Optimal Home Currency and the Curved International Equilibrium

Jussi Keppo

Helsinki University of Technology
Systems Analysis Laboratory
P.O. Box 1100
FIN-02015 HUT, Finland
http://www.hut.fi/Units/Systems.Analysis

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Jussi Keppo
Systems Analysis Laboratory
Helsinki University of Technology
Otakaari 1, 02150 Espoo, Finland
E-mail: Jussi.Keppo@hut.fi

ABSTRACT

In this paper we extend the choice of optimal portfolio and consumption to include also the selection of an optimal home currency for single agents in a segmented real international economy. We also derive an equilibrium that emerges if representative agents behave according to the optimization model. In this situation the market prices of risk do not reflect the risk attitudes of the investors in the particular currency, but are simply determined by the interplay of all investors in the international economy. This yields the curved international equilibrium.

KEYWORDS: Equilibrium, foreign exchange rate, market price of risk, optimization

1. INTRODUCTION

International asset markets are said to be integrated, if assets of equal risk yield equal expected returns in every currency. If the markets are segmented then, e.g., the cost of capital for an investment will depend on the currency in which it is to be raised. Also, if the markets are segmented, an individual investor has an optimal home currency, i.e., the currency in which the agent consumes.

In this paper we extend the framework on optimal portfolio and consumption choice to include also the selection of an optimal home currency for single traders in a real international economy. Under stochastic real foreign exchange rates, we show that market prices of risks differ between currencies and derive the explicit relationship linking the market price of risk vectors in different currencies. Using this dependence, we solve the optimal consumption problem for a single agent who is able to freely choose his home currency, i.e., we allow investors to take advantage of the diverging valuation of risks and real interest rates between currencies. Also, assuming the prescribed optimizing behavior on the part of investors, we show that the stochastic fluctuations of the exchange rates between the respective countries are entirely due to the differences in real interest rates between currencies.

Many other papers have studied the portfolio selection problem. The basic optimal portfolio selection by using the mean-variance analysis can be found, for instance, in Markowitz (1959) and Merton (1972). The optimal consumption and portfolio choice problem and its solution using the Hamilton-Jacobi-Bellman equation in finite and infinite horizon settings are found in Merton (1969, 1971). This method usually yields a nonlinear partial differential equation that is hard to solve and for which numerical methods must be applied. Especially, when there are constraints on portfolio and consumption this method becomes even more difficult. The martingale method to solve the optimal consumption and portfolio choice is developed in Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987). This approach leads to a linear partial differential equation, unlike the nonlinear partial differential equation appearing in dynamic programming. In this paper we use the martingale method, that utilizes the market's state-price deflators and gives explicit solution only to the optimal consumption problem. We are not interested in the form of the optimal trading strategy, because we only derive the optimal consumption process and prove that there exists a trading strategy that finances the consumption and a state-price deflator such that the deflated trading strategy is a martingale. However, an optimal trading strategy can be derived by using Malliavin calculus [see Ocone and Karatzas (1991)]. The basic framework for deriving market equilibrium in continuous-time setting can be found e.g. in Duffie (1992). Duffie and Zame (1989) have proved the existence of an Arrow-Debreu equilibrium in the case of smooth-additive utility function that is also used in this paper. Huang (1987) has derived an equilibrium model with a smooth-additive utility function. The consumption-based capital asset pricing model is derived in Breeden (1979). We utilize the Breeden's framework, because the models of this paper are in real terms. Market equilibrium in an international setting is studied e.g. in Uppal (1993). This model shows that if purchasing power parity does not hold, risk averse agents prefer foreign assets. Our model is similar to the framework of Uppal. However, we add to the analysis the selection of optimal home currency. The necessary conditions for an arbitrage-free international economy have been studied in Amin and Jarrow (1991).

The rest of the paper is organized as follows: Section 2 defines the framework used in the paper, Section 3 derives the relationship between different currencies' market prices of risk, Section 4 calculates the optimal consumption and home currency of a single agent as a function of her utility function, Section 5 derives an equilibrium in which representative agents are assumed to behave according to the optimization model, and Section 6 concludes.

