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Valuation of Asian Quanto-Basket Options

Bachelor's thesis

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ABSTRACT OF THE BACHELOR'S THESIS

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Abstract:

The probability distribution of a finite sum of log-normally distributed and mutually correlated random variables is unknown. This poses a mathematical problem when pricing arithmetic Asian and basket options and their combinations: no exact analytical pricing formulas exist.

Analytical solutions are desirable because they can be computed practically instantaneously as opposed to Monte Carlo methods that require a significant amount of computing capacity when pricing path-dependent options. Accordingly, several approximate analytical methods – so called analytical approximations – have been suggested in literature.

This thesis gives a literature review on what methods have been suggested to price Asian quanto-basket options when the underlying asset is the arithmetic average of a basket of stocks and where some of the stocks may be quoted in another currency than the option's final payoff. The accuracy of two analytical approximations is tested when model parameters are varied.

The results suggest that a reasonable accuracy ban be obtained using simple moment-matching approximations. Largest pricing errors seem to occur when the volatilities of the underlying assets are high or the test option's maturity time is long. The benefits of analytical approximations become more pronounced when the number of averaging points increases or the count of underlying assets grows.

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AALTO-YLIOPISTO PERUSTIETEIDEN KORKEAKOULU	KANDIDAATINTYÖN TIIVISTELMÄ

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Tiivistelmä:

Korreloituneiden ja log-normaalisti jakautuneiden satunnaismuuttujien äärellisen summan todennäköisyysjakaumaa ei tunneta analyyttisessa muodossa. Tämä vaikeuttaa aritmeettisten aasialaisten optioiden, korioptioiden ja niiden yhdistelmien hinnoittelua, koska tarkkaa hinnoittelumallia ei ole suljetussa muodossa esitettävissä.

Analyyttiset ratkaisut ovat toivottavia, koska tuotteiden hinnoittelu niitä käyttäen on nopeampaa kuin Monte Carlo-simulointiin perustuva hinnoittelu. Monte Carlo-simulointi vaatii runsaasti laskentakapasiteettia. Tästä syystä kirjallisuudessa on ehdotettu useita analyyttisia approksimatiivisia menetelmiä aasialais-tyyppisille optioille.

Kandidaatintyössä tarkastellaan kirjallisuuden pohjalta, mitä analyyttisia approksimaatioita on ehdotettu käytettäväksi aasialaisten kvanttokorioptioiden hinnoittelussa. Aasialaisissa kvanttokotioptioissa alla olevana instrumenttina voi olla esimerkiksi osakekorin arvon aritmeettinen keskiarvo jollakin aikavälillä. Lisäksi jotkin korin osakkeista voivat olla noteerattuja eri valuutoissa kuin missä option loppuarvo tilitetään sijoittajalle. Kahta approksimatiivista menetelmää kokeillaan testioptioille ja niiden tarkkuutta arvioidaan parametrien arvoja muuttelemalla.

Tulosten mukaan usein riittävän tarkkuuden saamiseksi riittää yksinkertainen approksimaatio, jossa käytetään parametreina todellisen jakauman momentteja. Suurimmillaan hinnoitteluvirhe näyttää olevan suurilla volatiliteetin arvoilla ja pitkillä maturiteeteilla. Analyyttisten menetelmien edut Monte Carlo-simulointiin verrattuna näkyvät erityisesti laskenta-ajan säästymisenä keskiarvon laskennassa käytettävien havaintopisteiden määrää tai koriin kuuluvien osakkeiden määrää lisättäessä.

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Avainsanat: aasialainen optio, kvantto-optio, korioptio, analyyttinen approksimaatio, Levyn approksimaatio, reciprocal gamma -approksimaatio

Contents

1. Introduction

Since the early 1990s, the spectrum of financial derivatives has grown quickly and today they are an important tool for many companies and investors. This thesis considers the pricing problem of arithmetic Asian quanto-basket options that are among the most popular exotic over-the-counter derivatives. As the name suggests, they combine the characteristics of Asian, basket and quanto options.

Asian options are options whose payoff depends on the average of the underlying security over a pre-defined interval. The average can be either arithmetic or geometric and the time interval over which the average is calculated can be either continuous or subdivided into discrete observation points. Also, the exercise style may differ: there are both American-style Asian options in which early exercise is possible and European-style Asian options that can only be exercised at a pre-defined exercise date. In this paper, by average, we mean arithmetic average calculated over a finite set of pre-defined observation points and we only consider European-style Asian options.

Basket options are options whose underlying asset is a basket, that is, a weighted average of two or more underlying assets. Again, there are both American-style and European-style basket options; in this paper, we only consider European-style options. For example, a basket could consist of ten different stocks that are equally weighted.

Assuming the underlying asset follows the geometric Brownian motion, which is a widely accepted assumption in the financial industry, the probability density function of the underlying security is log-normal and fully defined by its first two non-centered moments. Then, a closed-form solution exist for both call and put option prices and they can be obtained using the famous Black-76 [2] formula. In the case of an Asian-style option, the underlying security contains a term that is a sum of log-normally distributed random variables and is thus not log-normal.

Several forms of simple functions have been suggested in the literature to be used as approximate density functions of the real distribution such as log-normal, reciprocal gamma and functions of the Johnson family [2]. Based on these approximate density functions analytical solutions can be derived for option prices and also for the Greeks that are the sensitivities of the option value to different market parameters. The main benefit of closed-form approximations over Monte Carlo simulation is calculation speed: they are almost instantaneous and, accordingly, allow quick pricing.

The aim of this thesis is to give a review on different analytical approximations that have been suggested in the literature to value Asian quanto-basket options. Also, two of the approximations are implemented and their performance is compared to a reference price obtained by Monte Carlo simulation. The emphasis is put on sensitivity analysis on the main parameters affecting the option price such as volatility, time horizon, strike price, correlations and the number of fixing points.

2. Asian, Basket and Quanto Options

An Asian quanto-basket option consists of several components. It combines the characteristics of Asian and basket options. Moreover, if some underlying assets are not quoted in the currency of the option payout, a so called quanto adjustment needs to be accommodated in the pricing model. This section considers the pricing of an Asian quanto-basket by examining its components first individually.

2.1. The Price of a European vanilla call option

Let's assume that underlying S is a domestic, continuous-dividend paying stock quoted in euros and follows the geometric Brownian motion. Then, the dynamics of S are given by the stochastic differential equation

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ &= (r_d - \delta_d) S dt + \sigma S dW, \end{aligned} \tag{2.1.1}$$

where μ is the constant risk-neutral drift rate of the process, r_d is the risk-free interest rate in the euro economy, δ_d is the continuous dividend rate in euros, σ is the constant volatility of S and dW is an increment of a standard Wiener process. Let $V(S,t)$ represent the value of a European vanilla call option on S. As shown in [17], we can build a risk-free portfolio consisting of $V(S,t)$ and S by selecting the amount of S in a suitable manner. Let us short an amount of Δ of the underlying stock S. Then, the value of our replicating portfolio Π can be written

$$\Pi = V(S,t) - \Delta S. \tag{2.1.2}$$

The value of Π changes over time because the value of the components of Π change and because we have to pay a continuous dividend for the stock S that we have sold short. An infinitely small change in the value of Π can thus be written

$$\mathrm{d}\Pi = \mathrm{d}V(S,t) - \Delta dS - \Delta \delta_d S dt. \tag{2.1.3}$$

Using Itô's lemma, we can expand term $\mathrm{d}V(S,t)$ and obtain

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS - \Delta \delta_d S dt. \tag{2.1.4}$$

If we seek to make Π riskless, i.e., the change in the value of Π deterministic given a change in time dt, we must eliminate the stochastic terms in (2.1.4). This can be accomplished by setting the coefficient of dS to zero over $[0, dt]$

$$\frac{\partial V}{\partial S} - \Delta = 0. \tag{2.1.5}$$

Because Π is now riskless over time horizon $[0, dt]$, its return must match to the risk-free return rate r_d that is currently observed in the euro-zone economy. If we ignore transaction costs and if the return of the portfolio was bigger than r_d, we could borrow from a bank at rate r_d, invest it in portfolio Π and delta hedge it gaining a risk free profit of $r_\Pi - r_d$ and, thus, make an arbitrage profit. Similarly, if the return of the portfolio were lower than that of a risk free account, we could sell the option $V(S, t)$, delta hedge it and invest the remaining cash in bank [17]. The assumption of transaction costs being zero is an idealization. In practice, banks are market makers and there is a bid-offer spread in the interest rates they offer. However, based on these assumptions and substituting equations (2.1.2) and (2.1.5) into (2.1.4) we have

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial S}\delta_d S \right) dt = r_d \Pi dt$$

$$\Leftrightarrow \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial S}\delta_d S \right) dt = r_d (V - \frac{\partial V}{\partial S} S) dt$$

$$\Leftrightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}(r_d - \delta_d) - r_d V = 0. \tag{2.1.6}$$

Equation (2.1.6) is the famous Black-Scholes partial differential equation for a continuous-dividend paying asset. When the underlying security follows the geometric Brownian motion, (2.1.6) can be written in the form of the standard one-dimensional heat equation

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} \tag{2.1.7}$$

and solved using standard calculus [12], [17]. The closed-form solution for a European call option is then

$$C(t) = e^{-r_d(T-t)}\left[S(t)e^{(r_d - \delta_d)(T-t)}N(d_1) - KN(d_2)\right], \tag{2.1.8}$$

where $S(t)$ is the price of the underlying asset at time t, T is the maturity time, K is the strike price, and N is the cumulative normal distribution. The terms d_1 and d_2 are

$$d_1 = \frac{ln\left(\frac{S(t)}{K}\right) + (r_d - \delta_d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln\left(\frac{S(t)}{K}\right)+\left(r_d-\delta_d-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}\,. \tag{2.1.9}$$

The cumulative normal distribution functions $N(x)$ arise because the derivation of (2.1.8) includes integration of terms of type $\exp[-1/2\, x^2]$.

Formula (2.1.8) can be used e.g. to calculate the Greeks i.e. the sensitivities of the option price to different parameters. The most important of these sensitivities is delta, that is, the sensitivity of the option price to change in the underlying price. Other sensitivities are gamma, speed, vega and theta which respectively represent the second derivative with respect to the underlying price, the third derivative with respect to the underlying price, the derivative with respect to the volatility parameter and the derivative of the option price with respect to time. The Greeks are used when hedging option positions and they can be obtained using standard calculus if there exists a closed-form expression for price.

2.2. Asian Option

Asian options are path dependent derivatives. Their payoff at maturity depends on the path that the underlying asset has taken to arrive at its state at the maturity date. Several variations of Asian options are traded in exchanges and over-the-counter markets. This paper concentrates on Asian options that use arithmetic averaging and the payoff at maturity T is

$$Payoff = max(0, A - K), \tag{2.2.1}$$

where K is a constant strike price that is set when issuing the option contract and A is the arithmetic average of the underlying asset S over some pre-defined discrete and finite set of observation points.

