Abstract

This paper examines the pure-strategy subgame-perfect equilibrium payoffs in discounted supergames with perfect monitoring. It is shown that the payoff sets are typically fractals unless they are full-dimensional, which may happen when the discount factors are large enough. More specifically, the equilibrium payoffs can be identified as subsets of self-affine sets or graph-directed self-affine sets. We propose a method to estimate the Hausdorff dimension of equilibrium payoffs and relate it to the equilibrium paths and their graph presentation.

Keywords: subgame-perfect equilibrium, payoff set, fractal, sub-self-affine set, Hausdorff dimension

JEL Classification: C72, C73

1. Introduction

Repeated games are fundamental in the analysis of intertemporal cooperation among self-interested agents. The most important equilibrium concept for such games is the subgame perfection: the players should not be willing to deviate from the equilibrium strategy in any contingency.

In this paper, we consider the set of pure-strategy subgame-perfect equilibria in infinitely repeated discounted games also known as supergames. Our main result is that the payoff sets can be analyzed by identifying them as particular
fractals and using the appropriate measures designed for studying such objects. More precisely, it is shown that the equilibrium payoffs are sub-self-affine sets \cite{1, 2}. Since these sets are typically fractals, the Hausdorff dimension provides an objective means for measuring the relative amount of details or irregularities at different scales of the set. Since the Hausdorff dimension plays a central role in analyzing fractals, it is also of major interest for repeated games.

The fractal nature of equilibria has not received much attention in game theory. The reason why it has not been observed in the prior literature is that most papers assume that the players can use public randomization devices which makes the payoff set convex. But what happens when these devices are not available? Fudenberg and Maskin \cite{3} show that the Folk theorem \cite{4} still holds, i.e., any feasible and individually rational payoff in a stage game is an equilibrium payoff when the discount factor approaches one. This asymptotic result does not, however, hold in general. Yamamoto \cite{5} presents a game where the payoff set is not convex, no matter how large the discount factor is. This fact has been previously observed by Salonen and Vartiainen \cite{6}, who find that the payoffs can have irregularities, such as holes and gaps, which are typical for fractals. Moreover, it has been observed that the payoff set is not monotonic with respect to the discount factor \cite{7, 5}. We provide a more comprehensive treatment of these features by developing methodology for analyzing the equilibrium payoffs in general.

Our work builds upon the fixed-point characterization of equilibrium payoffs originally shown by Abreu, Pearce, and Stachetti \cite{8, 9} for the games with imperfect monitoring and by Cronshaw and Luenberger \cite{10} for the case of perfect monitoring. Perfect monitoring, which is assumed in this paper, means that the players observe perfectly the past history of play. The characterization result tells that the equilibrium payoffs are a fixed-point of a particular iterated function system. In other words, the payoffs are formed by taking suitable infinite sequences of affine mappings from the incentive compatible payoffs. Due to the incentive compatibility conditions, the equilibrium payoffs become a subset of a self-affine set obtained by allowing all action sequences. See \cite{11, 12} for generalizations of the fixed-point characterization to stochastic games.

We show that the Hausdorff dimension is given by a formula involving the
payoffs and the numbers of equilibrium paths of different length. However, the
discount factors are assumed to be smaller than one half. For larger discount
factors the formula gives an upper bound. To estimate the Hausdorff dimension
we need the paths of action profiles that are induced by equilibrium strategies,
i.e., the equilibrium paths. To find the equilibrium paths we utilize the recent
results of Berg and Kitti [13, 14], who identify the elementary subpaths of the
game, i.e., the fragments of which the equilibrium paths consist of. The ele-
mentary subpaths and the related methods have been generalized to stochastic
games in [15].

When the repeated game has a finite number of elementary subpaths, it is
possible to represent all the equilibrium paths with a finite graph. Consequently,
the payoff set is a graph-directed self-affine set [16]. This graph presentation
can be efficiently utilized in the computation of the Hausdorff dimension. Even
if a finite graph presentation cannot be found, it is always possible to form
approximating graphs which can be used in estimating lower and upper bounds
for the Hausdorff dimension.

