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GARCH MODELS FOR FOREIGN EXCHANGE RATES

Bachelor's thesis

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Aalto University School of Science	ABSTRACT OF THE BACHELOR'S THESIS
Author: Janne Kunnas	
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Abstract:

This thesis specified the structure of a generalized autoregressive conditional heteroskedasticity model. The GARCH model is widely used in forecasting time-varying volatility and volatility clustering in finance and economics. It has been modified and extended by many authors and one of its extensions is the exponential GARCH model, which responds asymmetrically to positive and negative excess returns. This thesis also presents the EGARCH and a modification to the error distribution of the GARCH model.

The empirical part of this thesis begins with pre-estimation diagnostic tests for USD/EUR exchange rates. By using auto- and partialautocorrelation functions, Ljung-Box-Pierce Q-test and Engle's ARCH-test it is shown that the data is heteroscedastic. Then GARCH(1,1) model is fitted and the validation of it is tested using LBP Q-test and Engle's ARCH test. At the end of this thesis the drawbacks of GARCH models are presented together with comments about fitting GARCH model to exchange rates.

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Keywords: volatility, heteroscedasticity, GARCH, EGARCH, autocorrelation function, partial autocorrelation function, regression model, log-likelihood function, Ljung-Box-Pierce Q-test, ARCH-test

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<tr><td colspan="2">Tiivistelmä:
Tässä työssä esitellään yleistetyn autoregressiivisen ehdollisen heteroskedastisen mallin määritelmä ja rakenne. GARCH-mallia käytetään paljon rahoitus- ja kansataloustieteessä ennustamaan ajassa muuttuvaa volatiliteettia sekä volatiliteettirykelmiä. Sitä on laajennettu ja muokattu monen tutkijan toimesta eri tavoin, mutta yksi tunnetuimmista laajennuksista on eksponentiaalinen GARCH-malli, joka reagoi eri suuruudella positiivisiin ja negatiivisiin epätavallisen suuriin voittoihin. Tässä työssä myös esitellään EGARCH-malli ja yksi tapa muokata GARCH-mallia käyttämällä erilaisia jakaumia virhetermeille.
Aluksi työn empiirisessä osassa tarkastellaan USD/EUR valuuttakurssien autokorrelaatio- ja osittaisautokorrelaatiofunktioita sekä esitetään Ljung-Box-Pierce Q-testin ja ARCH-testin tulokset, joiden perusteella datan todetaan olevan heteroskedastista. Tämän jälkeen sovitetaan dataan GARCH(1,1) malli, jonka sopivuutta tarkastellaan Ljung-Box-Pierce Q-testin ja ARCH-testin tulosten avulla. Lopuksi esitetään kommentteja mallin soveltuvuudesta USD/EUR valuuttakursseille ja mallin yleisesti tunnetuista epäkohdista.</td></tr>
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1 Introduction

The autoregressive conditional heteroskedasticity model, or the ARCH model, was introduced by Robert F. Engle in 1982. He assessed the validity of a conjecture of Milton Friedman (Friedman 1977). Friedman's hypothesis was that the uncertainty about future prices and costs prevented entrepreneurs from investing and leads to the economical downturn and a recession. In applied econometrics future variations were forecasted using a least squares model. The problem with the ARCH model is that it assumes that the expected value of all squared error terms is the same. In econometrics changing uncertainty is called heteroskedasticity. The ARCH model solves heteroscedasticity problem treating it as a variance to be modelled (Engle 2004). It forecasts future variance by taking weighted averages of past squared forecast errors.

The generalized autoregressive conditional heteroskedasticity model, GARCH, was presented by Tim Bollerslev (Bollerslev 1986). He generalized the ARCH model to an autoregressive moving average. The past squared residuals are weighted assuming that their importance declines geometrically respect to time and an estimate to the rate of decline is computed from data.

GARCH models are used to characterize and model observed time series. They are commonly employed in modelling financial time series. Simple ARCH models with conditionally normal errors have been found inadequate in capturing all the excess kurtosis for stock returns and exchange rates. Tim Bollerslev used GARCH to model short-run exchange rate movements (Bollerslev 1992). Modelling and forecasting time-varying variance in exchange rate returns have important implications for financial decision-making including the pricing of derivatives and portfolio risk management.

