Aalto University School of Science and Technology

Systems Analysis Laboratory Research Reports A105, December 2010

A LEARNING APPROACH FOR NONLINEAR PRICING PROBLEM

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Dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the Faculty of Information and Natural Sciences for public examination and debate in Auditorium E at Aalto University School of Science and Technology (Espoo, Finland) on the 14th of January, 2010, at 12 noon.

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This report is downloadable at

www.sal.hut.fi/Publications/r-index.html

ISBN 978-952-60-3538-3

ISBN (pdf) 978-952-60-3544-4

ISSN 0782-2030

Multiprint Oy

Espoo 2010

Title: A Learning Approach for Nonlinear Pricing Problem

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Date: December 2010

Abstract: Quantity discounts are frequent both in everyday life and in business. Take, for example, product pricing, gas and electricity pricing, transportation and postage pricing, telecommunications, cable TV and Internet access pricing. These are all examples of nonlinear pricing, where the selling firm designs differentiated products and prices them according to the firm's marketing strategy. Nonlinear pricing is also a general model of incomplete information and it has a plenty of applications, such as regulation, taxation and designing labor contracts.

This Dissertation develops a new learning approach for the nonlinear pricing problem, where the selling firm has limited information about the buyers' preferences. The main contributions are i) to show how the firm can learn what kind of products should be put up for sale, and what information the firm needs to do this, ii) to introduce a new approach in modeling incomplete information using optimality conditions, iii) to analyze mathematically the general pricing problem with many buyer types and multiple quality dimensions, and iv) to examine the computational issues of solving the pricing problem.

The learning method is based on selling the product repeatedly. The firm sets linear tariffs, from which the buyers select the product they wish to consume. This reveals the buyers' marginal valuations, which is exactly the information that is needed to evaluate the optimality conditions. By evaluating the different optimality conditions, the firm learns the buyers who get the same product at the optimum and the buyers who are excluded. Different learning paths are examined in terms of profit, learning time and the buyers' preferences.

Keywords: nonlinear pricing, incomplete information, learning, adjustment, mechanism design, computation

Otsikko: Oppimismenetelmä epälineaarisessa hinnoittelussa

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Päiväys: Joulukuu 2010

Tiivistelmä: Ostettuun määrään perustuvat alennukset ovat yleisiä sekä arjessa että liike-elämässä. Hyviä esimerkkejä ovat mm. tuotteiden hinnoittelu (ota kolme, maksa kaksi), kaukolämmön ja sähkön hinnoittelu, liikenteen ja kuljetusten hinnoittelu, telekommunikaatio-, kaapelitelevisio- ja Internet yhteyksien hinnoittelu. Nämä ovat kaikki esimerkkejä epälineaarisesta hinnoittelusta, missä myyvä yritys suunnittelee valikoiman erilaisia tuotteita ja hinnoittelee ne yrityksen markkinointistrategian mukaisesti. Epälineaarisen hinnoittelun matemaattinen malli on lisäksi yksi keskeisimpiä epätäydellisen informaation malleja, ja sillä on useita sovelluksia, kuten sääntely, verotus ja työsopimusten suunnittelu.

Tässä väitöskirjassa kehitetään uusi oppimiseen perustuva lähestymistapa epälineaarisen hinnoittelun tehtävässä, jossa yritys ei tarkalleen tiedä asiakkaiden mieltymyksiä. Työn päätavoitteet ovat
1) näyttää miten yritys voi oppia millaisia tuotteita sen tulisi myydä ja mitä informaatiota yritys tarvitsee tähän, 2) esitellä uusi epätäydellisen informaation mallinnustapa käyttäen optimaalisuusehtoja, 3) analysoida matemaattisesti yleistä hinnoitteluongelmaa, jossa on useita ostajia ja laatudimensioita, ja lisäksi 4) tutkia hinnoitteluongelman laskennan kysymyksiä.

Oppimismenetelmä perustuu tuotteiden toistettuun myymiseen. Yritys asettaa lineaarisia tariffeja, joista asiakkaat valitsevat haluamansa tuotteen. Asiakkaiden tekemä valinta paljastaa heidän marginaalisen hyödyn, mikä on juuri yrityksen tarvitsema informaatio optimaalisuusehtoja käytettäessä. Kokeilemalla erilaisia optimaalisuusehtoja, yritys oppii ne asiakkaat joille myydään samaa tuotetta ja ne asiakkaat joille ei kannata myydä tuotetta laisinkaan. Työssä tutkitaan erilaisia oppimismenetelmiä eri kriteerien valossa, kuten oppimisaika, yrityksen voitto ja ostajien mieltymykset oppimisaikana.

Avainsanat: hinnoittelu, epätäydellinen informaatio, oppiminen, mekanismin suunnittelu, laskenta

Academic Dissertation

Systems Analysis Laboratory
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Faculty of Information and Natural Sciences
Aalto University School of Science and Technology

A Learning Approach for Nonlinear Pricing Problem

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	PhD David Vengerov, Oracle, USA
Official opponent:	Associate Professor Tommy Andersson, Lund University, Sweden

Publications

The Dissertation consists of the present summary article and the following papers:

[I] Ehtamo, H., Berg, K., Kitti, M. (2010). An adjustment scheme for nonlinear pricing problem with two buyers. *European Journal of Operational Research*, Vol. 201, pp. 259-266.