This paper is structured as follows. In Section 2 we restate the characteri-
zations of equilibrium payoffs and observe that the equilibrium payoff set is a
subset of a self-affine set. The Hausdorff dimension is analyzed in Section 3.
Numerical examples are presented in Section 4.

2. Discounted Supergames and Subgame-Perfect Equilibria

2.1. Definitions

We assume that there are \( n \) players, and \( N = \{1, \ldots, n\} \) denotes the set of
players. The set of actions available for player \( i \) in the stage game is \( A_i \). Each
player is assumed to have finitely many actions. The set of action profiles is
denoted by \( A = \times_i A_i \). As usual, \( a_{-i} \) denotes the action profile of other players
than player \( i \), and the corresponding set of action profiles is \( A_{-i} = \times_{j \neq i} A_j \).
Function \( u : A \mapsto \mathbb{R}^n \) gives the vector of payoffs that the players receive in the
stage game when a given action profile is played; if \( a \in A \) is played, player \( i \)
receives payoff \( u_i(a) \).

In the supergame the stage game is repeated infinitely many times, and the
players discount the future payoffs with discount factors \( \delta_i \), \( i \in N \). We assume
perfect monitoring: all players observe the action profile played at the end of each period. A history contains the path of action profiles that have previously been played. The set of length $k$ histories or paths is denoted by $A^k = \times_k A$. The empty path is $\emptyset$, i.e., $A^0 = \{\emptyset\}$. The set of infinitely long paths is denoted by $A^\infty$. When referring to the set of paths beginning with a given action profile $a$, we use $A^k(a)$ and $A^\infty(a)$ for length $k$ and infinitely long paths, respectively.

Moreover, $A$ is the set of all paths, finite or infinite, and $A(a)$ is the set of all paths that start with $a$, i.e., union of $A^k(a)$, $k = 1, 2, \ldots$ and $A^\infty(a)$.

A strategy for player $i$ in the supergame is a sequence of mappings $\sigma^0, \sigma^1, \ldots$, where $\sigma^k : A^k \mapsto A_i$. The set of strategies for player $i$ is $\Sigma_i$. The strategy profile consisting of $\sigma_1, \ldots, \sigma_n$ is denoted by $\sigma$. Given a strategy profile $\sigma$ and a path $p \in A^k$, $k \geq 0$, the restriction of the strategy profile after $p$ is $\sigma|p$. The outcome path, simply path, that $\sigma$ induces is $(a^0(\sigma), a^1(\sigma), \ldots) \in A^\infty$, where $a^k(\sigma) = \sigma( a^0(\sigma), \ldots, a^{k-1}(\sigma))$ for all $k$.

The average discounted payoff for player $i$ corresponding to strategy profile $\sigma$ is

$$U_i(\sigma) = (1 - \delta_i) \sum_{k=0}^{\infty} \delta_i^k u_i( a^k(\sigma)) .$$

The strategy profile $\sigma$ is a subgame-perfect equilibrium (SPE) of the supergame if

$$U_i(\sigma|p) \geq U_i(\sigma', \sigma_{-i}|p)$$

for all $i \in N$, $p \in A^k$, $k \geq 0$, and $\sigma' \in \Sigma_i$.

**Definition 1.** A path $p \in A^\infty$ is a subgame-perfect equilibrium path (SPEP) if there is an SPE strategy profile that induces it.

### 2.2. Equilibrium Payoffs as Sub-Self-Affine Sets

It is well-known that the set of SPE payoffs is a fixed-point of a particular set-valued monotone operator [8, 9, 10]. We refer to this result as the APS theorem. In the following we reformulate the theorem for pure-strategy equilibria in supergames with perfect monitoring. In particular, the result will be formulated such that it becomes possible to identify the payoff sets as sub-self-affine sets.