The main objective of this thesis is to present generalized autoregressive conditional heteroscedasticity and to provide an example of its applications. Chapter 2 presents structure and specification of GARCH model in general together with regression model and log-likelihood function. Additionally this chapter gives a brief description of one of the many extension of GARCH and few alternative error distributions for error terms. In chapter 3 an empirical example is provided beginning with validating exchange rate data, continuing with model estimation and ending in model validation.

2 GARCH specification and structure

2.1 Volatility

Volatility is a statistical measure the dispersion of a quantifiable phenomenon. It is commonly defined by standard deviation σ of continuously compounded returns of an instrument. Continuously compounded logarithmic return during a day i is defined as

$$u_i = ln\left(\frac{S_i}{S_{i-1}}\right), \tag{1}$$

where S_i is the value of the market variable. When we use the most recent m observations on u_i we can write maximum likelihood estimate of variance σ_n^2

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m}(u_{i-1} - \bar{u})^2, \tag{2}$$

where $\bar{u}$ is the mean of the m observations. For an unbiased estimate of the variance σ_n^2, m is replaced by m-1 (Hull 2005).

2.2 Generalized autoregressive conditional heteroscedasticity

Let us denote a real-valued discrete-time stochastic process by ε_t and the information set by ψ_t. The information set has all information through time t. Bollerslev defined GARCH(p,q) (Bollerslev 1986) process as follows

$$\varepsilon_t | \psi_{t-1} \sim N(0, h_t), \tag{3}$$

$$\begin{aligned} h_t &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-1} \\ &= \alpha_0 + A(L)\varepsilon_t^2 + B(L)h_t, \end{aligned} \tag{4}$$

where

$$\begin{gathered} p \geq 0, \quad q > 0 \\ \alpha_0 > 0, \quad \alpha_i \geq 0, \quad i = 1, \dots, q, \\ \beta_i \geq 0, \quad i = 1, \dots, p. \end{gathered} \tag{5}$$

In equation (4), L is a time-series lag-operator and it produces *k*th previous element:

$$A(L)\varepsilon_t^2 = \left(\alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^q\right)\varepsilon_t^2 \tag{6}$$

$$B(L)h_t = \left(\beta_1 L + \beta_2 L^2 + \ldots + \beta_p L^p\right)h_t \tag{7}$$

GARCH consist of three different weighted variance forecasts, long-run average α_0 for constant variance, variance forecast made in the previous period for current period and the new information in this period. We assume that ε_t is normally distributed, but other distributions can also be applied. When we set $p = 0$ we have an ARCH(q) process.

The ARCH process takes into consideration differences between conditional and unconditional variances. The conditional variance changes over time as a function of past errors but unconditional variance remains constant. In the GARCH process lagged conditional variances are also included. This attribute makes the GARCH model some sort of adaptive learning mechanism (Bollerslev, 1986) and it can thought of as Bayesian updating. The GARCH process, as defined in (3) – (4), is wide-sense stationary if $E(\varepsilon_t) = 0, var(\varepsilon_t) = \alpha_0(1 - A(1) - B(1))^{-1}$ and $cov(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$ and if and only if $A(1) - B(1) < 1$. For proof see Bollerslev (1986).

The GARCH(p,q) process can be expressed as an infinite ARCH process. From (4) we get

$$
\begin{gathered}
[1-B(L)]h_t = \alpha_0 + A(L)\varepsilon_t^2 \\
h_t = \frac{\alpha_0}{1-B(L)} + \frac{A(L)}{1-B(L)}\varepsilon_t^2 \\
h_t = \alpha_0^* + \sum_{i=1}^{\infty} \alpha_i^* \varepsilon_{t-i}^2, \\
\text{and } 1 - B(z) \neq 0.
\end{gathered} \tag{8}
$$

An alternative parameterization by Pantula (1986) for GARCH(p,q) is

$$
\varepsilon_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \varepsilon_{t-j}^2 - \sum_{j=1}^{p} \beta_j v_{t-j} + v_t, \tag{9}
$$

with

$$
v_t = \varepsilon_t^2 - h_t = (\eta_t^2 - 1)h_t, \tag{10}
$$

where η_t is identically, independently and normally distributed random variable with mean zero.

The parameterization (9) – (10) is more meaningful from a theoretical point of view whereas (3) – (4) is more suitable for practical purposes (Bollerslev 1986).

