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Discrete Choquet integral and multilinear forms

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Background

- In decision making, a need to evaluate multiple alternatives w.r.t. different criteria is quite common
- To represent the value of an alternative, the information gained from multiple criteria needs to be *aggregated* into one single value
  → aggregation functions $f_A: \mathbb{R}^n \rightarrow \mathbb{R}$
- The most common aggregation function is the weighted arithmetic mean: $\mathrm{f_{WAM}}(x) = \sum_{i=1}^{n} w_i x_i$

Background

- The most basic aggregation functions require the criteria to be independent
- It might be desirable to also model the interactions between criteria: the Choquet integral and the Multilinear forms
    - Both can be seen as generalizations of the weighted arithmetic mean
    - With correctly selected weights, both functions give at least as good results as the WAM

Objectives and scope

- The main objectives:
  1) To provide a *comprehensible* introduction to the Choquet integral and the Multilinear forms
  2) Conduct a qualitative comparison between these two methods

The Choquet integral

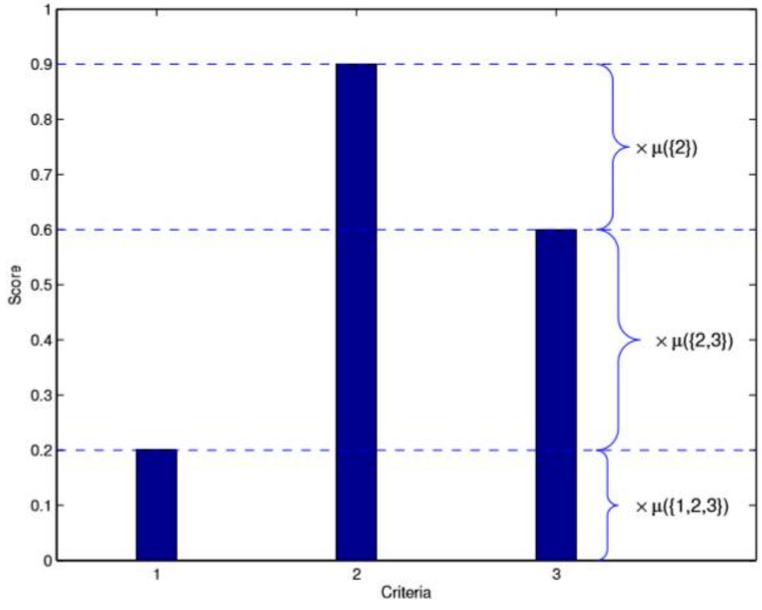
- A normalized *Capacity* is a monotone increasing set function $\mu: 2^N \to \mathbb{R}$ such that $\mu\{\emptyset\} = 0$ and $\mu\{N\} = 1$
- The values of the capacity can be used as weights for the Choquet integral defined as

$$C_\mu(x) := \sum_{i=1}^{n} [x_{\sigma(i)} - x_{\sigma(i-1)}]\mu(\{\sigma(i), \dots, \sigma(n)\}),$$

Where σ is a permutation on N such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$ and $x_{\sigma(0)} := 0$

The Choquet integral

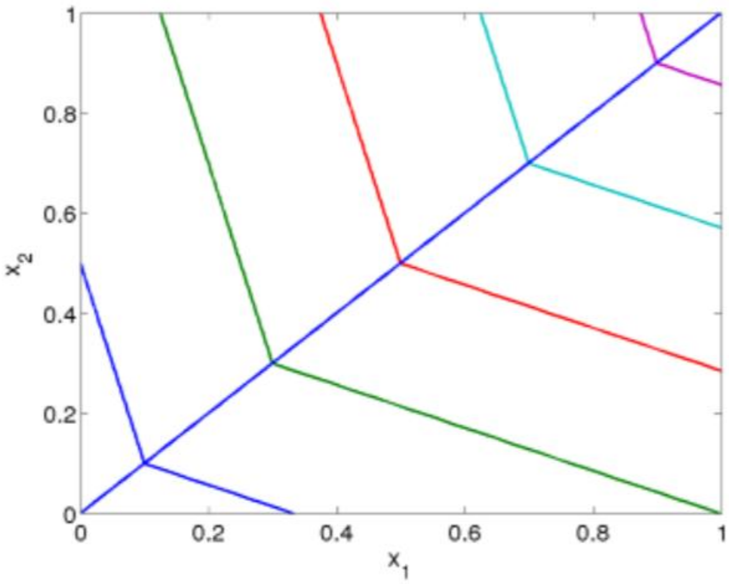
- Example: an alternative is given scores (0.2, 0.9, 0.6):



$$C_\mu = 0.2\mu(\{1,2,3\}) + 0.4\mu(\{2,3\}) + 0.3\mu(\{2\})$$

The Choquet integral

- The value of the Choquet integral of an alternative is always between the maximum and minimum evaluations of single criteria
- Contour lines are piecewise linear
- In figure $\mu(\{x_1\}) = 0.2$ and $\mu(\{x_2\}) = 0.3$