[II] Berg, K., Ehtamo, H. (2009). Learning in nonlinear pricing with unknown utility functions. *Annals of Operations Research*, Vol. 172, Num. 1, pp. 375-392.

[III] Berg, K., Ehtamo, H. (2008). Multidimensional Screening: Online Computation and Limited Information. ICEC '08: Proceedings of the *10th International Conference on Electronic Commerce*, ACM International Conference Proceedings Series **42**. Innsbruck, Austria.

[IV] Berg, K., Ehtamo, H. (2010). Interpretation of Lagrange multipliers in nonlinear pricing problem. *Optimization Letters*, Vol. 4, Num. 2, pp. 275-285.

[V] Berg, K., Ehtamo, H. (2010). Continuous Learning Methods in Two-Buyer Pricing Problem. *Systems Analysis Laboratory Research Report* E24, Aalto University School of Science and Technology.

Contributions of the author

Papers [II], [III], [IV] and [V] were initiated and primarily written by Berg. In these papers, Berg is also solely responsible for the development and implementation of the methods, the computation and the analysis of the results. An exception is the numerical example in Section 4.4 of Paper [III], which was computed by Arttu Klemettilä. Paper [I] was initiated by Ehtamo, and jointly written by Ehtamo, Berg and Kitti. In this paper, Berg is especially responsible for developing the algorithm and writing Sections 4 and 5.

Preface

This thesis has been carried out in the Systems Analysis Laboratory of Aalto University School of Science and Technology. I wish to express my gratitude and appreciation to my supervisor and the man of commonwealth, Professor Harri Ehtamo, for guidance, patience and encouragement during my thesis work. The academic path was made clear: from the summer job to the Master's thesis and continuing from there all the way to the doctoral dissertation. I would also like to thank Dr. Mitri Kitti, a co-author of mine, who always seemed to grasp the new ideas fast. I appreciate the positive comments of the preliminary examiners Professor Babu Nahata and PhD David Vengerov.

I am also grateful to Professor Raimo P. Hämäläinen, the founder and director of the laboratory, for all the work done behind the scenes in order for me to accomplish the thesis. I acknowledge the funding from the graduate school in Systems Analysis, Decision Making and Risk Management. Beside the extraordinary working environment, it has offered various activities including scientific writing seminars, summer school and conference trips.

It is the laboratory staff who have made the rich and pleasant working environment. We have shared ideas and intrinsic knowledge in daily quiz breaks, Monday seminars and sometimes simply having the door open and letting the *mouhuaminen* get in. Moreover, we have kept up the long tradition of Friday floorball matches between the mathematicians upstairs. Many thanks goes to the organizers of special events like orienteering, Christmas parties and dinners.

Finally, I want to thank my parents for the support during all these years and the inherited interest in maths. I am thankful to my brother Mikko for giving me a broader view on life and science. I have also received constant support from my grandparents, who have expressed the wish to see their grandchildrens' doctoral defence. I would also like to thank my friends and teammates in ultimate and online riddles.

Espoo, December 2010

Kimmo Berg

Contents

1 Introduction

'I should like to buy an egg, please,' she said timidly. 'How do you sell them?'
"Fivepence farthing for one -- Twopence for two,' the Sheep replied.
'Then two are cheaper than one?' Alice said in a surprised tone, taking out her purse.
'Only you MUST eat them both, if you buy two,' said the Sheep.
'Then I'll have ONE, please,' said Alice, as she put the money down on the counter. For she thought to herself, 'They mightn't be at all nice, you know.'
Through the Looking-Glass, Lewis Carroll (Carroll 1871, Chapter V)

The prices have very important role in society. The firms use pricing in marketing their products, and the prices affect the firms' production decisions. The prices also ease the exchange of goods and they carry information about the values of the products and services. The prices affect both the demand and the supply side of the economy, and thus the prices are associated with economic efficiency. But where do the prices come from?

There are almost as many mechanisms to trade a product as there are different kind of products. Bargaining processes can be used in selling or buying expensive or unique items. For example, in 1626 the Dutch colonizer Peter Minuit acquired Manhattan island from native Americans in exchange for trade goods worth 24 dollars. Modern alternatives for bargaining are different kinds of auctions, where the participants compete by bidding, i.e., offering a price for the product. The auctions are used, e.g., in selling antique, art, collectibles, estate and flowers, just to name a few. In electricity auctions, the bids to buy and the offers to sell determine the trading prices. The long-term contracts and the derivatives, such as futures and options, can be traded in exchange markets. For example, Nord Pool founded in 1996 is the world's first multinational exchange for trading electric power between Norway, Denmark, Sweden and Finland. Electricity is an example of a commodity that is difficult to store, which is one reason why it has a special trading mechanism.

The most common pricing mechanisms are, however, posted price mechanisms (Elmaghraby and Keskinocak 2003), where the seller sets the prices and the buyers choose the product they wish to consume or buy nothing at all. For example, a firm providing public transportation may set the fares based on the distance of the trip, zones or the period of time. The prices may be set to maximize the firm's revenue, recover costs, or if the firm is owned by the government then maximize the social welfare under budget constraints (Wilson 1993). Similar applications are mobile phone subscription and broadband Internet access pricing, where the prices may depend on the number of SMS messages sent, nominal data rate (Mbit/s), location and the technology used. These are examples of nonlinear pricing, where the seller designs differentiated products with suitable prices.