Let us now assume that underlying S follows the geometric Brownian motion described by (2.1.1). Then, S can be solved as a function of time t. This can be done correctly using Itô's lemma. However, Willmott and Luenberger [12], [17] suggest as a practical tool using Taylor's expansion of second order and then collecting all first-order terms. When using this approach, the terms involving dW^2 need to be replaced by dt. As Willmott explains [17], this method is mathematically incorrect but yields the correct result. If we write $f(s) = \ln S$, a small increment in $f(s)$ is

$$df(s) = f'(S)dS + \frac{1}{2}f''(S)dS^2 + O(dS^3)$$

$$= \frac{1}{S}dS - \frac{1}{2S^2}dS^2 + O(dS^3)$$

$$= \frac{1}{S}(\mu S dt + \sigma S dW) - \frac{1}{2S^2}(\mu S dt + \sigma S dW)^2 + \mathcal{O}(dS^3)$$

$$= \frac{1}{S}(\mu S dt + \sigma S dW) - \frac{1}{2S^2}\sigma^2 S^2 dW^2 + \mathcal{O}(dS^3). \tag{2.2.2}$$

When replacing dW^2 by dt and taking the limit as dt approaches zero, the higher order terms go to zero and equation (2.2.2) can be written

$$\xrightarrow{dt \to 0} \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW. \tag{2.2.3}$$

Now ln $f(s)$ can be solved from (2.2.3) by integrating both sides and we obtain an expression for S as a function of t.

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

$$= S(0)e^{\left(r_d - \delta_d - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}, \tag{2.2.4}$$

where $W(t)$ is a standard Wiener process. The arithmetic average of (2.2.1) can now be written

$$A = \frac{1}{n}\sum_{\tau=1}^{n} S(t_\tau) = \frac{S(0)}{n}\sum_{\tau=1}^{n} e^{\left(r_d - \delta_d - \frac{1}{2}\sigma^2\right)t_\tau + \sigma W(t_\tau)}. \tag{2.2.5}$$

If n is one, as is the case with a European vanilla call option, the probability density function of A at maturity time T is log-normal. When n is bigger than one, the probability density function of A still shows more log-normal behavior than e.g. normal behavior [13] but is no longer log-normal.

Let us assume that (i) the market is complete so that investors may purchase any contingent claim, (ii) there is no arbitrage and (iii) the law of one price holds. This guarantees the existence of a unique discount factor [5]. Also, let the probability density function of A at time T be $P(x)$. Then, the price of an arithmetic Asian call option – before the averaging period – on the underlying asset S can be obtained by evaluating integral [2]

$$C = e^{-r(T-t)} \int_{-\infty}^{\infty} P(x)\, max(0, A - K) dx. \tag{2.2.6}$$

Because the integrand in (2.2.6) is zero whenever $A - K$ is negative, this simplifies to

$$C = e^{-r(T-t)} \int_{K}^{\infty} P(x)(A - K) dx. \tag{2.2.7}$$

Unfortunately, for an Asian option, there is no closed-form solution for this integral, which is the central problem when pricing Asian options, basket options and their combinations. However, several approximate methods have been suggested in the literature [2], [3], [7], [8], [11]. They concentrate mainly on approximating the probability density function $P(x)$ by some other function so that the integral of (2.2.7) will have a closed-form solution.

2.3. Basket Option

A basket option is an exotic option whose underlying asset is a basket of stocks, commodities, or currencies for instance. The value of a basket is usually defined as the weighted average of a set of underlying assets at time t. Using different weights for different basket components, an investor can tailor a basket that suits her interest best. The payoff function of a basket option at maturity is

$$Payoff = max(0, A_B - K_B), \tag{2.3.1}$$

where A_B stands for the weighted average of the basket components at maturity and K_B is the strike price that can be set equal to the basket value at the issuance of the option for example.

If each individual underlying asset in a basket follows the geometric Brownian motion in (2.1.1) and there are m underlying assets that are all quoted in euros, A_B can be written

$$A_B = \frac{1}{m}\sum_{i=1}^{m} w_i S_i(0) e^{\left(\mu_i - \frac{1}{2}\sigma_i{}^2\right)T + \sigma_i W_i(T)}, \tag{2.3.2}$$

where w_i are the weights for each underlying asset and usually sum up to 1, $S_i(0)$ are the initial prices of the underlyings, μ_i are the risk-neutral drift rates, σ_i are the underlying volatilities and now $W_i(T), i\epsilon\{1,2,\ldots,m\}$ are correlated Wiener processes that satisfy correlation structure given by correlation matrix Σ. The correlations of the underlying processes can be estimated from historical data or implied correlations can be used. According to Walter [16], implied correlations are difficult to estimate, though. Again, if m is one, the basket consists only of one underlying asset and the probability density function of A_B becomes log-normal.

When a unique discount factor exists, the value of a basket option at time $t < T$ can be found by evaluating the integral in (2.2.7) when A has been replaced by A_B and $P(x)$ has been replaced by the probability density function of the basket $P_B(x)$. This leads to the integral

$$C = e^{-r(T-t)} \int_{K_B}^{\infty} P_B(x)(A_B - K_B)dx, \tag{2.3.3}$$

which cannot be evaluated in closed form because $P_B(x)$ is the probability distribution of a random variable that includes the sum of log-normally distributed random variables.

2.4. Quanto Option

Whenever an option payout is denominated in currency other than the underlying asset, the value of the option does not only depend on the performance of the underlying but also on the exchange rate between the payout currency and the currency of the underlying. Such options are often called quanto options. Geman [4] introduces in her book what she calls a commodity quanto option whose payoff function at maturity is

$$Payoff = max(0, S(T)X(T) - K_d). \tag{2.4.1}$$

Here $S(T)$ is the value of the underlying asset at maturity and $X(T)$ is the exchange rate so that the product $S(T)X(T)$ is in the same currency as the strike price K_d. For instance, if $S(T)$ is quoted in USD, $X(T)$ is the EUR-USD exchange rate at maturity and K_d is a pre-defined strike price in EUR, then the payoff of the above-mentioned quanto option is in EUR and is given by (2.4.1).

Another form of quanto options suggested in the literature [2], [17] has a payoff function

$$Payoff = X_0 max\big(0, S(T) - K_f\big), \tag{2.4.2}$$

where $S(T)$ is the value of the underlying asset at maturity in its own currency, K_f is a strike price in the same currency as the underlying asset and X_0 is a pre-defined exchange rate at which the payoff of the option is converted into the payment currency.

The dependence of the payoff function on two variables affects pricing. Willmott [17] suggests that this dependence can be derived by building a risk-free portfolio that consists of a quanto option, a short cash position of the currency in which the underlying is quoted and a short position in the underlying itself. Let us assume, for example, that dividend paying stock S quoted in USD and the EUR-USD exchange rate X both follow the geometric Brownian motion

$$\mathrm{d}S = \mu_S S dt + \sigma_S S dW_S$$

$$\mathrm{d}X = \mu_X X dt + \sigma_X X dW_X. \tag{2.4.3}$$

Let Π be a portfolio of the quanto option, cash in the currency of the underlying and the underlying itself

$$\Pi = \mathrm{V}(\mathrm{X}, \mathrm{S}, \mathrm{t}) - \Delta_{USD} X - \Delta_S SX. \tag{2.4.4}$$

Each term in (2.4.4) is in EUR, Δ_{USD} is the size of the USD cash holding in USD, and Δ_S is the amount of stock S in the portfolio. Let r_f be the risk-free interest rate in the USD economy. Then the value of Π changes over time because the value of S changes, the exchange rate changes, dividends have to be paid for the shorted underlying stock and interest has to be paid on the short USD cash position. Thus, an infinitely small change in the value of portfolio Π is

$$\mathrm{d}\Pi = \mathrm{dV} - \Delta_{USD}\big(dX + r_f X dt\big) - \Delta_S d(SX) - \Delta_S \delta_f SX dt$$

$$\begin{aligned} &= \left(\frac{\partial \mathrm{V}}{\partial \mathrm{t}} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 \mathrm{V}}{\partial X^2} + \rho\sigma_X\sigma_S XS \frac{\partial^2 \mathrm{V}}{\partial X\,\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \mathrm{V}}{\partial S^2} - \rho\sigma_X\sigma_S \Delta_S XS \; - r_f \Delta_{USD} X \right. \\ &\qquad \left. - \delta_f \Delta_S SX \right) dt + \left(\frac{\partial \mathrm{V}}{\partial \mathrm{X}} - \Delta_{USD} - \Delta_S S\right) dX + \left(\frac{\partial \mathrm{V}}{\partial \mathrm{S}} - \Delta_S X\right) dS \end{aligned} \tag{2.4.5}$$

where term $-r_f \Delta_{USD} X dt$ is the interest or cost of carry paid on the short USD position, term $-\delta_f \Delta_S SX dt$ represents the dividend that needs to be paid for the shorted stock and term $\rho\sigma_X\sigma_S\Delta_S XS dt$ arises when $d(SX)$ is expanded in (2.4.5). To make Π risk-free over time horizon $[0, dt]$, the coefficients of dX and dS must be equal to zero, so that

$$\Delta_S = \frac{1}{X}\frac{\partial \mathrm{V}}{\partial \mathrm{S}}$$

$$\Delta_{USD} = \frac{\partial \mathrm{V}}{\partial \mathrm{X}} - \Delta_S S = \frac{\partial \mathrm{V}}{\partial \mathrm{X}} - \frac{S}{X}\frac{\partial \mathrm{V}}{\partial \mathrm{S}}. \tag{2.4.6}$$

Because Π is in EUR, the return of Π must be equal to the risk-free return that prevails in the euro-zone economy. Let us mark this return r_d. Substituting (2.4.4) and (2.4.6) into (2.4.5) we get

$$\begin{aligned} &\left(\frac{\partial \mathrm{V}}{\partial \mathrm{t}} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 \mathrm{V}}{\partial X^2} + \rho\sigma_X\sigma_S XS \frac{\partial^2 \mathrm{V}}{\partial X\,\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \mathrm{V}}{\partial S^2} - (\delta_f + \rho\sigma_X\sigma_S) S \frac{\partial \mathrm{V}}{\partial \mathrm{S}} \right. \\ &\qquad \left. - r_f \left(\frac{\partial \mathrm{V}}{\partial \mathrm{X}} - \frac{S}{X}\frac{\partial \mathrm{V}}{\partial \mathrm{S}}\right) X \right) dt = r_d \Pi dt \end{aligned}$$

$$\begin{aligned} \Leftrightarrow &\frac{\partial \mathrm{V}}{\partial \mathrm{t}} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 \mathrm{V}}{\partial X^2} + \rho\sigma_X\sigma_S XS \frac{\partial^2 \mathrm{V}}{\partial X\,\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \mathrm{V}}{\partial S^2} + X \frac{\partial \mathrm{V}}{\partial \mathrm{X}}\big(r_d - r_f\big) \\ &\qquad + S \frac{\partial \mathrm{V}}{\partial \mathrm{S}}\big(r_f - \delta_f - \rho\sigma_X\sigma_S\big) - r_d V = 0. \end{aligned} \tag{2.4.7}$$