2. MODEL

We explore an economy where instruments are traded continuously within a time horizon $[0,t]$. The economy consists of $I + 1$ currencies each indexed by $i \in \{0, \ldots, I\}$ where currency number 0 is the current domestic currency. In each currency a set of assets is marketed. These sets are denoted by H_i, where $i \in \{0, \ldots, I\}$, and the set of all assets in the international economy, respectively, $H = \bigcup_{i \in \{0,\ldots,I\}} H_i$. In each currency there is also a set of agents M_i, and the set of all agents is denoted by M. An agent $m \in M$ is defined by a nonzero consumption endowment process e_m and a strictly increasing utility function U_m. We assume that agent m can consume only in one currency at a time. In each currency i there exists a consumption commodity whose price at time $t \in [0,t]$ is denoted by $d_i(t)$. It is assumed that $d_i(t)$ is positive for all $i \in \{0, \ldots, I\}$ and $t \in [0,t]$.

In describing the probabilistic structure of the economy, we refer to an underlying probability space (*W*, *F*, *P*). Here *W* is a set, *F* is a *s*-algebra of subsets of *W*, and *P* is a probability measure on *F*. We denote by *D* the space of processes with $E\left[\int_0^T x(t)^2\,dt\right] < \infty$, where *x* is a stochastic variable and $T \in [0, t]$, and by *L* the adapted processes in *D*.

The model is formulated in real terms, i.e., the real price of a tradable asset h_j in terms of currency *i* is

$$p_{i,h_j}(t) = \frac{\hat{p}_{i,h_j}(t)}{d_i(t)} \quad \text{for all} \quad i \in \{0, ..., I\}, \quad j \in \{0, ..., I\}, \quad h_j \in H_j, \quad t \in [0, t] \tag{1}$$

where $p_{i,h_j}(t)$ is the real price of h_j in terms of currency *i* and $\hat{p}_{i,h_j}(t)$ is the nominal price of h_j in terms of currency *i*.

After the normalization of (1) the consumption price is a state-price deflator [for the definition of state-price deflator or the equivalent martingale measure given by the state-price deflator see e.g. Duffie (1992), Harrison and Kreps (1979), and Harrison and Pliska (1981)]. Using the nominal state-price deflator $p_i(t)$ we get

$$0 = E_t\left\{\int_t^T d\left[p_i(s)\hat{p}_{i,h_j}(s)\right]\right\} \quad \text{for all} \quad i \in \{0, ..., I\}, \quad j \in \{0, ..., I\}, \quad h_j \in H_j, \quad t \in [0, T], \quad T \in [0, t] \tag{2}$$

where E_t is the conditional expectation operator with respect to *P*. Equations (1) and (2) yield

$$0 = E_t\left\{\int_t^T d\left[d_i(s)p^p_{i,h_j}(s)\right]\right\}, \tag{3}$$

where $p^p_{i,h_j}(s) = p_i(s)p_{i,h_j}(s)$.

This means that after the normalization to real prices d_i is a state price deflator in the sense of (3). Hereafter we assume that $p_i(t) = 1$ for all $i \in \{0, \dots, I\}$ and $t \in [0, t]$, i.e., $p^p_{i,h_j}(t) = p_{i,h_j}(t)$.

The following assumptions characterize more our economy.

ASSUMPTION A1: *The stochastic variables of the economy follow an Itô stochastic differential equation*

$$dx(t) = m(x,t)dt + \mathbf{e}(x,t)d\mathbf{z}(t) \quad \text{for all} \quad t \in [0, t] \tag{4}$$

where m: $\mathbf{R} \times [0, t] \rightarrow \mathbf{R}$ *and* $\mathbf{e}$: $\mathbf{R} \times [0, t] \rightarrow \mathbf{R}^n$ *are given functions that satisfy Lipschitz and linear growth conditions on x and* $\mathbf{z}(t)$ *is an n-dimensional Brownian motion on the probability space* (*W*, *F*, *P*), *along with the standard filtration* $\{F_t : t \in [0, t]\}$.

Assumption A1 guarantees the existence and uniqueness of the solution to (4).

We write the real process of a tradable asset in terms of currency *i* as follows

$$dp_{i,h_j}(t) = p_{i,h_j}(t)\left[r_i(t) + a_{i,h_j}(t)\right]dt + p_{i,h_j}(t)\mathbf{e}_{i,h_j}(t)d\mathbf{z}(t) \quad \text{for all} \quad i \in \{0, ..., I\}, \quad j \in \{0, ..., I\}, \quad h_j \in H_j, \quad t \in [0, t] \tag{5}$$

where r_i is the real instantaneous (risk-free) interest rate in currency *i*, $r_i + a_{i,h_j}$ is the expected return, and $\mathbf{e}_{i,h_j}$ is the volatility of h_j in terms of currency *i*. The volatility coefficients describe the sensitivity of a particular stochastic variable to each Brownian motion.