2.2.1 GARCH(1,1)

GARCH(1,1) is one of the most commonly employed models describing volatility dynamics of financial return securities. This is the simplest model and it has only one lag. Variance for GARCH(1,1) is

$$
h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}, \tag{11}
$$

and it satisfies wide-sense stationary if $\alpha_1 + \beta_1 < 1$ (Bollerslev 1986).

Usually this model is set to predict one period ahead but longer forecasts can also be made. GARCH models are mean reverting, meaning that the longer the forecast is the more closer it comes to the long-run average variance. The parameters α_1 and β_1 determine how quickly the variance changes with respect to new information and how quickly the variance estimate reverts to long-run mean (Engle 2004).

The distribution given in the equation (3) for ε_t is conditionally normal. Let us examine the unconditional distribution of the GARCH model. The unconditional variance of GARCH(1,1) is

$$
\begin{aligned}
\mathbb{E}[\varepsilon_t^2] &= \mathbb{E}[\mathbb{E}[\varepsilon_t^2 | \psi_{t-1}]\,] \\
&= \alpha_0 + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2] + \beta_1[\mathbb{E}[\varepsilon_{t-1}^2 | \psi_{t-2}\,]] \\
&= \alpha_0(1 - \alpha_1 - \beta_1)^{-1}.
\end{aligned} \tag{12}
$$

The fourth-order moment under the assumption of normally distributed ε_t is

$$\mathbb{E}[\varepsilon_t^4] = \frac{3\alpha_0^2(1+\alpha_1+\beta_1)}{(1-\alpha_1-\beta_1)(1-\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2)}, \tag{13}$$

which exists only if $\beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2 < 1$. Combining (12) – (13) we can write the coefficient of kurtosis of the GARCH(1,1)

$$\kappa = \frac{\mathbb{E}[\varepsilon_t^4]}{(\mathbb{E}[\varepsilon_t^2])^2} = \frac{3\alpha_0^2(1+\alpha_1+\beta_1)(1-\alpha_1-\beta_1)}{(1-\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2)}. \tag{14}$$

The GARCH(1,1) process shares a property of leptocurticity with ARCH(q) process. It means that there is a concentration of probability mass around the zero mean and also heavy tails. The third moment is zero because normal distribution is symmetric.

2.2.2 Autocorrelation and partial autocorrelations

Autocorrelation and partial autocorrelation functions are useful in terms of examining time series behaviour. These methods were well established by Box and Jenkins 1976. Autocorrelation and partial autocorrelation functions measures magnitude of linear dependence of two random variables generated by stationary process. For the squared error term ε_t^2, the covariance is

$$\begin{aligned} \gamma_n &= cov(\varepsilon_t^2, \varepsilon_{t-n}^2) \\ &= \mathbb{E}[(\varepsilon_t^2 - \mathbb{E}(\varepsilon_t^2)(\varepsilon_{t-n}^2 - \mathbb{E}(\varepsilon_{t-n}^2)] \\ &= \mathbb{E}[(\varepsilon_t^2 - \mu)(\varepsilon_{t-n}^2 - \mu)] \end{aligned} \tag{15}$$

Autocorrelation function is a series of autocorrelations γ_n. For GARCH(p,q) process we have covariance function (Bollerslev1986)

$$\gamma_n = \sum_{i=1}^{q} \alpha_i \gamma_{n-1} + \sum_{i=1}^{p} \beta_i \gamma_{n-1} = \sum_{i=1}^{m} \varphi_i \gamma_{n-1}, \quad n \geq p+1, \tag{16}$$

where $m = max(p,q)$, and $\varphi_i = \alpha_i + \beta_i, i = 1, \dots, p$,

and, additionally, $\alpha_i = 0$ when $i > q$and $\beta_i = 0$ when $i > p$. Thus we can write the following analogue to the Yule-Walker equations, for autocorrelations coefficient we have now

$$\rho_n = \frac{\gamma_n}{\gamma_0} = \sum_{i=1}^{m} \varphi_i \rho_{n-1}, \; n \geq p+1. \tag{17}$$

From equation (17) we see that the first p autocorrelations for process ε_t^2 depend directly on $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_p$ through $\varphi_1, \dots, \varphi_m$, and higher lags are determined uniquely by $\rho_p, \cdots, \rho_{p+1-m}$.

Partial autocorrelation function for is given by following equation

$$\rho_n = \sum_{i=1}^{k} \varphi_{ki} \rho_{n-1}, \; n = 1, \dots, k \tag{18}$$

Generally the partial autocorrelation function for ε_t^2 described above in non-zero but dies out. This behavior is identical to the AR(q) process (Granger and Newbold 1977).