Equation (2.4.7) can be used to price a quanto option when V is a function of X, S and t [17] and S is a dividend paying stock. In the special case when the payoff function is given by (2.4.2), the option value no longer depends on the exchange rate X directly [10]. Thus, $V = \mathrm{V(S,t)}$ and (2.4.7) reduces to

$$\frac{\partial \mathrm{V}}{\partial \mathrm{t}} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 \mathrm{V}}{\partial S^2} + S\frac{\partial \mathrm{V}}{\partial \mathrm{S}}\left(r_f - \delta_f - \rho\sigma_X\sigma_S\right) - r_d V = 0. \qquad (2.4.8)$$

Equation (2.4.8) can be interpreted as the Black-Scholes equation with constant dividend yield and, thus, a closed-form solution exists within the Black-Scholes framework. For the purposes of this thesis, we only need to find what dynamics need to be used for the underlying asset when pricing a quanto option. Kwok and Wong [10] note that the risk-neutral drift rate is the coefficient of the third term $r_f - \delta_f - \rho\sigma_X\sigma_S$. Thus, when pricing a quanto option on a dividend paying stock and the payoff function is as in (2.4.2)[1], the following dynamics can be used

$$\mathrm{dS} = \left(r_f - \delta_f - \rho\sigma_X\sigma_S\right)Sdt + \sigma_S S dW_S.$$

The asset value S can be solved as a function of time, as was done in (2.2.3) and (2.2.4)

$$S(t) = S(0)e^{\left(r_f - \delta_f - \rho\sigma_X\sigma_S - \frac{1}{2}\sigma_S{}^2\right)t + \sigma_S W(t)}. \qquad (2.4.9)$$

2.5. Pricing Framework for Arithmetic Asian Quanto-Basket Options

Let us summarize the results of the earlier sections and define the pricing framework for arithmetic Asian quanto-basket options that is to be used as a basis for Monte Carlo simulations and later in the discussion on analytical approximations.

$S_i = \{S_i(t): t \geq 0\}$	price of a share that is quoted in currency X_i
$X_i = \{X_i(t): t \geq 0\}$	exchange rate between the currency of underlying i and the option payoff
μ_i	risk-neutral drift rate of underlying asset i
σ_i	volatility of underlying asset i
ρ	correlation matrix that describes the mutual correlations between underlying assets S_i , $i\epsilon\{1, \dots, m\}$ and exchange rates $X_i, i\epsilon\{1, \dots, m\}$
Σ	correlation matrix of underlying assets i, $i\epsilon\{1, \dots, m\}$
r_i	risk-free interest rate in the currency of underlying asset i

[1] The dynamics given in (2.4.9) should apply when pricing quanto options with other forms of payoff functions as well [18].

δ_i continuous dividend rate of underlying asset i

r_d risk-free interest rate in the domestic (EUR) economy

The underlying assets S_i and the exchange rates X_i are assumed to follow the geometric Brownian motion as follows

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i \text{ , } i\epsilon\{1, \dots, m\} \tag{2.5.1}$$

$$dX_i = \mu_{X_i} X_i dt + \sigma_i X_i dZ_i \text{ , } i\epsilon\{1, \dots, m\}. \tag{2.5.2}$$

Furthermore, the underlying assets and exchange rates are assumed to have correlations

$$Corr\left(W_i, W_j\right) = \Sigma_{i,j} \tag{2.5.3}$$

$$Corr\left(Z_i, W_j\right) = \rho_{i,j}. \tag{2.5.4}$$

According to the results obtained in sections 2.1-2.4, pricing an Asian quanto-basket option is equivalent to using the dynamics of (2.4.9) for all underlying assets S_i

$$S_i(t) = S_i(0)e^{\left(r_i - \delta_i - \rho_{i,i}\sigma_{X_i}\sigma_i - \frac{1}{2}\sigma_i{}^2\right)t + \sigma_i W_i(t)}. \tag{2.5.5}$$

Combining equations (2.2.5), (2.3.2) and (2.5.5), we get

$$A_{AQB} = \frac{1}{n}\sum_{\tau=1}^{n} A_B(t_\tau)$$

$$= \frac{1}{n}\sum_{\tau=1}^{n}\sum_{i=1}^{m} w_i S_i(0)e^{\left(r_i - \delta_i - \rho_{i,i}\sigma_{X_i}\sigma_i - \frac{1}{2}\sigma_i{}^2\right)\tau + \sigma_i W_i(\tau)}. \tag{2.5.6}$$

As noted earlier, the term $\rho_{i,i}\sigma_{X_i}\sigma_i$ arises due to the quanto property of the option, or more mathematically, due to the $d(SX)$ term in (2.4.5). This result (2.5.6) will be used in our Monte Carlo simulations.

3. Approximate Pricing methods

No closed-form solutions exist for pricing arithmetic Asian quanto-basket options because the probability distribution of the underlying average A_{AQB} is not log-normal. Figure 1 illustrates the differences between the cumulative probability distribution function of A_{AQB} and that of a log-normal distribution when the two first non-centered moments of the distributions have been equalized. Figure 2, in turn, shows the probability density functions of the distributions.

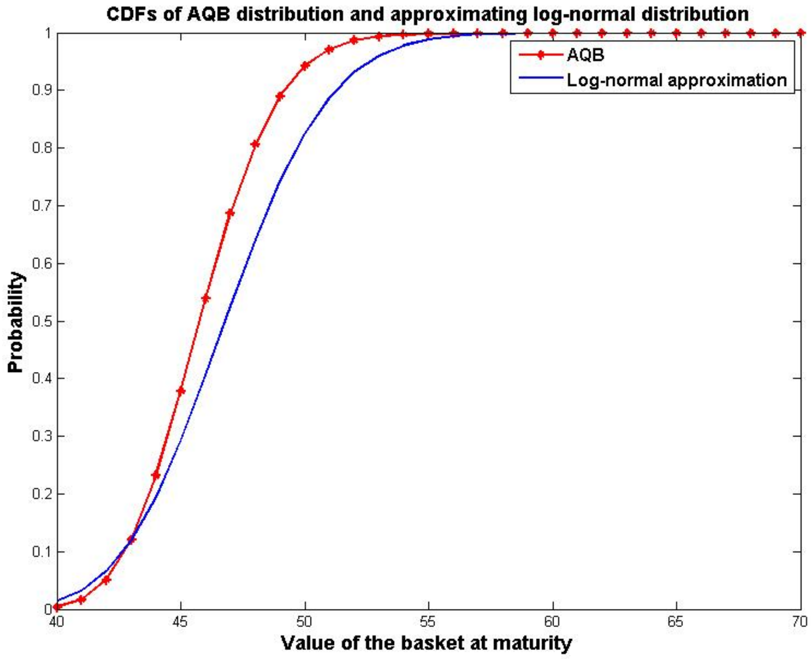

Figure 1: The CDFs of the real distribution and an approximate log-normal distribution. The first two moments of the approximating distribution have been set equal to those of the real distribution. The data concerning the real distribution was simulated.

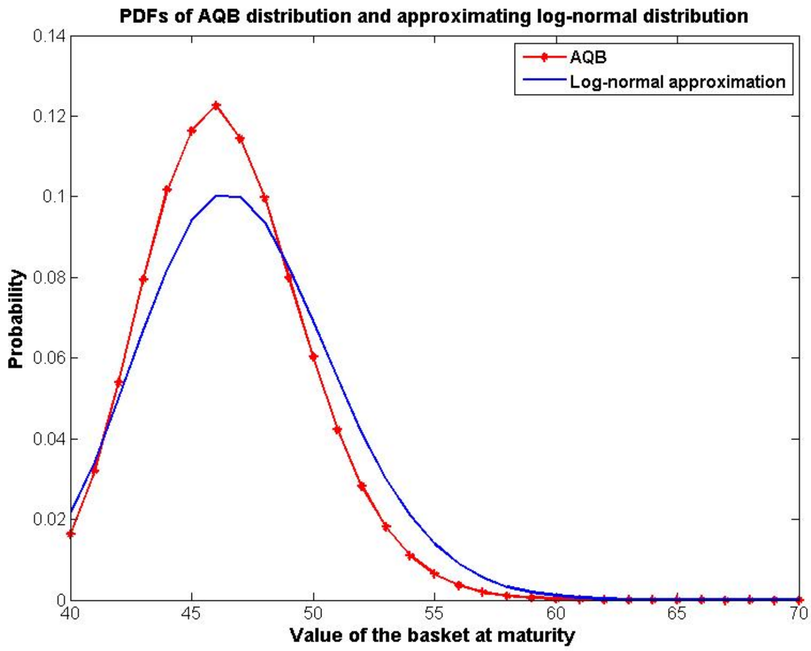

Figure 2: PDFs of log-normal approximation and the real distribution

Asian-style options are a very important derivative class and, consequently, several approximate methods have been developed. One of them was already illustrated in Figures 1 and 2. This section aims at introducing a few of the popular analytical approximate methods mentioned in the literature.

3.1. Levy's Approximation

Let $P(x)$ be the real probability density function of A_{AQB} that was defined in formula (2.5.6). The idea of Levy's method is to approximate $P(x)$ with a log-normal density function $P_{log}(x)$. This is done by setting the first two non-centered moments of $P(x)$ equal to the corresponding moments of $P_{log}(x)$. Then, Black's call option formula can be utilized.

The first two moments of $P(x)$ can be calculated based on formula (2.5.6) for A_{AQB} and using identity [2]

$$E[e^{a+bW}] = e^{a+\frac{1}{2}b^2}. \tag{3.1.1}$$

The first moment is

$$m_1 = E\left[A_{AQB}\right]$$

$$= E\left[\frac{1}{n}\sum_{\tau=1}^{n}\sum_{i=1}^{m} w_i S_i(0) e^{\left(r_i-\delta_i-\rho_i\sigma_X\sigma_i-\frac{1}{2}\sigma_i{}^2\right)t_\tau+\sigma_i W_i(t_\tau)}\right]$$

$$= \frac{1}{n}\sum_{\tau=1}^{n}\sum_{i=1}^{m} w_i S_i(0)\, E\left[e^{\left(r_i-\delta_i-\rho_i\sigma_X\sigma_i-\frac{1}{2}\sigma_i{}^2\right)t_\tau+\sigma_i W_i(t_\tau)}\right]$$

$$= \frac{1}{n}\sum_{\tau=1}^{n}\sum_{i=1}^{m} w_i S_i(0) e^{(r_i-\delta_i-\rho_i\sigma_X\sigma_i)t_\tau}. \tag{3.1.2}$$