ASSUMPTION A2: *In each currency the markets are complete and there is no arbitrage.*

A2 implies that for some $h^1, ..., h^n \in H$ the following $n \times n$ dimensional matrix [see e.g. Heath, Jarrow, and Morton (1992)]

(6) $$\mathbf{E}_i(t) = \begin{bmatrix} \mathbf{e}_{i,h^1}(t) \\ \vdots \\ \mathbf{e}_{i,h^n}(t) \end{bmatrix} \quad \text{for all} \quad i \in \{0, ..., I\}$$

is non singular $P \times \lambda$ - *a.s.* (λ is the Lebesgue measure) on the interval $[0, t]$.

Now there exists $P \times \lambda$ - almost unique solution vector $\mathbf{n}_i(t)$ to

(7) $$\mathbf{a}_i(t) = \mathbf{E}_i(t)\mathbf{n}_i(t),$$

where $\mathbf{a}_i(t) = \begin{bmatrix} a_{i,h^1}(t) \\ \vdots \\ a_{i,h^n}(t) \end{bmatrix}$, and, therefore, also a unique equivalent martingale measure exists in each currency *i*. We will refer to $\mathbf{n}_i(t)$ as the market price of risk vector in the *i*'th currency at time *t*.

Given the market price of risk vector it follows that the real state-price deflator is

(8) $$d_i(T) = \exp\left\{ -\int_0^T \left[r_i(t) + \frac{1}{2}\mathbf{n}_i(t)'\mathbf{n}_i(t) \right] dt - \int_0^T \mathbf{n}_i'(t) d\mathbf{z}(t) \right\}$$
$$\text{for all} \quad i \in \{0, \dots, I\}, \quad T \in [0, t]$$

where $\mathbf{n}'$ denotes the transpose of $\mathbf{n}$.

ASSUMPTION A3: *The utility functions of investors are smooth-additive.*

A3 means that utility function U: $\mathbf{R}_+ \to \mathbf{R}$ is defined by

(9) $$U(c) = E\left[\int_0^T u(c, t) dt \right] \quad \text{for all} \quad T \in [0, t]$$

where c is a non-negative consumption in L, u: $\mathbf{R}_+ \times (0,T) \to \mathbf{R}$ is smooth on $\mathbf{R}_+ \times (0,T)$, and for each $t \in [0,T]$, $u(\cdot,t)$: $\mathbf{R}_+ \to \mathbf{R}$ is increasing, strictly concave, with an unbounded partial derivative $\frac{\partial u(\cdot, t)}{\partial c}$ on $\mathbf{R}_+$ satisfying Inada conditions: $\inf_{c \in L} \frac{\partial u(c,t)}{\partial c} = 0$ and $\sup_{c \in L} \frac{\partial u(c,t)}{\partial c} = \infty$ for all *t*.

ASSUMPTION A4: *There exist frictions in trading consumption commodities between different currencies and future frictions are uncertain.*

The uncertainty about the future frictions can be due to an uncertainty about the availability of transporting capacity, the development of technology, and/or the wastage of consumption commodities during the transportation. A4 means that the relationship $d_0(t) = \hat{S}_i(t)\, d_i(t)$, where $\hat{S}_i(t)$ is the nominal price of the *i*'th currency in terms of the domestic currency, does not always hold. This gives a situation in which there can be stochastic real foreign exchange rates, since $p_{0,h_j}(t)$ does not always equal $p_{i,h_j}(t)$. The real exchange rate can be understood as a measure of the difference between $d_0(t)$ and $\hat{S}_i(t)\, d_i(t)$, since from equation (1) we get $\frac{\hat{p}_{i,h_j}(t)}{d_i(t)} S_i(t) = \frac{\hat{p}_{0,h_j}(t)}{d_0(t)}$ which gives $S_i(t) = \frac{\hat{S}_i(t)\, d_i(t)}{d_0(t)}$. That is, we assume that purchasing power parity does not hold because of the frictions. This does not lead to arbitrage opportunities, because of the frictions and the fact that agents can consume only in one currency at a time.