One important factor in pricing is the market structure, i.e., how many buyers and sellers there are in the market and what is their market power (Mas-Colell et al 1995). If the market consists of a monopoly and many buyers, then it is said that the monopoly is a price maker and has high market power whereas the buyers are price takers and have no market power. On the other hand, if the market allows free entry and there are many producers, then it is a perfectly competitive market and the firms are price takers. Between these two extremes there are different oligopolies, e.g., the classic Cournot and Bertrand models, cartel and imperfect competition models, where a small number of firms control the market.

Another important factor is information asymmetry. If a firm is selling a product to a group of buyers and is planning the price, then is it reasonable to assume that the firm knows how much the buyers are willing to pay for the product? The firms rarely have complete information about the buyers' preferences, but on the other hand that may not be needed to achieve the optimal pricing. There are many approaches to solve the problem of incomplete information. The firm may estimate the demand with different methods, i.e., get the probability distribution over the buyers' valuations, or the firm may learn good prices by selling the product repeatedly and adjusting the prices.

This Dissertation develops a new learning approach for the nonlinear pricing problem, where the seller has limited information about the buyers' preferences. Mathematical theory and numerical methods are developed, where the firm uses specific pricing schemes to reveal information about the buyers' valuations. The acquired information is then used in adjusting the prices towards the firm's objectives. More explicitly, Papers [I] and [II] develop the learning approach when so-called single-crossing property holds. Papers [III] and [IV] analyze the more general nonlinear pricing problem where the product has multiple characteristics or qualities. These papers also examine the computational issues of solving the problem numerically. Paper [V] compares the optimal learning path computed with complete information against the different methods that use only limited information. This paper gives a new estimate to the value of information and a suggestion for a good learning method when the whole learning period is considered.

This summary is structured as follows. Section 2 discusses the different functions of pricing. Section 3 introduces the basic pricing models. Nonlinear pricing is an application of a general model of asymmetric information, which is discussed in Section 4. The different approaches to model incomplete information and learning in pricing problems are discussed in Section 5. Section 6 summarizes the contributions of the Dissertation. Finally, future research directions and conclusions are presented in Section 7.

2 Role of Prices

Before money was invented the trading was based on barter and gift economics. Barter is based on the coincidence of wants, where goods or services are exchanged without the medium of exchange, such as money. The idea of gift economics can, however, be interpreted through social status and reciprocal altruism. You hand out gifts and do favors, and doing so you expect to gain higher status and get the same treatment back in similar situations. The role of money was formalized in Babylonia when debt and law codes were developed. The interest on debt is a compensation in money for breaking the law of not paying back in time. The money is also a solution to the coincidence of wants problem, and the prices give a measure of value to the goods and services.

The prices do not necessarily reflect the exact value of the good to the seller nor the buyer. For example, the Russians sold Alaska in 1867 to the United States for 7.2 million dollars. According to Bolkhovitinov (2003), the Russians were expecting 5 million dollars and probably the United States valued the land more than the final sum of 7.2 millions. The price, however, reflected more the Russian financial position and the military state after the Crimean War rather than the value of the land.

In neoclassical economics the prices and the market equilibrium is determined by the supply and the demand. The prices itself are just transfers that determine the redistribution of income between the parties in the economy. The important role of prices comes from the indirect effect. The prices influence the economic efficiency by affecting both the supply and demand side through the firms' production and the consumers' purchase decisions. From the society's point of view, it is important to design the markets so that the prices are formed and the parties behave in an efficient way. Mechanism design theory is a suitable framework for studying this kind of problems, where the emphasis is on incentives and private information. Mechanism design and its relation to nonlinear pricing is discussed more in Section 4.

Besides the efficient utilization of resources, pricing has several other roles in practice (Wilson 1993). Pricing is one aspect of the four Ps in the marketing mix, which also includes Product, Promotion and Place. Pricing can be used in cost recovery, firm's strategy, competition, market penetration and capturing market share, growth, product placement and positioning (Dobson and Kalish 1988), price skimming, revenue management and profit maximization, inventory clearance sales, and signaling the quality of the product (McConnell 1968), among other things. The price itself may also be the whole business idea of a firm. For example, a dollar store is a retail store that sells inexpensive items, usually with a single price for all items in the store. As there are several roles of prices, there are almost as many pricing models. Some of these are discussed in the next section.

3 Modeling the Pricing Situation

Pricing is a form of art and economic models will probably never beat a good car salesman in making the sales. But leaving psychological and sociological issues aside, the pricing models capture many important principles and practical considerations (Nagle 1984) including inventory pricing (Karlin and Carr 1962, Elmaghraby and Keskinocak 2003), capacity and peak load pricing (Oren et al 1985), road and congestion pricing (Vickrey 1952), priority pricing, price discrimination (Pigou 1932, Phlips 1988, Armstrong 2006, Stole 2007, Armstrong 2008), spatial pricing (Hotelling 1929), pricing durable goods, zone pricing, asset and stock pricing (Black and Scholes 1973, Merton 1973), retail pricing (Lazear 1986) and bundling products (Stigler 1963, Adams and Yellen 1976, Palfrey 1983), again to make the long list short.

