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# COMPUTATIONAL METHODS FOR DYNAMIC OPTIMIZATION AND PURSUIT-EVASION GAMES

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Tuomas Raivio

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Abstract: The thesis deals with the numerical solution of deterministic optimal control problems and pursuit-evasion games that emerge in the field of aircraft trajectory optimization. In the optimization framework, the thesis presents a continuation method for minimum time problems and compares the use of discretization and nonlinear programming with indirect approaches through numerical examples. Motivated by the benefits of the discretization and nonlinear programming, the thesis extends the underlying ideas into pursuit-evasion games. The approaches proposed here allow the computation of open-loop representations of feedback saddle point strategies for a class of pursuit-evasion games without explicitly solving the necessary conditions. Numerical results are computed for complex pursuit-evasion problems describing a missile chasing an aircraft and a visual identification of an unknown aircraft. Finally, a way to use the proposed approaches in the numerical solution of a game of kind is presented and applied to the estimation of the capture set of an optimally guided missile. Although the application examples are related to aeronautics, the presented methods are of a general nature and can be applied to different areas as well.

Keywords: optimal control, pursuit-evasion games, multipoint boundary value problems, discretization and nonlinear programming, aerospace applications.

Academic dissertation

Systems Analysis Laboratory
Helsinki University of Technology

Computational methods for dynamic optimization and pursuit-evasion games


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Publications

The dissertation consists of the present summary article and the following papers:

[I] Ehtamo, H., Raivio, T., and Hämäläinen R.P., A continuation method for minimum time problems. Helsinki University of Technology, Systems Analysis Laboratory, Research Report E3, 2000.

[II] Raivio, T., Ehtamo, H., and Hämäläinen, R.P., "Aircraft trajectory optimization using nonlinear programming," In: J. Dolezal and J. Fidler, eds., System Modeling and Optimization, Chapman & Hall, London, 1996.

[III] Raivio, T. and Ehtamo, H., "On the numerical solution of a class of pursuit-evasion games," In: J. Filar, K. Mizukami, and V. Gaitsgory, eds., Annals of the International Society of Dynamic Games, Vol. 5: Advances in Dynamic Games and Applications, Birkhäuser, Boston, accepted for publication.

[IV] Ehtamo, H. and Raivio, T., Applying nonlinear programming to pursuit-evasion games. Helsinki University of Technology, Systems Analysis Laboratory, Research Report E4, 2000.

[V] Raivio, T. and Ehtamo, H., "Visual Aircraft Identification as a Pursuit-Evasion Game," Journal of Guidance, Control and Dynamics, accepted for publication.

[VI] Raivio, T., Capture set computation of an optimally guided missile. Helsinki University of Technology, Systems Analysis Laboratory, Research Report E5, 2000.

Contributions of the author

The basic ideas for the continuation method reported in paper I, as well as for the decomposition method to solve the maxmin problems in papers III-VI were given by the author. He has also performed all the computations and analyzed the results in each of the papers. The mathematical analysis of the methods has been carried out together with professor Ehtamo. Papers I-V have been written jointly by the authors.

## Preface

This work has been carried out in the Systems Analysis Laboratory of the Helsinki University of Technology. I am grateful to PROFESSOR RAIMO P. HÄMÄLÄINEN, Head of the Laboratory, for the opportunity, guidance, and collaboration throughout my studies. I have learned a lot. I express my sincere thanks to my instructor and co-author, PROFESSOR HARRI EHTAMO, for the exuberant, unfailing enthusiasm and the overwhelming knowledge he has shared with me. Our discussions, covering all the important aspects of life, are warmly in mind. I am indebted to my teamworker, MR. KAI VIRTANEN, for the good sense of humor, constructive criticism, and the interesting moments we have shared. The whole personnel of the Systems Analysis Laboratory has provided a stimulating and excellent working atmosphere.

Cooperation with the Finnish Air Force is thankfully acknowledged. Especially I wish to thank MR. KARI RENKO for the warm and flexible collaboration. DR. JAAKKO HOFFREN has provided enlightening moments in the field of aerodynamics. The profound revision of the thesis by DR. JARI HÄMÄLÄINEN and DR. MARKO MÄKELÄ is gratefully appreciated.

The complete list of friends who have contributed to the thesis, directly or indirectly, is long. Great moments during hikes in Lapland, at sea, in quartet rehearsals and in international photography & biking are reminisced with pleasure. I might have been able to complete the thesis without the friends, but then my life would have been quite dull. Nevertheless, I could not have completed the thesis without my brother TANELI, who finally led the way, his spouse NANNE, MATIAS, who has taught me many facts of life, my parents VUOKKO and MATTI, to whom I owe my interest in knowledge, and ANU, the very woman of my life. Thank you for all the tolerating love and support.

*Espoo, March 2000*

## 1. Introduction

The theory of dynamic optimization, or equivalently, optimal control theory, provides a mathematical framework for a single decision maker to control a dynamic system in the best possible way with respect to a given performance index. The field originates from the calculus of variations. Modern theory is based on the principle of optimality and dynamic programming (Bellman, 1957), and the Pontryagin Maximum Principle (Pontryagin et al., 1962). For a comprehensive review, see Athans & Falb (1966), Bryson & Ho (1975) and Leitmann (1981). Besides engineering, optimal control is today applied to a variety of fields that range from biotechnology and human physiology to economics and space mission planning. For examples of applications on these areas, see Oberle & Sothmann (1999), J. Hämäläinen (1988), R.P. Hämäläinen (1992), Koslik & Breitner (1997) and Kluever & Pierson (1995).

Analytical solutions can be obtained for optimal control problems with linear dynamics and a quadratic performance index, see Kwakernaak & Sivan (1972). For other than linear-quadratic problems, open-loop solutions corresponding to a given initial state can be obtained numerically either by solving the necessary conditions given by the Pontryagin Maximum Principle, or by direct optimization of the performance index. Recently, direct approaches where the optimal control problem is discretized in time and approximated by a nonlinear programming problem, have received attention, see the survey papers by Hull (1997) and Betts (1998). The solutions obtained by discretization and nonlinear programming are approximative, but the computational effort is smaller compared to solving the necessary conditions. Feedback solutions could be computed by dynamic programming, but the approach is in practice restricted to problems of moderate size. In stochastic optimal control, however, the role of dynamic programming is more significant, see Bertsekas (1976).