For a compact set of payoffs $W \subset \mathbb{R}^n$, let us denote $v^-_i(W) = \min \{v_i : v \in W\}$, i.e., $v^-_i(W)$ is the smallest payoff for player $i$ over $W$. A pair $(a, v)$ of an action profile $a \in A$ and a continuation payoff $v \in W$ is admissible with respect to $W$
if it satisfies the incentive compatibility conditions

\[(1 - \delta_i)u_i(a) + \delta_i v_i \geq \max_{a'_i \in A_i} [(1 - \delta_i)u_i(a'_i, a_{-i}) + \delta_i v_i^{-}(W)] \quad \forall i \in N. \tag{2}\]

This condition means that it is better for player \(i\) to take action \(a_i\) and get the payoffs \(v_i\) than to deviate and then obtain \(v_i^{-}(W)\).

Given a set of \(W \subset \mathbb{R}^n\), we define the set of feasible action profiles, \(F(W)\), to consist of all action profiles \(a\) for which there is \(v \in W\) such that \((a, v)\) is admissible. For \(a \in F(W)\), we denote the set of possible continuation payoffs as \(C_a(W)\), i.e., \(v \in C_a(W)\) if \((a, v)\) is admissible.

We define an affine mapping \(B_a : \mathbb{R}^n \mapsto \mathbb{R}^n\) corresponding to an action profile \(a \in A\) by setting

\[B_a(v) = (I - T)u(a) + Tv,\]

where \(I\) is \(n \times n\) identity matrix and \(T\) is a \(n \times n\) diagonal matrix with discount factors \(\delta_1, \ldots, \delta_n\) on the diagonal. Now, we are ready to restate the APS theorem for equilibrium payoffs. It is possible that there are no subgame-perfect equilibria, i.e., the set of SPE payoffs \(V\) is empty. A usual assumption to guarantee \(V \neq \emptyset\) is that the stage game has pure-strategy equilibria. However, Proposition 1 holds without this assumption. To shorten the notation, let us denote \(BC_a(W) = B_a(C_a(W))\).

**Proposition 1.** The set of subgame-perfect equilibrium payoffs is the unique largest compact set \(V\) for which

\[V = \bigcup_{a \in F(V)} C_a(V) = \bigcup_{a \in F(V)} BC_a(V). \tag{3}\]

**Proof.** The result for the equality of \(V\) and the union of \(B_a(C_a(V))\), \(a \in A\), follows directly from the APS theorem according to which \(V \) is the largest set for which \(V = B(V)\), where

\[B(V) = \bigcup_{(a, v)} \{B_a(v) : (a, v) \text{ admissible w.r.t. } V\}.\]

Observe that \(B(W)\) could be defined for any compact set \(W \subset \mathbb{R}^n\) as

\[B(W) = \bigcup_{a \in F(W)} BC_a(W). \tag{4}\]
Let us now show the first equality. The inclusion that the right hand side is contained in $V$ is obvious. Hence, we are left to show that $v \in V$ implies that $v$ belongs also to the right hand side set. If this was not the case then there were no $a$ such that $(a, v)$ is admissible. This is in contradiction with $v$ being an SPE payoff.

Proposition 1 tells that $V$ is a fixed-point of the iterated function system defined by $B_a$, $a \in A$, and the incentive compatibility conditions. In the particular case when $C_a(V) = V$ for all $a \in F(V)$, the set $V$ is self-affine. However, in supergames the incentive compatibility conditions restrict the admissible action-payoff pairs, which breaks the self-affinity of the payoff set. Thus, the supergame payoffs are contained in the self-affine set defined by the action profiles, i.e., it is a sub-self-affine set.

**Proposition 2.** $V$ belongs to the self-affine set $W$, which is defined by the iterated function system consisting of contraction mappings $B_a$, $a \in F(W)$,

$$W = \bigcup_{a \in F(W)} B_a(W).$$

Hence, $V$ is a sub-self-affine set.

