Improving Construction of Conditional Probability Tables for Ranked Nodes in Bayesian Networks

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Abstract—This paper elaborates the ranked nodes method (RNM) that is used for constructing conditional probability tables (CPTs) for Bayesian networks consisting of a class of nodes called ranked nodes. Such nodes typically represent continuous quantities that lack well-established interval scales and are hence expressed by ordinal scales. Based on expert elicitation, the CPT of a child node is generated in RNM by aggregating weighted states of parent nodes with a weight expression. RNM is also applied to nodes that are expressed by interval scales. However, the use of the method in this way may be ineffective due to challenges which are not addressed in the existing literature but are demonstrated through an illustrative example in this paper. To overcome the challenges, the paper introduces a novel approach that facilitates the use of RNM. It consists of guidelines concerning the discretization of the interval scales into ordinal ones and the determination of a weight expression and weights based on assessments of the expert about the mode of the child node. The determination is premised on interpretations and feasibility conditions of the weights derived in the paper. The utilization of the approach is demonstrated with the illustrative example throughout the paper.

Index Terms—Bayesian networks, conditional probability tables, probability elicitation, influence diagrams, ranked nodes.

1 INTRODUCTION

Bayesian networks (BNs), see, e.g., [1], [2], are used in many areas to represent uncertain knowledge and to reason under uncertainty. The areas include, e.g., medical diagnosis [3], [4], [5], hardware troubleshooting and diagnosis [6], simulation metamodeling [7], [8] as well as military planning [9], [10]. A BN is a directed acyclic graph with nodes representing random variables and arcs representing dependencies between them. When the nodes have a finite number of states, the dependencies are defined by conditional probability tables (CPTs). A CPT specifies the conditional probability distributions of the descendant, the child node, for all combinations of the states of its direct predecessors, the parent nodes. The CPTs and the graphical structure of the BN encode the joint probability distribution of the discrete random variables. When the states of some nodes are known, the probability distributions of the others can be updated accordingly using effective algorithms, see, e.g., [1], [11]. Therefore, BNs are an effective tool to answer probabilistic queries about random variables, and they provide means to conduct, e.g., risk analysis, see, e.g., [12], [13]. Furthermore, the decision theoretical extension of BNs, influence diagrams [14], in which CPTs are also required, offer a way to support decision making under uncertainty, see, e.g., [15], [16].

In the absence of data, CPTs have to be constructed based on expert elicitation involving subjective assessments of a domain expert. The main difficulty therein is the vast amount of probability assessments needed. As the size of a CPT grows exponentially with the number of parent nodes, the amount of probability assessments needed for a single BN may be up to hundreds or thousands. Assessing so many probabilities coherently and without biases can be an insuperable problem for the expert due to cognitive strain or scarcity of time [17], [18]. For example, conventional probability elicitation techniques, such as probability wheel or reference lottery, are generally acknowledged to be too time-consuming for the probability elicitation of BNs [17], [19], [20].

In order to save time and elicitation effort of the expert, CPTs can be constructed using methods referred to as, e.g., parametric probability distributions [17], canonical models [2], or canonical distributions [11]. In these methods, it is assumed that the probabilistic relationship between the parent nodes and the child node fits some standard pattern [11]. Then, the CPT is generated based on the assessments of the expert on parameters whose amount grows only linearly with the amount of the parent nodes. Examples of well-known parametric methods are Noisy-OR [2] designed for binary variables and Noisy-MAX [21], [22], the generalization of the former to variables with multiple states.

This paper presents a novel approach for using a parametric method developed by Fenton et al. [23] to construct CPTs for specific types of random variables called ranked nodes. The states of a ranked node are expressed with an ordinal scale so that each of the states represents a range of values of a continuous quantity. Typically, ranked nodes are used to represent continuous quantities for which there are no well-established ratio or interval scales, see, e.g., [24], [25], [26], [27]. In such cases, the states of the nodes have descriptive labels associated with them. Whereas Fenton et al. do not name their method in [23], it is referred to as the “ranked nodes method” (RNM) in this paper for convenience.
RNM can be regarded complementary to many well-known parametric methods. For example, in both Noisy-OR and Noisy-MAX, each parent node represents a cause that can produce a certain effect represented by the child node [28]. The parameters elicited from the expert are probabilities concerning the ability of a single cause to produce the effect in the absence of other causes. In turn, in RNM, the child node represents a quantity that can be seen to aggregate quantities represented by the parent nodes. The parameters elicited from the expert include the type of an aggregation function, called a weight expression, and weights representing relative strengths by which the parent nodes define the central tendency of the child node on the ordinal scale. Furthermore, the expert has to assign a variance parameter that describes uncertainty about the central tendency of the child node and defines the flatness of the probability distributions generated. Note that the formal definition of “central tendency” is not given in the literature on RNM, but in practice, it reflects the mode of a distribution as well as the clustering of the distribution around the mode.

