How patient the players need to be to obtain all the relevant payoffs in discounted supergames?

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Abstract

This paper examines the subgame-perfect equilibria in the symmetric 2×2 supergames. We extend the folk theorem by solving the smallest discount factor values when the players obtain all the feasible and individually rational payoffs. This enables us to determine all the equilibrium payoffs for high discount factor values, which is in general a difficult task since the payoff sets are complicated for patient players. We study how the different assumptions affect the set of equilibria by comparing the payoff sets in pure, randomized and correlated strategies. Moreover, we analyze how exactly the stage game's payoffs affect the required level of patience and organize the games into few classes. We find that the bounds generally depend on how large the payoff set is compared to the set of feasible payoffs and that the bounds are quite moderate for many games. We also observe discontinuities in the bounds, which means that small changes in the stage game's payoffs may affect dramatically the equilibrium payoffs.

Keywords: repeated game, folk theorem, discount factor, 2x2 games, payoff set, correlated equilibrium
JEL: C72, C73, D99

1. Introduction

The folk theorem states that any feasible and individually rational payoff is an equilibrium, when the players are patient enough (Friedman, 1971; Fudenberg and Maskin, 1986; Abreu et al., 1994; Fudenberg and Tirole, 1991; Mailath and Samuelson, 2006); these payoffs are referred to relevant payoffs from now on. However, in many applications the players are not arbitrarily patient but they rather have some intermediate values of the discount factor. The question then

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arises: what is the set of subgame-perfect equilibria for these values? This is a difficult question in general and this paper finds the threshold value for the discount factor when the payoff set covers all the relevant payoffs in the symmetric 2×2 supergames. Thus, we extend the folk theorem in these games and find the games where extreme patience is required to fill the payoff set.

The theory of infinitely repeated games was developed by Abreu (1986, 1988) and Abreu et al. (1986, 1990). These papers characterize the subgame-perfect equilibria with a set-valued fixed-point equation in pure and correlated strategies, which forms the base of our analysis. Recently, this theory was extended to randomized strategies in Berg and Schoenmakers (2014), which enables us to compare the three different settings. Moreover, the fixed-point characterization has been utilized in the computation of the payoff set (Cronshaw and Luenberger, 1994; Cronshaw, 1997; Judd et al., 2003; Sannikov, 2007; Burkov and Chaib-draa, 2010; Salcedo and Sultanum, 2012; Abreu and Sannikov, 2014). These methods work well when the discount factors are small but in general they provide only approximations of the payoff set. They also assume public randomization, which makes the payoff set convex and simplifies the task of computation dramatically.

In pure strategies, Berg and Kitti (2012, 2013) have developed a novel method for producing the equilibrium paths beside the payoffs, i.e., the method also finds the possible action sequences that can be played in the game. However, it faces the same problem as the earlier methods; it is possible to find all the equilibria when the discount factors are small or moderate, but the equilibrium paths increase fast and they get more complicated when the players become more patient. This paper approaches the problem from the other end. We find the high discount factor values when the payoff set is full. We note, however, that even though the payoff set is then known for these values, it does not mean that the equilibrium paths are known, i.e., the folk theorem does not imply that all the action sequences are subgame-perfect equilibrium paths. Nevertheless, it is possible to find some of the equilibrium paths using our method.

This paper examines analytically when the sets of attainable payoffs (Abreu et al., 1986, 1990) cover all the relevant payoffs and this is the condition when the payoff set is full. This idea was used in Stahl (1991) who solved the bounds for the discount factors in a class of prisoner's dilemma games under public randomization; see also Sections 2.5.3 and 2.5.6 in Mailath and Samuelson (2006). Here, we examine systematically all the symmetric 2×2 supergames in pure, randomized and correlated strategies. The games are organized into a few classes based on the stage game's payoffs. We provide a figure of the bounds and the classes in these games. From this figure, it is easy to see when a high level of patience is required and to make the comparison between the different strategies.

The paper is structured as follows. In Section 2, the repeated game model is presented and the properties of equilibria is analyzed. Sections 3-5 examine the discount factor bounds in pure, correlated and randomized strategies. Section 6 is the conclusion.

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2. The repeated game

2.1. Stage games

In a repeated game, a stage game is played repeatedly by the same players. A stage game is defined by a finite set of players $N = \{1, \ldots, n\}$, a finite set of pure actions for each players A_i, $i \in N$, which form the set of pure action profiles $A = \times_{i \in N} A_i$, and the players' utilities for each action profile $u : A \mapsto \mathbb{R}^n$. Also, a pure action of player i is called $a_i \in A_i$ and a pure action profile is called $a \in A$.

Each player $i \in N$ may randomize over his pure actions $a_i \in A_i$. This defines a mixed action q_i such that $q_i(a_i) \geq 0$ for each $a_i \in A_i$ and $\sum_{a_i \in A_i} q_i(a_i) = 1$. The set of probability distributions over A_i is called Q_i and $Q = \times_{i \in N} Q_i$. A mixed action profile is denoted by $q = (q_1, \ldots, q_n) \in Q$. The support of a mixed action is the set of pure actions that is played with a strictly positive probability: $Supp(q_i) = \{a_i \in A_i | q_i(a_i) > 0\}$. We also define $Supp(q) = \times_{i \in N} Supp(q_i)$ and for each $a \in Supp(q)$, we let $\pi_q(a)$ be the probability that the action profile a is realized if the mixed action profile q is played: $\pi_q(a) = \prod_{j \in N} q_j(a_j)$. In pure strategies, we make the restriction that $q_i(a_i) = 1$ for one action $a_i \in A_i$. In correlated pure strategies, the players observe a public lottery and they can condition their pure action based on this signal. From now on, by correlated strategies we mean correlated pure strategies.

The stage game payoffs are given by the function $u : Q \mapsto \mathbb{R}^n$. For example, if the players choose a mixed action profile $q \in Q$, then player i receives an expected payoff of

$$u_i(q) = \sum_{a \in A} u_i(a) \pi_q(a). \tag{1}$$

Let $q_{-i} \in Q_{-i} = \times_{j \in N, j \neq i} Q_j$ denote player i's opponents' actions. Now, an action profile q is a Nash equilibrium in the stage game if no player has a profitable deviation, i.e.,

$$u_i(q) \geq u_i(q_i', q_{-i}) \text{ for all } i \in N \text{ and } q_i' \in Q_i. \tag{2}$$

2.2. Repeated games

We examine a model where the stage game is repeated infinitely many times and these games are sometimes called as supergames. We assume that the players observe all the past realized pure actions but not the possible randomizations. This public past play is denoted by the set of histories $H^k = A^k = \prod_k A$, where $H^0 = A^0 = \{\varnothing\}$ is the empty set and corresponds to the beginning of the game. Thus, the history contains all the pure actions that were played in the previous stages. The set of all possible histories is $H = \bigcup_{k=0}^\infty H^k$. A behavior strategy σ_i of player $i \in N$ is a mapping that assigns a probability distribution over player i's pure actions for each possible history $\sigma_i : H \mapsto Q_i$. The set of player i's strategies is Σ_i. The players' strategies form a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, a strategy profile of all players except player i is denoted by σ_{-i} and the set of strategy profiles is given by $\Sigma = \times_{i \in N} \Sigma_i$. A pure strategy assigns a pure action

for each possible history $\sigma_i : H \mapsto A_i$. In correlated pure strategies, the history also contains a public signal for each of the current and the previous stages and the correlated strategy assigns a pure strategy for each possible history.

We assume that the players discount the future payoffs with a common discount factor $\delta \in [0, 1)$. They have the same discount factor as we examine the symmetric games. The expected discounted payoff of a strategy profile σ to player i is

$$U_i(\sigma) = \mathbb{E}\left[(1 - \delta) \sum_{k=0}^{\infty} \delta^k u_i^k(\sigma) \right], \tag{3}$$

where $u_i^k(\sigma)$ is the payoff of player i at stage k induced by the strategy profile σ. A strategy profile σ is a Nash equilibrium if no player has a profitable deviation, i.e.,

$$U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i}) \text{ for all } i \in N, \text{ and } \sigma_i' \in \Sigma_i, \tag{4}$$

and it is a subgame-perfect equilibrium (SPE) if it induces a Nash equilibrium in every subgame, i.e.,

$$U_i(\sigma|h) \geq U_i(\sigma_i', \sigma_{-i}|h) \text{ for all } i \in N, \ h \in H, \text{ and } \sigma_i' \in \Sigma_i, \tag{5}$$

where $\sigma|h$ is the restriction of the strategy profile after history $h \in H$. From now on, by equilibrium we mean subgame-perfect equilibrium.

2.3. The characterization of equilibria

Let V be the compact set of SPE payoffs and we also use $V(\delta)$ when we want to emphasize the players' discount factor δ. By V^P, V^C and V^M we refer to the equilibria in pure, correlated and randomized strategies, respectively. We begin with the case of pure strategies, then shortly cover the correlated pure strategies and then consider the randomized strategies.

The player i's minimum equilibrium payoff, which is also called the punishment payoff, is denoted by $v_i^-(\delta) = \min\{v_i : v \in V(\delta)\}$, when $V(\delta)$ is non-empty; and this is the case in the symmetric 2×2 supergames. Similarly, the maximum equilibrium payoff is $v_i^+(\delta) = \max\{v_i : v \in V(\delta)\}$. The minimax payoff is

$$\underline{v}_i = \min_{q_{-i} \in Q_{-i}} \max_{q_i \in Q_i} u_i(q_i, q_{-i}). \tag{6}$$

Note that the minimax payoffs can be different in pure and randomized strategies. However, it holds under perfect monitoring that $v_i^-(\delta) \geq \underline{v}_i, \ \forall i \in N$ (Fudenberg and Tirole, 1991; Mailath and Samuelson, 2006). The player's minimum and maximum payoffs in a compact set W are denoted by $v_i^-(W)$ and $v_i^+(W)$, respectively. Moreover, $v^s(W) = \max\{v_i \in W, \ v_i = v_j, \forall j \in N\}$ is the maximum symmetric payoff in set W.

