Aalto-yliopisto
Perustieteiden
korkeakoulu

Missing preferences in pairwise comparison matrices: a numerical study

Vili Ojala

16.6.2014

Instructor: Ph.D Matteo Brunelli

Supervisor: Prof. Raimo P. Hämäläinen

Systeemianalyysin
laboratorio

Background

- The analytic hierarchy process (AHP) is a decision making tool and it was developed by Thomas L. Saaty in 1977
  - Pairwise comparisons are a major part of AHP
- Pairwise comparison matrix $\boldsymbol{A}$ can be formed by asking the decicion maker $\frac{n(n-1)}{2}$ pairwise comparisons, where n is the number of alternatives
  - a_{ij} ="How much better option i is compared to option j?"

$$A = \begin{array}{c|cccc} & A_1 & A_2 & \cdots & A_n \\ \hline A_1 & w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ A_2 & w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & w_n/w_1 & w_n/w_2 & & w_n/w_n \end{array}$$

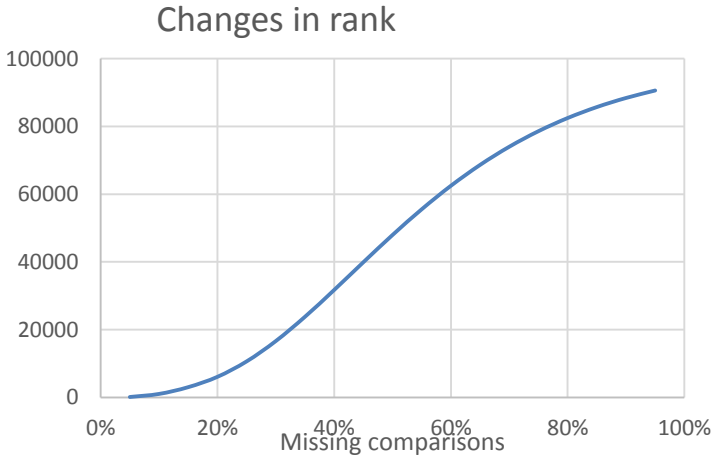
- The outcome of AHP are weights for different criteria and alternatives. Weights can be used to rank the different alternatives.
- One way to estimate the weights $\mathbf{w} = [w_1, w_2, \ldots, w_n]\hat{}T$ is by solving an equation $\boldsymbol{Aw} = \lambda_{max}\boldsymbol{w}$, where λ_{max} is the largest eigenvalue of $\boldsymbol{A}$

Background

- The number of questions can be overwhelming for the desicion maker
  - This can lead to inaccurate answers
- The decicion maker might not know, or be unwilling to tell their preference in some cases
  - Profit vs risk of injury on employees.
- What can be done when some comparisons are missing and $\boldsymbol{A}$ is incomplete?

Research objectives

- Analyzing the performance of different methods that can be used to estimate the weights even when there are missing comparisons
- How many comparisons can be left unanswered without significantly compromising the robustness of the results?
- How does the size of the matrix and the amount of missing information affect the results?
  - Performance can be measured by how often the ranking of the options change or how much the estimated weights vary from the weights that have been estimated using full information



Methods

- This is a numerical study in which the main tool will be Mathematica
- First we create normalized weights $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]\text{^}T$ and then use them to create a consistent pairwise comparison matrix $\boldsymbol{A}$
- Consistent matrix will give perfect results with both methods. Therefore we need to add random variations to $\boldsymbol{A}$
- This gives us $\acute{\boldsymbol{A}}$. We can use this to estimate weights $\acute{\boldsymbol{w}}$

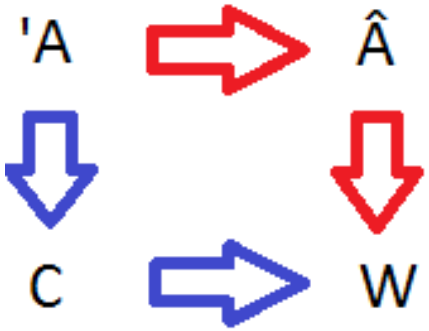
$$A = \begin{array}{c|cccc} & A_1 & A_2 & \cdots & A_n \\ \hline A_1 & w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ A_2 & w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & w_n/w_1 & w_n/w_2 & & w_n/w_n \end{array}$$

- We remove certain comparisons from $\acute{\boldsymbol{A}}$ and get $\overline{\boldsymbol{A}}$. Using two different methods we estimate the weights from the incomplete matrix $\overline{\boldsymbol{A}}$ and we get $\overline{\boldsymbol{w_1}}$ and $\overline{\boldsymbol{w_2}}$. We can use these weights to rank the alternatives and compare this with the rank induced by the weights $\acute{\boldsymbol{w}}$. The difference in the weights can be also examined directly.
- The size of the matrices and the amount of missing comparisons will be changed and their effects observed.

Estimating weights from incomplete pairwise matrices

- The first method is developed by P. Harker. The incomplete matrix $\acute{\boldsymbol{A}}$ is used to create an auxiliary matrix $\boldsymbol{C}$. The weights $\overline{\boldsymbol{w_1}}$ are then estimated directly from $\boldsymbol{C}$.

- The second method was developed by S. Shiraishi, T. Obata and M. Daigo. In it you first estimate the values of the missing comparisons and complete the matrix $\acute{\boldsymbol{A}}$. The outcome is matrix $\hat{\boldsymbol{A}}$ from which the weights $\overline{\boldsymbol{w_2}}$ are estimated from.



Timetable

22.5.2014 – 16.6.2014	Reading articles and forming an overall idea of the thesis
16.6.2014	Presentation of the subject
16.6.2014 – 30.6.2014	Learning to use Mathematica and clarifying the structure of the thesis
1.7.2014 – 20.7.2014	Performing the calculations
1.7.2014-26.7.2014	Writing the thesis
2.8.2014- 7.8.2014	#1 possibility to present final work if a seminar will be held
8.8.2014-1.1.2015	Semester in Eindhoven
12.1.2015	#2 possibility to present final work

Overall goal is to become a Bachelor of Science before applying to summer jobs in 2015

Sources

- Analytic hierarchy process
  - T. Saaty, A scaling method for priorities in hierarchical structures, Journal of Mathematical Psychology, Volume 15, Issue 3, 1977

- Harker's method
  - P. Harker, Alternative modes of questioning in the analytic hierarchy process, Mathematical Modelling, Volume 9, Issues 3–5, 1987

- Shiraishi, Obata & Daigo method
  - S. Shiraishi, T. Obata & M. Daigo, Properties of Positive reciprocal matrix and their application to AHP, Journal of the Operation Research Society of Japan , Volume 41, Issue 3, 1997