The concept of optimality is no more clear if several independent decision makers or players with different objectives control a common dynamic system. Such situations can be modeled using the theory of dynamic games. For an overview, see Basar & Olsder (1995), and for a review of the state of the art, see Hämäläinen & Ehtamo (1991a,b), Basar & Haurie (1994) and Olsder (1995). In dynamic games, each player wishes to find a course of actions that optimizes his or her own performance index. The performance indices are usually conflicting and depend on the state of the game and hereby on the actions of the other players as well. Different solution concepts correspond to different behavior of the players. For example, by cooperation each player can often achieve more than by acting in a noncooperative manner. Nevertheless, a cooperative solution is not an equilibrium in the sense that for some players it could be even more beneficial to deviate from the cooperation during the game. On the other hand, with the knowledge on the other players' actions, the players can form various strategic games where game solution is an equilibrium. For studies on dynamic game theory based on cooperative, or Pareto optimal, Nash and Stackelberg solution concepts, see Ehtamo (1988), Ruusunen (1990) and Verkama (1994).

Pursuit-evasion games are two-player zero-sum dynamic games with an unspecified final time. Since the performance index of one player is the opposite of the performance index of the other one, cooperation is impossible, and a pursuit-evasion game only admits one type of solution called a saddle point equilibrium. Neither of the players wishes to deviate from it, provided that the other player retains his solution. The saddle point equilibrium is a special case of the Nash equilibrium solution. Pursuit-evasion games were first introduced by Isaacs

(1975; first edition 1965). Blaquière et al. (1969) and Friedman (1971) provide a rigorous mathematical analysis, and much of the theory is summarized by Basar & Olsder (1995).

Numerical solutions of pursuit-evasion games are computed by solving the necessary conditions of an open-loop representation of a feedback saddle point solution, given by an approach resembling the Pontryagin Maximum Principle (Isaacs, 1975; see also Basar & Olsder, 1995). Recently also numerical solution methods for the Isaacs' partial differential equation (Isaacs, 1975) have been presented, see Bardi & Capuzzo-Dolzetta (1997). These methods can be used to compute approximate feedback solutions for small pursuit-evasion game problems. Despite the similarity between pursuit-evasion games and optimal control problems, only few methods comparable with direct solution approaches of optimal control problems have been reported.

The primary aim of this thesis is to study the numerical solution of pursuit-evasion games by discretization and nonlinear programming. Two approaches are presented. The first one, described in paper III, decomposes the solution of the necessary conditions of the saddle point. For a class of games, the decomposition leads to a sequence of optimal control problems that can be solved by discretization and nonlinear programming. In the other approach, described in paper IV, the dynamics of the players in the original pursuit-evasion game are first discretized. The resulting parameterized saddle point problem is then interpreted as a bilevel programming problem (see e.g. Falk & Liu, 1995) and solved using a first order feasible direction method. Although the basis of the approaches is different, they both lead to similar subproblems, and the bilevel programming approach can also be used to justify the decomposition approach.

As the subproblems of the approaches are optimal control problems, knowledge on the efficient solution of them is essential. Paper I of the thesis provides a continuation method which makes the numerical solution of many time-optimal problems easier. Paper II, on the other hand, provides computational experience on the use of discretization and nonlinear programming in solving optimal control problems. In fact, this experience has also been one of the key motivations of papers III and IV.

The applications considered in this thesis are related to aircraft trajectory optimization and aeronautical pursuit-evasion game problems. Aircraft trajectory optimization has its roots already in the 1940's, when calculus of variations was used to optimize the flight of Messerschmitt Me-262, the world's first jet fighter (Lippisch, 1943; Kaiser, 1944). Overall, aerospace applications are an inherent part of the history and motivation of optimal control theory and especially pursuit-evasion games.

An aircraft or a missile is a complicated dynamic system. Simulation models with six degrees of freedom involve a system of 12 highly nonlinear first order differential equations. Fortunately, for most trajectory optimization and game problems the model dimension can be reduced to six. The structure of the models used in the computations of this thesis is taken from the literature, and the aerodynamic data corresponds to generic aircraft models that are obtained either from public sources or by modifying real data.

In optimal control problems of military aviation, the elapsed time is a commonly used performance index. In a minimum time climb problem, the total time needed to achieve a certain altitude and a certain velocity is minimized (e.g., Bryson & Denham, 1962; Kelley &

Cliff, 1986; Ehtamo et al., 1994). In a minimum time interception problem (e.g., Shinar & Spitzer, 1988; Imado et al., 1990; Glizer, 1997), a minimum time trajectory to a (possibly) moving point is sought. A problem where the time is of less importance is the optimal avoidance of a missile that uses a known feedback guidance law (e.g. Imado & Miwa, 1989; Shinar & Guelman, 1994; Ong & Pierson, 1996). Problems such as maximum range flight (Seywald et al., 1994), minimum fuel flight (Pierson & Ong, 1989) and safe landing in the presence of a windshear (Miele at al., 1987; Bulirsch et al., 1991; Kaitala et al., 1991; Berkmann & Pesch, 1995) are also of interest in civil aviation.

Central applications of pursuit-evasion games in aeronautics are duels between a missile and an aircraft (Green et al., 1992; Breitner et al., 1993a,b; Imado, 1993; Shinar et al., 1994; Lachner et al., 1995), between two aircraft equipped with guns or missiles (Merz & Hague, 1977; Shinar & Gutman, 1980; Järmark, 1985; Moritz et al., 1987; Grimm & Well, 1991), and lately also between two missiles (Lipman & Shinar, 1995; Shinar & Shima, 1996; Breitner at al., 1998). In the last two papers of the thesis, the decomposition method presented in paper III is applied to similar problems. Paper V analyzes a case where an aircraft that has entered the airspace of a country is being pursued by a defender who wants to identify the aircraft before it again leaves the airspace. In paper VI, the set of initial states from which an optimally guided missile can capture an aircraft irrespective of the aircraft's actions is estimated.