2.2.3 Regression model and log-likelihood function

In order to estimate parameters for the GARCH(p,q) in (3) - (4), we rewrite the model (Bollerlev 1986)

$$\begin{gathered} \varepsilon_t = y_t - x_t{}' b, \\ \varepsilon_t | \psi_{t-1} \sim N(0, h_t), \\ h_t = z_t' \omega, \end{gathered} \tag{19}$$

where $z_t' = (1, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-q}^2, h_{t-1}, \dots, h_{t-p})$, $\omega' = (\alpha_0, \alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_p)$ and $\theta \in \Theta, \theta = (b', \omega')$, where Θ is a compact subspace of Euclidean space such that ε_t possesses finite second moments. In the economics literature, forecast errors ε_t are called innovations.

Maximization of the log-likelihood function is often used in estimating θ, under the assumption of conditional normality (3). Let us denote the log-likelihood function for a sample of T observations with

$$l_t(\varepsilon_t, \theta) = \sum_{t=1}^{T} \left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(h_t) - \frac{1}{2}\frac{\varepsilon_t^2}{h_t} \right]. \tag{20}$$

The parameter θ cannot be solved analytically, it requires iterative optimization routines.

2.3 Exponential generalized autoregressive conditional heteroscedasticity

The GARCH model has several limitations due to its simple structure. It assumes that only the magnitude and not the positivity or negativity of unanticipated excess returns determine feature h_t. Researchers, beginning with (Black 1976), have found evidence that stock returns, for example, are negatively correlated with changes in returns volatility. In response to good news about the economy, volatility tends to decrease and in response to bad news it tends to increase. Also GARCH models essentially specify the behaviour of the square of the data. In this case a few large observations can dominate the sample. The GARCH models are not able to explain the observed covariance between ε_t^2 and ε_{t-j}. To do this conditional variance has to expressed as an asymmetric function of ε_{t-j}.

The exponential GARCH model was introduced by Nelson (1991) to correct the problems associated with linear GARCH. The EGARCH provided the fist explanation for the h_t depending on both the magnitude and the sign of lagged residuals. The result was asymmetric model defined as follows

$$ln(h_t) = \alpha_0 + \sum_{i=1}^{p} \beta_i ln(h_{t-1}) + \sum_{i=1}^{q} \alpha_i [\varphi z_{t-i} + \gamma(|z_{t-i}| - \mathbb{E}|z_{t-i}|)], \quad (21)$$

where $\beta_1 = 1$, $z_t = \frac{\varepsilon_t}{\sqrt{h_t}}$, $\mathbb{E}|z_{t-i}| = \sqrt{\frac{2}{\pi}}$ when $z_t \sim N(0,1)$, $\alpha_i, \beta_i, \varphi, \gamma$ are coefficients and in exception to the GARCH parameters α_i, β_i do not have nonnegative constraints. The component $\gamma(|z_{t-i}| - \mathbb{E}|z_{t-i}|)$ represents the magnitude effect. If $\gamma > 0$ and $\varphi = 0$, the innovation ε_t in $ln(h_{t+1})$ is positive (negative) when the magnitude of z_t in larger (smaller) than its expected value. If $\gamma = 0$ and $\varphi < 0$, the innovation ε_t in conditional variance is positive (negative) when returns innovations are negative (positive). (Nelson 1992)

The advantage of EGARCH is that conditional variances are always positive. But due to exponential structure of EGARCH it may tend to overestimate the impact of outliers on volatility. (Engle and Ng 1993)

2.4 Other error distributions

One of the common modifications is to use other than normal distribution for error terms ε_t. The reason for this is to better account for the deviations from normality in the conditional distributions of returns in financial markets. The usage of Student's t-distribution (Bollerslev 1987) and General Error Distribution (Nelson 1991) among other distributions has been widely studied by many researchers. The GED distribution family includes normal distribution as a special case and many other distributions, some of which are fatter tail or thinner tail than normal distribution. In the 2001 the Normal Inverse Distribution was introduced by Jensen and Lunde (2001) who showed with daily stock market data that not only NIG distributed error terms fit better at the tails but also at the centre of the distribution.