Let us now consider the pure strategies. Abreu (1988), see also Abreu et al. (1986, 1990); Cronshaw and Luenberger (1994), has shown that all the equilibrium outcomes can be obtained in simple strategies. These strategies consist of $n+1$ paths: an initial path that the play follows and a punishment path for each

player that gives the player's minimum equilibrium payoff. The players follow the current path unless a single player deviates from it. If this happens, the play switches to the punishment path of the deviator. If more than one player deviates, then the play stays on the current path and there is no punishment. Since we examine a non-cooperative game, we do not need to consider deviations by more than one player. Abreu has shown that a path is an equilibrium if and only if it does not have any profitable deviations when the deviations leads to the players' smallest equilibrium payoffs. Furthermore, the payoff set is characterized by a set-valued fixed-point equation and the set is self-generating. These results are now shortly presented and for more details we refer to Section 2 in Mailath and Samuelson (2006) and Berg and Kitti (2014).

A pair (a, w) of an action profile $a \in A$ and a continuation payoff $w \in W$ is admissible with respect to W if it satisfies the incentive compatibility conditions

$$(1 - \delta)u_i(a) + \delta w_i \geq \max_{a_i' \in A_i} \left[(1 - \delta)u_i\left(a_i', a_{-i}\right) + \delta v_i^-(W) \right], \ \forall i \in N. \quad (7)$$

This condition means that it is better for player i to take action a_i and get the payoff w_i than to deviate and then obtain $v_i^-(W)$.

For a set of continuation payoffs W, the set of feasible action profiles is denoted by

$$F^\delta(W) = \{a \in A \text{ such that } (a, w) \text{ is admissible for some } w \in W\}. \quad (8)$$

For $a \in F^\delta(W)$, we denote the set of possible continuation payoffs as

$$C_a^\delta(W) = \{w \in W \text{ such that } (a, w) \text{ is admissible}\}. \quad (9)$$

Let $D_a^\delta : \mathbb{R}^n \mapsto \mathbb{R}^n$ be an affine mapping that corresponds to an action profile $a \in A$ and a discount factor δ

$$D_a^\delta(w) = (I - T)u(a) + Tw, \quad (10)$$

where I is an $n \times n$ identity matrix and T is an $n \times n$ diagonal matrix with the discount factor δ on the diagonal. The mapping D_a^δ is also defined for sets; then the addition is the Minkowski sum and $D_a^\delta(\varnothing) = \varnothing$. Finally, we denote the attainable payoffs that start with an action profile $a \in A$ by $B_a^\delta(W) = D_a^\delta(C_a^\delta(W))$.

Theorem 1. *The set of pure-strategy subgame-perfect equilibrium payoffs V^P is the unique largest compact set satisfying the fixed-point*

$$W = B^\delta(W) \doteq \bigcup_{a \in F^\delta(W)} B_a^\delta(W). \quad (11)$$

For proof, see Theorem 2 in Abreu et al. (1990), Theorem 1 in Cronshaw and Luenberger (1994), Corollary 2.5.1 in Mailath and Samuelson (2006) or Berg and Kitti (2014). The theorem tells that the payoff set V^P is a fixed-point of the iterated function system defined by D_a^δ, $a \in A$ and the incentive compatibility

conditions (7). Note that the sets V^P and $B_a^\delta(V)$ are compact (Abreu et al., 1986, 1990; Mailath and Samuelson, 2006). The results of this paper are based on analyzing the sets $B_a^\delta(V)$ in the symmetric 2×2 supergames.

Now, we shall consider the correlated pure strategies. Let $co(W)$ denote the convex hull of W. The payoff set under public correlation V^C is given by the largest set satisfying

$$V = \mathfrak{B}^\delta \doteq co \left(\bigcup_{a \in F^\delta(V)} B_a^\delta(V) \right). \tag{12}$$

Finally, we describe the equilibria in randomized strategies. The first observation with the randomized strategies is that the player must be indifferent between the actions in his support. For example, assume that the players are supposed to play an action profile $q = (q_1, q_2, \ldots, q_n)$ as a part of an equilibrium, then all of the actions in $Supp(q_i)$ must give the same expected payoff to player i. Otherwise, he could just play an action profile in $Supp(q_i)$ that gives the highest payoff and such a deviation could never be detected. Let us denote the most profitable deviation outside $Supp(q_i)$, which is given by the pure action that yields

$$d_i(q) = \max_{a_i' \in A_i \backslash Supp(q_i)} u_i(a_i', q_{-i}).$$

If $Supp(q_i) = A_i$, then there are no outside deviations for player i and $d_i(q) = \max \emptyset = -\infty$. Moreover, if $d_i(q) < u_i(q)$, then there are no profitable outside deviations, since both the current stage and the continuation payoffs, given by the punishment payoffs after an observable deviation, are lower.

We say that a pair (q, w) consisting of an action profile q and an expected continuation payoff w is admissible with respect to W if it satisfies the incentive compatibility (IC) conditions

$$(1 - \delta)u_i(q) + \delta w_i \geq (1 - \delta)d_i(q) + \delta v_i^-(W),$$

for all $i \in N$. Now, every action profile in $a \in Supp(q)$ has a positive probability of being realized and each of them could be followed by a different continuation play and corresponding payoff $x(a)$. Thus, given an action profile q, an expected continuation payoff w can be formed by taking for each $a \in Supp(q)$ a continuation payoff $x(a) \in W$ such that

$$w = \sum_{a \in Supp(q)} x(a)\pi_q(a). \tag{13}$$

Consider a stage game where the payoff of action profile $a \in A$ is given by

$$\tilde{u}_\delta(a) \doteq (1 - \delta)u(a) + \delta x(a),$$

i.e., the continuation payoffs $x(a)$ are included in the stage game payoffs. Let $M(x)$ be the set of all equilibrium payoffs in this stage game. Now, we are ready to state the characterization for the subgame-perfect equilibrium payoffs (Berg and Schoenmakers, 2014).

Theorem 2. *The payoff set V^M is the largest fixed point of $\mathcal{B}$:*

$$W = \mathcal{B}(W) \doteq \bigcup_{x \in W^{|A|}} M(x), \qquad (14)$$

where (q, w) is admissible with respect to W, w is formed by the continuation payoffs x as in Eq. (13) and q is an equilibrium in the stage game defined by the continuation payoffs x.

Note that the complexity of computing all randomized equilibria is much higher compared to the pure-strategy equilibria. Each iteration of $\mathcal{B}$ goes through all Nash equilibria of all permutations of all possible continuations payoffs after each action profile over all action profiles in A.

Proof. We show that i) there are no profitable unilateral deviations from the profiles in the fixed-point construction and ii) the strategies outside the construction are not equilibria.

Notice that we only need to consider deviations in the stage games. If no players has a profitable deviation in any of the stage games, then there are no profitable deviations in the continuation play and all the continuation payoffs $x(a)$, $a \in A$, are produced by an equilibrium. Indeed, there are no profitable deviations in any stage game, since

1. The incentive compatibility conditions guarantee that no player should ever deviate to an action outside the support of his prescribed mixed action. Such a deviation by player i will be observed and will be followed by the punishment strategies and the minimum equilibrium payoff to player i.

2. No player can profit from deviating to a mixed action within the support of his prescribed mixed action. This is guaranteed by the fact that the prescribed profile is a Nash equilibrium in the stage game and thus each player must be indifferent between the pure actions that are in the carrier of his mixed action.

Thus, there are no profitable deviations from the profiles produced by the construction.

The strategies that are not produced by the fixed-point characterization are either not Nash equilibrium in the stage game or they use continuation payoffs that are not admissible. If the continuation payoffs are not admissible then it means that the strategies are not subgame-perfect and thus not equilibria. Similarly, if the strategies are not Nash equilibria in the stage game then some players have profitable deviation and these strategies cannot be equilibria either. □

2.4. Monotonicity results

The aim of this paper is to study when the payoff set covers all the relevant payoffs. In the previous subsection, we gave the necessary and sufficient conditions for this to happen; the sets $B(V)$, $\mathfrak{B}(V)$ and $\mathcal{B}(V)$ need to cover the

relevant payoffs. Now, we show the monotonicity result, i.e., once the condition holds for a given discount factor, it also holds for more patient players. Then the remaining task is to find the smallest discount factor when this happens.

In this subsection, by $B(V)$ we refer to the sets $B(V)$, $\mathfrak{B}(V)$ and $\mathcal{B}(V)$, depending on which strategies are in question. A set W is called self-generating if $W \subseteq B^\delta(W)$. The following result follows directly from Theorems 1 and 2 and Eq. (12).

Proposition 1. *If a bounded set W is self-generating then $B^\delta(W) \subseteq V(\delta)$.*

The following shows that the payoff set is monotone in the discount factor when it is convex (Abreu et al., 1990; Sorin, 1986; Mailath and Samuelson, 2006; Berg, 2013).

Theorem 3. *Suppose $V(\delta_1)$ is convex then $V(\delta_1) \subseteq V(\delta_2)$ for $\delta_2 \geq \delta_1$.*

Proof. By Proposition 1, it is enough to show that for all $v \in V(\delta_1)$ it holds that $v \in B^{\delta_2}(V(\delta_1))$. In pure and correlated strategies, it is enough to show that there is an admissible pair (a, w^2) of an action profile a and a continuation payoff $w^2 \in V(\delta_1)$ such that $v = (1 - \delta_2)u(a) + \delta_2 w^2$ for every $v \in V(\delta_1)$, i.e., (a, w^1) is admissible for a continuation payoff $w^1 \in V(\delta_1)$ and $v = (1 - \delta_1)u(a) + \delta_1 w^1$. By denoting $\delta_2 = \delta_1 + \epsilon$, we can solve

$$(\delta_1 + \epsilon)w^2 = \delta_1 w^1 + \epsilon u(a).$$

This means that w^2 is a convex combination of w^1 and $u(a)$. Combining with the result that v is between $u(a)$ and both w^1 and w^2, we get that w^2 is between w^1 and v. Thus, it follows that $w^2 \in V(\delta_1)$ by convexity and $v, w^1 \in V(\delta_1)$.

In randomized strategies, it is enough to show that the same stage game payoffs can be achieved with δ_2, i.e., the suitable continuation payoffs are in $V(\delta_1)$ and admissible. The proof is the same as above but applied to all action profiles in the stage game.