The second moment is [2]

$$m_2 = E\left[A_{AQB}{}^2\right]$$

$$= E\left[\left(\frac{1}{n}\sum_{\tau=1}^{n}\sum_{i=1}^{m} w_i S_i(0) e^{\left(r_i-\delta_i-\rho_i\sigma_X\sigma_i-\frac{1}{2}\sigma_i{}^2\right)t_\tau+\sigma_i W_i(t_\tau)}\right)^2\right]$$

$$= \frac{1}{n^2}\sum_{i_1,i_2=1}^{m}\sum_{\tau_1,\tau_2=1}^{n} w_{i_1} w_{i_2} S_{i_1} S_{i_2}\, \mathrm{Exp}[\left(r_{i_1}-\delta_{i_1}-\sigma_X\sigma_{i_1}\rho_{i_1}+\sigma_{i_1}\sigma_{i_2}\Sigma_{i_1,i_2}\right)$$
$$*\min(t_{\tau_1},t_{\tau_2}) + \left(r_{i_2}-\delta_{i_2}-\sigma_X\sigma_{i_2}\rho_{i_2}\right)*\max(t_{\tau_1},t_{\tau_2})]. \tag{3.1.3}$$

The derivations of the second and higher moments are omitted from this paper. If A_{AQB} is assumed log-normal, the approximating log-normal probability density function of A_{AQB} can be written

$$P_{log}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, \tag{3.1.4}$$

where μ and σ^2 are the mean and variance parameters of the normally distributed natural logarithm of A_{AQB}. By definition the first two non-centered moments of the approximating log-normal distribution are

$$m_1^{log} = \int_0^\infty x \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx = e^{\mu+\frac{1}{2}\sigma^2} \tag{3.1.5}$$

$$m_2^{log} = \int_0^\infty x^2 \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx = e^{2\mu+2\sigma^2}. \tag{3.1.6}$$

For the numerical purposes, it is sufficient to solve for the parameters μ and σ^2 as functions of the moments of the real distribution. The real moments m_1 and m_2 can be calculated on the fly using (3.1.2) and (3.1.3). Let us set the first two moments of the approximating distribution and the real distribution equal:

$$\begin{cases} e^{\mu+\frac{1}{2}\sigma^2} = m_1 \\ e^{2\mu+2\sigma^2} = m_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma^2 = \ln m_2 - 2\ln m_1 \\ \mu = 2\ln m_1 - 1/2 \ln m_2 \end{cases}. \tag{3.1.7}$$

Now, the parameters of the approximating log-normal distribution are known, and the approximate closed-form solution for the option price within the Black-Scholes framework is

$$C = e^{-rT}\left[m_1 N(d_1) - K_{AQB} N(d_2)\right], \tag{3.1.8}$$

where K_{AQB} is the strike price of the option and d_1 and d_2 are

$$d_1 = \frac{\ln\frac{m_1}{K_{AQB}} + \frac{\sigma^2}{2}}{\sqrt{\sigma^2}}$$

$$d_2 = d_1 - \sqrt{\sigma^2}. \tag{3.1.9}$$

3.2. Vorst's Approximation

Vorst and Kemna [8] suggest that Asian-style options can be priced by replacing the arithmetic average of the payoff function of (2.2.1) with geometric average. If this is done, (2.2.1) can be re-written

$$Payoff = \max(A, 0) = \max\left(\prod_{t=t_1}^{t_n} S(t)^{\frac{1}{n}}, 0\right). \tag{3.2.1}$$

Assuming that the underlying asset S follows the geometric Brownian motion in (2.1.1) the probability density function of A is also log-normal at any time t. This can be seen by taking the logarithm of A:

$$\ln(A) = \ln\left(\prod_{t=t_1}^{t_n} S(t)^{\frac{1}{n}}\right) = \frac{1}{n}\ln\big(S(t_1)\big) + \cdots + \frac{1}{n}\ln\big(S(t_n)\big). \tag{3.2.2}$$

Because the sum of normally distributed random variables is also normal, the sum of the logarithms on the right side of (3.2.2) is normal. Thus, the logarithm of A is normal, which implies that A is log-normally distributed and there exists a closed form solution within the Black-Scholes framework.

Vorst's approximation has been applied also to Asian basket options by modifying it accordingly. According to several sources in the literature [4], [7], [9], however, it does not perform as well as the best approximations for arithmetic Asian-style options. Thus, it will not be considered further here. For options, whose underlying security is the geometric average of a time series, Vorst's method is a natural choice.

3.3. Edgeworth expansion: The Turnbull and Wakeman's Approximation

Edgeworth expansions were first introduced by Jarrow and Rudd in [6] to value options whose underlying security had an arbitrary probability distribution. Turnbull and Wakeman suggest an Edgeworth expansion of fourth order to value Asian options [15]. This method is applied by Datey, Gauthier and Simonato in [2] to value Asian quanto-basket options. According to their research the approximation works well. However, Ju points out [7] that an Edgeworth expansion may not converge and, thus, the approximation may not always reliable results.

Let us assume that underlying asset S has an unknown probability distribution function $P(x)$ and C is the price of a call option on S. Edgeworth expansion is a method where the centered and non-centered moments of the real distribution $P(x)$ and an approximate distribution $P_{app}(x)$ are used to price an option on S.

Basically, an Edgeworth expansion consists of the Black-Scholes option price summed with correction terms.

Let the non-centered (m_k) and centered (l_k) moments of the real distribution be defined as

$$m_k = \int_{-\infty}^{\infty} x^k P(x)dx$$

$$l_k = \int_{-\infty}^{\infty} (x - m_1)^k P(x)dx. \qquad (3.3.1)$$

Turnbull and Wakeman suggest approximating the real probability distribution of an Asian-style underlying with a log-normal distribution $P_{log}(x)$ by using the first four moments of the distributions. Log-normal probability density function is completely defined when the mean and variance parameters μ and σ^2 are known in (3.1.4). The first two non-centered moments of the approximating distribution and the real distribution are thus set equal as in Levy's approximation. Then, the price of a call option can be approximated

$$C = C_1 - e^{-rT}\,\frac{l_3 - l_3^{log}}{3!}\frac{dP_{log}\left(K_{AQB}\right)}{dx} + e^{-rT}\,\frac{l_4 - l_4^{log}}{4!}\frac{d^2P_{log}\left(K_{AQB}\right)}{dx^2}, \qquad (3.3.2)$$

where C_1 is the price given by the Levy approximation

$$C_1 = e^{-rT}\left[m_1 N(d_1) - K_{AQB}N(d_2)\right] \qquad (3.3.3)$$

and d_1 and d_2 are as in (3.1.9).

As can be seen from (3.3.2), moments of third and fourth order are needed to calculate the approximation. The first two moments are given in this paper by equations (3.1.2) and (3.1.3). Datey, Gauthier and Simonato [2] present the third and fourth non-centered moments of the real underlying security A_{AQB}

$$m_3 = E\left[{A_{AQB}}^3\right]$$

$$= \frac{1}{n^3}\sum_{\tau_1,\tau_2,\tau_3=1}^{n}\;\sum_{i_1,i_2,i_3=1}^{m} w_{i_1}w_{i_2}w_{i_3}S_{i_1}(0)\,S_{i_2}(0)S_{i_3}(0)$$

$$* Exp\begin{bmatrix}\left(r_{i_1} - \sigma_{X_{i_1}}\sigma_{i_1}\rho_{i_1,i_1} + \sigma_{i_1}\sigma_{i_2}\Sigma_{i_1,i_2} + \sigma_{i_1}\sigma_{i_3}\Sigma_{i_1,i_3}\right) * \min\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}\right) \\ + \left(r_{i_2} - \sigma_{X_{i_2}}\sigma_{i_2}\rho_{i_2,i_2} + \sigma_{i_2}\sigma_{i_3}\Sigma_{i_2,i_3}\right) * \mathrm{med}\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}\right) \\ + \left(r_{i_3} - \sigma_{X_{i_3}}\sigma_{i_3}\rho_{i_3,i_3}\right) * \max\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}\right)\end{bmatrix} \qquad (3.3.4)$$

$$m_4 = E\left[{A_{AQB}}^4\right]$$

$$= \frac{1}{n^3} \sum_{\tau_1,\tau_2,\tau_3,\tau_4=1}^{n} \sum_{i_1,i_2,i_3,i_4=1}^{m} w_{i_1} w_{i_2} w_{i_3} w_{i_4} S_{i_1}(0)\, S_{i_2}(0) S_{i_3}(0) S_{i_4}(0)$$

$$* Exp\begin{bmatrix} \left(r_{i_1} - \sigma_{X_{i_1}}\sigma_{i_1}\rho_{i_1,i_1} + \sigma_{i_1}\sigma_{i_2}\Sigma_{i_1,i_2} + \sigma_{i_1}\sigma_{i_3}\Sigma_{i_1,i_3} + \sigma_{i_1}\sigma_{i_4}\Sigma_{i_1,i_4}\right) * \min\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\right) \\ + \left(r_{i_2} - \sigma_{X_{i_2}}\sigma_{i_2}\rho_{i_2,i_2} + \sigma_{i_2}\sigma_{i_3}\Sigma_{i_2,i_3} + \sigma_{i_3}\sigma_{i_4}\Sigma_{i_3,i_4}\right) * \min_2\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\right) \\ + \left(r_{i_3} - \sigma_{X_{i_3}}\sigma_{i_3}\rho_{i_3,i_3} + \sigma_{i_3}\sigma_{i_4}\Sigma_{i_3,i_4}\right) * \min_3\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\right) \\ + \left(r_{i_4} - \sigma_{X_{i_4}}\sigma_{i_4}\rho_{i_4,i_4}\right) * \max\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\right) \end{bmatrix}, \qquad (3.3.5)$$

where e.g. $\min_2\left(t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\right)$ means the second smallest of the set of numbers $\{t_{\tau_1}, t_{\tau_2}, t_{\tau_3}, t_{\tau_4}\}$.

3.4. Reciprocal gamma approximation

Milevsky and Posner [13] show that under certain conditions, an infinite sum of correlated log-normally distributed random variables converges asymptotically to an inverse gamma distribution. Datey, Gauthier and Simonato apply this result to price Asian quanto-basket options [2] when the number of underlying assets is limited. Their method uses an Edgeworth expansion where the approximating distribution is the inverse gamma distribution. In this thesis we evaluate the two-moment version of the inverse gamma approximation without the correction terms that Edgeworth expansion gives.

Let $P_\gamma(x)$ be the probability density function of the gamma distribution [2]

$$P_\gamma(x) = \frac{\beta^{-\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)}, x > 0. \qquad (3.4.1)$$

Then, the probability density function of $y = 1/x$ is [2]

$$P_{1/\gamma}(x) = \frac{\beta^{-\alpha} e^{-\frac{1}{y\beta}}}{y^{\alpha+1}\Gamma(\alpha)}, y > 0, x \sim Gamma(\alpha, \beta). \qquad (3.4.2)$$

For an inverse gamma distributed random variable y moments can be calculated using the formula [18]

$$E\left[y^i\right] = \frac{1}{\beta^i(\alpha-1)(\alpha-2)\ldots(\alpha-i)}. \qquad (3.4.3)$$

Matching the first two moments of $P_{1/\gamma}(x)$ to those of the real distribution $P(x)$ both α and β are defined. Then, the price of a call option is

$$C = e^{-rT}\left[m_1 G\left(\frac{1}{K_{AQB}}\middle|\,\alpha-1,\beta\right) - K_{AQB} G\left(\frac{1}{K_{AQB}}\middle|\,\alpha,\beta\right)\right] \qquad (3.4.4)$$

where G is the cumulative gamma distribution function.