Pricing can be modeled on different levels of abstraction, including industry, market and transaction levels. The industry level examines the supplier side price changes and the customer demand changes. The market level focuses on the competition between the products on the market, differentiation and customization issues. Pricing at the transaction level examines the discounts off the list prices. For example, a supplier may set different discount percentages for a customer on the different product lines depending on the volume of the sales of each line.

The simplicity of the tariff may also be an important aspect of pricing. A monthly flat rate may be easier to implement and more convenient than complicated tariffs based on multiple factors. The more complicated tariffs allow, however, more efficient pricing where the costs are distributed based on the service usage. For example, in pay-per-view the customer pays only for the chosen television shows and the residential water and electricity costs may be divided based on water and electricity meters in housing cooperatives rather than dividing the costs based on flat rate per person or household.

Market equilibrium and market behavior depend strongly on the market structure (Mas-Colell et al 1995). In perfectly competitive market, the goods are traded at publicly known prices and the sellers and the buyers act as price takers. According to the fundamental theorems of welfare economics, the equilibria of competitive markets are Pareto efficient. The assumptions of competitive market do not, however, hold in real markets and the allocations may not be efficient, which is called as market failure. The market failure originates often from externalities, asymmetric information and non-competition, where the firms may have barriers to enter the market or some firms have market power. Examples of such are monopoly (Spence 1977b, Mussa and Rosen 1978, Maskin and Riley 1984) and oligopoly pricing models (Spence 1977a, Oren and Wilson 1983, Ivaldi and Martimort 1993), which include Cournot and Bertrand duopoly models, collusion and cartels modeled with repeated games (Green and Porter 1984, Abreu et al 1986; 1990) and supply function equilibria (Klemperer and Meyer 1989).

In this Dissertation the main assumptions are that the seller can set the prices and differentiate the product, e.g., sell different quantities or qualities of the product. When the tariff is not strictly proportional to the quantity purchased, the pricing situation is called as nonlinear pricing (Wilson 1993). The main focus is to study incomplete information in a monopoly model, even though the model could be extended to include competition by making small changes to the model. In the next section, it is discussed that the mathematical model is a general model of contracting under asymmetric information, and thus the results of this Dissertation apply as well to the other applications, such as taxation and regulation.

4 Models of Asymmetric Information

Information, uncertainty and ignorance are one of the most important aspects of modeling in economics (Stigler 1961, Arrow 1963). The cornerstone of modeling the incomplete information was laid in 1967 when John C. Harsanyi defined the Bayesian game (Harsanyi 1967–1968). The theory of uncertainty spread to the applications of economics such as the market for lemons (Akerlof 1970), i.e., the market of used cars, taxation (Mirrlees 1971), screening (Stiglitz 1975), monopoly pricing (Spence 1977b; 1980, Mussa and Rosen 1978, Harris and Raviv 1981), insurance (Stiglitz 1977), auctions (Myerson 1981, Riley and Samuelson 1981), credit rationing (Stiglitz and Weiss 1981) and regulation (Baron and Myerson 1982). What is most surprising about these models is that they all have similar mathematical models. They can all be modeled with contract theory (Bolton and Dewatripont 2005) and principal-agent framework (Ross 1973, Grossman and Hart 1983).

The principal agent models can be divided into two broad categories: adverse selection (Riley 2001, Stiglitz 2002) and moral hazard (Holmstrom 1979; 1982). Moral hazard is also known as the model of hidden action, where the principal cannot perfectly monitor the agent's action. For example, a firm may condition the manager's wage based on the firm's profit but not on the manager's actual effort. Adverse selection is also known as the model of hidden information, and it can be modeled with signaling (Spence 1973) and screening games. In job market signaling, a worker signals her competence to the employer, e.g., by acquiring educational credentials. The employer assumes a good signal is correlated with greater ability to work and offers a higher wage.

An example of a screening or self-selection application is the nonlinear pricing model. A monopolistic seller produces a product to a market with two types of buyers: a high type that values the quality more and is willing to pay more for the product and a low type with lower valuation for quality. The monopoly designs two products with different qualities so that the profit is maximized and the buyer types choose the products intended for them, i.e., the high type chooses the high quality bundle and the low type the low quality bundle. The buyers may choose any bundle they wish or buy nothing at all, and the firm must take this into account when designing the bundles, that is, the qualities and their prices. The incomplete information here means that the monopoly may not give individual offers to the different buyer types, i.e., the monopoly does not distinguish the buyers.

Another example is monopoly regulation (Baron and Myerson 1982). A government regulates a firm so that it does not behave as a monopoly. The government has, however, incomplete information about the firm's costs. The government designs a payment scheme to the firm which is based on the firm's production level so that the social surplus is maximized. A higher production level means a bigger payment to the firm, and the firm chooses the production level based on its true costs and the designed payment scheme.

The screening model is also an instance of mechanism design (Mas-Colell et al 1995, Nisan and Ronen 2001, Conitzer and Sandholm 2002, Dash et al 2003), which examines different mechanisms with which desirable outcomes could be achieved. The focus of mechanism design is on identifying desirable goals, the players' private information, the players' incentives to act in a desirable way and the implementation of the goals with a mechanism. The study of mechanism design originates from resource allocation problems (Hurwicz 1960; 1972; 1973, Hurwicz et al 1975).