Finally, we need to check the admissibility with δ_2. The only remaining thing to check is that the punishment payoff is not increasing and it is not since $v_i^-(V(\delta_1)) \in V_i(\delta_1)$ for all $i \in N$ and together with the above result we have $v^-(V(\delta_2)) \leq v^-(V(\delta_1))$. □

Since the set of reasonable payoffs is convex, the above result means that once the reasonable payoffs are obtained, they are obtained for more patient players as well. This also implies that the payoff sets are always monotone in correlated strategies. Furthermore, it is also possible to show that convex self-generating sets are monotone. The proof is similar to Theorem 3.

Proposition 2. *Suppose a self-generating set $W \subseteq V(\delta_1)$ is convex then $W \subseteq V(\delta_2)$ for $\delta_2 \geq \delta_1$.*
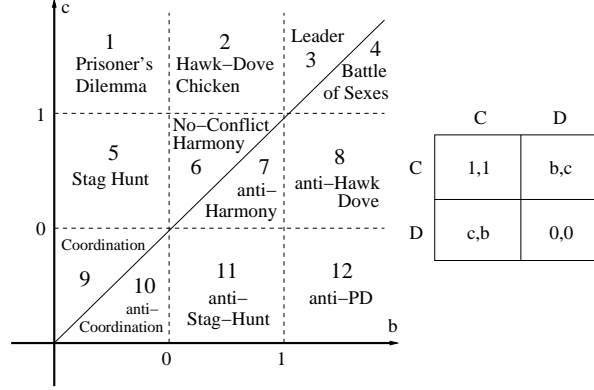
Figure 1: Symmetric ordinal 2×2 games with parameters b and c.

2.5. Reasonable and punishment payoffs in symmetric 2×2 games

The twelve symmetric strict ordinal 2×2 games are presented in Figure 1, see Robinson and Goforth (2005) for the taxonomy. The two actions are C (cooperate) and D (defect), and they give the players the payoffs $\mathfrak{a} = 1$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d} = 0$; the corresponding action profiles are also denoted by letters a, b, c, d. For example, if the players choose actions (C, D), i.e., the action profile b, the players receive payoffs $\mathfrak{b}$ and $\mathfrak{c}$. Each of the twelve regions represents a certain class of games: 1. prisoner's dilemma, 2. chicken, 3. leader, 4. battle of the sexes, 5. stag hunt, 6. no conflict, 9. coordination and their anti-games.

Let $V^\dagger = co\,(v \in \mathbb{R}^n : \exists q \in A \text{ s.t. } v = u(q))$ be the set of feasible payoffs and the reasonable payoffs

$$V^* = \{v \in V^\dagger : v_i \geq \underline{v}_i, \ i \in N\}$$

contains only the individual rational payoffs. Note that the minimum equilibrium payoffs $v^-(\delta)$ may be strictly higher than the minimax payoffs $\underline{v}$ and then it is impossible to obtain all the reasonable payoffs. We could have defined the reasonable payoffs using the minimum payoffs, but this does not make a difference in the symmetric 2×2 supergames, as will be shown below. The minimum payoffs in pure strategies are studied in Berg (2013), where it is shown that the equilibrium paths need not be monotone in the discount factor; see also Berg and Kärki (2014) for an improved algorithm for finding the punishment paths and payoffs for different discount factors. This non-monotonicity result means that there are games where a path is an equilibrium for one discount factor but is no longer an equilibrium for a higher discount factor value.

Let us denote the discount factor bound by

$$\delta^F = \min\{\delta : V(\delta) = V^*\}, \tag{15}$$

which gives the smallest discount factor value when the payoff set covers all the

reasonable payoffs. Note that V^* is convex and thus $V(\delta) = V^*$ for all $\delta \geq \delta^F$ by Theorem 3.

The minimum payoffs in pure strategies are equal to the minimax payoffs for most of the symmetric 2×2 supergames for all discount factors, because the minimax payoff is given by a stage game Nash equilibrium. No conflict, its anti-game and anti-stag hunt are the only exceptions. In these games, the minimum payoff is between the minimax and the maximum payoff depending on the discount factor (Berg, 2013; Berg and Kärki, 2014), i.e., $\underline{v}_i \leq v_i^-(\delta) \leq v_i^+(V^\dagger)$. For small discount factor values the minimum payoff is high, but when the players are patient enough it is possible to punish the other player with the minimax value. The following shows sufficient conditions for the discount factor such that the minimum payoffs are the minimax values.

In no conflict game, it is possible to play the punishment path b^∞, or symmetrically c^∞, giving the minimax payoff if it holds that the payoff $\mathfrak{c} \geq (1-\delta) + \delta\mathfrak{b}$; the right-hand side gives the best deviation to payoff 1 that is followed by the punishment payoff $\mathfrak{b}$. From this we get that $v^-(\delta) = \underline{v}$ when $\delta \geq (1-\mathfrak{c})/(1-\mathfrak{b})$. This value does not affect the results since it is lower than the bound δ^F that we will later find for the no conflict games.

In anti-no conflict and anti-stag hunt games, it holds that $v^-(\delta) = \underline{v}$ when $\delta \geq 1/(2-\mathfrak{b})$. This is the discount factor value when $B_d(V^*)$ intersects with $B_a(V^*)$ and the line from $u(d)$ to $u(a)$ belongs to the payoff set, and this guarantees that the minimax value is exactly obtained in the game. Again, this value does not affect the results since in these games $\delta^F \to 1$ and the reasonable payoffs are not obtained for any $\delta < 1$.

The minimax payoffs in mixed strategies are the same as in pure strategies, except in leader, battle of the sexes, coordination and anti-coordination games. In these games, the minimax payoff is given by a mixed strategy that is a Nash equilibrium in the stage game. Thus, $v^-(\delta) = \underline{v}$ for all δ in mixed strategies in these games. We will show in Section 5 that the reasonable payoffs are not obtained for any $\delta < 1$ in these games.

2.6. Sets of attainable payoffs

We give few more definitions and show some results related to the sets $B_a(W)$, $a \in A$. Let $v^A(W)$, $v^B(W)$, $v^C(W)$ and $v^D(W)$ be the corners of the set W corresponding to payoffs $u(a)$, $u(b)$, $u(c)$ and $u(d)$. For example, if $u(a)$ and $u(b)$ are the payoffs in the northeast and northwest corners of $V^\dagger$, then $v^A(B_b(\delta))$ and $v^B(B_b(\delta))$ are the northeast and northwest corners of the set $B_b(\delta)$; see Fig. 4(a). Moreover, let

$$v^M(\delta) = (1-\delta)\max_{a \in A} u_i(a) + \delta v_i^-(\delta). \tag{16}$$

This is the right-hand side of Eq. (7) for the column that contains the maximum payoff in the game. The following remark gives the relation of v^M to certain sets B_a, $a \in A$; see, e.g., B_a and B_c in Fig. 4(a).

Remark 1. *If $B_a(\delta) \neq \varnothing$, $a \in A$, then $v^M(\delta) = v_i^-(B_a(\delta))$ for all a and i where player i can deviate to the maximum payoff of the stage game.*

The following result describes how the sets B_a, $a \in A$, may cover the boundaries of V^*. The result implies that we need as many sets B_a, $a \in A$, to cover the reasonable payoffs as there are corner points in V^*.

Proposition 3. *The set $B_a^\delta(V^*)$, $a \in A$, may only cover the corner point of V^* closest to $u(a)$. It cannot cover the other corner points or the boundary of V^* between these corner points.*

Proof. By the definition of D_q^δ, the set V^* is contracted by δ and is thus strictly smaller than V^*. The translation part $(I - T)u(a)$ moves the set towards $u(a)$. $\square$

Finally, we note that it is enough to cover the corner points of a polygon in order to cover it. It follows from the fact that a convex set contains all of its convex combinations.

Proposition 4. *Let S be a convex set. If $\{v_1, \ldots, v_k\} \subseteq S$, then $co(\{v_1, \ldots, v_k\}) \subseteq S$.*

3. Equilibria in pure strategies

Theorem 1 implies that the payoff set covers the reasonable payoffs when the sets B_a^δ, $a \in A$, cover all the payoffs in V^*. To find the bound δ^F, we need to find the smallest discount factor for this to happen. The results are based on determining the sets B_a^δ analytically for the four action profiles in the 2×2 supergames. The main idea is to find the last payoff point v^F in the payoff set that is covered when the discount factor is increased to δ^F. The bound δ^F is typically solved from a condition that two sets, say B_b and B_c intersect. In the proofs, we show two parts: the point v^F is not covered for a smaller discount factor value and all the other reasonable payoffs are covered for the given value. Thus, we get the necessary and sufficient conditions for δ^F.

The main results[1] are given in Table 1; it gives the discount factor δ^F in the symmetric 2×2 supergames in pure strategies. The games can be classified into five classes; see the left parts of Figs. 2 and 3. The classes are based on the location of the payoff v^F: in Class I, v^F is found on the upper edge of V^* between $u(a)$ and $u(b)$, on the bottom edge between $u(b)$ and $u(d)$ in Class II, and in the middle in Classes IIIa-d. The games in Class IV do not get covered for any $\delta < 1$ and the games in Class V are covered for all $0 \leq \delta < 1$. Note that the class boundaries cross the game boundaries. For example, the triangle-shaped prisoner's dilemmas and chicken games all belong to Class IIIb. The proofs are given in the following subsections. Note that the $\mathfrak{b}$ and $\mathfrak{c}$-axes are scaled with a tangent function in Figure 3 in order to show both small and large values at

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[1]FOR REVIEWERS: see http://tinyurl.com/q8c9x3e for mathematica applets that help to check the results. If you do not have mathematica, you can install free CDF player.
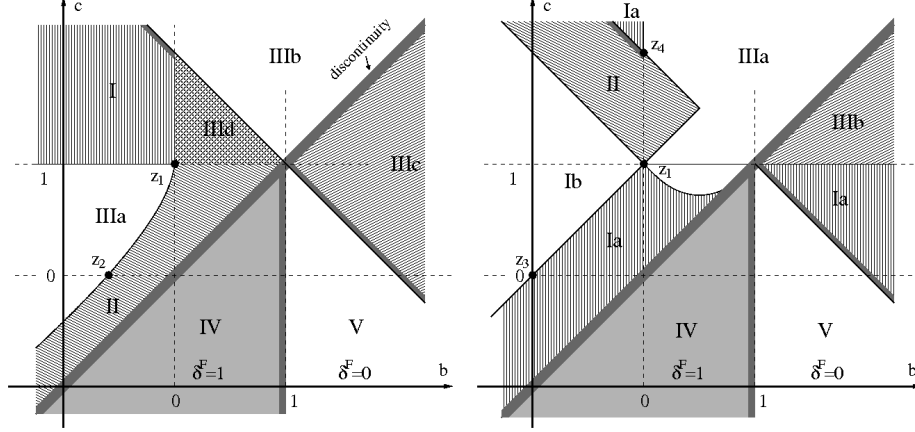
Figure 2: Classification based on δ^F with pure (left) and correlated pure (right) strategies.