3.5. Approximating the real distribution with a Johnson family distribution

Datey, Gauthier and Simonato [2] also suggest approximating the real probability density function $P(x)$ with a function of the Johnson family $P_J(x)$. Let N be a standard normal random variable. If variable X is defined as

$$X = \gamma + \delta \phi^{-1}\left(\frac{N-\alpha}{\beta}\right), \qquad (3.5.1)$$

it is said to have a Johnson distribution where α, β, γ and δ are the parameters of the distribution and ϕ is selected to be of form $\ln(.)$ or of form $\sinh^{-1}(.)$.

As discussed earlier in (2.2.6), (2.2.7) and (2.3.3), if $P(x)$ is the probability density function of the real distribution of A_{AQB}, the price of a call option is [2]

$$C = e^{-rT} \int_{K_{AQB}}^{\infty} P(x)(A_{AQB} - K_{AQB})dx$$

$$= e^{-rT}\left(\int_{-\infty}^{\infty} P(x)(A_{AQB} - K_{AQB})dx - \int_{-\infty}^{K} P(x)(A_{AQB} - K_{AQB})dx\right)$$

$$= e^{-rT}\left(m_1 - K_{AQB} - \int_{-\infty}^{K}\int_{-\infty}^{x} P(y)\,dy\,dx\right). \qquad (3.5.2)$$

Approximating function $P(y)$ with a function of form $P_J(x) = \sinh^{-1}(.)$, Datey, Gauthier and Simonato [2] obtain

$$\int_{-\infty}^{K}\int_{-\infty}^{x} P_J(y)\,dy\,dx = (K_{AQB} - \gamma)N(q) + \frac{\delta}{2} e^{\frac{1}{2\beta^2}}\left[e^{\frac{\alpha}{\beta}} N\left(q + \frac{1}{\beta}\right) - e^{-\frac{\alpha}{\beta}} N\left(q - \frac{1}{\beta}\right)\right]$$

where

$$q = \alpha + \beta \sinh^{-1}\left(\frac{K_{AQB} - \gamma}{\delta}\right). \qquad (3.5.3)$$

4. Implementation and computational results

The details of the test option used in this section are elaborated in Table 1. There are 5 underlying assets that are assumed to be continuous-dividend paying stocks. The strike price is set equal to the option's intrinsic value at the beginning, that is, the intrinsic value of the option at the time when it is issued. The domestic risk-free interest rate is assumed to be constant at 2 % per annum and the maturity of the test option is set to 6 months unless otherwise mentioned. The foreign risk-free interest rates are assumed to be 5 % per annum for each underlying asset in the basket. Also, the weights, initial prices, volatilities, exchange rate volatilities and dividend rates are assumed to be the same for each underlying asset for simplicity. The correlations between the returns of the underlying assets and exchange rates are assumed individual for each underlying asset. Lastly, Table 1 presents the correlation matrix that is to be used in the simulations to capture the correlations between each pair of underlying assets.

Table 1: The parameter set used in simulations

Number of underlying assets	5				
Strike as per cents of the initial basket price	100 %				
Domestic interest rate	0.02				
Maturity	6 months				
	Underlying assets				
	1	2	3	4	5
Foreign interest rate	0.05	0.05	0.05	0.05	0.05
Weights	0.20	0.20	0.20	0.20	0.20
S(0)	100	100	100	100	100
Volatilities	0.20	0.20	0.20	0.20	0.20
FX rate volatilities	0.14	0.14	0.14	0.14	0.14
Dividend	0.03	0.03	0.03	0.03	0.03
Correlation between instruments and their currencies	0.07	0.10	0.10	0.15	0.30
Correlation matrix	1.00	0.10	0.35	0.35	0.15
	0.10	1.00	0.10	0.20	0.35
	0.35	0.10	1.00	0.25	0.06
	0.35	0.20	0.25	1.00	0.00
	0.15	0.35	0.06	0.00	1.00

4.1. Convergence analysis of Monte Carlo

Let us first test the convergence of the selected benchmark pricing method by gradually increasing the number of simulated paths N and calculating the option price each round. The purpose of the analysis is to find a sufficient number of simulated paths. We can estimate the simulation error SEM using the standard deviation of the simulated option price as follows

$$SEM = \frac{STD_{Sample}}{\sqrt{N}}, \tag{4.1.1}$$

in which STD_{Sample} is the standard deviation of the simulated option price and N the number of simulated paths. The convergence analysis is performed using Monte Carlo with antithetic variance reduction. The results are shown in Figure 3 below.

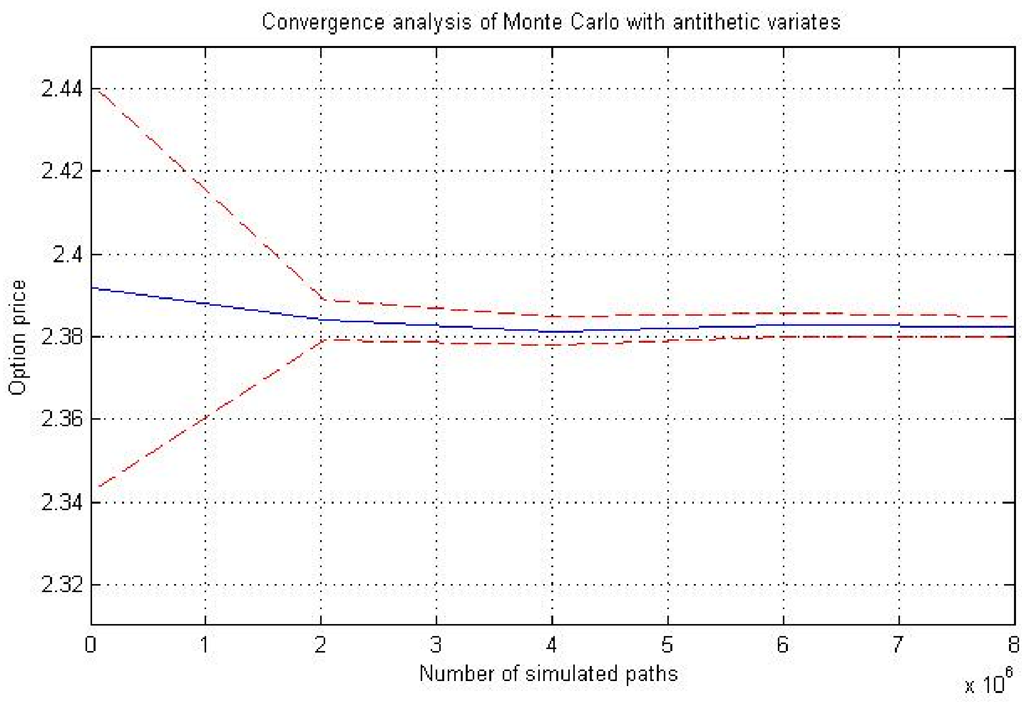

Figure 3: Convergence of simulated option price using normal Monte Carlo method. The red lines are the mean +/- two standard deviations. The simulation was performed with 6 fixing dates.

Reasonable accuracy can be obtained when the number of simulated paths is over 6 million. In the simulations of Figure 3 the number of fixing dates was set to 6. In the following analysis the number of fixings does not exceed 6 except for section 4.2.4 in which the number of fixing dates are varied from 2 to 12. The variance reduction method does not seem to give a clear advantage over ordinary Monte Carlo methods.

4.2. Comparison of accuracy

In this section the accuracy of Levy's approximation and Milevsky and Posner's two-moment approximation are compared in different scenarios. It is required of an analytical approximation that pricing error is not sensitive to parameter values, that is, the approximations produce correct prices and price changes within the parameter space observed in real life. For example, good accuracy with large parameter set is beneficial for the purposes of hedging based on the Greeks.

4.2.1 Varying volatilities of the underlying instruments

In this thesis we employ the Black-Scholes framework where each underlying asset is assumed to have a constant volatility. In practice, if the underlying assets are publicly traded stocks, the constant volatilities can be replaced by volatilities implied by market data. Hedging can then be performed based on the implied volatilities. However, in order for the model to produce correct values of vega, it is essential that the model produces correct prices when volatility is varied within a reasonable range. Figures 5 and 6 show how the two models compare against Monte Carlo when the volatility of each underlying stock is gradually increased. Figures 6 and 7, in turn, show how accurate the two pricing methods are when volatilities are varied asymmetrically.

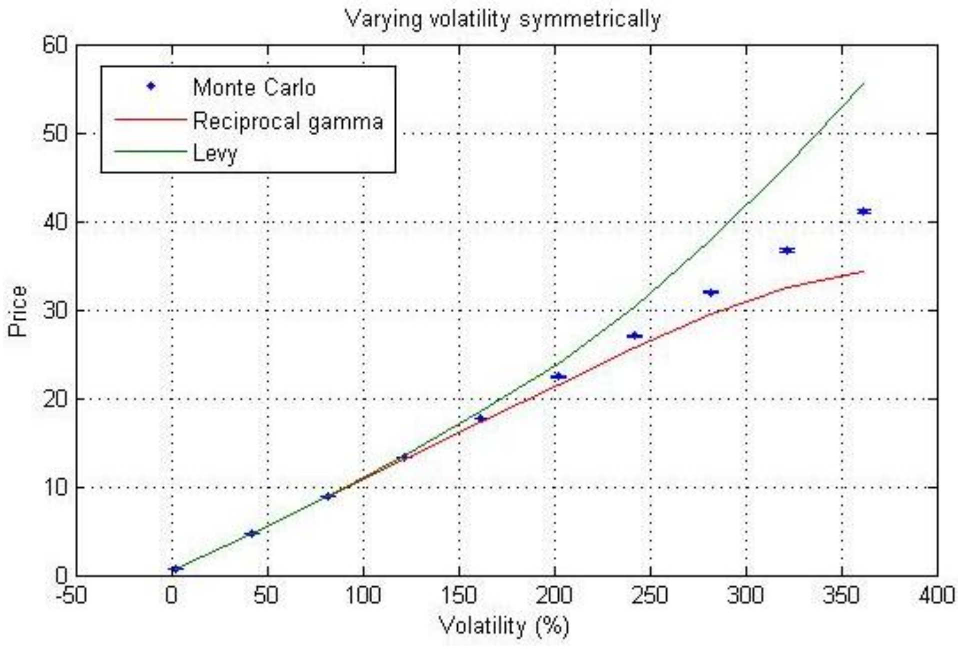

Figure 4: Option price when the volatilities of the underlying assets are gradually increased. The maturity was 6 months and number of simulated paths 6 million.