2.3. Equilibrium Payoffs as Graph-Directed Self-Affine Sets

Berg and Kitti [14] have shown that the SPE paths consist of elementary subpaths, which define the suitable sequences of action profiles that can be played in the game. This gives a more detailed characterization for equilibrium paths and payoffs. It is always possible to form approximations for SPE paths and payoffs using graphs, which allows a simple way to generate and analyze equilibria. Sometimes, it is possible to represent all the equilibrium paths with the graph, and then the payoff set is a graph-directed self-affine set.

First, we introduce some notation related to the paths. For a path $p \in A$, the path that starts from $j + 1$-th element is denoted by $p_j$. Respectively, $p_k$ is the path of first $k$ elements of $p$. This means that if $p = a^0a^1\cdots$, then $p_1 = a^1a^2\cdots$, $p_2 = a^0a^1$ and $p_j^k = a^ja^j+1\cdots a^{j+k−1}$. The initial and the final elements of $p$ are denoted by $i(p)$ and $f(p)$, respectively. If $p$ is infinitely long, then $f(p) = \emptyset$. Furthermore, the length of path $p$ is denoted by $|p|$. 
Definition 2. A path $p' \in A(a)$ is an SPE subpath if there is an SPE path $p \in A^\infty(a)$ such that $p[p'] = p'$.

Let $W(p)$ denote the set of continuation payoffs that can follow path $p$. This set satisfies the recursion

$$W(p) = C_{i(p)}(V) \cap B_{i(p_1)}(W(p_1)),$$

which means that the continuation payoff for the first action $i(p)$ should belong to $C_{i(p)}(V)$ and it should be produced by $i(p_1)$ from $W(p_1)$. Note that for $W(p)$ we need $W(p_1)$ for which we need $W(p_2)$, and so on. If the path $p$ is infinitely long, the recursion is infinite and $W(p)$ becomes a singleton, i.e., the vector of average discounted payoffs corresponding to path $p$. The definition is completed for one length paths by setting $W(\varnothing) = V$ and $B_\varnothing = I$.

The elementary subpaths satisfy three conditions. First, a path $p$ should have a non-empty set of possible continuation payoffs, i.e., $W(p) \neq \emptyset$. Second, any SPE subpath starting from the final element of $p$ is a possible continuation for $p$, i.e.,

$$B_{i(p_1)}(W(p_1)) \subseteq C_{i(p)}(V).$$

Finally, the path should be minimal, i.e., if $W(p_1) \neq \emptyset$ and $p$ satisfies condition (6) then there is no $k < |p|$ such that $p^k$ satisfies these conditions.

Definition 3. If $p \in A(a)$ satisfies $W(p_1) \neq \emptyset$, condition (6), and is minimal, then $p$ is an elementary subpath, and we denote $p \in P|p|_1(a)$.

The infinitely long subpaths can be elementary and then they are equilibrium paths themselves. These subpaths are denoted by sets $P^\infty(a)$, $a \in A$. All the SPE paths are characterized by the elementary subpaths [14].

Proposition 3. A path $p \in A^\infty(a)$ is an SPE path if and only if for all $j \in \mathbb{N}$ either $p_j^k \in P^k(i(p_j^k))$ for some $k$ or $p_j \in P^\infty(i(p_j))$.

When there are finitely many elementary subpaths then they can be represented by a finite graph. The graph consists of nodes, whose labels give the action profiles that are played when the nodes are visited. The arcs give the possible moves between the nodes. Fig. 1(a) gives an example of such graph. The graph contains sufficient information for creating all the equilibrium paths and
payoffs. Hence, it is a useful tool for generating and analyzing the equilibrium outcomes, as will be shown in the next section.

**Proposition 4.** When $P^k(a)$ and $P^\infty(a)$, $k \in \mathbb{N}$ and $a \in A$, contain finitely many subpaths, then all SPE paths can be represented with a graph and the payoff set is a graph-directed self-affine set.

Berg and Kitti [14] present a method for computing the elementary subpaths and forming the graph presentation. As they point out, in general there is no guarantee that there would be finitely many elementary subpaths. However, even when there are infinitely many of them or finding all of them is computationally intractable, it is always possible to form an approximation for equilibrium payoffs using a graph constructed from finitely many elementary subpaths.