3 Application to foreign exchange rates

The autoregressive conditional heteroscedasticity models can be applied to any time series and they are relevant when the stochastic process that is not white noise. Financial time series usually exhibit varying variance or volatility clustering. In this chapter GARCH(1,1) model is employed to USD/EUR exchange rates. The literature covers quite well GARCH fitting into stock market returns and some exchange rates, but because EUR is relatively new currency it has not been much used.

We use a sample of 3050 daily observations of USD/EUR exchange rates covering the period 1 January 1999 to 29 November 2010. The exchange rates and logarithmic returns are presented in Figure 1. Plots indicate that returns might not be uncorrelated. The returns exhibit higher and lower volatility periods. Between 2000 and 2002, the volatility is higher, between 2006 and 2008 lower, and during the year 2009 higher than on average. This phenomenon is called volatility clustering as noted by (Mandelbrot 1963), and it is very common for speculative returns.

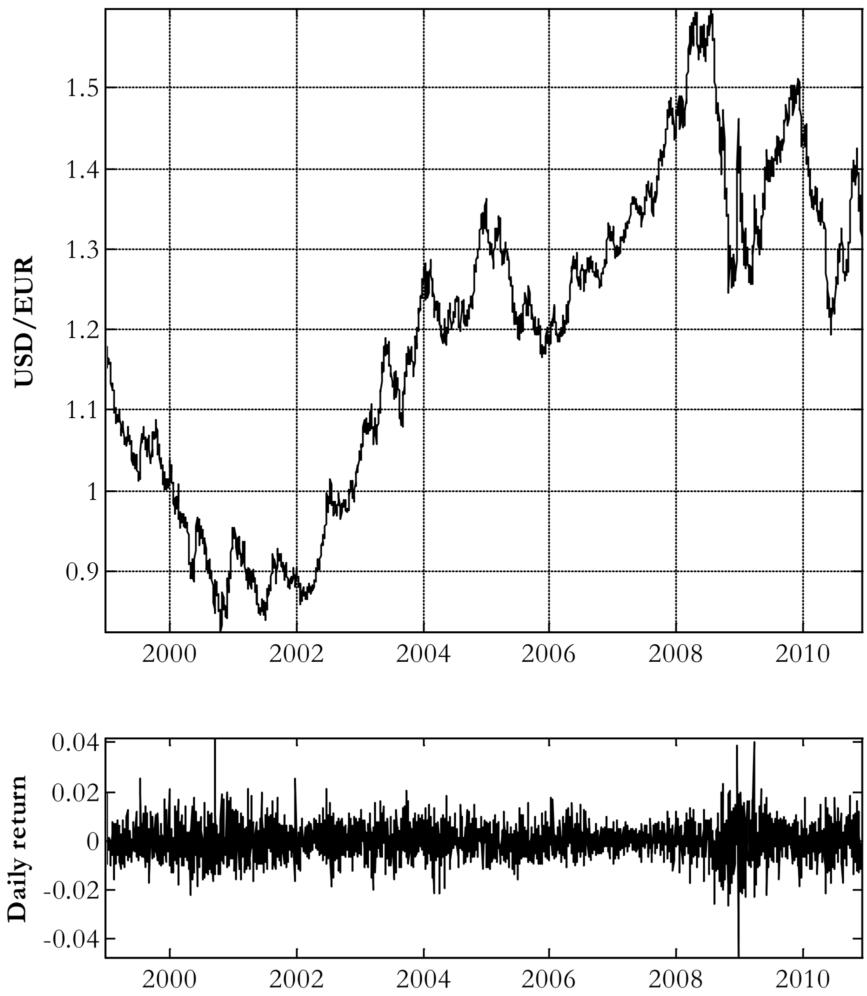

Figure 1: Daily exchange rates and returns of USD/EUR.

Returns of USD/EUR exhibit the following statistics: mean $\mu = 0.0000357$, standard deviation $\sigma = 0.0067$, skewness $s = 0.1095$ and kurtosis $k = 5.6791$. The normal probability distribution has kurtosis of 3. Therefore our USD/EUR return distribution has so called excess kurtosis of $5.6791 - 3 = 2.6791$. This means that the return distribution exhibits excess mass around mean and fatter tails compared to normal distribution.

The GARCH model provided an adequate description of second-order dynamics for most exchange rates. But the assumption of normally distributed residuals does not capture the excess kurtosis of daily return distribution (Wang et al. 2001). Also the problem associated with GARCH is that it does not capture the asymmetric second moment, or so-called leverage effect (Black 1976), which means that negative shocks often increase volatility to a greater extent than positive shocks.