The applications of pursuit-evasion games are not necessarily restricted to antagonistic settings. For instance, in collision avoidance problems (Merz, 1991; Lachner et al., 1996; Yavin et al., 1997) it is assumed that the other party actually tries to collide with the other one who then designs optimal worst case avoidance maneuvers. More generally, dynamic games and related ideas can be utilized in worst case control of plants with uncertainties or disturbances by assuming that they behave in the worst possible way (Hämäläinen, 1976; Basar & Bernhard, 1991; Breitner & Pesch, 1994; Breitner, 1995; Raivio et al., 1996).

The present summary is organized as follows. The next section gives a description of the aircraft and missile models used in the optimization and game modeling. Section 3 contains an overview of numerical methods for optimal control problems and pursuit-evasion games and further enlightens the contribution of the thesis. Section 4 explains the game models applied in the thesis and presents selected results from papers IV, V and VI. Concluding remarks appear in section 5.

## 2. Aircraft optimization models

In modeling an aircraft or another flying vehicle for trajectory optimization, the following assumptions, usually associated with Miele (1962), are common:

1. The thrust and the drag forces, as well as the velocity vectors, are assumed parallel with the reference line of the vehicle.
2. The lift force is assumed to be orthogonal to the velocity vector and to point upwards in the frame of reference of the vehicle.
3. The inertias of a vehicle are assumed negligible.
4. The Earth is assumed flat and the gravitational acceleration is assumed constant.

These assumptions lead to a set of six nonlinear differential equations that describe the motion of an aircraft as a point mass in the three dimensional space. The state variables of the model are shown in Figure 1. The *x* and *y* coordinates and the altitude *h* specify the position of the aircraft, whereas the flight path angle $\gamma$ and the heading angle $\chi$ specify the direction of the velocity *v*. If necessary, the mass of the aircraft, that decreases due to the fuel consumption, can also be included in the state variables. A complete model is presented, e.g., in paper II. In the model, the motion of the aircraft is controlled using three control variables: the throttle setting *u*, the loadfactor *n* and the bank angle $\mu$, see again Figure 1. The throttle setting selects the fraction of the maximal thrust force available. The loadfactor is the ratio of the gravitational and the lift force affecting the aircraft. Together with the bank angle it is used to control the flight direction. In reality, an aircraft is steered with aerodynamic controls - ailerons, elevators and rudder - which alter the balance of forces on the aircraft causing it to change direction. If needed, near optimal aerodynamic controls corresponding to an optimal trajectory of the point mass model can be computed afterwards using inverse simulation (see Gao & Hess, 1993) and an aircraft model with an appropriate amount of degrees of freedom. The optimization model also contains several constraints that concern both the control and the state variables. For example, to prevent structural damages, the loadfactor cannot be chosen arbitrarily large.

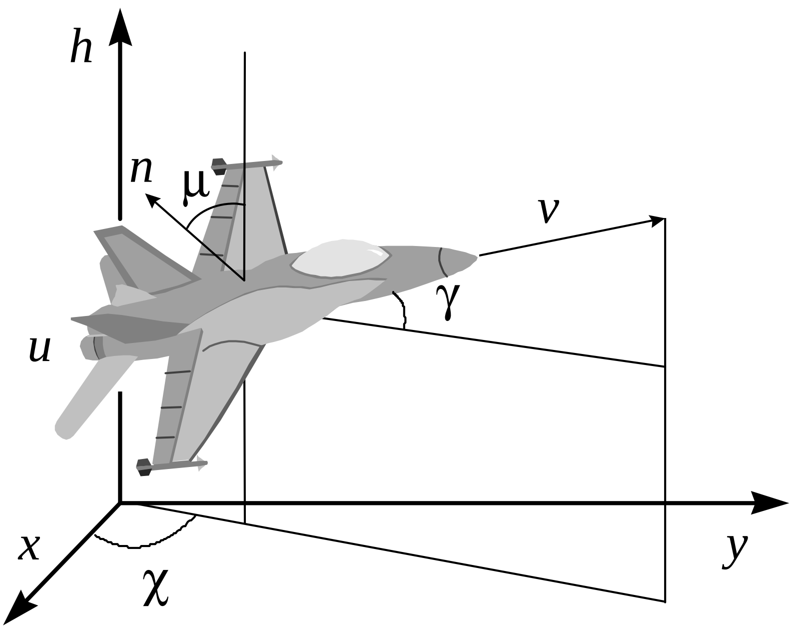


***Figure 1.*** *The state and the control variables of the point mass model.*

For some problems it is sufficient to consider flight in the vertical or the horizontal plane only. Consequently, the model can be reduced. In the former case, the equations for the y-range and the heading angle are ignored and the bank angle is fixed to 0°. In the latter case, the equations for the altitude and the flight path angle are ignored, and instead of the loadfactor and bank angle, the time derivative of the heading angle can be used as a control variable. For details, see Raivio et al. (1996).

The drag and the thrust forces of an aircraft play an important role in the optimization. The drag force of an aircraft is characterized by a drag polar that relates the drag coefficient to the applied lift force and the Mach number, i.e., the ratio of the velocity and the speed of sound. Besides the drag coefficient, the drag force depends on the reference wing area of the aircraft, the velocity, and the density of the air. The latter, as well as the Mach number, can be obtained as functions of altitude from an appropriate atmosphere model, such as the ISA Standard Atmosphere (Anon., 1993). The maximal thrust force of a jet engine is a complicated function that depends on a variety of factors. Nevertheless, once the atmosphere model is fixed, modeling the effect of the Mach number and the altitude is usually considered sufficient.

The thrust force and the drag polar of a given aircraft are usually obtained by flight tests and announced as tabular data. For optimization purposes, an estimate for the values between the data points is needed. All gradient-based direct optimization algorithms, as well as the solution of the necessary conditions in practice require that this estimate is continuous and smooth. The basic approaches are either to use a smooth two-dimensional interpolation scheme or to approximate the data by a suitable surface that is fitted with least squares. Both approaches have their benefits and drawbacks. Interpolants may fluctuate between data points, which can cause convergence difficulties in the optimization. Fitted surfaces can be made smoother but they do not in general coincide with the tabular data. If the data points are considered exact, the use of fitted surfaces thus means accepting a small error. In practice, a tradeoff between the accuracy and the smoothness may be needed.