### 3. The Hausdorff Dimension of Equilibrium Payoffs

It was observed in Proposition 2 that the payoff sets in discounted supergames are sub-self-affine sets, which are typically fractals. Now, we examine how to estimate the Hausdorff dimension of the payoff set, which is the most elementary concept in analyzing such sets. Intuitively, it measures how much space the payoff set takes. Formally, the Hausdorff dimension of a set $V$ is the real number

$$\dim_H(V) = \inf\{\beta \geq 0 : \mathcal{H}_\beta(V) = 0\},$$

where

$$\mathcal{H}_\beta(V) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} |W_i|^\beta : V \subseteq \bigcup_{i=1}^{\infty} W_i, \ |W_i| \leq \varepsilon \text{ for all } i \right\},$$

and $|W|$ denotes the diameter of $W \subseteq \mathbb{R}^n$.

Folk theorem [4, 3] tells that for repeated games the payoff set becomes filled as the discount factors increase above certain threshold. This means that the Hausdorff dimension of the payoff set is $n$, the number of players, if the feasible payoffs in the stage game are full-dimensional and the discount factors are large enough. For smaller discount factors, the sub-self-affinity means that the payoff set is a fractal, i.e., the Hausdorff dimension is not an integer.
3.1. Analysis with the Graph

The graph presentation gives a simple way to analyze the equilibrium outcomes. It was shown in [14] that the Hausdorff dimension of the payoff set and the asymptotic growth rate of the number of equilibrium paths are both related to the principal eigenvalue of the adjacency matrix of the graph.

Let us denote the $m \times m$ adjacency matrix of the graph by $D$, where $m$ is the number of nodes in the graph and $D_{ij} = 1$ if there is an arc from node $i$ to node $j$ and otherwise $D_{ij} = 0$. The element $d_{ij}^{(k)}$ of the matrix $D^k$ equals the number of $k$-length walks from node $i$ to node $j$. Thus, the number of $k$-length equilibrium paths is given by the elements of $D^k$, and the asymptotic growth rate of the number of paths is $\rho(D)$, i.e., the principal eigenvalue of matrix $D$, see [17]. This measure tells how large the set of equilibrium paths is.

In general, it is difficult to determine the exact Hausdorff dimension of a set, which is generated by an iterated function system whose different parts may overlap [18]. Typically, the open set condition is assumed, which guarantees that the mapped sets in the iterated function system are separated. In repeated games, a sufficient condition is that the discount factors are less than one half. For larger discount factors, the same measures can be calculated for an approximation, but there is no guarantee that it is the exact Hausdorff dimension due to the possible overlaps. However, there are methods to estimate lower and upper bounds for the Hausdorff dimension [19].

When the players have a common discount factor $\delta \leq 1/2$, it is possible to determine the Hausdorff dimension of the payoff set with the graph [14]. The Hausdorff dimension is given by the affinity dimension

$$\dim_H(V) = \dim_A(V) = -\log \rho(D)/\log \delta,$$

i.e., $\dim_A(V)$ solves $\rho(\delta^s D) = 1$ for $s$; see, e.g., [16, 18]. Thus, the Hausdorff dimension is directly related to the principal eigenvalue $\rho(D)$ of the graph. The affinity dimension [20] is sometimes called the similarity dimension [18] when the iterated function system is self-similar rather than self-affine, or the singularity dimension [2] which highlights the fact that the dimension is given by the singular values of the iterated function system.
3.2. The Case of Unequal Discount Factors

When the players have different discount factors, the payoff set is generated by contractive affine mappings, which have diverse scaling factors in different dimensions. This means that we have to adopt more general methods for analyzing the payoff set using the matrix $T$ that contains the discount factors $\delta_1, \ldots, \delta_n$ on the diagonal. These methods are similar to what was presented in the last section as they enumerate the possible equilibrium paths and utilize the singular values.