Table 1: The bounds of the discount factor δ^F in pure strategies.

Game	Class 1	Class 2	Class boundary
Prisoner's dilemma	I: $\delta^F = \frac{c-b}{1+c-b}$	IIIb: $\delta^F = \frac{2c}{b+3c}$	$c = 2 - b$
Chicken	IIId: $\delta^F = \frac{c}{1+c-b}$	IIIb: $\delta^F = \frac{2c}{3c-b}$	
Leader	IIIb: $\delta^F = \frac{2c}{3c-b}$		
Battle of the sexes	IIIc: $\delta^F = \frac{2b-2}{3b-c-2}$		
Anti-PD	V: $\delta^F = 0$	IIIc: $\delta^F = \frac{2b-2}{3b-c-2}$	$c = 2 - b$
Anti-chicken			
Stag hunt	II: $\delta^F = \frac{1-b}{1-2b+c}$	IIIa: $\delta^F = \frac{3-3b-2c+b(b+c)}{5-4b+b^2-3c+bc}$	$c = 1 + (b-1)\sqrt{\frac{b}{b-2}}$
Coordination			
No conflict	II: $\delta^F = \frac{1+b(c-2)}{1+b(b-3)+c}$		
Anti-coordination			
Anti-no conflict	IV: $\delta^F = 1$		
Anti-stag hunt			

the same time, i.e.,

$$b' = \frac{1}{2\sqrt{1-2/\sqrt{5}}} \tan\left(\pi b - \pi/2\right) + 1/2. \tag{17}$$

Let us analyze the results. The smallest value $\delta^F = 0$ is obtained in Class V. These include anti-games, where $V^\dagger$ is an obtuse-angle quadrilateral. These are not particularly interesting games, since V^* is a single point. Thus, the payoff set is full for all discount factors in these games.

The smallest meaningful value $\delta^F = 1/2$ is in point z_1, between Classes I, II, IIIa and IIId; see Fig. 2. These are the games where V^* is close to a rectangle and the player's own action has a little effect on the payoff. Moreover, the smallest value over the coordination games is in point z_2, between Classes II and IIIa, where $\delta^F \approx 0.687$ and $b \approx -0.84$. Finally, the smallest value is
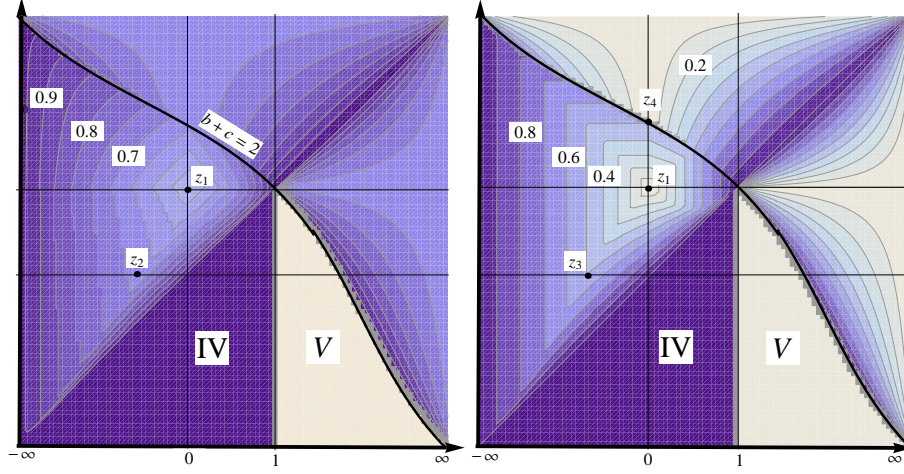
Figure 3: Values of δ^F with pure (left) and correlated pure (right) strategies.

$\delta^F = 2/3$ in all triangular games. These are the games in Classes IIIb and IIIc, where V^* is a triangle. The bounds are achieved in the limit when $\mathfrak{b} \to \infty$ or $\mathfrak{c} \to \infty$. The smallest values are achieved near $\mathfrak{b} = 0$ in prisoner's dilemma and chicken games, near $\mathfrak{b} = 1$ in leader games, near $\mathfrak{c} = 1$ in battle of the sexes and anti-chicken games, and near $\mathfrak{c} = 0$ in anti-PD games.

For all anti-coordination, anti-no conflict and anti-stag hunt games, it holds that $\delta^F \to 1$. This means that the payoff set never gets covered and the payoff set is not full for any δ. The same result holds for some payoff parameters in all regions even if they are divided both by the classes and the games, except in Class V where $\delta^F = 0$ for all payoffs. For example, $\delta^F \to 1$ holds in the following cases: near the $\mathfrak{c} = \mathfrak{b}$ diagonal, on the limit when $\mathfrak{b} \to -\infty$ for prisoner's dilemma, stag hunt and coordination games, and near the line $\mathfrak{c} = 2 - \mathfrak{b}$ for anti-PD and anti-chicken games. Thus, we cannot find a limit for δ^F for any class or game, except for Class V. This means that we cannot extend the folk theorem unless these extreme payoff parameters can be ruled out.

We can observe that the value of δ^F depends on how large V^* is compared to $V^\dagger$. When V^* is small, it is difficult to play certain actions in the game and the value of δ^F is large, i.e., a high level of patience is required. For example, it is difficult to play the actions b and c in a prisoner's dilemma when $\mathfrak{b} \to -\infty$. Geometrically, this means that V^* stays almost the same but $V^\dagger$ keeps increasing, making the proportion of V^* to $V^\dagger$ smaller. On the other hand, if V^* is large then δ^F is smaller. Note that Classes IV and V are exceptions, where δ^F is a constant and thus independent of V^* and $V^\dagger$.

What about the boundaries between the classes? On the $\mathfrak{c} = \mathfrak{b}$ diagonal, $\delta^F = 1/2$ when $\mathfrak{b} = \mathfrak{c} \leq 0$ and $\delta^F = 0$ when $\mathfrak{b} = \mathfrak{c} > 0$. This means that there is a discontinuity of δ^F on the diagonal in all directions. There is also a

discontinuity on line $\mathfrak{c} = 2 - \mathfrak{b}$ between Classes I and IIIb (prisoner's dilemma), and between IIIc and V. On this line, the boundary belongs to Class IIIb when $\mathfrak{b} < 0$ and to Class V when $\mathfrak{b} > 1$. The discontinuity between the prisoner's dilemma games is surprising and it shows that a small geometric change from the triangle-shape to the quadrangle-shape may affect the filling of the Pareto efficient frontier of the payoff set. Furthermore, the boundary belongs to Class V when $\mathfrak{b} = 1$ and $\mathfrak{c} < 1$ and thus there is a discontinuity between Classes IV and V. The other class boundaries are continuous in δ^F. The discontinuity means that small changes in the payoffs can affect dramatically the filling of the payoff set and how large δ^F is. The discontinuities are shown in Figure 2 by thick grey lines.

3.1. Class I

This class is defined by parameters $1 < \mathfrak{c} < 2 - \mathfrak{b}$ and $\mathfrak{b} < 0$. These are the prisoner's dilemmas where V^* is a quadrilateral with an obtuse angle in $u(a)$ corner; see Fig. 4(a). It is enough to examine the upper half of V^* where $v_2 > v_1$ since V^* is symmetric with respect to the center line. Thus, the last point v^F that is covered is only defined in the upper half of V^*.
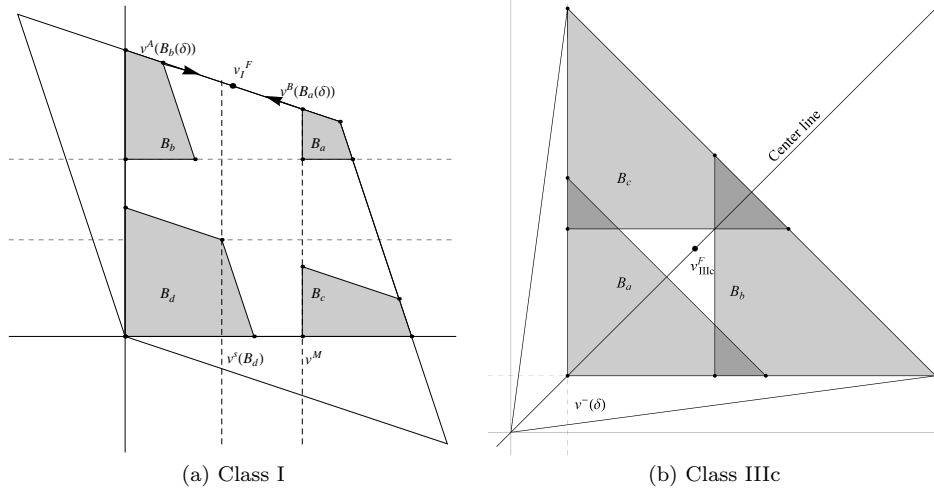

(a) Class I   (b) Class IIIc

Figure 4: Attainable payoffs in Classes I and IIIc.