We write the real foreign exchange rates as follows

(10) $$dS_i(t) = S_i(t) m_{S_i}(t) dt + S_i(t)\mathbf{e}_{S_i}(t) d\mathbf{z}(t) \quad \text{for all} \quad i \in \{1, \dots, I\}, \quad t \in [0, t]$$

where $S_i(t)$ is the real price of the i'th currency in terms of the domestic currency. $\mathbf{e}_{S_i}$ and $m_{S_i}(t)$ can be viewed as results of the frictions given in A4. A reasonable candidate for $m_{S_i}(t)$ is $1-S_i(t)$. In this case the foreign exchange process follows same kind of mean reverting process as in the Cox-Ingersoll-Ross (1985) model of the term structure and we get from (10) [see e.g. Duffie (1992)]

(11) $$E_t[S_i(T)] = 1 + [S_i(t)-1]\exp[-S_i(t)(T-t)]$$

Thus $E_t[S_i(T)] \to 1$ exponentially as $T \to \infty$. This is consistent with Dumas (1992).

ASSUMPTION A5: *A risk-free real asset is marketed in each currency.*

3. MARKET PRICE OF RISK

In this section we derive the relationship between domestic and foreign market price of risk vectors.

Now we can state the following theorem.

THEOREM 1: *The foreign and domestic market price of risk vectors are related to each other by the equation*

(12) $$\mathbf{n}_0(t) = \mathbf{n}_i(t) + \mathbf{e}_{S_i}(t)' \quad \text{for all} \quad i \in \{1, \dots, I\}, \quad t \in [0, t]$$

PROOF: From (8) we get the process of real state-price deflator

(13) $$d\, d_i(t) = -d_i(t) r_i(t) dt - d_i(t) \mathbf{n}_i(t)' d\mathbf{z}(t)$$

Since $d_i(t) p_{i,h_j}(t)$ is martingale and $S_i(t) p_{i,h_j}(t) = p_{0,h_j}(t)$ we get that $\frac{d_i(t)}{S_i(t)}$ is a state-price deflator in currency 0, since $d_i(t) p_{i,h_j}(t) = \frac{d_i(t)}{S_i(t)} p_{0,h_j}(t)$. Using Itô's lemma and $d_0(t) = \frac{d_i(t)}{S_i(t)}$ we get

(14) $$\begin{aligned} d\, d_0(t) &= -d_0(t)\{r_i(t) + m_{S_i}(t) - \mathbf{e}_{S_i}(t)[\mathbf{n}_i(t) + \mathbf{e}_{S_i}(t)']\}dt - \\ &\quad d_0(t)[\mathbf{e}_{S_i}(t) + \mathbf{n}_i(t)']d\mathbf{z}(t) \end{aligned}$$

With (13) this gives (12). We also get $r_0(t) - r_i(t) = m_{S_i}(t) - \mathbf{e}_{S_i}(t)\mathbf{n}_0(t)$. *Q.E.D.*

Theorem 1 indicates that the real domestic and foreign market price of risk vectors are never equal in the presence of stochastic real foreign exchange rates. In this situation, international real asset markets are segmented. Hypothesizing that the market price of risk in any individual currency is entirely determined by the interplay of all investors in the international economy, the market price of risk in any other currency are simultaneously determined by the volatility process of the foreign exchange rates.

Theorem 1 also states that the real international asset markets are incomplete although every national market is complete according to A2. This is because there exist I + 1 different equivalent martingale measures in the international markets and in each currency the martingale measure is unique. Further, nominal foreign exchange rates are nonstochastic in our economy, because Theorem 1 also holds with nominal market price of risk vectors and foreign exchange rates and because we have earlier assumed that $p_i(t) = 1$ for all $i \in \{0, \dots, I\}$ and $t \in [0, t]$, i.e., nominal market price of risk vectors are zero vectors.