The drag polar is often assumed quadratic, i.e., the drag force is assumed to be proportional to the square of the loadfactor. Then, instead of two-dimensional interpolation or approximation of the drag coefficient, two one-dimensional coefficients, the zero-lift and the induced drag coefficient, need to be treated. If curve fitting is preferred, the shape of the coefficients suggests the use of rational polynomials (Breitner et al., 1993a; Seywald et al., 1994). Also splines can be used, see Virtanen et al. (1999). The thrust force is usually approximated by fitting a two-dimensional polynomial of Mach number and altitude to the tabular data.

Although missile aerodynamics is somewhat different from aircraft aerodynamics, a similar point mass model is commonly used. The drag polar is modeled as above, but the thrust force and the mass of a missile are usually fixed functions of time that do not significantly depend on altitude or velocity and cannot be controlled.

Some evidence on the accuracy of the point mass model can be obtained from Hoffren & Raivio (2000). In the summer of 1998, a Finnish trainer fighter BAe-Hawk Mk.51, suffered an engine fault. The pilot and the copilot survived but the aircraft was lost. The accident inquiry board raised a question whether the plane could have been taken to an airfield. The question was analyzed by first optimizing the glide trajectory to the nearest airfield using a point mass model. Then, the trajectory was reproduced with a simulation software with five degrees of freedom, using the same aerodynamic data.

In the simulation, the largest achieved energy altitude deviated 7.6 % from the optimization result. The difference is largely due to the large nonoptimal initial bank angle of the aircraft that remains unmodeled in the point mass model. Other sources of deviation are the reproduction of the optimal trajectory that mimics the pilot's behavior, and the approximate nature of the applied direct optimization technique. As for the result itself, it turned out that the plane could have been saved in theory but hardly in practice.

In problems of short duration and excessive maneuvering, the model may become deficient. For example, some preliminary computational experiences with the optimal avoidance maneuvers against a missile employing Proportional Navigation (see Zarchan, 1996) indicate that a point mass model may be inadequate. For a model with finite pitch, yaw and roll rates, see Fan et al. (1995).

In the analysis of next fighter aircraft generations, model modifications will be needed. Features like thrust vectoring systems will allow maneuvers like flying with large angles of attack that contradict the assumptions in the beginning of the section. Such maneuvers cannot necessarily be captured meaningfully by point mass models and conventional drag models. Some modeling work can be found in Goman & Khrabrow (1994).

## 3. Numerical solution of optimal control problems and pursuit-evasion games

The Pontryagin Maximum Principle (Pontryagin et al., 1962; see also Bryson & Ho, 1975) provides a set of necessary conditions that a solution satisfies if it is optimal. A similar approach yields the necessary conditions for an open-loop representation of a feedback saddle point solution of a pursuit-evasion game, see Isaacs (1975), Bryson & Ho (1975) and Basar & Olsder (1995).

The necessary conditions constitute a nonlinear multipoint boundary value problem that can be solved by methods like multiple shooting (Bulirsch, 1971; Breitner et al., 1993b; Bulirsch & Kraft, 1994; Pesch, 1994), quasilinearization (Bryson & Ho, 1975), collocation by piecewise polynomials (Ascher, 1988) or finite differences. The methods that solve an optimal control problem via satisfying the necessary conditions are commonly referred to as indirect methods. The methods require a rather precise initial estimate of the solution to converge. A less accurate initial estimate will suffice, if a continuation method (Allgower & Georg, 1990; Bulirsch et al., 1991; Pesch, 1994) is used. Paper I of the thesis provides such a method for minimum time optimal control problems. In continuation methods, the problem is embedded into a family of problems characterized by a continuation parameter. If the parameter is chosen appropriately, the family will contain at least one problem, corresponding to a certain value of the parameter, that can be solved with a simple initial estimate. The solution of the original problem is then found via gradually changing the value of the parameter and hence following a homotopy path through the family. In this process, the solution of the previous problem serves as the initial guess for the following one.

In the presented continuation method the minimum time problem is turned into a sequence of terminal cost minimization problems with a fixed final time. The final time is used as the continuation parameter and updated by a one-dimensional search until the original terminal condition becomes satisfied. In addition to the continuation approach, the determination of the final time is decomposed from the determination of the optimal trajectory. The dimension of the problems used in the continuation thus becomes lower, which may improve their convergence.

Another approach for solving optimal control problems, being today actively investigated especially in the field of aerospace applications, to discretize the dynamics of the system under consideration in time and describe it with a set of equality and inequality constraints. The performance index of the original problem is optimized subject to these constraints using an appropriate nonlinear programming algorithm (e.g., Bazaraa et al., 1993; Bertsekas, 1995). These methods are usually referred to as direct methods, since they do not explicitly deal with the necessary conditions.

The discretization can be carried out in a number of ways. For a review, see Hull (1997) and Betts (1998). Paper II of the thesis applies two discretization schemes, direct collocation with piecewisely defined cubic polynomials (Hargraves & Paris, 1987; see also von Stryk, 1995) and a method based on differential inclusions (Seywald, 1994) to some typical aircraft trajectory optimization problems. The nonlinear programming problems are solved with the widely used NPSOL software package (Gill et al., 1986) that is a sophisticated implementation of the Sequential Quadratic Programming algorithm. The solution process and the results are compared with an indirect solution method. For simplicity, equidistant grids are used in the discretization. For complex problems, more efficient computation could possibly be achieved with nonequidistant and adaptive grids (see von Stryk, 1995; Betts & Huffman, 1998).