The point v_I^F in Class I is located on the upper edge between $u(a)$ and $u(b)$. This point and the corresponding discount factor δ_I^F are solved from the

intersection of B_a and B_b. It is enough to consider only player 1's payoffs:

$$
\begin{aligned}
v_I^F &= v^B(B_a(\delta^F)) = v^A(B_b(\delta^F)) \\
&\Rightarrow (1-\delta^F)\mathfrak{c} + \delta^F \cdot 0 = (1-\delta^F)\mathfrak{b} + \delta^F \cdot 1 \\
\Rightarrow \delta_I^F &= \frac{\mathfrak{c}-\mathfrak{b}}{1+\mathfrak{c}-\mathfrak{b}}, &(18) \\
v_I^F &= \left(\frac{\mathfrak{c}}{1-\mathfrak{b}+\mathfrak{c}}, \frac{\mathfrak{b}-2\mathfrak{c}}{\mathfrak{b}-\mathfrak{c}-1} \right). &(19)
\end{aligned}
$$

On the second line, the first payoff $v_1^B(B_a(\delta^F))$ can be solved from the right-hand side of the incentive compatibility conditions.

We first show that $v^F \notin B_a^\delta(V^*)$, $a \in A$, if $\delta < \delta^F$. Since the sets B_a and B_b intersect at δ^F, it means that v^F does not belong to either B_a or B_b for $\delta < \delta^F$. Moreover, the sets B_c and B_d cannot cover v^F on the boundary of V^* by Proposition 3.

Now, let us show that every $v = \{v_2 \geq v_1, v \in V^*\} \in B_a^\delta(V^*)$ for some $a \in A$ and $\delta = \delta^F$. We divide V^* into three regions. If $v_1 \geq v^M(\delta^F)$ then $v \in B_a(\delta^F)$ since $v^M(\delta)$ coincides with $v_1^-(B_a(V^*))$ for all $\delta \geq \delta^F$ by Remark 1. If $v_1 \leq v^M(\delta^F)$ and $v_2 \leq v^M(\delta^F)$ then we show that $v \in B_d(\delta^F)$. We apply Proposition 4 and show that all corner points $(v^M(\delta^F), v^M(\delta^F))$, $(0,0)$ and $(0, v^M(\delta^F))$ belong to B_d. $(0,0)$ is the Nash equilibrium and belongs to B_d for any δ. Also, $(0, v^M(\delta^F))$ belongs to B_d if $(v^M(\delta^F), v^M(\delta^F))$ belongs to B_d, because $v_2^B(B_d)$ is higher than $v_2^A(B_d)$ due to $\mathfrak{c} > 1$. So for the last corner, we need to show that $v^s(V_d(\delta^F)) \geq v^M(\delta^F)$. It holds that $v^s(V_d(\delta^F)) = \delta^F$ and $v^M(\delta^F) = (1-\delta^F)\mathfrak{c}$ and the above condition holds if $\delta^F \geq (1-\delta^F)$. Using Eq. (18), this is equal to $\mathfrak{b} \leq 0$ and this is true in this class.

Finally, if $v_1 \leq v^M(\delta^F)$ and $v_2 \geq v^M(\delta^F)$ then we show that $v \in B_b(\delta^F)$ by Proposition 4. The corner points are $v^B(V^*)$, $(0, v^M(\delta^F))$, $(v^M(\delta^F), v^M(\delta^F))$ and $v^B(B_a(\delta^F))$. $v^B(B_a)$ belongs to B_b trivially by the definition of v^F. $v^B(V^*) \in B_b$ since v^F is at the upper edge and B_b covers the edge all the way to $v^B(V^*)$. $(0, v^M(\delta^F)) \in B_b$ since $v^M(\delta)$ coincides with $v_2^-(B_b(V^*))$ for all $\delta \geq \delta^F$ when $\mathfrak{c} > \mathfrak{b}$ by Remark 1. Again, $\mathfrak{c} > 1$ ensures that $v_1^C(B_b) > v_1^A(B_b)$ and thus $(v^M(\delta^F), v^M(\delta^F)) \in B_b$ since $v_1^A(B_b(\delta^F)) = v_1^F = v^M(\delta^F)$.

3.2. Classes II and IIIa

Classes II and IIIa include no conflict, stag hunt and coordination games. The payoffs satisfy $\mathfrak{b} < \mathfrak{c} < 1$ and V^* is a quadrangle with an acute angle in $u(a)$ corner; see Fig. 5(a). In these games, δ^F is a maximum of Eqs. (21) and (23). In Class II, the maximum is Eq. (21) and the point v_{II}^F is located at the bottom edge between $u(b)$ and $u(d)$. In Class IIIa, the maximum is Eq. (23) and v_{IIIa}^F is in the middle of V^*. The boundary between the classes is given when the two equations are equal, i.e.,

$$
\mathfrak{c} = 1 + (\mathfrak{b}-1)\sqrt{\frac{\mathfrak{b}}{\mathfrak{b}-2}}. \tag{20}
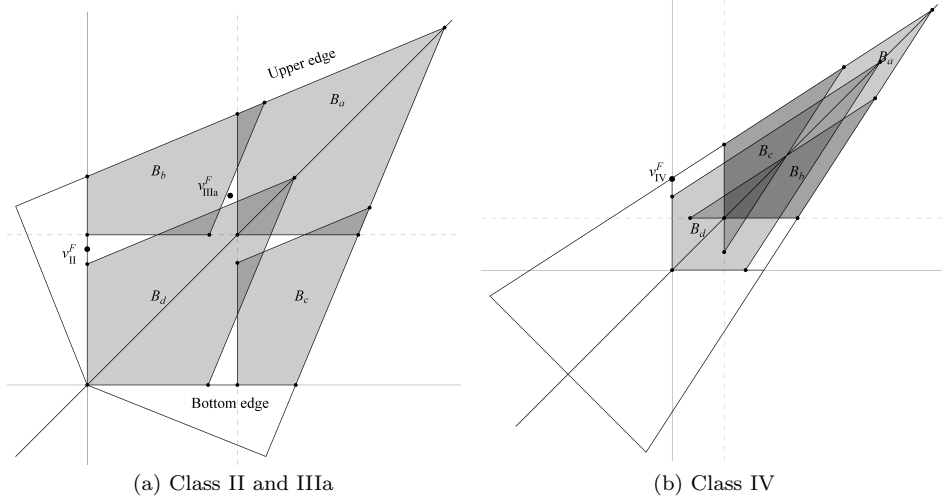$$

(a) Class II and IIIa   (b) Class IV

Figure 5: Attainable payoffs in Classes II, IIIa and IV.

This means that δ^F is continuous on the boundary.

The point v_{II}^F and δ_{II}^F are solved from the intersection of B_b and B_d:

$$v_{II}^F = v^D(B_b(\delta^F)) = v^B(B_d(\delta^F))$$

$$\Rightarrow (1-\delta^F)\cdot 1 + \delta^F v_2^-(\delta^F) = \frac{\mathfrak{c}-\mathfrak{b}}{1-\mathfrak{b}}\delta^F + \frac{1-\mathfrak{c}}{1-\mathfrak{b}}v_2^-(\delta^F)$$

$$\Rightarrow \delta_{II}^F = \frac{1-\mathfrak{b}-v^-(\delta^F)+\mathfrak{c}v^-(\delta^F)}{1-2\mathfrak{b}+\mathfrak{c}-v^-(\delta^F)+\mathfrak{b}v^-(\delta^F)}, \tag{21}$$

$$v_{II}^F = \left(0, \frac{\mathfrak{c}-\mathfrak{b}+(1-\mathfrak{c})v_2^-(\delta^F)+(\mathfrak{c}-1)v_2^-(\delta^F)^2}{1+\mathfrak{c}+\mathfrak{b}(v_2^-(\delta^F)-2)-v_2^-(\delta^F)}\right). \tag{22}$$

On the second line, the payoff $v_2^D(B_b(\delta^F))$ is again solved from the incentive compatiblity conditions. The term $v_2^B(B_d(\delta^F))$ can be solved from the equations: $v_1^B(B_d(\delta^F)) = (1-\delta^F)\cdot 0 + \delta^F y_1 = v_1^-(\delta^F)$, $v_2^B(B_d(\delta^F)) = (1-\delta^F)\cdot 0 + \delta^F y_2$, where (y_1, y_2) is a payoff on the line between payoffs (b,c) and (a,a), defined by z_1 and z_2: $bz_1 + z_2 = c$, $az_1 + z_2 = a$ and $y_1 z_1 + z_2 = y_2$.

First, let us show that $v^F \notin B_a^\delta(V^*)$, $a \in A$, if $\delta < \delta^F$. This is exactly the same as before: $v^F \notin B_d(\delta)$ or $B_b(\delta)$ when $\delta < \delta^F$ and $v^F \notin B_a(\delta)$ or $B_c(\delta)$ by proposition 3.

It is enough to check that $B_d(\delta^F)$ covers $[0, v^M] \times [0, v^M]$, i.e., all the corner points $(0,0)$, $(0, v^M(\delta^F))$ and $(v^M(\delta^F), v^M(\delta^F))$. It will be shown with Class IIIa that the other parts of V^* are covered if $\delta \geq \delta_{IIIa}^F$ and in Class II it holds that $\delta_{II}^F > \delta_{IIIa}^F$. Now, $(0, v^M(\delta^F)) \in B_d(\delta^F)$ by definition of δ_{II}^F and since v^M coincides with $v_2^-(B_b(\delta^F))$ by Remark 1. By the shape of V^*, $v^s(B_d) > v_2^B(B_d)$ since $\mathfrak{c} < 1$. Thus, $(0,0)$ and $(v^M(\delta^F), v^M(\delta^F))$ belong to $B_d(\delta_{II}^F)$.