3.1 Pre-estimation diagnostics

We now calculate autocorrelation function (ACF) and partial-autocorrelation (PACF) for returns and squared returns. Often the returns of a financial instrument show no correlation but squared returns do (Box et al. 1994). ACFs and PACFs are presented in Figure 2 with the upper and the lower standard deviation confidence bounds assuming that all autocorrelations are zero beyond lag zero. Not much can be said based on ACF and PACF of returns but in the case of the squared returns, the ACF indicates that variance process exhibits autocorrelation. The ACF of the squared returns dies out very slowly. This might indicate that the variance process is not stationary.

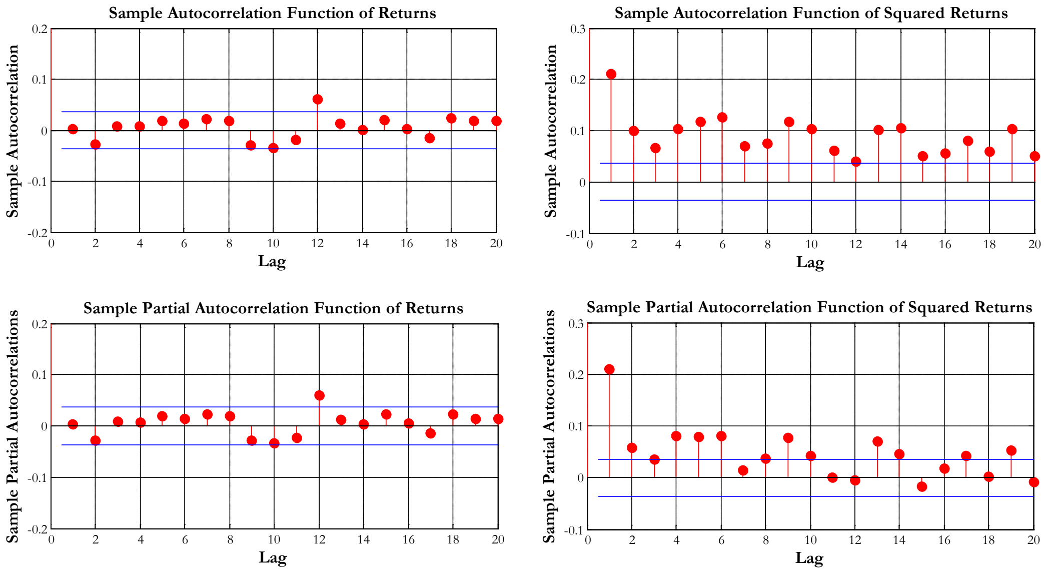

Figure 2: ACFs and PACFs of returns and squared returns of USD/EUR.

To verify whether there is correlation or not we employ Ljung-Box-Pierce Q-test under the null hypothesis of no serial correlation (Box et al. 1994). The LBP-test statistic is calculated as

$$Q_{LBP} = T(T+2)\sum_{k=1}^{s}\frac{r_k^2}{T-k}, \qquad (22)$$

where T is the number of observations, *s* is number of coefficients to test autocorrelation and r_k the autocorrelation coefficient (for lag *k*). The null hypothesis is that none of the autocorrelation coefficients up to lag *s* are statistically different from zero at the specified significance level.

The test results are presented in Tables 1 and 2. The test is performed using lags of the ACF up to 10, 15 and 20 with 0.05 level of significance.

Table 1: Ljung-Box-Pierce Q-test results for daily returns of USD/EUR exchange rates.

Lags	H_0	p-Value	Statistic	Critical Value
10	0	0.23	12.86	18.307
15	1	0.0254	27.4316	24.9958
20	1	0.041	32.2242	31.4104

Table 2: Ljung-Box-Pierce Q-test results for squared daily returns of USD/EUR exchange rates.

Lags	H_0	p-Value	Statistic	Critical Value
10	1	0.00	409.8744	18.307
15	1	0.00	499.8081	24.9958
20	1	0.00	581.1894	31.4104

The null hypothesis holds only for LBP Q-test for daily returns with lags up to 10 and there is significant serial correlation in the squared daily returns.

