>Size of the crack in the beginning (0.01):
```

The size of the crack in the beginning should be something small, but not zero or below zero.

```

>Parameter c (0.005):
>Parameter beta (1):
>Parameter mm (1.3):
>Parameter sigma (s^2) (1.7):

```

Parameters *c*, *beta*, *mm* and *sigma* are designed for this case, so it is not necessary to change them.

```

>The number of states (101):

```

The number of states in this program is the number of integer between zero and the final size of the crack. So if the maximum size of the crack is 100, then the number of states should be 101, because state 0 exists in this model.

```

>Rounding (0):

```

If you would like to have a smoother transition probability matrix, you can use rounding. The rounding makes the probabilities under the defined number zero. You should not use a rounding number bigger than 0.01!

```

>Do you want to save the transition probability matrix? Y/N
[Y]: y
>Give the name of the file (tfmatrix1.txt): tmatrix.txt

```

The program asks if you would like to save the new matrix. The name should be `.txt`, because at this moment it is not possible to save a file in excel-form. But you can later open with Excel the file you saved in `.txt`-form and save it as `.xls`-file.

A.5.3 Creating measurement accuracy matrix

The program prints:

```

>2.Creating measurement accuracy matrix
>
>Do you want to create a new measurement accuracy matrix? Y/N
[Y]:

```

If you choose 'n', the program asks you to give the name of the measurement accuracy matrix. The program does not accept false names, or files that do not exist.

```
>Do you want to create a new measurement accuracy matrix? Y/N
[Y]: n
>Give the name of the excel-file that contains
>the measurement accuracy matrix: accmatrix.xls
```

If you choose to create a new matrix, the program prints the following:

```
>Do you want to create a new measurement accuracy matrix? Y/N
[Y]: y
>
>Parameters for measurement accuracy
>If you press enter (ctrl), default value will be used.
>
>Parameter b0 (0.06):
>Parameter b1 (1.25):
>Parameter sigma (s^2) (0.22):
>Rounding (0):
```

These parameters describe how the measured values depend on the actual crack lengths. By using rounding, you can make the measurement accuracy matrix smoother so that all the numbers below the rounding number will be rounded to zero.

```
>Do you want to save the measurement accuracy matrix? Y/N [Y]:
y
>Give the name of the file (accmatrix1.txt): accmatrix2.txt
```

The program asks you if you would like to save the matrix. The name should be .txt, because at this moment it is not possible to save a file in excel-form. But you can later open with Excel the file you saved in .txt-form and save it as .xls-file.

A.5.4 The measurement data

The program asks you for your measurement data, which should be in an excel-file:

```
>3. The measurement data
>
>Measurement data should be in a n*2 matrix so that
>    measurement times are in the first column and
>    measurement values are in the second column
>Give the name of the excel-file that contains the measurement
data:
```

The data should be in two columns so that the measurement times are in the first column and the measurement values are in the second column.

```
>Give the name of the excel-file that contains the measurement
data: mitt
>Give the name of the excel-file that contains the measurement
data: mittaus0.xls
```

The program does not accept false names, or files that do not exist.

A.5.5 The probability distribution of the initial state

The program asks you to give the probability distribution of the initial state:

```
>4. The probability distribution of initial state
>
>The width of the initial state vector should be 101 columns
>Do you want to use the default vector [1,0,...,0]? Y/N [Y]:
```

You can choose to use the default vector, where the initial state is zero with probability $p=1$. If you choose to use your own initial state vector, it should be in excel-form and contain in the one row as many states as the size of your transition probability matrix, and the sum of the cells should be exactly 1.

A.5.6 Plotting

The program asks for some plotting parameters. If you want to use the ones given in brackets, press enter (ctrl).