In Class IIIa, it holds that $\delta^F_{IIIa} > \delta^F_{II}$. The point δ^F_{IIIa} is located in the middle of V^* at the intersection of B_a, B_b and B_d:

$$v^F_{2,IIIa} = v^+_2 \left(B_d \cap B_b\right) = v^-_2 \left(B_a \cap B_b\right)$$

$$\Rightarrow \delta^F_{IIIa} \quad = \frac{3 - 3\mathfrak{b} - 2\mathfrak{c} + \mathfrak{b}(\mathfrak{b} + \mathfrak{c})}{5 - 4\mathfrak{b} + \mathfrak{b}^2 - 3\mathfrak{c} + \mathfrak{b}\mathfrak{c}}, \tag{23}$$

$$v^F_{IIIa} \quad = \left(\frac{2 - \mathfrak{b} - \mathfrak{c}}{5 - 3\mathfrak{c} + \mathfrak{b}(\mathfrak{b} + \mathfrak{c} - 4)}, \frac{2 + \mathfrak{b}(\mathfrak{b} - 2) - \mathfrak{c}^2}{5 - 3\mathfrak{c} + \mathfrak{b}(\mathfrak{b} + \mathfrak{c} - 4)}\right), \tag{24}$$

see Fig. 5(a). First, we show again that $v^F \notin B^\delta_a(V^*)$, $a \in A$, if $\delta < \delta^F$. $v^F \notin B_a$ by definition of v^F. v^F is not only located at the boundary of the intersection of B_b and B_d but also on the boundary of the union of these sets. Thus, $v^F \notin B_b$ or B_d. Finally, $v^F \notin B_c(\delta)$, $\delta < \delta^F$, since $v^-_1(B_c(\delta)) > v^-_1(B_c(\delta^F)) = v^-_1(B_a(\delta^F))$.

If $v_1 \geq v^F_1$ then $v \in B_a(\delta^F)$. If $v_1 \leq v^F_1$ and $v_2 \geq v^M(\delta^F)$ then v belongs to either B_b or B_d because the slope between $v^A(B_d)$ and $v^B(B_d)$ is greater than the slope between $v^A(B_b)$ and $v^B(B_b)$. Finally, the region where $0 \leq v_2 \leq v^M(\delta^F)$ is examined with Class II and it is covered since $\delta^F_{IIIA} > \delta^F_{II}$ in this class.

3.3. Class IIIb

In this class, the payoffs satisfy $\mathfrak{b} + \mathfrak{c} > 2$ and $\mathfrak{c} > \mathfrak{b}$. These are triangular versions of prisoner's dilemma, chicken and leader games. The set V^* is a triangle since $\mathfrak{b} + \mathfrak{c} > 2$ and $u(a)$ is inside the set V^*. The point v^F is located in the middle of V^* and it is solved from the intersection of B_d, B_b and B_c:

$$v^F_{IIIb} \quad = \quad v^s(B_d(\delta^F)) = \left(v^M(\delta^F), v^M(\delta^F)\right)$$

$$\Rightarrow \quad (1 - \delta^F) \cdot 0 + \delta^F \frac{\mathfrak{b} + \mathfrak{c}}{2} = (1 - \delta^F)\mathfrak{c} + \delta^F v^-(\delta^F)$$

$$\Rightarrow \delta^F_{IIIb} \quad = \quad \frac{2\mathfrak{c}}{\mathfrak{b} + 3\mathfrak{c} - 2v^-(V^*)}, \tag{25}$$

$$v^F_{IIIb} \quad = \quad \left(\frac{\mathfrak{c}(\mathfrak{b} + \mathfrak{c})}{\mathfrak{b} + 3\mathfrak{c} - 2v^-(\delta^F)}, \frac{\mathfrak{c}(\mathfrak{b} + \mathfrak{c})}{\mathfrak{b} + 3\mathfrak{c} - 2v^-(\delta^F)}\right). \tag{26}$$

First, we examine $\delta < \delta^F$. $v^F \notin B_a$ since $v^-_i(B_a(\delta)) \geq v^M(\delta) > v^F(\delta^F)$, $i = 1, 2$. $v^F \notin B_b$ since $v^-_2(B_b(\delta)) \geq v^M(\delta) > v^M(\delta^F)$. $v^F \notin B_c$ since $v^-_1(B_c(\delta)) \geq v^M(\delta) > v^M(\delta^F)$. Finally, $v^F \notin B_d$ by the definition of v^F.

Let $v = \{v_2 \geq v_1, v \in V^*\}$. If $v_2 \geq v^M(\delta^F)$ then $v \in B_b(\delta^F)$ by the geometry. Similarly, if $v_2 \leq v^M(\delta^F)$ then $v \in B_d(\delta^F)$.

3.4. Class IIIc

This class is the reversed version of Class IIIb so that $\mathfrak{b} > \mathfrak{c}$. These are battle of the sexes, triangular versions of anti-prisoner's dilemma and anti-chicken games. The set V^* is triangular as in Class IIIb but the sets B_b and B_c are in

reverse order. Again, the point v^F is located in the middle of V^* and it is in the intersection of sets B_a, B_b and B_c:

$$
\begin{aligned}
v^F_{IIIc} &= v^s(B_a(\delta^F)) = \left(v^-_1(B_b(\delta^F)), v^-_2(B_c(\delta^F))\right) \\
&\Rightarrow (1-\delta^F) \cdot 1 + \delta^F \frac{\mathfrak{b}+\mathfrak{c}}{2} = (1-\delta^F)\mathfrak{b} + \delta^F v^-(\delta^F)
\end{aligned}
$$

$$
\Rightarrow \delta^F_{IIIc} = \frac{2\mathfrak{b}-2}{-2+3\mathfrak{b}-\mathfrak{c}} \tag{27}
$$

$$
v^F_{IIIc} = \left(\frac{\mathfrak{b}^2+\mathfrak{b}\mathfrak{c}-2v^-(\delta^F)}{3\mathfrak{b}+\mathfrak{c}-2(1+v^-(\delta^F))}, \frac{\mathfrak{b}^2+\mathfrak{b}\mathfrak{c}-2v^-(\delta^F)}{3\mathfrak{b}+\mathfrak{c}-2(1+v^-(\delta^F))}\right), \tag{28}
$$

see Fig. 4(b). Let $\delta < \delta^F$. $v^F \notin B_a(\delta)$ by the definition. $v^F \notin B_b$ since $v^-_1(B_b(\delta)) > v^-_1(B_b(\delta^F))$. $v^F \notin B_c$ because $v^-_2(B_c(\delta)) > v^-_2(B_b(\delta^F))$. Finally, $v^F \notin B_d$ since $v^s(B_d(\delta^F)) < v^s(B_a(\delta^F))$ due to $\mathfrak{d} < \mathfrak{a}$.

Now, $v_2 \geq v^-_2(B_c(\delta^F))$ implies that $v \in B_c(\delta^F)$. Also, $v_2 \leq v^-_2(B_c(\delta^F))$ implies $v \in B_a(\delta^F)$.

3.5. Class IIId

This class contains the quadrilateral versions of chicken such that $\mathfrak{b}+\mathfrak{c} > 2$ and the $u(a)$ corner is obtuse due to $\mathfrak{c} > 1$. These games are only slightly different from Class I. Since $\mathfrak{b} > \mathfrak{d} = 0$, B_d needs a higher discount factor to reach B_a than B_b does. The point v^F is in the middle and solved as an intersection of B_a and B_d:

$$
\begin{aligned}
v^F_{IIId} &= v^A(B_d(\delta^F)) = v^D(B_a(\delta^F)) \tag{29} \\
&\Rightarrow (1-\delta^F) \cdot 0 + \delta^F \cdot 1 = (1-\delta^F)\mathfrak{c} + \delta^F \mathfrak{b}
\end{aligned}
$$

$$
\Rightarrow \delta^F_{IIId} = \frac{\mathfrak{c}}{1+\mathfrak{c}-\mathfrak{b}}, \tag{30}
$$

$$
v^F_{IIId} = \left(\frac{\mathfrak{c}}{1+\mathfrak{c}-\mathfrak{b}}, \frac{\mathfrak{c}}{1+\mathfrak{c}-\mathfrak{b}}\right).
$$

Let $\delta < \delta^F$. $v^F \notin B_a$ or B_d by the definition of v^F. $v^F \notin B_b$ since $v^-_2(B_b(\delta)) = v^-_2(B_a(\delta)) > v^-_2(B_a(\delta^F))$. Similarly, $v^F \notin B_c$ since $v^-_1(B_b(\delta)) = v^-_1(B_a(\delta)) > v^-_1(B_a(\delta^F))$.

Now, $v_1 \geq v^M(\delta^F)$ implies that $v \in B_a(\delta^F)$. $v_2 \leq v^M(\delta^F)$ implies that $v \in B_d(\delta^F)$ due to the shape and $\mathfrak{c} > 1$ as in Class I. Next, we show that $v_1 \leq v^M(\delta^F), v_2 \geq v^M(\delta^F)$ belong to $B_b(\delta^F)$. We examine all the corner points: $p^1 = (b, v^M(\delta^F)), p^2 = (v^M(\delta^F), v^M(\delta^F)), p^3 = v^B(B_a(\delta^F))$ and $p^4 = u(b)$. $u(b)$ is a Nash equilibrium of the stage game and thus $u(b) \in B_b(\delta)$, $\forall \delta < 1$. By Remark 1, $p^k_2 \geq v^-_2(B_b(\delta^F)) = v^M(\delta^F)$. Also, it holds that $v^C_1(B_b(\delta^F)) > v^A_1(B_b(\delta^F)) > v^A_1(B_d(\delta^F)) \geq p^k_1$. Thus, $p^k \in B_b(\delta^F)$ for all k.

3.6. Classes IV and V

Class IV contains the anti-coordination, anti-no conflict and anti-stag hunt games. The payoff set is full only in the limit when $\delta \to 1$. By Proposition 3 the

Table 2: The bounds of discount factor δ^F in correlated strategies.