5 Pricing under Incomplete Information and Learning

Nonlinear pricing is an application of the general screening model. It is not just one mathematical model but multiple models that differ slightly depending on whether the buyer type is modeled with continuous or discrete distribution, and whether the product has multiple or only one quality dimension. For example, the model of Spence (1980) is a discrete type, multidimensional model where the dimension is interpreted as quantity, whereas the model of Mussa and Rosen (1978) is a continuous type model with a product of single quality. The multidimensional models are examined in Wilson (1991; 1993), Armstrong (1996), Rochet and Chone (1998), Armstrong (1999), Armstrong and Rochet (1999), Armstrong and Vickers (2000), Rochet and Stole (2003), Nahata et al (2004), Basov (2005); see Räsänen et al (1997) for an application in electricity markets.

The mathematical model has many interpretations. The model can be interpreted as the seller's uncertainty about the buyer's preferences. The probability distribution describes the seller's belief over the possible buyer types. The model can also be interpreted as self-selection model where there is no incomplete information but a pricing rule that enforces public prices. The seller designs a public tariff, and the buyers self-select the bundle they wish to consume from the tariff. The distribution now describes the fractions of different buyer types in the population. When the pricing situation is examined as a single decision problem, the interpretation does not play a big role, but it does when the pricing situation is repeated. It is a different situation if there is a population of buyers rather than one buyer whose valuation is unknown.

When the seller has limited information and the pricing situation is repeated, the question arises whether the seller can learn the optimal pricing or not. And if the seller can, then what is the best way to learn it under different assumptions. There are many approaches to model learning (Fudenberg and Levine 1999) and incomplete information. These include Bayesian techniques (Keller and Rady 1999), auctions (Myerson 1981), multiagent learning (Sandholm 2007), reinforcement and Q-learning (Tesauro and Kephart 2002), different heuristic methods such as hill climbing methods (Brooks et al 2002), active and passive learning (Balvers and Cosimano 1990, Braden and Oren 1994, Bischi et al 2008), tatonnement and Cournot adjustment (Kitti 2010), dynamic programming (Bertsimas and Perakis 2006), dynamic pricing (Elmaghraby and Keskinocak 2003, Garcia et al 2005), stochastic programming and robust optimization (Adida and Perakis 2006) and different nonparametric methods (Carlier 2002).

When learning is modeled it is important to define what the players know, how they choose the strategies, how they gain more information and what is the interpretation (Camerer 2003). The most simple models that do not require much sophistication from the players are evolutionary, imitation and reinforcement approaches. In more sophisticated rule and belief-based models the players update their beliefs about what others will do and choose the strategies based on these beliefs. The sophistication allows the players to experiment actively and produce information about the other players. In pricing the tradeoff in experimentation is between the gain of information and higher profits in the future against the lower profit now.

In this Dissertation it is assumed that a monopolistic firm sells a product to a large population of buyers with different valuations. The firm does not know exactly the buyers' preferences but segments the buyers with similar preferences into groups. For simplicity, it is assumed that the firm knows the number and the sizes of the groups, i.e., the number of buyers in a group, but does not know the utility functions that represent each group. The firm designs pricing schedules that produce information about the utility functions so that the firm can learn how to sell the product more profitably. The learning is based on the assumption of buyers' myopicity. A myopic behavior means that the buyers choose the bundles from the pricing schedule by maximizing their utilities.

The learning approach is nonparametric in the sense that the firm needs not assume any probability distribution over the utility functions nor assume any specific shape of utility functions. The good thing about this is that it allows generalization and avoids making wrong assumptions when the utility functions are unknown. On the contrary, if the firm knows the shape of utility functions, then it should be taken into account in the method and it may speed up the learning process. The learning approach can also be seen as gradient or reinforcement learning, where the firm estimates the direction of profit increase and adjusts the pricing schedule towards this direction.

6 Contributions

Papers [I] and [II] show how the firm can learn the optimal solution in a pricing problem where the product has a single quality dimension. Paper [I] studies a pricing problem with two buyer types and suggests an adjustment approach using discrete steps. It is reasonable to assume only two types in some applications, e.g., in pricing phonecalls where there are two natural customer segments of business and personal use (Jain et al 1999). Paper [II] is an extension to more than two customer segments.

Papers [III] and [IV] examine the multidimensional problem where the buyers' utility functions need not be ordered. Paper [III] analyzes the problem mathematically and examines what modifications need to be done in the learning method. Paper [IV] gives an interpretation to the Lagrange multipliers of the problem and studies the computational side of the problem.

Paper [V] examines continuous learning paths instead of using discrete steps. The methods that use limited information are compared with each other and the optimal path which is computed with complete information. The main idea of the paper is to find good methods under different criteria when the whole learning period is considered.

The contributions of each paper are now explained more thoroughly.