```
>5. Plotting  
>  
>Give the time span to be plotted (1000):  
>Give the number of realizations to be plotted (5):  
>Give the time for the state probability vector to be plotted  
(1): 400
```

Next the function calculates and plots three figures for each measurement and for the case with no measurements. The case with no measurements is in figures 1-3, the case with one measurement in figures 4-6, the case with two measurements in figures 7-9, and so on.

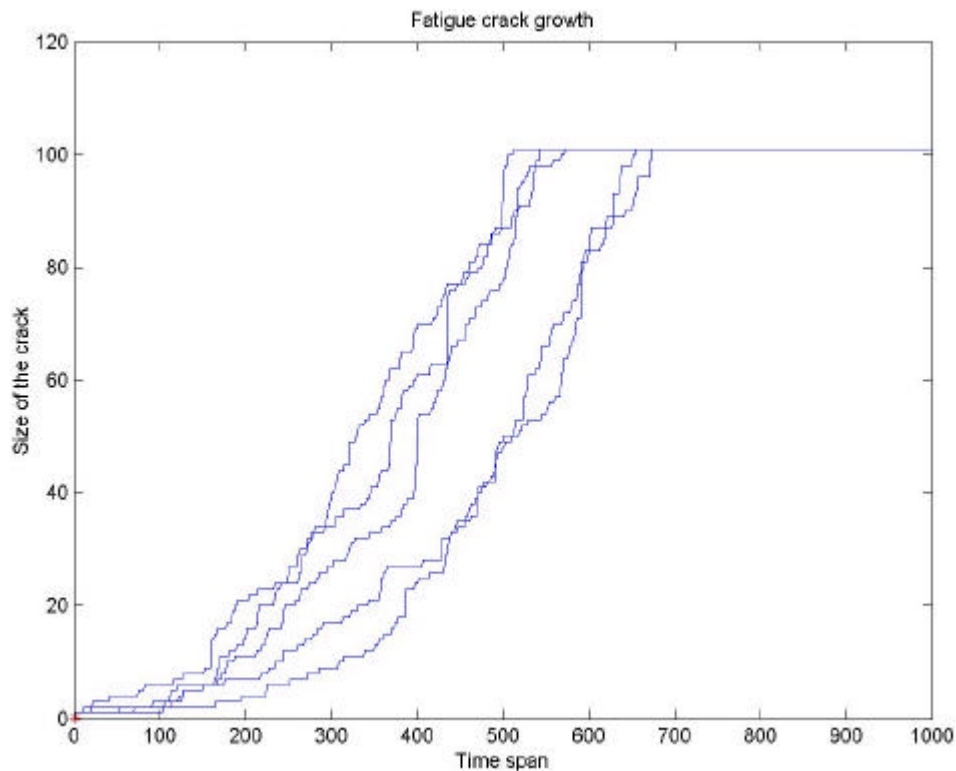


Figure 19: Five realisations of fatigue crack growth with one measurement.

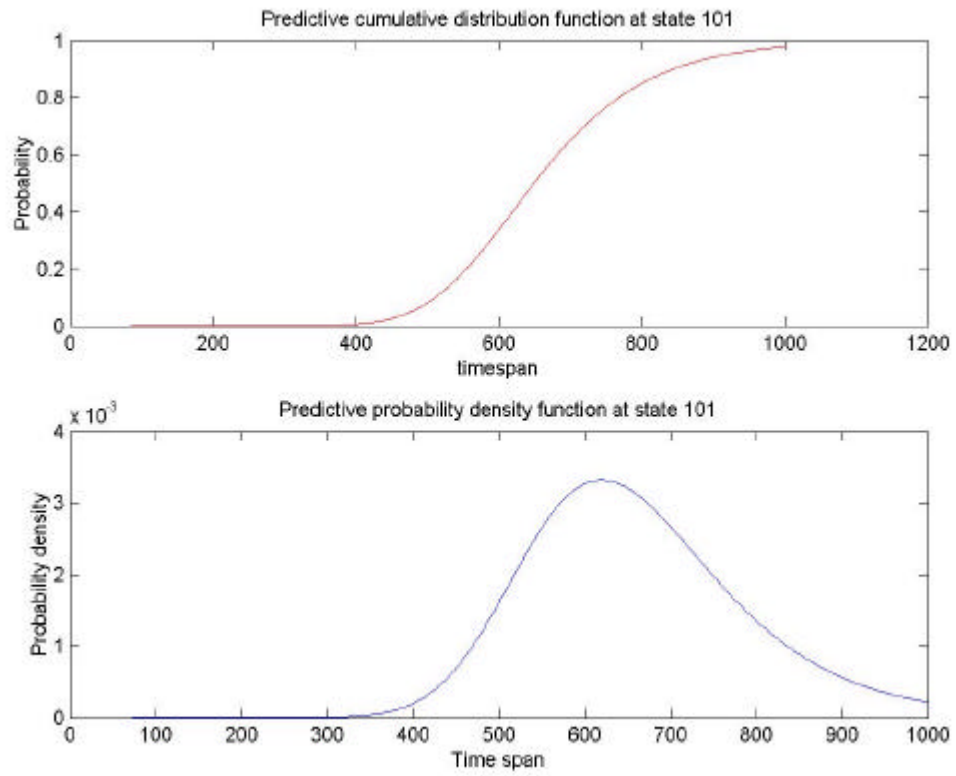


Figure 20: Predictive cumulative distribution and probability density functions of fatigue crack growth with one measurement.

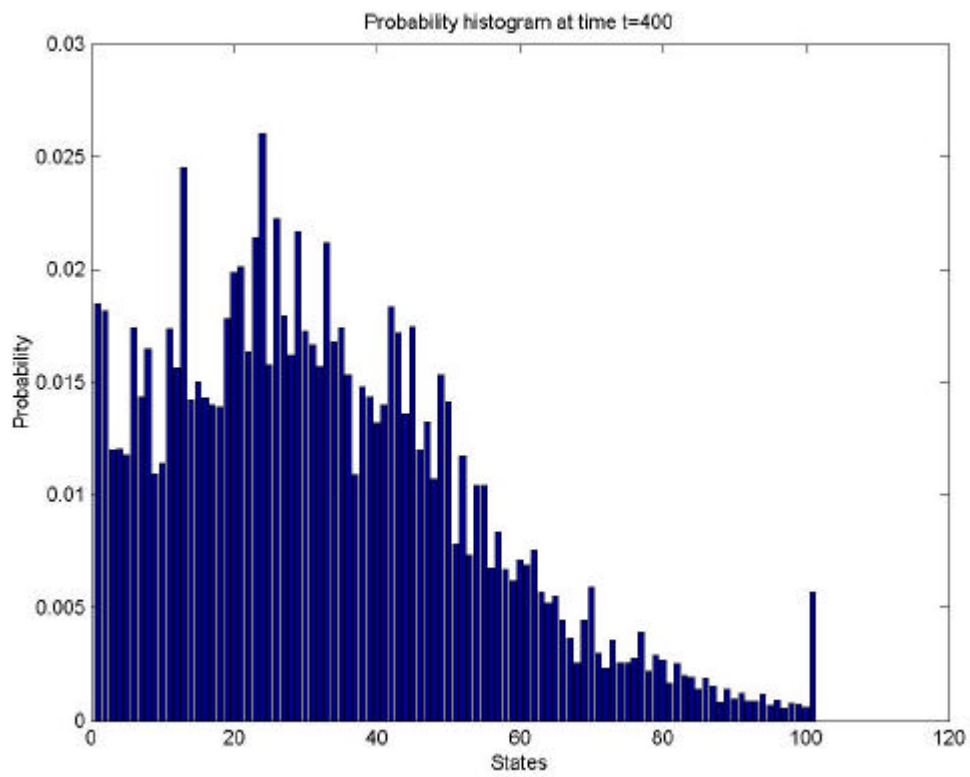


Figure 21: Probability histogram at time 400 with one measurement.