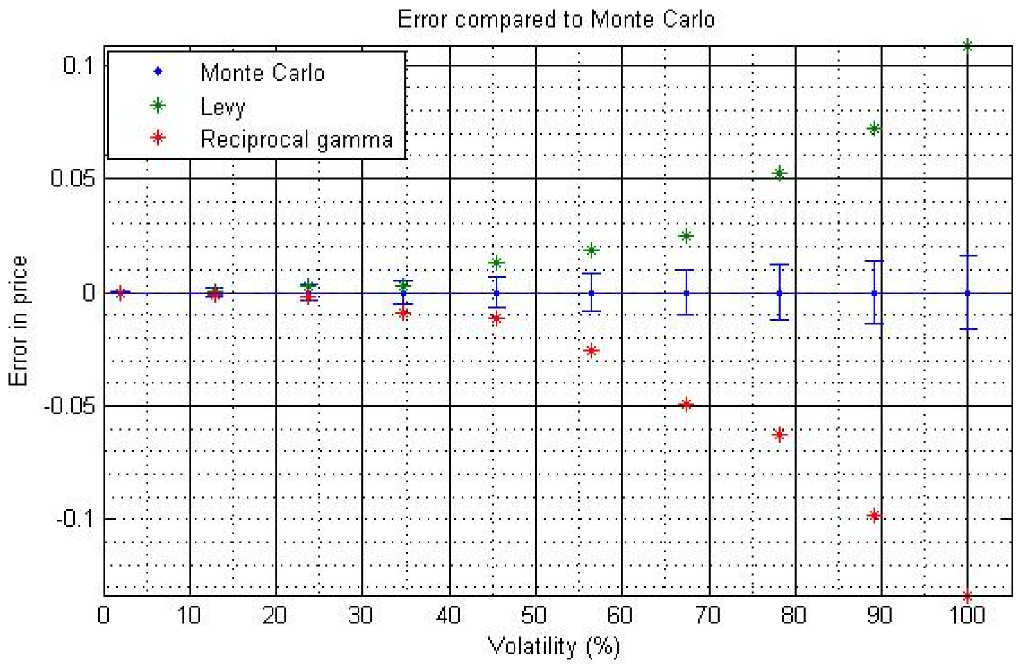

Figure 5: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when volatility was varied symmetrically. The maturity was 6 months and number of simulated paths 6 million.

Judging by the results of Figure 4, when volatility increases, Levy’s approximation starts to diverge up compared to Monte Carlo as opposed to the reciprocal gamma approximation that seems to give too low prices compared to the reference price. Figure 5 shows the absolute errors of the approximations, when volatility is restricted under 100 %. It seems that Levy’s approximation produces errors of less than 0.01 (0.2 %) when volatility does not exceed 45 %. However, when volatility does exceed 78 %, Levy’s error is over +0.05 (0.6 %). Prices obtained using the reciprocal gamma approximation diverge quicker downwards but are still reasonably accurate with small values of volatility. The two methods perform almost equally well when volatility is under 45 %. After this, the error of the reciprocal gamma approximation starts to grow quicker than that of the Levy approximation.

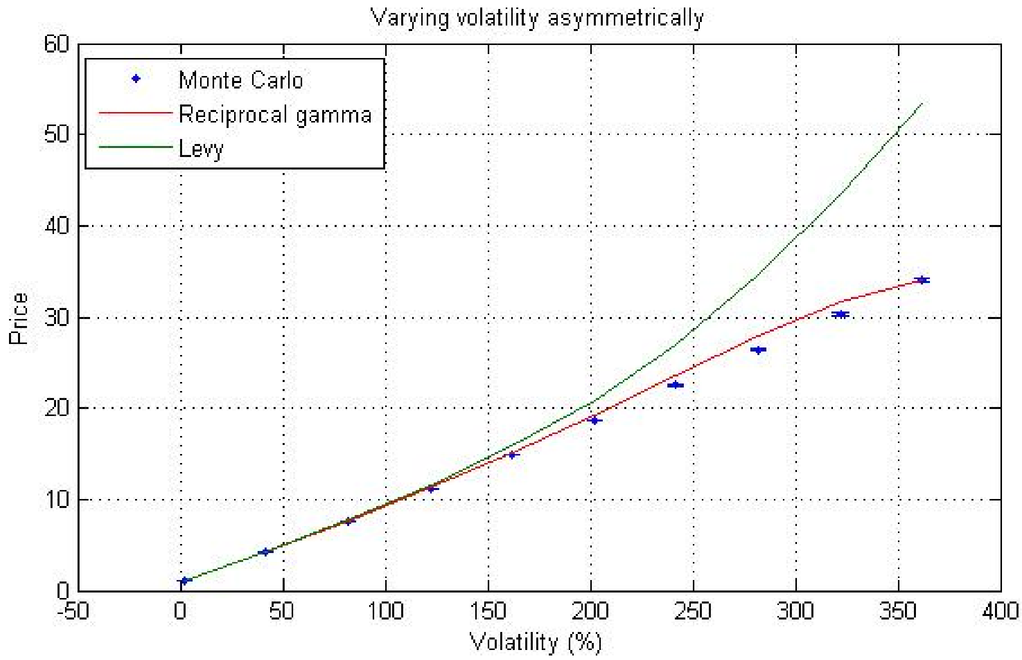

Figure 6: Option price when the volatilities of 4 of the 5 underlying assets are gradually increased. The maturity was 6 months and number of simulated paths 6 million.

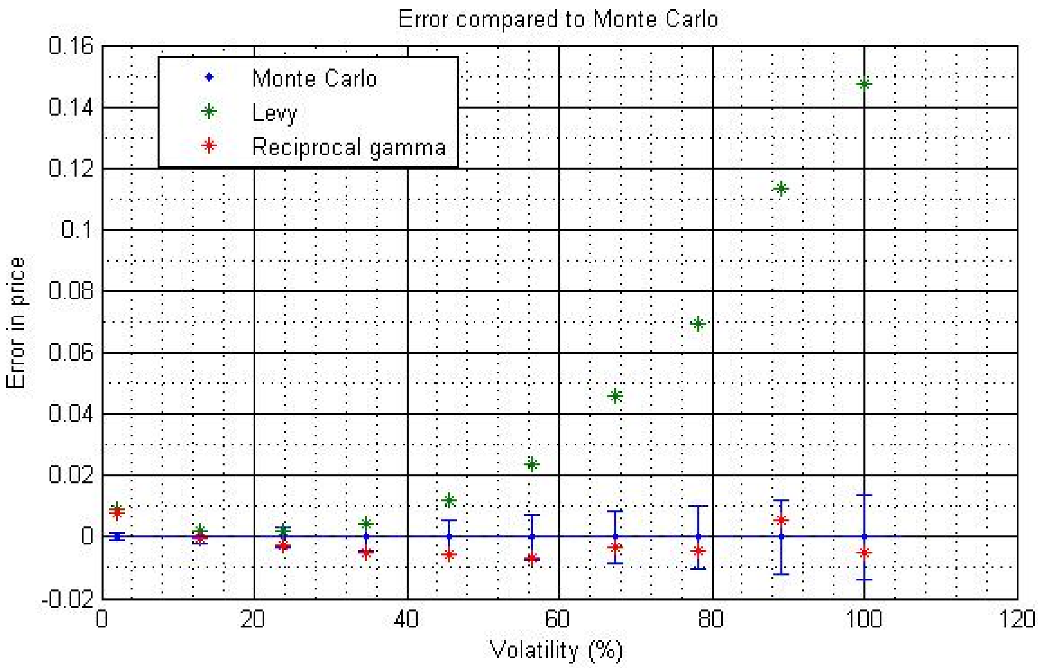

Figure 7: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when volatilities were varied asymmetrically. The maturity was 6 months and number of simulated paths 6 million.

When volatilities are varied asymmetrically so that they are gradually raised for four stocks and kept constant for one stock, the results are as shown in Figures 6 and 7. In this specific scenario the accuracy of the Levy approximation seems to be worse than that of the reciprocal gamma distribution. In fact, Milevsky and Posner’s method turns out to be very accurate so that the absolute pricing error is of size 0.01 which roughly translates to 0.05 %. The Levy approximation error is about the same size as in the symmetric case when volatilities are at most 57 % in

Figure 8. For volatilities bigger than that Levy's error is larger in the asymmetric case.

4.2.2 Varying strike prices

An option can be in-the-money, at-the-money or out-of-the-money depending on what the current price of the option's underlying asset is relative to the strike price. In section 4.2.1 the option that was studied was set at-the-money. In this section focus is put on testing how accurate the two approximate pricing methods are when strike prices are varied.

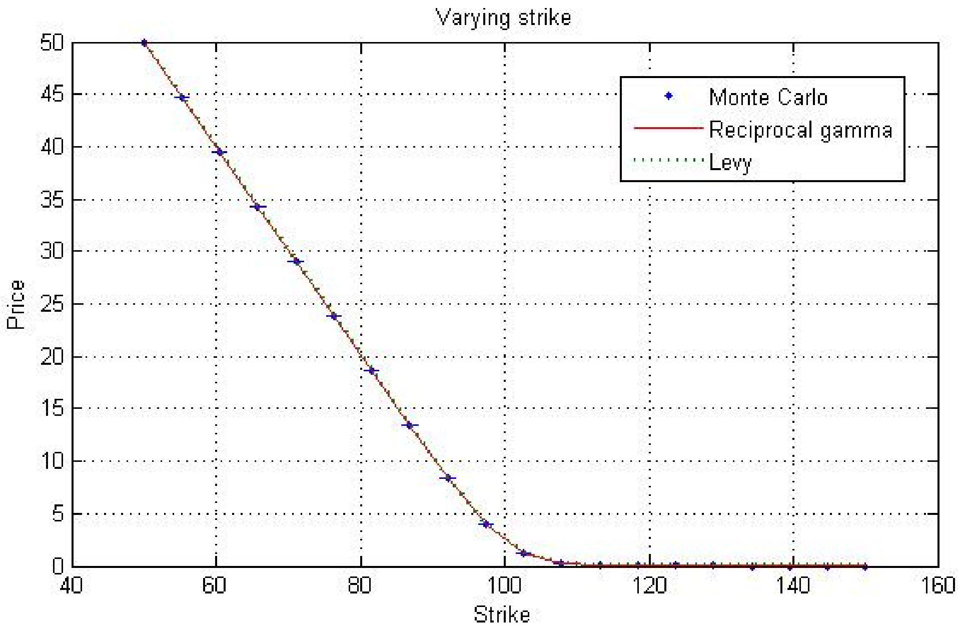

Figure 8: Option price when the strike price is varied. The maturity was set to 6 months, number of fixings once per month and the number of simulated paths to 6 million.