Paper II, being a concise overview, does not quantitatively assess the quality of the solutions. For the sake of completeness, discretization errors were reinvestigated. In this process, a minor discrepancy between the aircraft models used in the reference solution and in the original optimizations was noted. Table 1 presents the final times recomputed with the corrected model, and two measures for the discretization error, the maximal relative error and the average relative error. The former is here defined as

$$\varepsilon_{\max} = 100 \max_i \max_j \frac{|x_{ij} - x_{ref,i}(t_j)|}{|x_{ref,i}(t_j)|} [\%],$$

where $i$ and $j$ range over the state trajectories and discretization points, respectively, and the latter is defined as

$$\varepsilon_{ave} = 100 \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=2}^{m-1} \frac{|x_{ij} - x_{ref,i}(t_j)|}{|x_{ref,i}(t_j)|} [\%],$$

where $n$ is the dimension of the state vector and $m$ refers to the number of discretization points. The reference solution is denoted by $x_{ref}(t)$, and the discretization gridpoints by $t_j$. The fixed initial and final conditions and cases where $x_{ref}(t)=0$, are omitted. If the solutions are judged based on their accuracy only, and the geometric increase of the computational workload is omitted, both methods produce better results with increasing $m$, as should be expected.

| | ***Collocation*** | | | | ***Differential Inclusion*** | | | |
|---|---|---|---|---|---|---|---|---|
| ***m*** | 5 | 10 | 15 | 20 | 5 | 10 | 15 | 20 |
| ***$T_f$ ($T_{ref}$=60.2s)*** | 61.06 | 60.40 | 60.21 | 60.21 | 60.35 | 59.96 | 60.11 | 60.15 |
| **$\varepsilon_{\max}$ ,%** | 56.3 | 29.1 | 9.2 | 6.4 | 80.0 | 30.15 | 8.58 | 8.47 |
| **$\varepsilon_{ave}$ ,%** | 7.76 | 2.56 | 0.98 | 0.42 | 36.2 | 5.39 | 1.18 | 1.16 |

Table 1. Iteration results.

It should be noted that the maximal error mainly stems from the flight path angle that receives quite small absolute values on the reference trajectory. For example, with equal models, the maximal relative error of altitude in collocation with five discretization points is 7%.

In the method based on differential inclusions, the analytic elimination of the control variables is not necessarily possible. If a numerical scheme is needed, the computational burden may grow unacceptably large (Hull, 1997). Kumar & Seywald (1996) shed some light on the issue by comparing analytical and numerical elimination in a practical example. Conway & Larson (1998) compare a higher order discretization method to the method of differential inclusions using the same numerical example, and end up recommending the higher order scheme irrespective of the control elimination. One of the original arguments of Seywald (1994) was that in the presence of singular solution arcs the control elimination speeds up the computation. Some evidence to support the hypothesis was obtained in the last numerical example of paper II. Nevertheless, in a small experiment that was conducted in addition to the results presented in paper II, nothing conclusive was found.

Overall, the obtained computational experience supports the nowadays common view that direct methods have several benefits compared to indirect methods: First, the complex and problem dependent necessary conditions need not be solved explicitly. Second, the correct sequence of the unconstrained, constrained and singular arcs of the solution, also referred to as the switching structure of the solution, needs not be specified in advance. Third, an initial guess for the adjoint variables needs not be provided. Fourth, the convergence domain of discretized problems is larger than that of indirect methods. Finally, several sophisticated implementations of nonlinear programming algorithms are today available. Note that although the necessary conditions are not explicitly involved in the solution process, the solution and the Lagrange multipliers of the optimization problem satisfy the necessary conditions with an accuracy that depends on the discretization interval and the order of the discretization (von Stryk, 1995).

The robustness of direct methods makes them suitable for interactive computation. Virtanen et al. (1999) present an automated trajectory optimization software VIATO. It is easy to use even by nonexperts, as no previous knowledge on the methods of optimal control or mathematical modeling is needed. The underlying theory is hidden behind a menu driven graphical user interface. Although the optimization in VIATO is implemented with algorithms designed for dense matrices, the structure of the resulting optimization problems in principle allows the use of sparse nonlinear programming algorithms (see Betts & Huffman, 1993). Accordingly, very large problems can be solved with moderate computational effort.

For pursuit-evasion games, approaches based on discretization are scarce. The introduction of paper III offers a short review of existing numerical methods. Motivated by the benefits of the direct approaches of optimal control and the similarity between optimal control problems and pursuit-evasion games this thesis extends the ideas of discretization and nonlinear programming into pursuit-evasion games. The first work in this direction is presented in paper III. It provides a solution method for games in which the players control their own state equations and have perfect information on the state of the game. In such games, the regular necessary conditions of an open-loop representation of a feedback saddle point solution are coupled only via the terminal payoff and the capture condition. Consequently, the saddle point problem can be decomposed into two subproblems that are solved iteratively. The first subproblem is a minimum cost interception of the evader moving along a fixed trajectory, and the other one is a correction to the evader's trajectory. Both subproblems are optimal control problems that can be solved using either indirect methods or, as is done here, discretization and nonlinear programming. In particular, it is shown that if the iteration converges, the limit solution satisfies the regular necessary conditions of an open-loop representation of a feedback saddle point trajectory.

Singular surfaces (Isaacs, 1975; Basar & Olsder, 1995) are frequently encountered in saddle point solutions of pursuit-evasion games. One of the benefits of the presented decomposition method is that when discretization and nonlinear programming is used to solve the subproblems, the switching structure of the subproblems needs not be specified in advance. Thus, the method can automatically treat such singular surfaces that are encountered in the subproblems. These surfaces include at least the universal surface and the switching surface. Nevertheless, a saddle point solution can involve also such singular surfaces whose tributaries are not solutions to the subproblems. An example of this can be found in the first numerical example of paper III, in a classical game of 'homicidal chauffeur' concerning a pedestrian and a car. There, an equivocal surface remains undetected and instead of the saddle point solution, a maxmin solution is located. It is suspected that this would happen with the switching envelope surface as well, although it has not been verified. These surfaces require additional necessary conditions to be satisfied that cannot be expressed as optimality conditions of the subproblems. To date, there does not exist a systematic way to construct these surfaces, and they have to be analyzed separately (Basar & Olsder, 1995).

In the approach described above, the saddle point problem was first decomposed and the subproblems were discretized. Paper IV of the thesis provides yet another approach for the same class of games as above. The saddle point problem is first discretized and then solved using a feasible direction method. The discretized problem is in fact a special case of bilevel programming problems that recently have been studied extensively, see Falk & Liu (1995), Shimizu et al. (1997), and the literature cited therein. The minimization and the maximization are identified with the lower level and the upper level problems, respectively. The maxmin problem is solved iteratively. At each iteration, gradient information on the solution of the lower level problem is first acquired using basic sensitivity results, see Fiacco (1983). Then, an ascent method together with a line search is applied to the upper level problem.