We illustrate Theorem 1 with an example. Let us assume that there exist two currencies and one tradable asset, h, in our international economy. The process of the asset in terms of currency 1 is the following

(15) $$dp_{1,h}(t) = p_{1,h}(t) r_1(t) dt + p_{1,h}(t) dz(t); \quad p_{1,h}(0) = 1$$

where z is a standard Brownian motion and the process of real foreign exchange rate is given as follows

(16) $$dS_1(t) = S_1(t) dz(t); \quad S_1(0) = 1$$

Using equation (7) and Theorem 1 we get the following solutions for the market prices of risk

(17) $$n_0 = 1$$ $$n_1 = 0$$

4. CURRENCY SELECTION

In this section we consider the optimal consumption and home currency selection problem of a single agent in the international economy. The optimal home currency selection is equivalent to the finding of the most appropriate state-price deflator in the international economy. First we make the following assumption.

ASSUMPTION A6: *There is free investor mobility.*

The assumption A6 implies that all investors are allowed to freely choose their location of residence in any of the available currencies. However, we assume that investors change their home currency only once during the optimization interval.

Given the consumption endowment process e_m in L there exists a dynamic portfolio $f_i = \mathbf{g}\mathbf{p}_i'$, where $\mathbf{p}_i = \begin{bmatrix} p_{h^1,i} & \ldots & p_{h^k,i} \end{bmatrix}$, $i \in \{0,\ldots, I\}$, $h^y \in H$ for all $y \in \{1,\ldots, k\}$, and $\mathbf{g} = \begin{bmatrix} q^1 & \ldots & q^k \end{bmatrix}$ is a trading strategy process in *L*, financing a consumption process *c* in *L* if

(18) $$d_i(t) f_i(t) = \int_0^t d_i(s)\left[e_m(s) - c(s)\right]ds + \int_0^t \mathbf{g}(s) d\left[d_i(s)\mathbf{p}_i(s)'\right] \quad \text{for all} \quad t \in [0,T]$$

and $d_i(T) f_i(T) = 0$, i.e., the terminal consumption is zero, where $T \in [0,t]$.

Now we can state the following lemma.

LEMMA 1: *Given the endowment process e_m in L and any c in L, there exists a process $\mathbf{g} = \begin{bmatrix} q^1 & \ldots & q^k \end{bmatrix}$ in L financing c if and only if*

(19) $$E\left\{\int_0^T d_i(t)\left[c(t) - e_m(t)\right]dt\right\} = 0$$

where $T \in [0,t]$.

PROOF: Since $d_i(s)$ is a real state-price deflator in currency *i*, $E\left\{\int_0^T \mathbf{g}(s) d\left[d_i(s)\mathbf{p}_i(s)'\right]\right\} = 0$. Equation (18) gives (19).

Conversely, if (19) holds, then by martingale representation property of the Brownian filtration [see e.g. Øksendal (1995)] $E\left\{\int_0^T d_i(s)\left[c(s) - e_m(s)\right]ds \mid F_t\right\} = \int_0^t \mathbf{e}_{d_i[c-e_m]}(s) d\mathbf{z}(s)$ for all $t \in [0,T]$, because $d_i(t)$, $c(t)$, and $e_m(t)$ follow Itô processes. Because the markets are complete in each currency and $d_i(s)$ is a real state-price deflator, there exists $\mathbf{g} = \begin{bmatrix} q^1 & \ldots & q^k \end{bmatrix}$ such that $\mathbf{g}(t) d\left[d_i(t)\mathbf{p}_i(t)'\right] = \mathbf{e}_{d_i[c-e_m]}(t) d\mathbf{z}(t)$ for all $t \in [0,T]$. Using (18) we see that $\mathbf{g}$ finances *c*.

Q.E.D.

Given Lemma 1 single agent $m \in M$ faces the following problem

(20) $$\sup_{(c,i)\in\Lambda} U_m(c)$$

subject to

(21) $$E\left\{\int_0^T d_i(t)\left[c(t) - e_m(t)\right]dt\right\} = 0$$

where $\Lambda = \{(c,i) : c \in L, i \in \{0,\ldots,I\}\}$.