6.1 Adjustment in a Unidimensional Problem

The adjustment approach was introduced in Ehtamo et al (2002) and Kitti and Ehtamo (2009), where it is shown that the equilibrium arises as a long run outcome of an adjustment process. In Ehtamo et al (2002), the players who grope their way towards the Pareto optimal outcome have only one type. They also postulate an extra player, a mediator, who could help the principal and the agent in the negotiations. The mediator could find the equlibrium by using linear contracts without knowing the parties' utility functions.

Paper [I] takes another view on the adjustment approach. It is assumed that the monopolistic seller can set the prices, and there are two types of buyers. The adjustment is now more complicated as the equilibrium is not a single negotiable variable and its price but two quantity-price bundles, that is, one for each buyer type. The aim of the adjustment is also different. The seller adjusts the price schedule towards the profit-maximizing solution, which may not be Pareto optimal. The method to reveal information about the buyers' preferences is similar to Ehtamo et al (2002). When the seller offers the buyers a linear tariff, the buyer's choice of utility-maximizing amount will reveal the slope of the utility function at that chosen quantity. With this information the seller may adjust the bundles towards the optimal solution.

Paper [I] develops the optimality conditions for the pricing problem under standard assumptions made in the literature, and it shows how these conditions can be used in adjusting the price schedule under limited information. The assumptions eliminate some pathological pricing situations, and they make it possible to learn the optimal bundles using only local information about the buyers' preferences. It is examined in Paper [III] that the relaxation of the assumptions adds little complexity to solving the problem with only two buyer types. Paper [I] assumes the standard single-crossing property, which restricts the shape of buyer types' utility functions. This combined with the other assumptions mean that it is optimal to sell positive amounts to both buyers, the solution is never Pareto optimal and certain constraints are active in the optimum. It is discussed in Paper [III] that the optimal bundles may actually be efficient and the utility functions need not be peculiar for this to happen. When the more general utility functions are allowed, also the adjustment method needs to be modified a little from what is presented in Paper [I].

The optimality conditions in Paper [I] give the equations that determine the optimal bundles. There are two equations for the optimal prices. The first equation means that the buyer type who values the product less, the low type, is indifferent between having the bundle or not, i.e., the price equals the valuation. The second equation means that the buyer type who values the product more, the high type, is indifferent between the high and low bundles, i.e., the price difference equals the valuation difference of the bundles. There are also two equations for the optimal quantities. The equation for high type means that the marginal valuation equals the marginal cost at the optimal quantity. The equation for low type means that the marginal profit of the low bundle equals the difference of marginal valuations at the optimal quantity. So, from the seller's point of view the optimal quantities depend on the marginal valuations, and the optimal prices depend on the valuations itself. Furthermore, the optimal prices depend on the optimal quantities and not vice versa, and thus the optimal quantities should be solved first. Also, the optimal price of high bundle depends on the optimal price of the low bundle. This means that there is a natural order in which to solve the optimal bundles.

To solve the optimal quantities, the seller needs to know the buyers' marginal valuations. For the high bundle, the marginal valuation should equal the marginal cost. The seller can learn this quantity by offering linear tariffs as was initially suggested in Ehtamo et al (2002). The seller sets a slope for the tariff and adjusts it so that the optimality condition is met. The seller learns the marginal valuations since the buyers choose profit-maximizing quantities from the linear tariff. There is, however, a better way to find the optimal quantity in one iteration. The seller can use its cost function plus constant as a nonlinear tariff, and the buyers now choose automatically quantities so that the marginal valuations equal the marginal costs.

Learning the optimal quantity for the low bundle is a bit more complicated and it is the main idea of Paper [I]. Since the optimality condition involves both the marginal valuations of low and high types, the equation consists of two unknown terms for the seller. The seller could offer multiple linear tariffs and adjust the slopes so that both types choose the same quantity. This way the seller could evaluate the optimality condition at a certain quantity, and learn whether this quantity is lower or higher than the optimal amount. But again there is another way to evaluate the optimality condition in just two iterations. The idea is that the seller may first solve the low type's marginal valuation at some quantity and then solve what the high type's slope should be in order to satisfy the optimality condition. The seller then sets a tariff with this computed slope and tests whether it is the real marginal valuation for the high type or not. This way the seller learns whether the quantity is lower or higher than the optimal amount, and it gives the direction for adjustment.

Once the optimal quantities are found, the final step is to find the optimal prices. The prices can be learned by raising and lowering the prices and giving the buyers take it or leave it offers. The seller learns that the price is too high when the buyer refuses to buy its bundle. The firm can now find the optimal price with a simple method.

Paper [II] is a generalization to more than two buyer types. It analyzes the problem mathematically, examines what happens when there are many buyer types and shows how the learning method should be modified. The most fundamental change with many buyer types is that some types may get the same bundle at the optimum and this is called as bunching. It may also be optimal that the firm does not sell the product to all buyer types, which is called as exclusion. This means that different types are bunched and excluded when the buyers have different utility functions. From the learning point of view the firm does neither know the active constraints at the optimum nor the correct optimality conditions to be solved. But it is shown that when the single-crossing and appropriate convexity assumptions hold the seller can learn the optimal structure, i.e., who to bunch and who to exclude.