The main result of this section is a formula for the Hausdorff dimension when the unequal discount factors are less than one half. It is a consequence of a recent result by Kämäki and Vilppolainen [2], who study sub-self-affine sets using the topological pressure and the singular value function. These concepts are now defined in the framework of repeated games.

Let us assume that the players are indexed according to the order of discount factors such that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. The singular value function of matrix $T^j$, i.e., $j$-times product of $T$, is then

$$\phi^j(T^j) = \begin{cases} \delta_1 \delta_2 \cdots \delta_{m-1} \delta_m^{(t-m+1)}, & 0 \leq t < n, \\ \delta_1 \delta_2 \cdots \delta_n \delta_j^{t/n}, & t \geq n, \end{cases}$$

where $m$ is the integer such that $m-1 \leq t \leq m$. Let $K$ denote the set of all subgame-perfect equilibrium paths, $K_j = \{p_j \in A^j : p \in K\}$, and $\#K_j$ is the number of elements in $K_j$. When there are a finite number of elementary subpaths, the paths $K$ and $K_j$, $j \geq 1$, can be determined with the graph as was explained in Section 3.1. The topological pressure [1, 2] takes the form

$$P(t) = \lim_{j \to \infty} \frac{\log \left[ \phi^j(T^j) \right] \left( \#K_j \right)}{j}.$$ (8)

In the following $s(u, T)$ denotes the zero of the topological pressure for given $u$ and $T$. This is the generalized affinity dimension and it is equal to Eq. (7) in the case of equal discount factors. The set $V(u, T)$ stands for the equilibrium payoffs corresponding to the discount factors given by $T$, and the stage-game payoffs given by function $u$ for a fixed set of available action profiles $A$.

**Proposition 5.** Let us assume that $\delta_i < 1/2$ for all $i \in N$. Then the Hausdorff dimension of $V(u, T)$ is $\min\{n, s(u, T)\}$ for Lebesgue-almost all payoff functions $u$ for which $V(u, T) \neq \emptyset$. 

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Proof. The result follows from Theorem 5.2 of [2]. For the assumptions of the theorem we need three properties. First, $T$ should satisfy $\|T\| < 1/2$, where $\|T\|$ is the largest singular value of $T$, i.e., the square root of largest eigenvalue of $T \times T$. It easy to observe that $\|T\| = \max_i \delta_i$. Consequently, $\|T\| < 1/2$ when $\delta_i < 1/2$ for all $i \in N$. Second, $p_1$ should be a subgame-perfect equilibrium path (SPEP) whenever $p$ is an SPEP. This is obviously the case. The third property that we need is that the set of SPEPs should be compact in the topology of the metric defined by the distance

$$|p - r| = \begin{cases} \alpha \min\{k-1: p_k \neq r_k\}, & p \neq r, \\ 0, & p = r, \end{cases}$$

where $\alpha \in (0, 1)$.

To obtain compactness we first associate a common element to all action profiles that yield the same payoff. For example, if $u(a) = u(b)$ for $a, b \in A, a \neq b$, we can simply replace every $b$ on all paths with $a$. Now take a sequence of SPEPs $p(j), j = 1, 2, \ldots$. The sequence of payoffs corresponding to these paths has a convergent subsequence. This is because $V = V(u, T)$ is a compact set when it is non-empty. Let $p(j_k)$ denote the subsequence of SPEPs corresponding to this subsequence of payoffs, and let $p$ denote a path corresponding to the limit. Because $\delta_i < 1/2$ it holds that $B_a(V) \cap B_b(V) = \emptyset$ when $a, b \in A$ and $u(a) \neq u(b)$. This together with the assumption that all the action profiles on equilibrium paths have different payoffs implies that the limit path is unique. Moreover, the no-overlapping property guarantees that only the final elements of $p(j_k)$ can differ from $p$, and when $k$ is increased the threshold for $k$ above which the elements are different increases. This means that $|p(j_k) - p|$ goes to zero as $k$ increases, which proves the compactness.