THEOREM 2: *The optimal consumption and home currency choice for agent $m \in M$ on a time period $[0,T]$, where $T \in [0,t]$, is given by*

(22) $$c_m^*(t) = I_m\left[\gamma_{i_m^*}\delta_{i_m^*}(t), t\right] \quad \text{for all} \quad t \in [0,T]$$

and

(23) $$E\left\{\int_0^T u_m\left(I_m\left[\gamma_{i_m^*}\delta_{i_m^*}(t),t\right],t\right)dt\right\} \geq E\left\{\int_0^T u_m\left(I_m\left[\gamma_i\delta_i(t),t\right],t\right)dt\right\} \quad \text{for all} \quad i \in \{0,\ldots,I\}$$

where $I_m(\cdot,t)$ inverts $\frac{\partial u_m(\cdot,t)}{\partial c}$*, meaning that* $I_m\left[\frac{\partial u_m(x,t)}{\partial c},t\right] = x$ *for all x and t,* $\gamma_i > 0$ *is a Lagrange multiplier in currency* $i \in \{0,\ldots,I\}$ *satisfying* $E\left(\int_0^T \delta_i(t)\{I_m[\gamma_i\delta_i(t),t] - e_m(t)\}dt\right) = 0$, $i_m^* \in \{0,\ldots,I\}$ *is the optimal home currency, and* $c_m^* \in L$ *is the optimal consumption process.*

PROOF: From (9), (20), and (21) we get the following optimal consumption choice problem for agent $m \in M$ for all $i \in \{0,\ldots,I\}$

(24) $$\sup E\left(\int_0^T \{u_m(c_{i,m},t) - \gamma_i\delta_i(t)[c_{i,m}(t) - e_m(t)]\}dt\right)$$

where $c_{i,m}$ denotes the consumption of agent *m* in currency *i*.

The first order condition for optimality of $c_{i,m}^* > 0$ is

(25) $$\frac{\partial u_m(c_{i,m}^*,t)}{\partial c_{i,m}} - \gamma_i\delta_i(t) = 0 \quad \text{for all} \quad t \in [0,T]$$

where $c_{i,m}^*$ denotes the optimal consumption of agent *m* in currency *i*.

Solving (25) gives

(26) $$c_{i,m}^*(t) = I_m[\gamma_i\delta_i(t),t] \quad \text{for all} \quad t \in [0,T].$$

Under Assumption A3, the Dominated Converge Theorem [see e.g. Rudin (1987)] implies that $J(\gamma_i) = E\left\{\int_0^T I_m[\gamma_i\delta_i(t),t]dt\right\}$ is continuos, and $J(\gamma_i)$ is decreasing because $I_m[\gamma_i\delta_i(t),t]$ is for all $t \in [0,T]$ and $i \in \{0,\ldots,I\}$. Because $J : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is a bijective mapping, there exists a unique $\gamma_i \in \mathbf{R}_+$ such that $J(\gamma_i) = E\left(\int_0^T e_m(t)dt\right)$. Now we have a set of suboptimal consumption processes for the agent *m*, $\Phi = \{c_{i,m}^* : c_{i,m}^* \in L, m \in M, i \in \{0,\ldots,I\}\}$. Given A6 the optimal consumption strategies and home currencies are the ones that satisfy the equation (23) implying the optimality of (c_m^*, i_m^*).

Q.E.D.

Among other things the proof of Theorem 2 shows that the optimal consumption and home currency choice do not have to be unique. In each currency, an agent has a unique optimal consumption strategy, because his utility function is given by (9) and the national markets are complete. However, international markets are incomplete and in different currencies different consumption strategies can give equal utility for the agent.

5. INTERNATIONAL EQUILIBRIUM

In this section we derive an equilibrium that emerges if the representative agents of all currencies optimize continuously their consumption and home currency according to the optimization model presented in the previous section. That is, we analyze the equilibrium relationships between the foreign exchange rate volatility processes, market price of risk vectors, and real interest rates.

The international equilibrium is a collection

(27) $$\left\{d_i; (c_m, i_m), i \in \{0, \ldots, I\}, m \in M_{i_m}, i_m \in \{0, \ldots, I\}\right\},$$

such that, given the state-price deflators d_i for all $i \in \{0, \ldots, I\}$, for each agent m, c_m and i_m solves (20), and markets clear $\sum_{m=M} [e_m(t) - c_m(t)] = 0$ for all $t \in [0, t]$. The market clearing condition ensures that the commodity market clears. Using equation (18) we see that also asset market clears.

From equation (7) we have the following relationship for a portfolio process

(28) $$a_{f_i}(t) = \mathbf{e}_{f_i}(t)\mathbf{n}_i(t) \quad \text{for all} \quad i \in \{0, \ldots, I\}, \quad t \in [0, t]$$

where $d f_i(t) = f_i(t)\left[r_i(t) + a_{f_i}(t)\right]dt + f_i(t)\mathbf{e}_{f_i}(t)d\mathbf{z}(t)$.