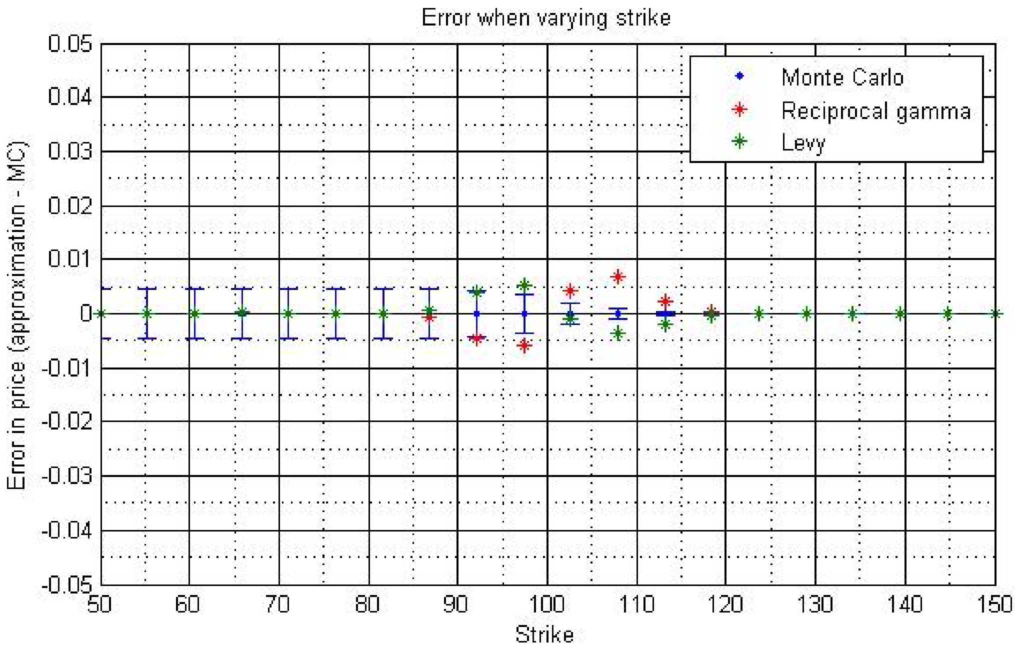

Figure 9: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when strike price was varied. The maturity was 6 months and number of simulated paths 6 million.

The results are shown in Figures 8 and 9. Both methods perform well for all strike prices. Levy's method yields an error whose absolute value is less than 0.005 (0.1 %). The maximum error of the reciprocal gamma method is 0.006 (0.1 %) in terms of absolute value. Funnily, pricing errors are only observed near the at-the-money point. Also, when one method overprices, the other underprices and vice versa.

4.2.3 Varying maturity

In this scenario, we vary the maturity time of the test option while keeping the number of fixing dates constant. The results are shown in Figures 10 and 11. Based on them, varying maturity does have an effect on accuracy. This is intuitive because the greater the maturity the greater is the variance of the probability distribution representing the option payoff. That is, increasing maturity should affect accuracy similarly as increasing volatility.

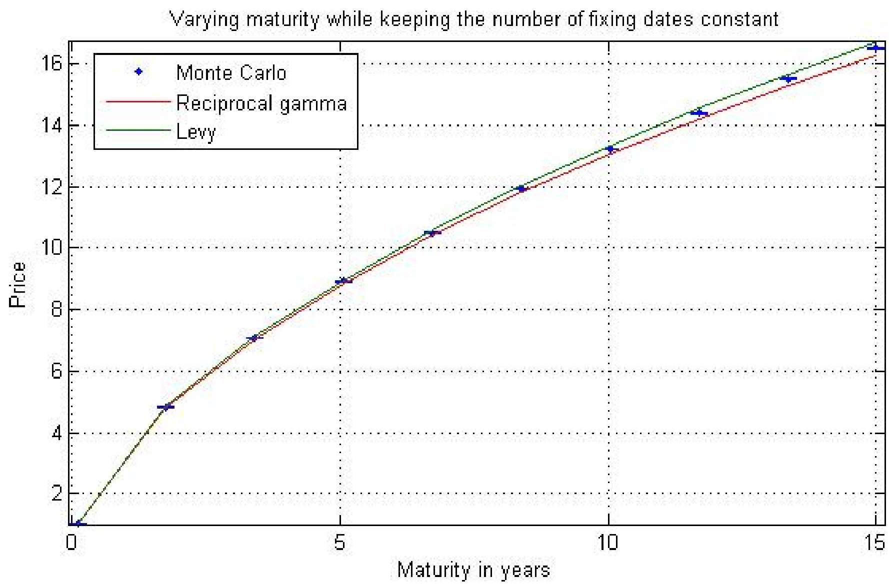

Figure 10: Option price when maturity time was varied. The number of simulated paths was set to 6 million.

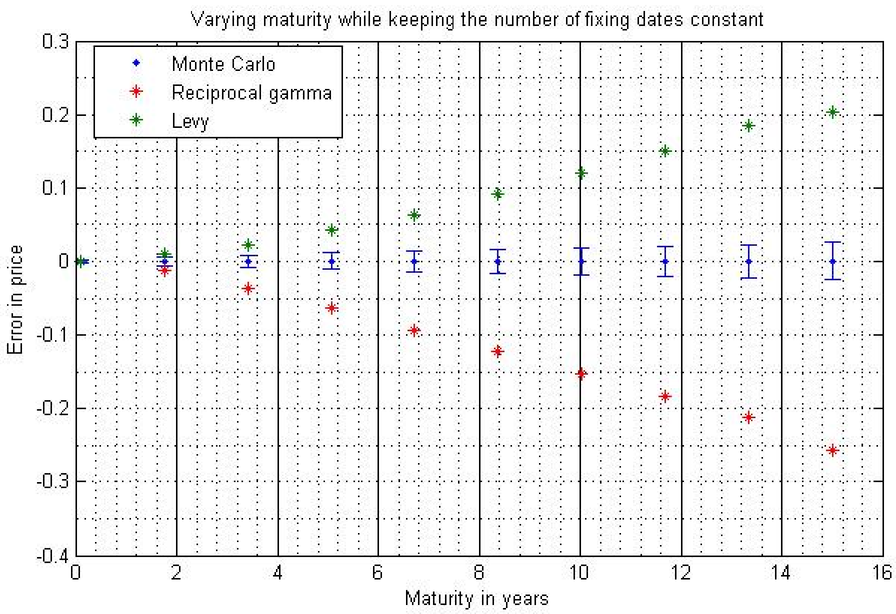

Figure 11: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when maturity time was varied and the number of fixing points was held constant.

As was the case in Figure 5, Levy's approximation yields more accurate results but overprices more and more when maturity increases. The maximum error of Levy's approximation is about +0.2 (1 %) when maturity time is set to 15 years. The maximum error of the reciprocal gamma approximation is likewise attained when maturity is at its highest: -0.26 (-2 %).

4.2.4 Varying the number of fixing dates

Fixing dates are the dates over which the average is taken in Asian options. Generally, when the number of fixing dates increases, an Asian option becomes cheaper. This scenario tests if varying the number of fixings has an effect on pricing error.

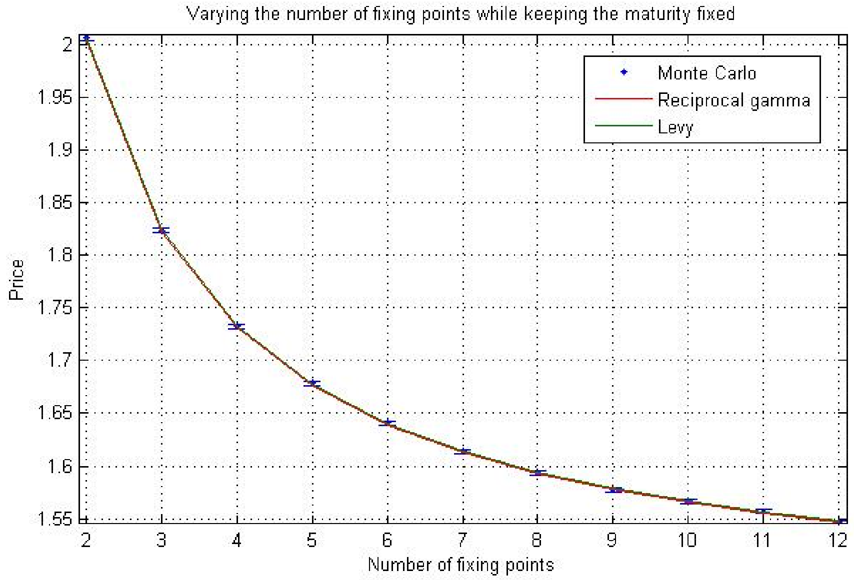

Figure 12: Option price when the number of fixing dates was varied. The maturity was set to 3 months and the number of simulated paths to 6 million.

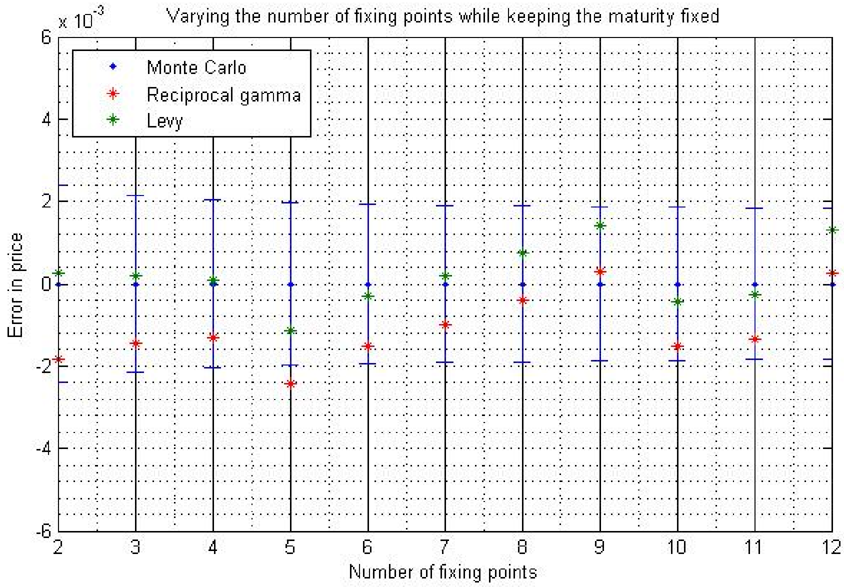

Figure 13: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when maturity time was held constant and the number of fixing points was varied.

As shown in Figure 12, price decreases as the number of fixings increases. There is a limit that the price approaches when the number of fixings approaches infinity, that is, the price of a corresponding option that uses continuous averaging. According to Geman [4] there exist a Laplace transform method that can be used to calculate this asymptotic limit.

In view of Figure 13 both methods yield good results: The approximate prices are within the Monte Carlo 95 % confidence interval apart from one exception. No clear distinction can be done between the two methods in this scenario.

4.2.5 Varying correlations

Correlations between underlying assets have a significant effect on the price of an option whose underlying asset is a basket. There are baskets of high-correlated assets such as distillates of oil and baskets with low-correlated assets such as stocks from different industries. Also, correlations might change quickly according to market movements so it is essential that a model is able to price correctly a basket irrelevant of what the level of correlation is. In this section the effect of varying correlations is studied so that the average correlation is gradually increased. The correlation matrix of Table 1 is replaced with an identity matrix that is then modified by increasing the correlations of each pair of assets at the same pace. The results are shown in Figures 14 and 15.