The first observation is that the optimality conditions consist of a marginal valuation of the lowest type in the bunch and a marginal valuation of the type above the highest type in the bunch. So again, the conditions consist of two unknown terms for the seller. Proposition 1 in Paper [II] shows a way to learn which types should be bunched and excluded. This adds another step in the learning method. The seller first learns who to bunch while the product's quality is adjusted. This is done by evaluating multiple optimality conditions. When the optimal bunch is known, the learning method is similar to the method in Paper [I] as only one condition needs to be evaluated. Paper [II] also offers some improvements to the learning method by introducing intervals and areas of uncertainty. It is also suggested that the buyers' utility functions could be approximated and estimated collectively rather than one by one, which could improve the learning method when there are many buyer types.

6.2 Multiple Dimensions and General Utility Functions

All theories have limiting assumptions. In nonlinear pricing, one of these assumptions is the single-crossing property and the related Spence-Mirrlees condition (Edlin and Shannon 1998). This condition restricts the shape of buyers' utility functions and assumes that the valuations can be ordered. The single-crossing condition was introduced in the multidimensional problem by McAfee and McMillan (1988). They showed that the multidimensional problem can be reduced to the single dimensional problem and thus it can be solved the same way provided that the single-crossing condition is satisfied. It has been later examined what happens when the assumption is not valid anymore (Wilson 1993; 1995, Araujo and Moreira 1999, Nahata et al 2001; 2003), i.e., the buyers' utility functions can be of general shape and the valuations need not be ordered the same way in all dimensions.

From the mathematical point of view, the relaxation of the single-crossing condition is dramatic as the assumption simplifies the problem considerably. Under the assumption, only small number of constraints, i.e., the local downward constraints (Maskin and Riley 1984), can be active at the optimum. This means that the structure of the solution is of chain type (Nahata et al 2004). From the economic point of view, the assumption affects the efficiency of the solution (Andersson 2005, Nahata et al 2006, Andersson 2008). Under the assumption, only the highest buyer type gets the efficient bundle, whereas the whole solution may be efficient when the valuations are appropriate, for example, when the buyers are not interested in each others' bundles.

From the learning point of view, the assumption has significance for two reasons. Firstly, the seller can learn the active constraints easily, as was shown in Paper [II], since there are not so many combinations as there can be without the single-crossing assumption. Secondly, the optimality conditions consist of no more than two types' marginal valuations, whereas the conditions may have many marginal valuations when the assumption is violated. This means that the optimality conditions are more complicated to solve under limited information.

Papers [III] and [IV] generalize the pricing problem to multiple dimensions and general shapes of utility functions. This means that the seller designs for each buyer type a bundle consisting of a price and multiple qualities that define the product. Paper [III] develops an important notion of directed graph (digraph) presentation which helps in representing and analyzing the solution; see Nahata et al (2004) for related digraphs in more general problem with type-splitting and general cost structure. The digraph basically consists of the buyer types and the active constraints between the types. The Lagrange multipliers can be interpreted as flows between the buyer types and the multipliers together form a flow network. The Lagrange multiplier interpretation is discussed more thoroughly in Paper [IV].

The digraph presentation makes it easy to analyze the solution. First, the digraph represents the relation of the bundles, i.e., which bundles are distorted in order to gain better profits from the other bundles and how the prices are related to each other. It is also possible to do sensitivity analysis with respect to changes in the buyer's preferences. With small changes it may happen that the active constraints do not change, and with bigger changes it is possible to guess the new active constraints and the corresponding digraph. Second, the bundles position in the digraph is associated with the profitability and efficiency of the bundle. The digraph gives a partial order to the bundles in terms of profit. The most profitable bundles are at the end of the digraph, and these must also be the efficient bundles in terms of quality.

The structure of the digraph can be used in solving the optimization problem more efficiently, which is explained in Paper [IV]. If the digraph consists of parts that do not have active constraints between them, then these parts can be solved in parallel, i.e., independent of each other. Also, some other features of a specific pricing problem can be used in enhancing the optimization. For example, the number of constraints can be reduced dramatically when the buyers' utility functions are known approximately and the Lagrange multipliers can be deduced when they have distinctive values.

The most important part of solving the pricing problem is finding the active constraints as it creates considerable complexity of solving the problem. When the active constraints and the Lagrange multipliers are known, the optimization problem reduces to solving a set of independent nonlinear equations. From the seller's point of view these equations consist of the buyers' marginal valuations depending on the active constraints. These equations can basically be solved in the same way as in Papers [I] and [II] under limited information. The problem is to know the active constraints and the fact that there are enormous number of combinations when the utility functions can be of a general shape. It is calculated in Paper [III] that there are about 100 different digraphs when there are only three buyer types, and with around 15 types the number of digraphs is over 10^{100}. This means that it may be difficult or nearly impossible to guess the correct active constraints when there are many buyer types.

The roles and interpretations of Lagrange multipliers are examined in Paper [IV]. The multipliers can be interpreted as flows between the buyer types. The optimality conditions represent a general conservation law. This law means that in each node of the digraph the incoming flows plus the weight of the corresponding buyer type must equal the outgoing flows. The multipliers also have the standard sensitivity interpretation by approximating how much the optimal profit would change if the constraints were changed a little. Paper [IV] also shows how the non-uniqueness of the multipliers is related to the stability of the solution. If some buyer types are bunched together, then the range of possible multipliers is connected to how much the buyers' preferences need to change in order to break the bunch and change the digraph.