In general, the Hausdorff dimension increases as the affine mappings become less contractive, i.e., when the discount factors increase, if the smallest payoffs do not change. This observation follows directly from the definition of topological pressure and the fact that the equilibrium paths are monotone in the discount factor [14]. In particular, by varying the discount factors the Hausdorff dimension can have values between zero and $n$. Consequently, for small discount factors the Hausdorff dimension can be larger than the topological dimension,
i.e., $V(u, T)$ is a fractal. On the other hand, the Folk theorems tell that the payoff set becomes filled when the discount factors are large enough. An interesting question is then when does this exactly happen and how to analyze the payoffs when the discount factors are below this limit.

For discount factors larger than one half, it is known that the Hausdorff dimension can be below the affinity dimension. Das and Ngai [18] show how to determine the Hausdorff dimension when the overlaps can be modeled and how far apart these two dimensions are. On the other hand, Jordan et al. [21] explain why the Hausdorff dimension suddenly drops and differs from the affinity dimension when the overlaps occur. They show that for self-affine sets, the value $\min\{n, s(u, T)\}$ is "almost surely" the Hausdorff dimension even when $\delta_i > 1/2$ for some $i \in N$. However, we are not aware of the same result for sub-self-affine sets. Since this question is beyond the scope of this paper, we propose the following conjecture. We emphasize that the result stated below holds when the payoff set is self-affine.

**Conjecture 1.** Let us consider the truncated game in which

1. the players’ payoffs are

   $$ (1 - \delta_i) \sum_k [u_i(a_k^u) + \varepsilon_k^i], $$

   where $\varepsilon_k^i$ are iid random variables with absolutely continuous distributions $\eta_i$, $i \in I$, bounded support, and mean zero, and

2. the players condition their actions only on the past history of play.

Then the Hausdorff dimension of the set of expected equilibrium payoffs is equal to $\min\{n, s(u, T)\}$ for $P$ almost all sequences of $(\varepsilon_1^k, \ldots, \varepsilon_n^k)$, $k = 0, 1, \ldots$, where $P$ is the infinite product measure of $\eta \times \cdots \times \eta \times \cdots$ with $\eta = (\eta_1, \ldots, \eta_n)$.

Note that when we add a random component to the payoffs and assume perfect monitoring, the players could condition their actions on this random component. This obviously increases the set of equilibrium payoffs. Hence, we restrict to equilibria in which there is no conditioning on the random disturbance of the payoffs. This conjecture suggests that the overlaps can be neglected and the difference between the Hausdorff and affinity dimensions vanishes when the payoffs are disturbed arbitrarily little.
4. Numerical Examples

4.1. No Conflict Game

Let us examine the following no conflict game.

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We assume that the players have a common discount factor $\delta$. When $\delta < 0.3$, the only subgame-perfect equilibrium path is $a^\infty$ and it gives the average payoffs 3 to the players. When $\delta \geq 1/2$, the players’ smallest payoffs are $v_i^-(V) = 1$, $i = 1, 2$, and the corresponding punishment paths are $b^\infty$ and $c^\infty$.

When $\delta = 1/2$, there are a finite number of elementary subpaths: $a, ba, bb, ca, cc$ and $da$. All the SPE paths of the game can be constructed from these fragments and they can be represented with the graph in Fig. 1(a), where the labels of the nodes give the actions that are played when the node is visited. In the graph, the game can start from any node. Fig. 1(b) shows the payoff set.

The principal eigenvalue of the graph, $\rho = 2.618$, gives the asymptotic growth rate of the finite length equilibrium paths. This means that when the path length is increased by one, the number of finite length paths increases by $\rho$. For

![Figure 1: SPE paths as a graph and the payoff set.](image-url)
example, there are four one-length paths \((a, b, c\) and \(d)\), nine two-length paths \((aa, ab, ac, ad, ba, bb, ca, cc\) and \(da)\), and 25 three-length paths. The fraction \(9/4, 25/9, \ldots\), approaches \(\rho\) when the path length is increased. Moreover, the Hausdorff dimension of the payoff set is then \(\dim_H(V) = -\log(\rho)/\log(\delta) \approx 1.4\).