It should be noted that all the methods described above produce open-loop solutions or, in the case of games, open-loop representations (Basar & Olsder, 1995) of feedback solutions. Feedback solutions are optimal solutions that map every possible state of the system to a control variable. Such solutions indeed exist and can, in principle, be obtained by dynamic programming (Bellman, 1957), or the tenet of transition (Isaacs, 1975), the equivalent in the game context. In the continuous time case this reduces to solving the Hamilton-Jacobi-Bellman or, in case of games, the Isaacs equation of the problem. Nevertheless, these equations are high-dimensional nonlinear partial differential equations that rarely can be solved. A solution methodology presented by Isaacs (1975) is based on filling up the state space with saddle point solutions by integrating them in retrograde time from systematically varied final states. It is suitable for simple and low-dimensional models for which analytical solutions can be obtained. More complex game models are often simplified by linearization or singular perturbation techniques (e.g., Shinar & Farber, 1984) to allow for such solutions. Recently the theory of viscosity solutions has provided some results for more complex models, see Bardi & Capuzzo-Dolzetta (1997) and the literature cited therein.

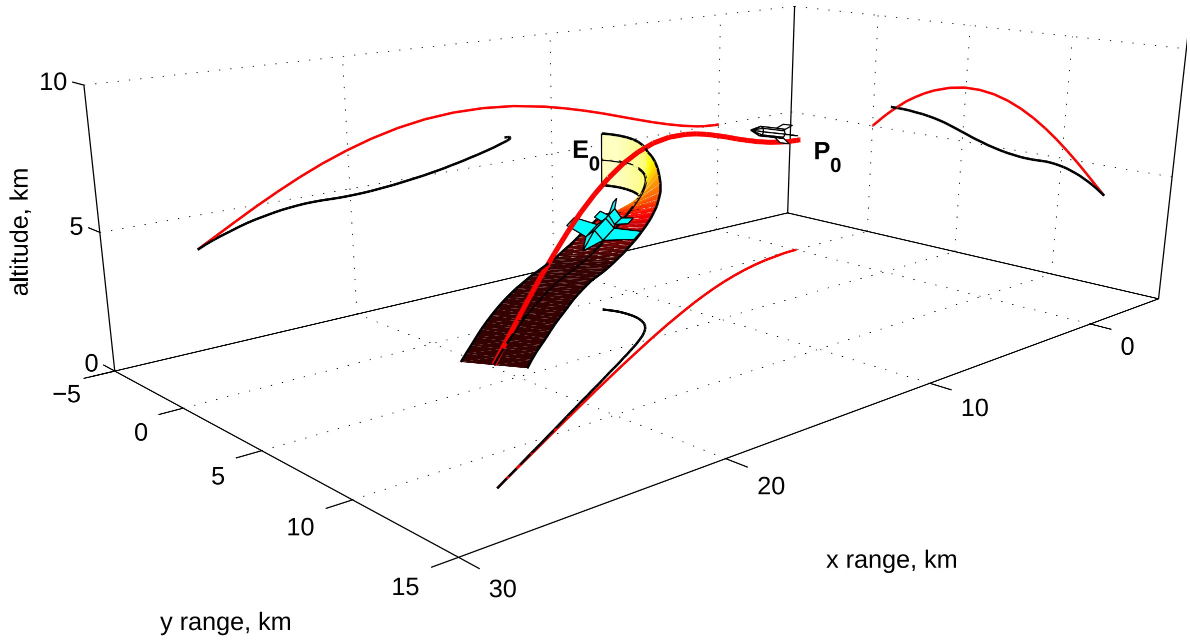


*__Figure 2.__ Optimal saddle point trajectories of the missile (red) and the aircraft (black) corresponding to the initial positions $E_0$ and $P_0$. The projections of the trajectories are presented on the coordinate planes. The ribbon describes the aircraft's bank angle and its color specifies the dynamic pressure. Yellow corresponds to approximately 50 kPa and dark red to the allowed maximum, 80 kPa.*

In practice, near optimal feedback solutions can be produced, e.g., by neighboring extremals (Bryson & Ho, 1975; Pesch, 1989). Another possibility is to compute a cluster of open-loop solutions from various initial states and then map the state of the system onto them using neural networks (Goh & Edwards, 1993; Järmark & Bengtsson, 1994; Pesch et al., 1995, Lachner et al. 1996).

## 4. Application examples

The numerical example in paper IV analyzes a missile-aircraft duel, where a medium range air-to air missile pursues an aircraft and wishes to capture it in minimum time. The aircraft maximizes the capture time. A capture occurs when the distance of the parties becomes smaller than a prescribed value. The ultimate goal of the aircraft, of course, is to avoid capture, but, if the missile can enforce a capture against any action of the aircraft, it is reasonable that the aircraft delay the capture as long as possible. The question whether the capture is possible or not will be addressed in the last paper of the thesis.

A saddle point solution of the problem, corresponding to one initial state, is shown in Figure 2. The missile first climbs to utilize the smaller drag force in the thinner atmosphere. Meanwhile, the aircraft turns away from the missile and dives until the dynamic pressure constraint becomes active. The constraint forces the aircraft to reduce its descent rate. Both trajectories lead to a common vertical plane.