We define the supremum of the expected returns with given standard deviation as follows

(29) $$b_i(s, t) = \sup\left[r_i(t) + \mathbf{e}_{f_i}(t)\mathbf{n}_i(t)\right] \quad \text{for all} \quad i \in \{0, \ldots, I\}, \quad t \in [0, t]$$

where the supremum is taken over all portfolio processes such that $s = \sqrt{\mathbf{e}_{f_i}(t)\mathbf{e}_{f_i}(t)'}$.

We call $b_i(\cdot, t)$ the efficient line in currency *i* at time *t*. $b_i(\cdot, t)$ is linear because of A5 [see e.g. Copeland and Weston (1992)]. Now we can prove the following lemma that defines the international efficient contour.

LEMMA 2: *The function* $b(\cdot, t) = \max_{i \in \{0, \ldots I\}} b_i(\cdot, t)$ *is a convex function on* $[0, \infty)$ *for all* $t \in [0, t]$.

PROOF: Because $b_i(\cdot, t)$ is linear, it is a convex function. This gives $b(\cdot, t)$ is a convex function on $[0, \infty)$ for all $t \in [0, t]$. *Q.E.D.*

Given equation (9) the indifference utility curves for all representative agents are convex on $\{r_i + a_{f_i}, s\}$ [see e.g. Copeland and Weston (1992)]. This leads to the following theorem.

THEOREM 3: *There exists* $s_i \in [0, \infty)$ *such that*

(30) $$b(s_i, t) = b_i(s_i, t) \quad \text{for all} \quad i \in \{0, \ldots I\}, \quad t \in [0, t].$$

PROOF: If (30) does not hold for some currency $i \in \{0, \ldots I\}$ then the currency *i* can not be optimal to any agent in our international economy. *Q.E.D.*

Theorem 3 states that for all currencies the national efficient line must equal the international efficient contour at least at one point. Because the international efficient contour is convex, our international equilibrium is curved. Figure 1 illustrates Theorem 3. The thick contour is the international efficient contour. The optimal point for a representative agent is the point where *b* touches the indifference curve. The line b_3 under *b* contradicts Theorem 3 and this cannot exist.

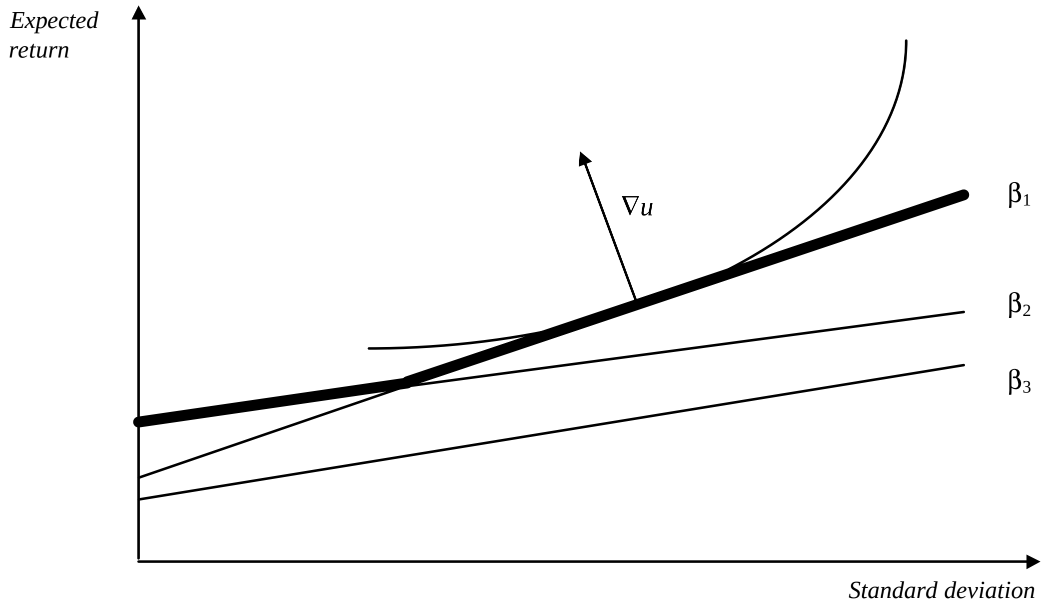

Figure 1: Curved international equilibrium (Ñ*u* is the utility gradient of the agent, the thick contour is the international efficient set, *b*$_i$ is the efficient line in currency *i*, and *i* = 1, 2, 3)