Multilinear forms

- Also known as multilinear extension (MLE)
- Linear w.r.t every criteria
- Definition with three criteria and weights λ:

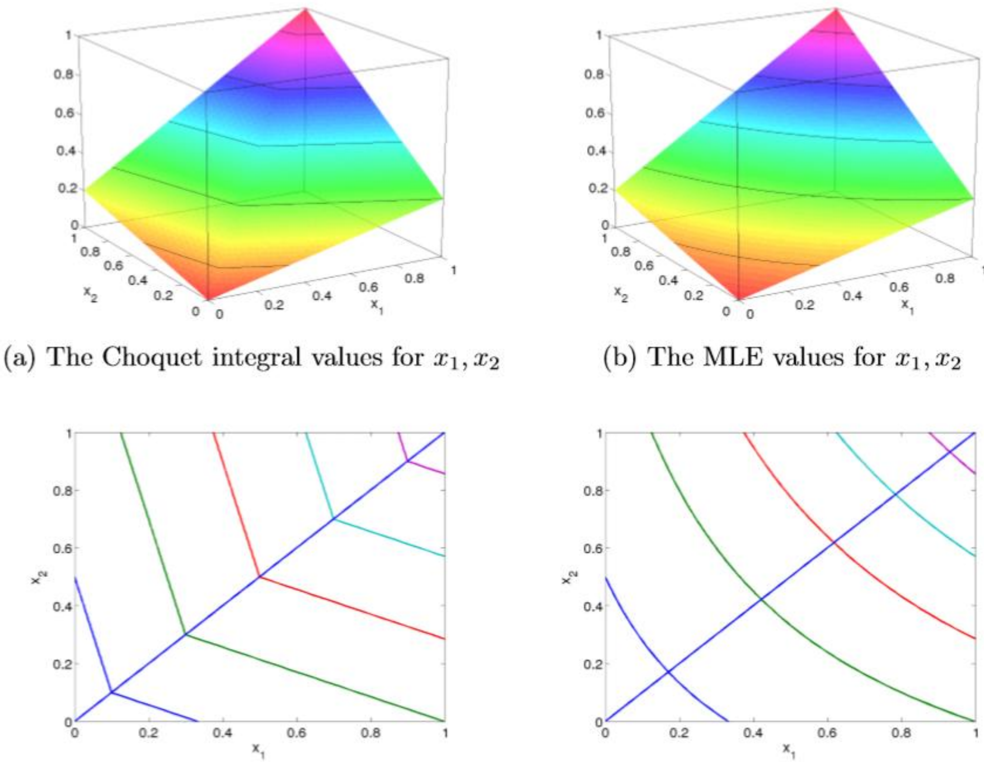
$$\begin{aligned}&\mathrm{MLE}(x_1, x_2, x_3)\\&= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3\\&+ \lambda_{12} x_1 x_2 + \lambda_{13} x_1 x_3 + \lambda_{23} x_2 x_3\\&+ \lambda_{123} x_1 x_2 x_3\end{aligned}$$

The Choquet integral compared to the Multilinear forms

- Criteria interaction can be taken into account, but the amount of weights grows exponentially
  - Sub-models can be used to reduce the amount of weights
- Can be defined with same weights
  - Using capacity produces monotone increasing aggregation functions
- Do not always produce same preference relations
- $C_\mu(x) = \mathrm{MLE}(x)$ if
  - Weights are set in such way, that the functions are reduced to the weighted arithmetic mean
  - At the edges of the value space

The Choquet integral compared to the Multilinear forms

- The value surfaces and contour plots for both the Choquet integral and MLE when
  $\mu(\{x_1\}) = 0.2$
  $\mu(\{x_2\}) = 0.3$



(a) The Choquet integral values for x_1, x_2

(b) The MLE values for x_1, x_2

(c) Contour plot for the Choquet integral

(d) Contour plot for the MLE

Conclusions

- Surprising connection between the methods despite the seemingly different starting points
- Using the methods gets laborious if the number of criteria grows → sub-models

Literature

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