Paper V deals with the final phase of an optimal visual identification of a scout aircraft that has entered a state's air space and wishes to escape before being identified. The encounter is modeled as a pursuit-evasion game where the intruder corresponds to the evader and the

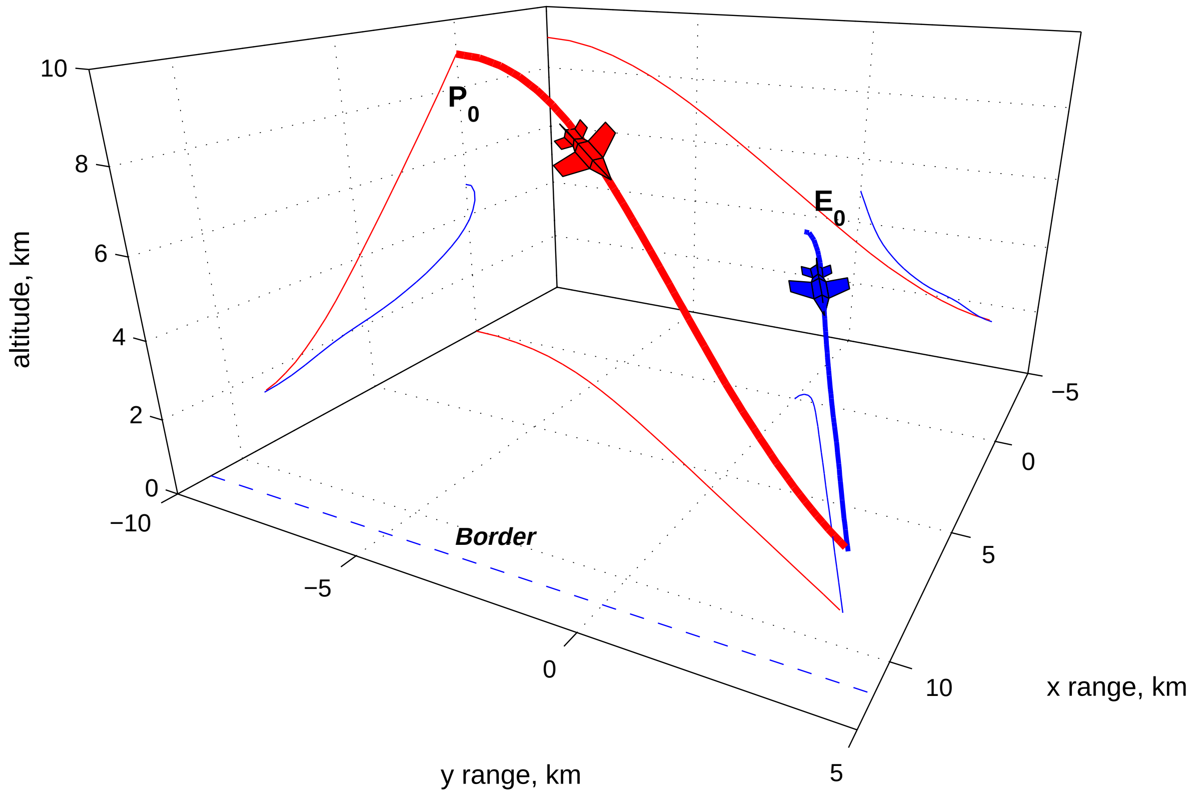


***Figure 3.*** *A saddle point solution representing the optimal behavior of the scout (initial position $E_0$) and the identifying aircraft (initial position $P_0$) in the final phase of an identification. The border is described a straight line parallel to the y-axis.*

identifying aircraft to the pursuer. The evader minimizes its final distance to the border while the pursuer tries to maximize this distance.

The study is motivated by the different goals and constraints compared to those aircraft pursuit-evasion problems in which the payoff is the final time and the terminal condition only involves the relative distance of the parties. Namely, it is not reasonable to assume that an intruder would aim at maximizing the capture time without considering the escape direction. From some initial constellations, a time optimal maneuver would require escaping inland. Moreover, a visual identification requires that the identifier achieves a similar attitude and velocity with the intruder in its vicinity. These requirements have been studied to some extent in the context of optimal control, but seemingly not in a game framework.

In this problem it is obvious that the velocity adaptation of the identifier requires deceleration. Due to the properties of the drag force of an aircraft, the deceleration can give rise to a chattering control arc, see Seywald (1994). For this reason, the models of the aircraft are revised so that the set of admissible state rates becomes convex. Saddle point trajectories corresponding to selected representative initial states are computed using the decomposition method described in paper III.

Figure 3 presents one saddle point solution corresponding to the case where the intruder is initially in level flight in the altitude of 5 km with the velocity of 200 m/s and flies inland, perpendicularly to the border that is assumed to be a straight line parallel to the y axis. The identifier approaches in level flight in the altitude of 9 km with the velocity of 550 m/s along the negative y-axis. The initial separation of the parties is 10 km in the xy-plane. In the saddle point solution the intruder turns rapidly back and dives to the dynamic pressure limit. The identifier dives, and in the end applies an airbrake to match the velocity with the intruder. Unlike in the missile-aircraft duel, the saddle point solution trajectories do not end up in a common vertical plane. In this scenario, the intruder can increase its x coordinate by approximately 9 km before getting identified.

In a given pursuit-evasion game it is not clear whether the pursuer can always capture the evader. There may exist such states in the state space that when the game starts from these states, the evader can avoid a capture infinitely long. The collection of these initial states is called the escape zone of a game. Consequently, the remaining states form the capture zone, also referred to as the capture set (Isaacs, 1975; Basar & Olsder, 1995). Identifying the capture zone is termed as the game of kind, whereas determining the optimal strategies inside the capture zone is referred to as the game of degree (Isaacs, 1975). Isaacs provides a unified procedure for solving the game of kind by identifying the surface that envelops the capture zone, also known as the barrier. The barrier is a collection of saddle point solution trajectories that, loosely speaking, lead to a momentary touch of the players instead of a true capture. An infinitesimal change of the initial state, or a deviation of either player from the optimal strategy immediately results in a capture or escape.

The approach of Isaacs for determining the barrier is useful with simple and low-dimensional game models. Nevertheless, a numerical solution approach is needed for settings with more complicated dynamics. Paper VI considers the numerical computation of barrier solutions for the missile-aircraft duel analyzed in paper IV.

A missile cannot capture an aircraft from an arbitrary initial state. A finite shooting range is a consequence of the finite and relatively short burn time of a missile's rocket motor. In the coasting phase the kinetic energy of the missile is rapidly dissipated by the aerodynamic drag force. In contrast to a missile, an aircraft can maintain its velocity as long as it has any fuel left. The maximal shooting range depends on many factors, such as the initial energy and constellation of the missile and target, their performance, the guidance law of the missile, the geometry of the shoot and moreover, on the maneuvering of the target. An estimate of the shooting range is crucial, for example, in assessing the threat related to each opponent in an air combat, and furthermore, in considering actions to be taken. On the other hand, the unit cost of a single missile is usually significant, which calls for minimizing the number of premature shoots. A worst case estimate of the shooting range is provided by the solution to the game of kind.