An interesting paradox emerges from the results above. From Figure 1 it can be seen that in equilibrium, the currency with the highest riskless real rate will be populated by the most risk averse investors, because the indifference curve is highly convex. According to Theorem 3, however, the currency with the highest real rate will have the lowest excess return for any given value of the volatility parameter. Otherwise purely dominated currencies would exist. The paradox is due to the fact that the market prices of risk do not reflect the risk attitudes of the investors in any particular home currency, but are simply determined by Theorem 1. That is, stochastic real foreign exchange rates force the market price of risk vectors to differ from each other and, in equilibrium, the real interest rates must be such that each state-price deflator is optimal at least to one agent. This paradox can be proved as follows.

COROLLARY 1: *The following relationship holds*

(31) $$r_0(t) < r_i(t) \quad iff \quad \mathbf{e}_{f_0}(t)\mathbf{n}_0(t) > \mathbf{e}_{f_i}(t)\mathbf{n}_i(t) \quad \text{for all} \quad i \in \{1,\ldots,I\}, \quad t \in [0,t]$$

where $\sqrt{\mathbf{e}_{f_0}(t)\mathbf{e}_{f_0}(t)'} = \sqrt{\mathbf{e}_{f_i}(t)\mathbf{e}_{f_i}(t)'} \in (0,\infty)$, f_0 *is an efficient portfolio in currency* 0, *and* f_i *is an efficient portfolio in currency i.*

PROOF: Since $b_i(\cdot,t)$ is linear for all $i \in \{0,\ldots,I\}$ and Theorem 3 holds we get

(32) $$r_0(t) < r_i(t) \quad iff \quad b_0(s,t) - r_0(t) > b_i(s,t) - r_i(t)$$

where $s \in (0,\infty)$. Because the portfolios are efficient, we have $b_0(s,t) - r_0(t) = \mathbf{e}_{f_o}(t)\mathbf{n}_0(t)$ and $b_i(s,t) - r_i(t) = \mathbf{e}_{f_i}(t)\mathbf{n}_i(t)$, where $\sqrt{\mathbf{e}_{f_0}(t)\mathbf{e}_{f_0}(t)'} = \sqrt{\mathbf{e}_{f_i}(t)\mathbf{e}_{f_i}(t)'} = s$, by the definition of the market price of risk. *Q.E.D.*

This corollary means that the difference between market price of risk between different currencies can be seen as a consequence of the difference between domestic and foreign instantaneous real (risk-free) interest rates. Given Theorem 1 this also gives stochastic real foreign exchange rates. Corollary 1 also sets restrictions to the diffusion processes of foreign exchange rates, since using Theorem 1 we get from equation (31)

(33) $$r_0(t) < r_i(t) \quad iff \quad \mathbf{e}_{f_0}(t)\mathbf{e}_{S_i}(t)' > 0 \quad \text{for all} \quad i \in \{1,\ldots I\}, \quad t \in [0,t]$$

where $\sqrt{\mathbf{e}_{f_0}(t)\mathbf{e}_{f_0}(t)'} \in (0,\infty)$.

Equation (33) implies that if the domestic real risk-free interest rate is lower than the corresponding foreign rate then an efficient portfolio must have positive correlation with the foreign exchange rate.

Let us continue our example [equations (15) – (17)] and set $r_0(t) = 0$. By assuming that the volatility of an efficient portfolio is strictly positive in currency 0, we get $0 < r_1(t)$ from Corollary 1.

6. SUMMARY

In this paper we have shown that given stochastic real foreign exchange rates, international markets are segmented. The stochastic real foreign exchange rates force the market price of risk vectors in different currencies to differ from each other. By using the relationship between the market prices of risk, we have derived the single investor's optimal consumption and home currency and a curved equilibrium in which representative agents behave according to the optimization model. In equilibrium, the real interest rates must be such that each currency is optimal at least to one agent.

If there is free investors mobility and the investors maximize only the utility from their consumption, the currency with the highest riskless real rate will be populated by the most risk averse investors. Also in this situation the market prices of risk do not reflect the risk attitudes of the investors in the particular home currency, but are simply determined by the interplay of all investors in the international economy.

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