Game	Class 1	Class 2	Class 3
Prisoner's dilemma	II: $\delta^F = (\mathfrak{c}-1)/\mathfrak{c}$		Ia: $\delta^F = -\mathfrak{b}/\mathfrak{c}$
Stag hunt		Ib: $\delta^F = \mathfrak{b}/(\mathfrak{b}-1)$	
Coordination	Ia: $\delta^F = (1-\mathfrak{c})/(1-\mathfrak{b})$		
No conflict		IIIa: $\delta^F = \mathfrak{b}$	
Chicken	IIIa: $\delta^F = 2\mathfrak{b}/(\mathfrak{b}+\mathfrak{c})$		II: $\delta^F = \frac{\mathfrak{c}-1}{\mathfrak{c}-\mathfrak{b}}$
Leader			
Anti-PD	V: $\delta^F = 0$	Ia: $\delta^F = \frac{\mathfrak{c}-1}{1-\mathfrak{b}}$	
Anti-chicken			
Battle of the sexes	IIIb: $\delta^F = (2-2\mathfrak{c})/(2-\mathfrak{b}-\mathfrak{c})$		
Anti-coordination			
Anti-no conflict	IV: $\delta^F \to 1$		
Anti-stag hunt			

corner point $v^C(V^*)$ can only be covered with B_c. However, this is not covered with any $\delta < 1$ because $v_1^-(B_c(\delta)) = v^M(\delta) = (1-\delta)a + \delta v^-(V^*) > v^-(V^*)$.

Class V contains the trivial anti-games, where $\mathfrak{b} > 1$ and $\mathfrak{b} + \mathfrak{c} < 2$. In these games $\mathfrak{a}$ is both the minimax and Pareto-efficient payoff. Thus, the set $V^* = (1,1)$ is a single point and the payoff set is always full, i.e., for all $0 \le \delta < 1$.

4. Equilibria in correlated pure strategies

The conditions for the bounds δ^F in correlated strategies are more simple. Equation (12) means that V^C is a convex combination of B_a, $a \in A$. In order to show that $V^C(\delta) = V^*$, we need to prove that the last corner point of V^* is covered when $\delta = \delta^F$. Proposition 3 helps this task since any one of B_a, $a \in A$, can cover only one corner. Note that the symmetry ensures that $v^B(V^*) \in V^C \Leftrightarrow v^C(V^*) \in V^C$. The following subsections examine each corner separately and δ^F is simply the maximum over each corner.

Table 2 gives the values of δ^F in the symmetric 2×2 supergames with correlated pure strategies. There are five classes and they are based on the last corner point to be filled: in Class I, the last corner is in the northwest, corresponding to $u(b)$ or $u(c)$, the $u(a)$ corner in Class II, and the $u(d)$ corner in Class III. Classes IV and V are exactly the same as in pure strategies; certain anti-games are not filled for any $\delta < 1$ and $\delta^F = 0$ in Class V.

See the right parts of Figs. 2 and 3 for the classification and the values of δ^F. We can see that the equations and the classes are different in pure and correlated strategies. However, the locations of the high and low values of δ^F are the same. The smallest value of $\delta^F = 0$ is achieved in Class V, in point z_1, and in the limits when $\mathfrak{b} \to \infty$ or $\mathfrak{c} \to \infty$. The minimum value over coordination games is achieved in point z_3, where $\mathfrak{b} = -1$, $\mathfrak{c} = 0$ and $\delta^F = 1/2$.

With public randomization, the values of δ^F are much lower. In pure strategies, the scale is between $1/2$ or $2/3$ to 1, and between 0 and 1 with correlated strategies. Thus, the smallest values are dropped to zero and this widens the range of possible values. The borders of discontinuity are the same as before. Now, it holds that $\delta^F = 0$ on the whole $\mathfrak{c} = \mathfrak{b}$ diagonal. Moreover, the same result holds that $\delta^F \to 1$ in all classes and games, except in Class V and chicken

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Class II games. The maximum value within chicken Class II games is in point z_4, where $\mathfrak{b} = 0$, $\mathfrak{c} = 2$ and $\delta^F = 1/2$; see Fig. 3.

4.1. Corner I

Let us examine the northwest corner of V^*. The set that fills this corner may only be B_b when $\mathfrak{c} > \mathfrak{b}$ and B_c otherwise, by Proposition 3. There are three conditions that are needed: the set B_b (or B_c) should reach south, east and west enough to cover the corner. The first condition is that $v^M(\delta) \leq v_2^B(V^*)$ and from this we can solve

$$\delta_{Ia}^F = \frac{v_2^+(V^\dagger) - v_2^+(V^*)}{v_2^+(V^\dagger) - v_2^-(\delta)}. \tag{31}$$

This is the maximum of the three values and thus a sufficient condition in the triangular games which make up Class Ia.

In quadrilateral games, where $\mathfrak{c} + \mathfrak{b} < 2$, the second condition is $v_1^+(B_b) \geq v_1^-(\delta)$, which gives

$$\delta_{Ib}^F = \frac{\mathfrak{b} - v_2^-(\delta)}{\mathfrak{b} - 1}. \tag{32}$$

This condition is the maximum in Class Ib games.

The third condition is that $v_1^-(B_b) = v_1^-(\delta)$. This is satisfied for all δ in Class Ia and Ib games. In Class IV games, this condition does not hold for any $\delta < 1$, regardless of correlated strategies. These are the only games where the column maximum that limits the set is different from the punishment of the game.

4.2. Corner II

The corner $v^A(V^*)$ exist only if the set V^* is quadrilateral, i.e., $\mathfrak{c} + \mathfrak{b} < 2$. This point is filled up when $v^M(\delta) \leq \mathfrak{a}$, from which we get

$$\delta_{II}^F = \frac{v_2^+(V^\dagger) - 1}{v_2^+(V^\dagger) - v_2^-(\delta)}. \tag{33}$$

Note that this condition means that $\delta_{II}^F = 0$, i.e., a is a Nash equilibrium, if $\mathfrak{c} < 1$.

4.3. Corner III

The corner $v^D(V^*)$ is filled by the set B_d if $\mathfrak{b} < \mathfrak{c}$, and by B_a if $\mathfrak{b} > \mathfrak{c}$. In the case where B_a fills the corner, the discount factor is solved from the equation $(1 - \delta)\mathfrak{a} + \delta v^S(V^*) = v_2^-$, which gives

$$\delta_{IIIa}^F = \frac{v_2^-(\delta) - \mathfrak{a}}{v^s(V^*) - \mathfrak{a}}. \tag{34}$$

In the B_d case, the bound is solved from $(1 - \delta)\mathfrak{d} + \delta v^S(V^*) = v_2^-$ and we get

$$\delta_{IIIb}^F = \frac{v_2^-(\delta) - \mathfrak{d}}{v^s(V^*) - \mathfrak{d}}. \tag{35}$$

These equations simplify a bit in different games, since $v^s(V^*) = \mathfrak{a}$ in quadrilateral games and $v^s(V^*) = (\mathfrak{b} + \mathfrak{c})/2$ in triangular games. Note that $\delta_{III}^F = 0$ if the punishment is a or d.

5. Equilibria in randomized strategies

The necessary and sufficient conditions for δ^F are more difficult to prove in randomized strategies, since there are so many ways to get payoffs in mixed strategies. The only chance is to go through systematically all the possible 2×2 stage games and check the payoffs that they produce (Berg and Schoenmakers, 2014). The following three examples demonstrate the variety of equilibria in randomized strategies. The equilibria in Game 1 is the square from $(1,1)$ to $(3,3)$. In Game 2, the set of equilibria consists of three lines: $(1,3)$ to $(3/4,3)$ to $(3/4,1)$ to $(3,1)$. In Game 3, the equilibria consists of three points $(2/3,2/3)$, $(1,2)$ and $(2,1)$. By Theorem 2, we need to go through the stage games where each of the eight stage game payoffs can be selected from the attainable payoffs B_a, $a \in A$, for each action profile.

Game 1				Game 2				Game 3	
$3,3$	$1,3$			$1,3$	$0,3$			$2,1$	$0,0$
$3,1$	$1,1$			$0,1$	$3,1$			$0,0$	$1,2$

Table 3 and Figure 6 give the results for the randomized strategies. The classes are based on which condition determines the value for δ^F. In Class Ia, the last point to be filled is on the boundary between $u(b)$ and $u(a)$, i.e., the intersection of B_a and B_b. Class Ib corresponds to the triangle games, where B_b and B_c intersect. In Class II, a corner point determines the value of δ^F: $u(b)$ corner is last to be covered in Class IIa and $u(d)$ corner in Class IIb. The intersection of B_d and B_b determines the value for Class III. The Classes IV and V are the same as before. Note that the values in Classes IIb and III may only be upper bounds, i.e., the conditions are sufficient and the values of δ^F may be a little bit lower if certain regions of payoffs can be obtained with more complicated mixed strategies than examined here.

It should also be noted that $\delta^F = 1$ in all leader, battle of the sexes, coordination and anti-coordination games. For comparison, the values in Table 3 and Figure 6 correspond to the discount factor values when the pure-strategy reasonable payoffs are obtained. In leader and battle of the sexes games, it is impossible to obtain the payoff on the punishment line between $u(d)$ and $u(b)$, and between $u(a)$ and $u(b)$ in coordination and anti-coordination games. This is simply because these payoffs can only be obtained by playing the pure strategies and the minimum payoff in pure strategies is strictly higher than the mixed-strategy punishment payoff. For this reason, we could define the reasonable payoffs to be strictly higher than the minimax payoffs, but this would not help since obtaining a payoff close to the (mixed-strategy) minimax payoff in the corner requires a discount factor close to one. Thus, one of the problems in

Table 3: The bounds of discount factor δ^F in randomized strategies.

Game	Class 1	Class 2	Class 3
Prisoner's dilemma	Ib: $\delta^F = (\mathfrak{c} - \mathfrak{b})/2\mathfrak{c}$	Ia: $\delta^F = (\mathfrak{c} - \mathfrak{b})/(1 + \mathfrak{c} - \mathfrak{b})$	
Chicken	III*: $\delta^F = (\mathfrak{b} + \mathfrak{c})/2\mathfrak{c}$	Ia: $\delta^F = (\mathfrak{c} - \mathfrak{b})/(1 + \mathfrak{c} - 2\mathfrak{b})$	III*: $\delta^F = \frac{1}{2-\mathfrak{b}}$
Leader**			
Battle of the sexes**	III*: $\delta^F = \frac{\mathfrak{b}+\mathfrak{c}-2}{2(\mathfrak{b}-1)}$		
Stag hunt	IIa: $\delta^F = (1 - \mathfrak{c})/(1 - \mathfrak{b})$	Ia: $\delta^F = (\mathfrak{b} - 1)/(\mathfrak{b} - 2)$	
Coordination**			
No conflict		Ia: $\delta^F = 1/2$	IIb*: $\delta^F = \mathfrak{b}$
Anti-no conflict	IV: $\delta^F = 1$		
Anti-coordination**			
Anti-stag hunt			
Anti-chicken	V: $\delta^F = 0$	Ib: $\delta^F = (\mathfrak{b} - \mathfrak{c})/2(\mathfrak{b} - 1)$	
Anti-PD			

* possibly only an upper bound.
** $\delta^F = 1$, the values correspond to obtaining reasonable payoffs in pure strategies.