In paper VI, an auxiliary game of degree is set up. It is then shown that the necessary conditions of the solution to the auxiliary game are identical to those of a saddle point trajectory lying on the barrier. To locate points on the barrier, the auxiliary game is solved from systematically varied initial states by the decomposition approach presented in paper III. Figure 4 presents the maximal shooting range of the missile as a function of its initial altitude and the direction of the shoot when the aircraft is initially in the altitude of 3 km and flies 400 m/s in level flight. The initial velocity of the missile is fixed to 300 m/s, and it is allowed to choose the initial direction of flight freely.

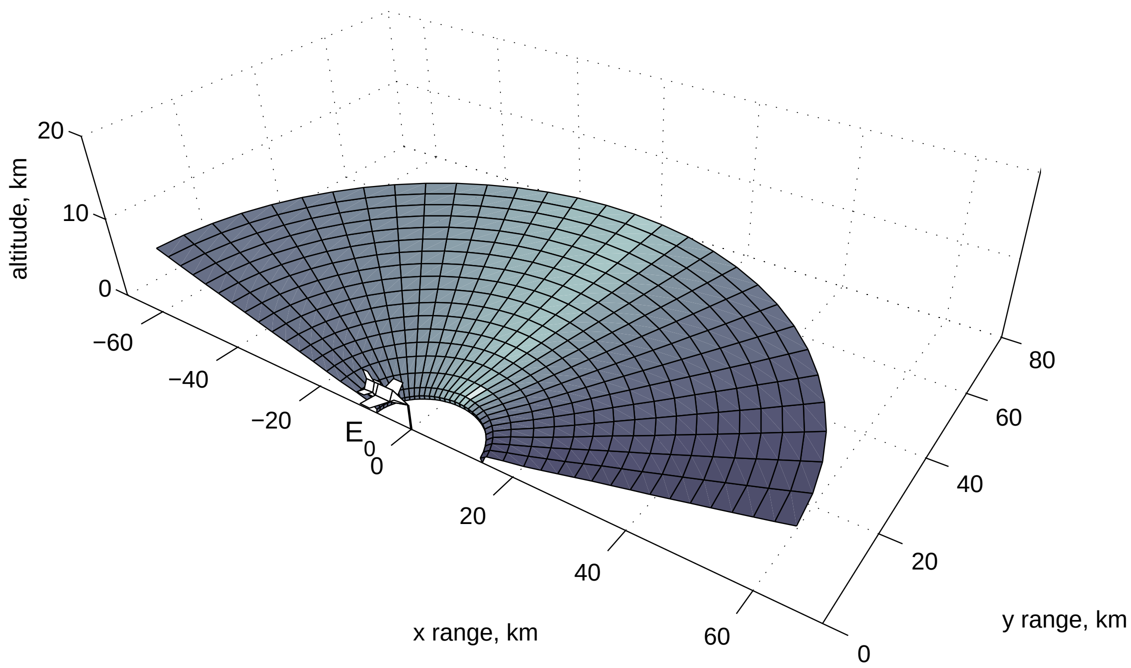


***Figure 4.*** *The surface represents the maximal shooting range of a missile with an initial velocity of 300 m/s against an aircraft flying in the altitude of 3 km with a velocity of 400 m/s in level flight. The aircraft is initially situated in the origin and is flying along the positive x-axis.*

The effect of the atmosphere is accentuated in the shooting range. When the missile starts from a low altitude, it stays in a denser part of the atmosphere during the flight. As a result, the maximal shooting range is relatively small. However, when the initial altitude of the missile increases, it can reach thinner atmosphere, and the shooting range grows almost linearly with the initial total energy of the missile.

It should be marked that in these computations the information structure of the game is assumed perfect. Nevertheless, an aircraft usually detects the missile only when the missile's radar is activated, which does not necessarily happen at the time of the launch. Therefore, an aircraft can in practice commence evasive maneuvers only later, and the actual shooting range is larger.

## 5. Concluding remarks

The thesis consists of studies that concern the numerical solution of optimal control problems and pursuit-evasion games. The main focus is on the use of discretization and nonlinear programming, although the continuation method presented in paper I can be used in connection with any solution method. The application examples of the thesis are related to atmospheric flight.

The obtained computational experiences indicate that discretization and nonlinear programming provides a robust way to approximately solve aircraft trajectory optimization problems. Perhaps the most important practical benefit of the technique is that a rough initial estimate of the solution is sufficient to achieve convergence. For a detailed investigation of the necessary conditions, however, an indirect solution method might be the suitable choice.

As a remark, it may be stated that a hybrid approach is possible as well: a solution obtained with discretization and nonlinear programming provides an estimate of the switching structure of the solution as well as a likely convergent starting point of iteration for an indirect method.

Game problems are, by definition, more difficult than optimization problems. To obtain a saddle point or a maxmin solution, a sequence of optimization problems has to be solved in the decomposition and the discretization approaches introduced in the thesis. Nevertheless, numerical game examples give evidence that these approaches have similar benefits as discretization and nonlinear programming in solving optimal control problems. The approaches are thus closing the gap between the present solution methods of optimal control problems and pursuit-evasion games.

The work reported in the thesis can possibly form a basis for extending the recent interactive automated solution framework of optimal control problems (Virtanen et al., 1999) into pursuit-evasion games as well. Automatically computed open-loop representations of feedback solutions would allow one to easily study the worst-case performance of different systems. Automatic computation would also be advantageous in the synthesis of optimal feedback strategies for complex models, where large amounts of open-loop solutions corresponding to different initial states are required.

The work also raises open research problems. For example, alternative approaches for the solution of the maxmin bilevel programming problem, like the use of exact penalty functions should be explored. In the present approaches, higher order methods for the max subproblem could be tested. On the other hand, possibilities to broaden the presented approaches to automatically treat the now undetected singular surfaces should be investigated.

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