We can see from Fig. 1(b) that there are holes in the payoff set, and the payoffs fill the space irregularly. For example, it is possible to play \(a\) and \(b\) (\(c\)) in any given order and the line between the payoffs \(u(a)\) and \(u(b)\) (\(u(c)\)) is fully covered. This tells intuitively that the dimension must be at least one, since the set covers more than one dimensional line. On the other hand, a large area of the payoffs between 1.5 and 2 are not achieved in this game when \(\delta = 1/2\).

4.2. Stag Hunt Game

Let us examine the set of equilibria for the following stag hunt game. The affinity dimension and the asymptotic growth rate are computed for different discount factors between 0.5 and 0.7. For common discount factors, the affinity dimension can be determined by Eq. (7) and it equals the zero of the topological pressure in Eq. (8). For unequal discount factors, the sequence of topological pressure is computed up to length 500 as the sequence converges fast for small graphs. The maximum length of elementary subpaths is restricted to 10 for computational reasons, which means that only a subset of equilibrium paths is used in the calculations.

\[
\begin{array}{cc|cc}
  & L & R \\
 T & 3, 3 & 0, 2 \\
 B & 2, 0 & 1, 1 \\
\end{array}
\]

The approximate dimensions and the asymptotic growth rates are shown in Table 1. The first number is the affinity dimension and the second number is the asymptotic growth rate of paths. The star (*) indicates that the Hausdorff dimension is always below two, even though the zero of the topological pressure used in the definition of the affinity dimension is above the dimension of payoffs. We can see that the dimension increases fast as the discount factors change from 0.5 to 0.7. The increase is mainly due to the growth in discount factors, since the asymptotic growth rates do not increase dramatically. For example, if all the paths that are equilibrium paths for discount factor 0.71 were also
paths for a game with the discount factor 0.51, then the dimension would be $-\log 3.2 / \log 0.51 \approx 1.7$ and this is not that much larger than 1.5, i.e., the dimension of a game with discount factor 0.51.

Table 1: The affinity dimensions and asymptotic growth rates.

<table>
<thead>
<tr>
<th>$\delta_1=0.51$</th>
<th>$\delta_2=0.61$</th>
<th>$\delta_3=0.71$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5/2.7</td>
<td>1.7/2.8</td>
<td>2.1*/2.9</td>
</tr>
<tr>
<td>2.1*/2.9</td>
<td>2.6*/3.0</td>
<td></td>
</tr>
<tr>
<td>3.4*/3.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since the affinity dimension is above two, i.e., $s(u, T)$ is at least two, the results suggest that the payoff set in the stag hunt game becomes full-dimensional for some part of the payoffs around discount factor 0.6. It should be noted that the values in Table 1 are lower bound estimates since the elementary subpaths are limited to maximum length 10. However, it was tested that the results do not change even though this limit would be increased a little.

5. Conclusions

This paper formulates the pure-strategy subgame-perfect equilibrium payoffs as sub-self-affine sets. The connection between fractal geometry and game theory provides new ways for analyzing repeated interactions. For example, the density of the payoff set can be measured by the Hausdorff dimension. It was shown that for small discount factors it is possible to find the Hausdorff dimension and it coincides with the affinity dimension. For moderately large discount factors, it is difficult to determine the Hausdorff dimension due to the possible overlaps, and it may differ from the affinity dimension. For larger discount factor values, the payoffs fill the corresponding space for many games, like the folk theorems suggest, and then it is no longer meaningful to consider the payoff set as a fractal. However, its boundary may still be fractal when the discount factors are not equal.

The affinity dimension is related to the growth rate of equilibrium paths, and it can be easily computed if the graph presentation is available for the equilibrium paths. The affinity dimension is monotone in the discount factor if
the smallest payoffs do not change, and it does not behave irregularly like the Hausdorff dimension may behave when the payoffs overlap. Furthermore, the number of equilibrium paths may increase even though the payoff set is already full-dimensional.

References