In addition we perform Engle's ARCH test which tests the conditional heteroscedasticity of residuals. The null hypothesis of ARCH test is that the time series follows Gaussian distribution. The results in the table 3 show clear evidence that residuals are heteroscedastic. The critical values of Engle's ARCH and Ljung-Box-Pierce Q-test results are the same. Both test statistics are Chi-Square distributed.

Table 3: Engle's ARCH test results for daily returns of USD/EUR exchange rates.

Lags	H_0	p-Value	Statistic	Critical Value
10	1	0.00	232.4917	18.3070
15	1	0.00	252.8194	24.9958
20	1	0.00	266.9501	31.4104

3.2 Model estimation and validation

After quantifying the serial correlation of our daily return of USD/EUR exchange rates we begin to estimate GACH models. We first estimate the GARCH(1,1) model using Matlab. The function garchfit produces estimates for our regression model in (19). The estimates and the statistic results for GARCH(1,1) are given in Table 4. Substituting these to the equation we get

$$\begin{gathered} \hat{y}_t = 0.00016 + \varepsilon_t, \\ h_t = 2{,}00 * 10^{-7} + 0.031879 * \varepsilon_{t-1}^2 + 0.96396 h_{t-1}, \\ \varepsilon_t | \psi_{t-1} \sim N(0, h_t). \end{gathered} \tag{23}$$

The sum $\hat{\alpha}_1 + \hat{\beta}_1 = 0.9958 < 1$, the model is stationary.

Table 4: Estimates and Statistic results for Gaussian GARCH(1,1).

Parameter	Value	Standard Error	T Statistic
α_0	2,00E-07	7,34E-08	2,7240
α_1	0,031879	0,0041538	7,6747
β_1	0,96396	0,0047467	203,0785

The value of the estimate α_0 is quite small but the long-run variance will contribute significantly and eventually dominate as the length for forecasting periods grows. Let us now check standardized innovations (Figure 3), the innovations are divided by their conditional standard deviation.

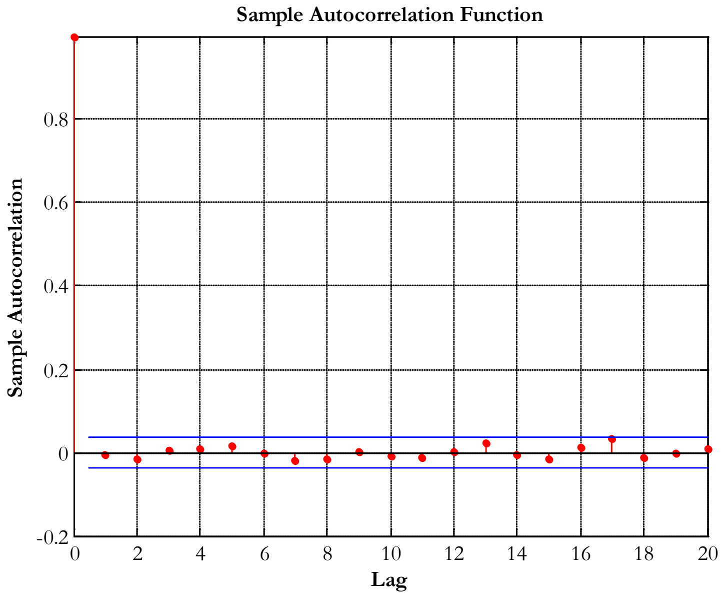

Figure 3: ACF of the Squared Standardized Innovations of GARCH(1,1).

The squared standardized innovations do not show correlation, and neither do standardized innovations. LBP- and Engle's ARCH –test statistics for correlation are show in Tables 5 and 6.

Table 5: Ljung-Box-Pierce Q-test results for squared standardized GARCH(1,1) innovations.

Lags	H0	p-Value	Statistic	Critical Value
10	0	0.96	3.81	18.307
15	0	0.9693	6.5351	24.9958
20	0	0.9402	11.2226	31.4104

Table 6: Engle's ARCH test results for standardized GARCH(1,1) innovations.

Lags	H0	p-Value	Statistic	Critical Value
10	0	0.95	3.86	18.307
15	0	0.9705	6.4816	24.9958
20	0	0.9402	11.2236	31.4104

Both LBP- and Engle's ARCH test results shows no evidence of correlation. Tests also shows that the null hypothesis holds confirming that the GARCH(1,1) model sufficiently explains the heteroscedasticity in the raw USD/EUR returns. The kurtosis of standardized GARCH innovations is 4.0028 and it is less than the sample kurtosis $k = 5.6791$. This means that our GARCH model does not fully capture the leptokurtosis of the sample data. The skewness of standardized GARCH innovations is 0.135 and it is greater than the sample skewness $s = 0.1095$.