6.3 Optimization over the Learning Period

Papers [I] and [II] study how the seller can learn the optimal solution under limited information. These papers do not, however, examine how well the optimum is reached, i.e., what happens during the learning period. Paper [V] defines different learning paths and analyzes these paths with respect to suitable criteria. The learning paths are defined by heuristics that use only limited information. The paths are compared to the optimal learning path in terms of discounted profit, which is computed with complete information and dynamic programming. Besides the profit, the other criteria used in evaluation are the learning time and the buyers' utilities over the learning period.

The learning dynamics of Paper [V] are the gradient and different modified methods. The methods assume that the seller knows the buyers' marginal valuations locally around the currently sold bundles. The difference to Papers [I] and [II] is that the adjustment is done continuously rather than taking some discrete steps. This means that the step lengths need not be defined in the methods of Paper [V], which makes it easier to do the comparison.

The gradient method uses the steepest ascent direction to the seller's profit. The numerical experiments show that the gradient method improves the profit fast initially but it takes long time to learn the optimal solution. Paper [V] defines a class of learning methods, which use directions that both improve the seller's profit and are acceptable for the buyers as well. Two methods are examined from this class of methods: price raise method and constant direction method. The former is similar to the gradient method, except when there are no active constraints for a bundle. Only the price is raised when this happens. The numerical results show that the price raise method finds the optimum faster than the gradient method and gives better profits in the end of the learning period. The idea of the constant direction method is to update the quality of a bundle towards the optimal value. This method finds the optimal bundles faster and gives better utilities to the buyers than the other two methods. The method is, however, a bit problematic as it is assumed that the optimal structure of the solution is known.

The optimal learning path is computed using complete information and dynamic programming (Bertsekas 2005). The quality-price space is discretized and the optimal path is solved in a regular grid. The idea of the method is to define a value, or a profit-to-go function, in each point of the grid. With these values the optimal path can be solved by determining locally where the next step should be taken. The profit-to-go function is solved by repeating the value iteration, which takes into account the future profits and discounting. The numerical results show that the optimal path may be far off from the learning dynamics due to jumps, where some buyer types switch from one bundle to another. The jumps are difficult to include in the learning dynamics since some bundles are updated even though none of the buyers buys them. If the buyers do not buy the bundle, the seller does not get information about the preferences around the bundle. But if the optimal path does not involve jumps, it can be approximated with appropriate learning methods. When the discount factor is high, the gradient method is close to the optimal path. On the other hand, if the seller wants to minimize the learning time, the constant direction type of methods can be used.

7 Conclusions and Directions for Future Research

'Living backwards!' Alice repeated in great astonishment. 'I never heard of such a thing!'
'-- but there's one great advantage in it, that one's memory works both ways.'
'I'm sure mine only works one way,' Alice remarked.
'I can't remember things before they happen.'
'It's a poor sort of memory that only works backwards,' the Queen remarked.
'What sort of things do you remember best?' Alice ventured to ask.
'Oh, things that happened the week after next,' the Queen replied in a careless tone.
Through the Looking-Glass, Lewis Carroll (Carroll 1871, Chapter V)

This Dissertation develops a new learning approach for the nonlinear pricing problem. The main contributions are i) to show how the firm can learn how many products and what kind of products should be put up for sale when the demand is uncertain, and what information the firm needs to do this, ii) to analyze mathematically the general pricing problem with multiple quality dimensions and more general utility functions, and iii) to examine the computational questions of solving the pricing problem numerically. The learning method is based on the use of linear tariffs and the revelation of the buyers' marginal valuations. These valuations allow the firm to evaluate the optimality conditions and adjust the pricing towards greater profits.

The developed methods help firms in marketing questions such as pricing, product placement and differentiation. The approach, however, leaves aside important practical issues like advertising, competition, sociology and psychology (Wertenbroch and Skiera 2002, Liechty et al 2005, Voelckner 2006). Some of these aspects could be included in the model with small modifications, like the brands and competition (Bonatti 2010).

The methods extend to a variety of applications as the pricing model is an instance of a general model of incomplete information. The pricing model is also a Stackelberg game and these games offer possible extensions and applications to the methods. The requirement for the learning approach is that the situation is repeated. This allows the players to learn about each other's preferences and make the adjustment to their actions.

One interesting future research direction is applying the methods to real-life problems. This means modifying the model and matching the available data to the model. One important aspect of the problem is data collection and data mining (Chen et al 1996, Kantardzic 2002), i.e., the extraction of patterns from possibly huge data sets. Take, for example, Google who collects enormous data sets from visitors. This data can be used in finding current trends, customer segmentation, or creating personalized ads based on Internet usage and spatial information.

Another research direction is to study further the computational questions that were raised in Papers [III] and [IV]. What are good algorithms and heuristics to solve the multidimensional pricing problem when all customer data is available and what about when the firm has limited information? The model could also be modified to include, e.g., inventory, capacity and integer constraints. It may, for example, be that some quality dimensions in the pricing problem have only few possible quality levels, and this could be modeled with mixed integer nonlinear programming framework. Moreover, it would be interesting to study real-time and nonlinear pricing as an alternative to combinatorial auctions (Sandholm 2002, de Vries and Vohra 2003).

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