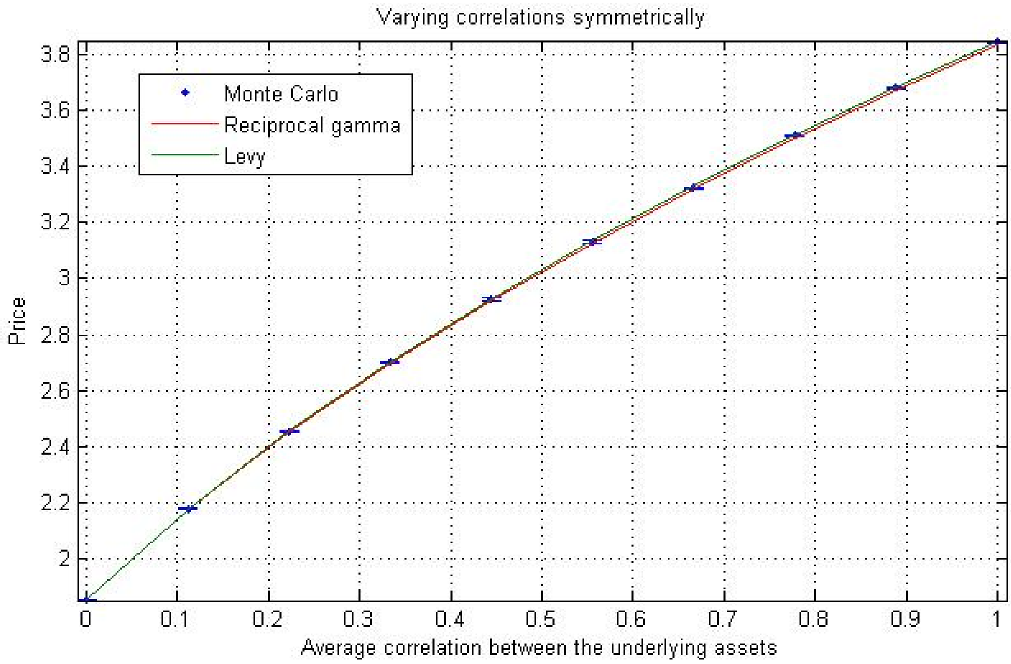

Figure 14: Option price when the correlation between each pair of assets was increased at the same pace. The maturity was set to 6 months and the number of simulated paths to 6 million.

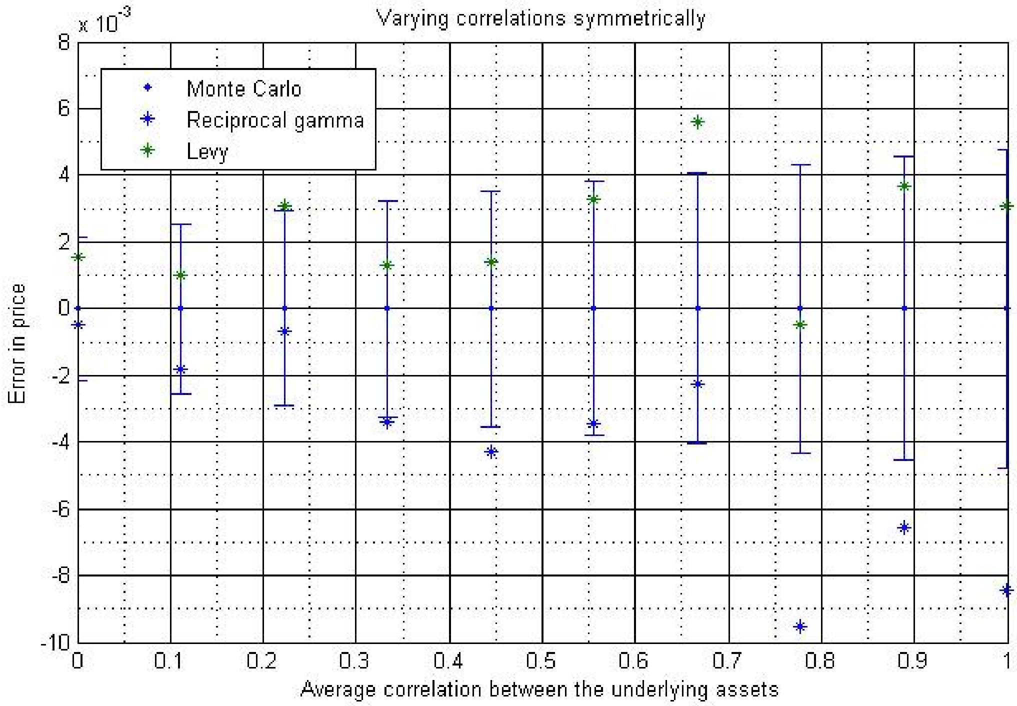

Figure 15: Comparison of the errors of Levy's approximation and the reciprocal gamma approximation when correlations are gradually increased at the same pace.

Both methods seem to perform well. The maximum error of the Levy approximation is now +0.0055 (0.2 %) and the maximum error of the reciprocal gamma method -0.0095 (-0.2 %). As expected, Levy's method overprices slightly on average and the reciprocal gamma method underprices. When correlations are very high, the error of the reciprocal gamma approximation is a little higher than that of the Levy approximation. Otherwise, the test does not show clear distinction between the compared methods. From Figure 15 we can see that there exist a maximum price for the test option and it is achieved when the underlying assets are perfectly correlated.

5. Conclusion

The first objective of this thesis was to review what methods have been suggested in the literature to price Asian, quanto and basket options and their combination Asian quanto-basket option. The second objective was to test their performance against Monte Carlo pricing. The distribution of a finite sum of correlated log-normally distributed random variables is yet unknown [13], which is the reason why no exact analytical pricing methods exist. However, several approximate tools were found. Two moment matching methods were implemented in Matlab and their accuracy was compared in several scenarios. The scenarios were meant to test sensitivity against changes in model parameters.

In the first scenario, volatilities of underlying assets were varied both symmetrically and asymmetrically. In the symmetric scenario the volatility of each of the five underlying assets was increased gradually at the same pace. The asymmetric scenario was otherwise the same but the volatility of the first underlying asset was held constant. In the symmetric case the Levy approximation seemed to yield more accurate pricing. Its maximum error the volatility being at most 100 % was +0.1 which in relative terms translates to 1 %. For all values of volatility, Levy's approximation seemed to give slightly too high option premiums. The Milevsky and Posner's reciprocal gamma approximation seemed to underprice slightly for small values of volatility in the symmetric scenario but when volatilities exceeded 45 %, the underpricing became more pronounced. However, in the asymmetric scenario, the reciprocal gamma approximation turned out to be more accurate. For volatilities smaller than 100 % its error was of size 0.01 in absolute terms which was only 0.1 % of the test option's Monte Carlo price. The error was not negative for all values of volatility but seemed to change its sign as opposed to the symmetric case. Levy's approximation overpriced also in the asymmetric case the maximum error being approximately 0.155 (1 %) when volatility was 100 %.

The second scenario was meant to test if varying strike prices would have an effect on the accuracy of the selected pricing methods. Both approximations performed well and errors were only observed near the at-the-money point. The maximum error of the reciprocal gamma approximation was 0.006 (0.1 %) and Levy's 0.005 (0.1 %) in terms of absolute value.

In the third scenario, the maturity time of the test option was varied. The results were quite similar to the results of the first test: Levy's approximation produced more accurate pricing. Both methods were accurate when maturity was short but the greater the maturity time the bigger was the error. The maximum error of the Levy approximation was +0.2 (1 %) when maturity was set to 15 years.

The fourth scenario tested how varying the number of fixing dates would affect accuracy. Both methods seemed to perform equally well. However, it took a considerable amount of time to run Monte Carlo simulation when the number of observation points exceeded 10.

In the fifth and final scenario the correlations of the underlying assets were increased symmetrically up until 0.999999999. Both methods performed well for weak correlations. For very high values of correlation the Levy's method turned out to give slightly more accurate results.

Based on the tests performed in this paper, the Levy approximation yields slightly more accurate results on average and – with few exceptions – overprices consistently. The only scenario in which Levy's method turned out to be inferior was the asymmetric volatility test that is elaborated in Figure 7. Both approximations are quick to implement on a personal computer and can be used to approximate different underlying distributions without modifying the calculation formulas. Figure 16 summarizes the main shortcomings of Levy's approximation: overpricing when volatility is high or maturity is long.

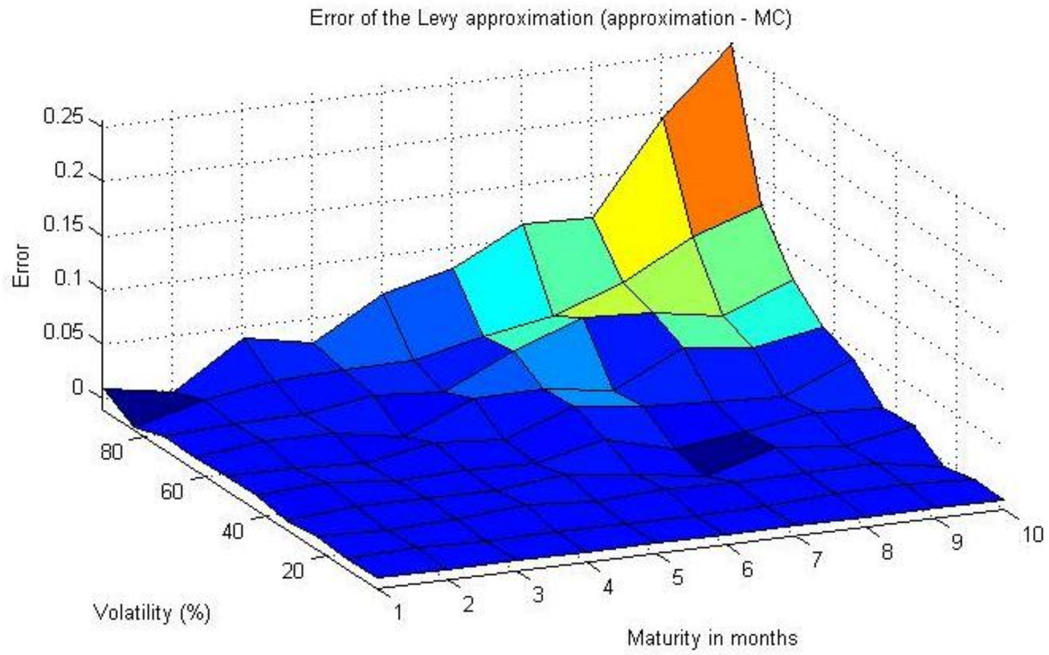

Figure 16: The absolute error of the Levy approximation as a function of volatility and maturity. The bigger is the volatility or longer the maturity, the larger is the error of Levy's approximation.

In the future Ju's approximation could be compared with Levy's in the case of an Asian quanto-basket option. Ju's approximation has been praised in many articles as probably the most accurate analytical method to price arithmetic Asian and basket options. However, implementing the method for an underlying security that is the average value of a basket over some discrete set of points involves challenges such as re-deriving some of the formulas. This paper will not address those challenges.

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