Figure 4 presents the innovations of the estimated GARCH(1,1) process and corresponding standard deviations and returns of used data. The plot of innovations and the plot of returns look similar. Volatility clustering and extreme values are found from innovations plot. The conditional standard deviation plot shows that volatility rises sharply when extreme returns occur.

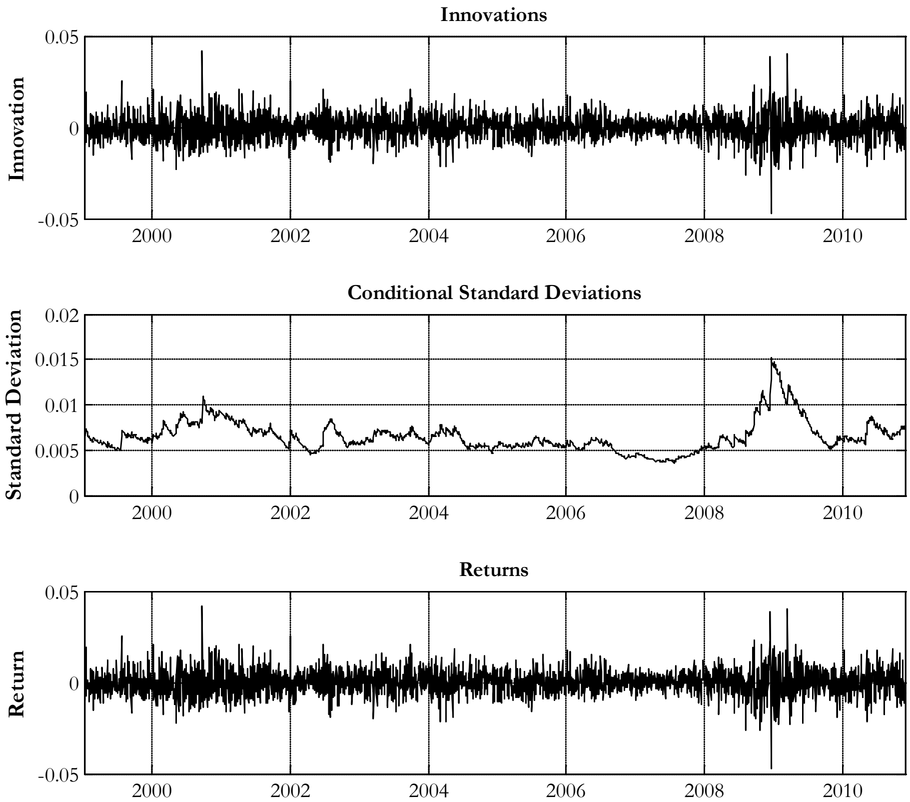

Figure 4: GARCH(1,1) innovations, corresponding standard deviations and returns of the data.

4 Conclusions

This thesis has presented the generalized autoregressive conditional heteroskedasticity model and its definition. Autocorrelation functions and partial autocorrelations functions were presented. The regression model and log-likelihood function were described as they are needed in estimating the GARCH model.

GARCH(1,1) was employed to a data sample of 3050 daily observations of USD/EUR exchange rates covering the period from 1 January 1999 to 29 November 2010. We saw that the raw returns were not serially correlated but squared returns were and LBP Q-test and Engle's ARCH test provided evidence for serial correlation. The exponential autoregressive conditional heteroskedasticity model was described briefly and alternative error distributions for error terms were also discussed.

The GARCH model explained satisfactorily the heteroscedasticity in the raw USD/EUR returns but standardized residual were leptokurtic and skewed, in comparison to the normal distribution. Many researchers have written that stock returns, exchange rates and other financial time series are not normally distributed. This is indeed the case with USD/EUR rates in our sample. Although the assumption of normality is highly questionable the GARCH models are amongst the most commonly used methods in estimating time varying volatilities. According to the empirical literature on GARCH processes, it turns out that conditional normality of speculate returns is more of an exception than the rule.

Speculative returns are nearly always skewed due to investors' tendency to avert big losses especially during periods of high volatility. Advanced developments of GARCH have led to asymmetric models like Exponential GARCH and other GARCH models with non-normal error distributions.

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