(a) Classification of games
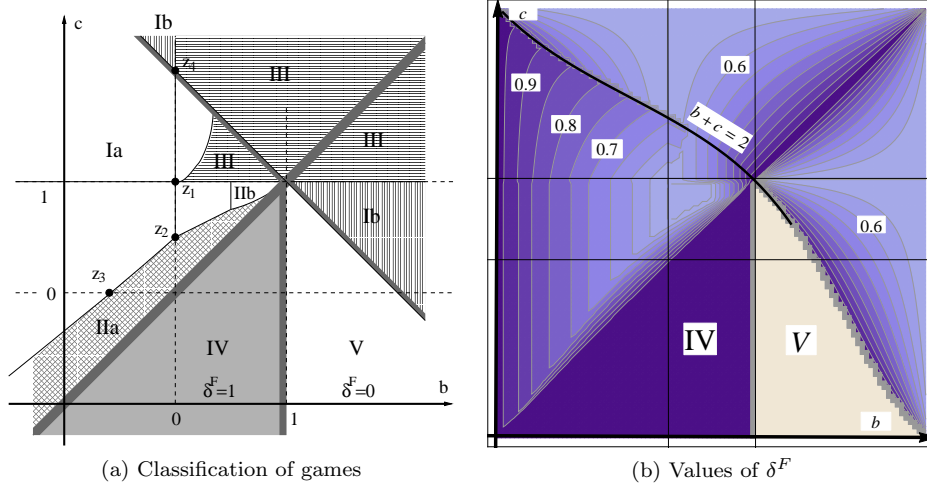
(b) Values of δ^F

Figure 6: Classification and bounds in randomized strategies.

obtaining all the reasonable payoffs in randomized strategies is the fact that the punishment payoffs may be strictly lower in mixed strategies.

We can see from the results that the differences between pure and randomized strategies are very small. In some games, the values of δ^F are exactly the same when the last payoff to be filled is obtained in pure strategies. When the last payoff is in the middle, then it is possible to fill these payoffs sooner in mixed strategies. The difference in the value of δ^F is typically smaller than 0.05 in quadrilateral games and smaller than 0.15 in triangle games.

5.1. Class I

In Class Ia, the necessary and sufficient condition for δ^F is that B_a and B_b intersect. In prisoner's dilemma, this is obvious since the last point to be filled in pure strategies is on the boundary and this payoff cannot be obtained in non-pure mixed strategies; thus, the condition in PD is the same as in pure strategies. The same argument for the necessity of the condition also holds for the other games in Class Ia. However, we need to prove the sufficiency for the other games.

In quadrilateral chicken, a sufficient condition is that 1) B_a and B_b intersect and 2) B_d and B_b intersect. It is possible to fill the payoffs in the middle that do not belong to the (pure-strategy) attainable sets B_a, $a \in A$, with mixed strategies. For example, we can construct a stage game with payoffs $(u_1^+, z_1) \in B_a$, $(u_1^-, z_2) \in B_b$, $(u_1^+, u_2) \in B_c$ and $(u_1^-, u_2) \in B_d$, where $u_1^+ > u_1^-$ and z_1 and z_2 do not matter as long as the payoffs are in B_a and B_b. Now, there is a Nash equilibrium where the first player plays bottom and the second player can use any randomization. Thus, they can obtain any payoff on the line between (u_1^-, u_2) and (u_1^+, u_2). By going through the sets B_c and B_d, these lines cover all the payoffs in the middle that are left between the sets B_a, $a \in A$. Note that this is only a sufficient condition, and it is possible that all these payoffs can be obtained with lower discount factors with some other mixed strategies.

u_1^+, z_1	u_1^-, z_2
u_1^+, u_2	u_1^-, u_2

In stag hunt and coordination games, the necessary and sufficient condition is that 1) B_a and B_d intersect and 2) $v^B(V^*)$ is covered with B_b. The reason is the fact that the line between $u(a)$ and $u(b)$ can only be obtained by playing pure strategies a and b. In Class Ia, condition 1) implies 2) which makes condition 1) a necessary and sufficient condition. In Class IIa, condition 2) implies 1), which makes condition 2) a necessary and sufficient condition. In other words, the condition is the maximum over 1) and 2) in all these games. The gaps between the pure-strategy attainable payoffs B_a, $a \in A$, can be filled with similar strategies as explained above.

In no conflict games, there is an additional condition 3) B_d should be non-empty. The value of δ^F is a maximum over the three conditions, and these give the three regions Ia (B_a and B_b), IIa (b corner covered) and IIb (d corner

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covered). Again, the gaps between the attainable payoffs B_a, $a \in A$, are filled with suitable mixed strategies.

In triangle PD, anti-PD and anti-chicken, the necessary and sufficient condition is that B_b and B_c intersect. This condition implies that B_d and B_b has intersected. Thus, it is possible to fill the gaps between the sets B_a, $a \in A$ with suitable mixed strategies as explained before. The condition is necessary since the payoffs between $u(b)$ and $u(c)$ can only be obtained by playing the pure strategies b and c.

5.2. Class II

In Class IIb, a necessary and sufficient condition is that the payoffs near $v^D(V^*)$ are covered. A sufficient condition for this is that B_d is non-empty. This is satisfied when it is possible to play d: $(1 - \delta) \cdot 0 + \delta \cdot 1 = (1 - \delta)\mathfrak{b} + \delta\mathfrak{b}$ and thus $\delta^F = \mathfrak{b}$ for Class IIb. We are not sure if this condition is necessary as all the payoffs near $v^D(V^*)$ may be obtained with some mixed strategies with lower discount factor value.

In Class IIa, a necessary and sufficient condition is that $v^B(V^*)$ is covered with B_b. This implies that the sets B_a and B_d have intersected, as explained before. Note that the conditions are the same as in correlated strategies of Class Ia.

5.3. Class III

A sufficient condition for this class is that B_d and B_b intersect. This guarantees that the gaps between the pure-strategy attainable sets B_a, $a \in A$, are covered with suitable mixed strategies. We can solve the value δ^F for triangle chicken and leader games in the following way. We first solve the value of δ when it is possible to play d: $(1 - \delta) \cdot 0 + \delta(\mathfrak{b} + \mathfrak{c})/2 = (1 - \delta)\mathfrak{b} + \delta\mathfrak{b}$, from which we get $\delta_1 = 2\mathfrak{b}/(\mathfrak{b} + \mathfrak{c})$. Now, $v^B(B_d(V^*))$ moves linearly from $v^D(V^*)$ to $v^B(V^*)$ when δ goes from the above value to 1: $z_1\delta_1 + z_2 = \mathfrak{b}$ and $z_1 \cdot 1 + z_2 = \mathfrak{c}$, from which we get $z_1 = \mathfrak{b} + \mathfrak{c}$ and $z_1 = -\mathfrak{b}$. Finally, we can solve the value when it intersects $v^D(B_b)$: $z_1\delta^F + z_2 = (1 - \delta^F)\mathfrak{c} + \delta^F\mathfrak{b}$ and we have $\delta^F = (\mathfrak{b} + \mathfrak{c})/2\mathfrak{c}$. Again, we are not sure if this condition is necessary but it provides an upper bound for δ^F.

6. Conclusions

This paper examines the discount factor values when the subgame-perfect equilibrium payoffs cover all the relevant payoffs in the symmetric 2×2 supergames. The main motivation is to study if the folk theorem could be extended in a class of games and find out the reasons why a high level of patience such as $\delta \to 1$ is required in some games. We find that the main reason is that it is impossible to obtain payoffs close to the minimax values: 1) this happens in Class IV in all strategies, 2) it is a result of the fact that the mixed-strategy punishment payoff is strictly smaller than the pure-strategy punishment in leader, battle of the sexes, coordination and its anti-game in randomized strategies,

and 3) it is due to geometrical reasons in some games, i.e., how the stage game payoffs are located and how large the reasonable payoffs V^* is in proportion to feasible payoffs $V^{\dagger}$. If V^* is small, which also means that it is difficult to play certain actions in the game, then a high level of patience is required. On the other hand, if V^* is large and square-shaped then the payoff set is filled already with a small discount factor value.

We also examine how the randomized and correlated strategies affect the results. The games are organized into a few classes based on the equation that determines the smallest discount factor value as a function of the stage game payoffs. The classes and the equations are a bit different under different strategies, but the overview looks similar. Even though a lower level of patience is required with public randomization, the highest and the lowest values are obtained in the same regions: 1) the payoff set is a single point for all discount factors in the obtuse-angle quadrilateral anti-games, 2) the lowest value for two dimensional payoff sets is in the square-shaped games in point z_1, 3) the limit $\delta \to 1$ is achieved when $b \to -\infty$, when $c \approx b$, when $b \geq 1$ and $b + c \approx 2$, or in certain anti-games. Thus, it is not possible to extend the folk theorem in any of the typical game classes, such as prisoner's dilemma, chicken and stag hunt games, unless certain extreme payoffs can be ruled out. Furthermore, the randomized nor correlated strategies do not provide a remedy in these games and it holds that $\delta^F \to 1$ under all strategies.

The results of this paper help in determining the payoff set for high discount factor values. If the discount factor is higher than the value δ^F, then all the reasonable payoffs are subgame-perfect equilibrium payoffs. This is a good result since these discount factor values are challenging for the computational methods since then the set of equilibria is huge. Moreover, it is a bit surprise how small δ^F can be with correlated strategies. It was also observed that there are certain boundaries where δ^F is discontinuous, which means that small changes in the payoffs may affect dramatically the equilibrium payoffs. Finally, it should be noted that this kind of analysis could be extended to asymmetric games with more than two actions and more than two players.

Acknowledgements

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