The aggregation principle and the probability generation are realized in RNM by associating the states of each node with consecutive subintervals of a normalized scale $[0, 1]$. The aggregation function maps points from the normalized scales of the parent nodes to the normalized scale of the child node. The resulting points are then used as mean parameters of doubly truncated normal distributions that are utilized in the generation of the CPT.

According to Fenton et al. [23], the use of weight expressions aids experts to understand and express probabilistic relationships between nodes in a BN. This may be the reason why ranked nodes are used in some applications, see, e.g., [29], [30], [31], to represent quantities with well-established interval scales, even though alternative approaches, such as those based on dynamic discretization [32], [33], exist for modeling probabilistic relationships with such types of nodes.

When RNM is applied to nodes with interval scales, the ordinal scales of the ranked nodes are defined by discretizing the interval scales into consecutive subintervals. While the ordinal scales may be preferable to describe and represent the probabilistic relationship between the nodes, the generation of CPTs in RNM is always based on the normalized scales $[0, 1]$ associated with the nodes. If the use of the normalized scales is not well understood, the original interval scales may be discretized so that the resulting ordinal scales are incompatible with the functioning of RNM. This prevents the formation of sensible CPTs — i.e., those that portray the views of the expert — with any configuration of parameters elicited. This is a challenge that is not addressed in the existing literature but is demonstrated through an illustrative example in this paper. In addition, the paper discusses through the illustrative example how the lack of exact interpretations for the weights of parent nodes complicates their elicitation in RNM. A note is also made about possible difficulty concerning the selection of a weight expression. While recognized by Baraldi et al. [34], these two themes are not thoroughly addressed in the existing literature either.

This paper is motivated by the recognition of the above challenges concerning the application of RNM in the context of BNs representing quantities measured on interval scales. The contribution of the paper is the presentation of a novel approach for using RNM which overcomes these challenges. The approach consists of three phases: 1) a guideline for discretizing the interval scales into ordinal scales that are compatible with the functioning of RNM, 2) a guideline for the elicitation of a weight expression and weights, and 3) suggestions of ways to refine a generated CPT after its verification.

In the first phase, an initial discretization is obtained by dividing the interval scales of all nodes into an equal amount of subintervals. The discretization is then refined by asking the expert revision questions about the boundary points of the subintervals. In the second phase, a weight expression and weights are determined by utilizing expert assessments about the mode of the child node on the interval scale in various scenarios. Here, a scenario refers to a given combination of values of the parent nodes on the interval scales. The determination of the weights is based on their interpretations concerning the mode of the child node in specific types of scenarios. In turn, these interpretations are based on interpretations of the weights regarding the values of the parent nodes and the mode of the child node on the normalized scales as well as on mappings defined between the normalized and interval scales. The determination of the weight expression is based on feasibility conditions of the weights that reflect the correspondence of the mode assessments with the forms of the weights expressions. The interpretations, the mappings, and the feasibility conditions are all derived in the paper. In the third phase, performing sensitivity analysis on a BN, see, e.g., [35], [36], and [37], is recognized as a good supportive act for any refinement of a CPT. In addition, different ways to refine the CPT are suggested by considering various types of situations in which the CPT initially generated is considered faulty. Among the ways are the manual editing of the CPT as well as the adjustment of the variance parameter and the mode assessments. Furthermore, the construction of the CPT in parts with alternate values of the parameters, see [23], is elaborated.

Throughout the paper, the execution of the approach is demonstrated by using the same example BN with which the challenges inspiring the development of the approach are illustrated. This exemplifies how the approach facilitates the use of RNM and allows it to be applied more efficiently. The example also showcases how the interpretations of weights in RNM alleviate the elicitation in a similar manner as the interpretations of criteria weights [38] do it in the context of multi-criteria decision analysis, see also, e.g., [39], [40].

The paper is organized as follows. Section 2 gives a short introduction to ranked nodes and RNM. Section 3 discusses through an illustrative example challenges related to the application of RNM to BNs representing quantities with interval scales. Section 4 presents the novel approach for applying RNM to such BNs. Section 5 discusses practical means to support the execution of the approach, its limitations, and topics for further research. Finally, Section 6 provides concluding remarks.
2 RANKED NODES METHOD (RNM)

The following introduction to RNM is based on the description presented by Fenton et al. [23] as well as on private communication on the technical details of the method with Professor Norman E. Fenton, one of the developers of RNM. The implementation of RNM in AgenaRisk software [41] is also referred to in order to illustrate the ranges of parameters used in the method.

Recall that RNM is typically used to construct CPTs that represent probabilistic relationships between quantities that are considered continuous but lack a well-established interval scale. Liveliness of a person or disturbance level in an office are examples of such quantities. In RNM, each of the quantities is represented as a ranked node whose states are expressed on an ordinal scale. Thus, any ranked node has a discrete set of states that can be sorted according to the order of magnitude or rank order, e.g., \{Low, Medium, High\}. The construction of a CPT with RNM consists of six steps:

1. Associate the states of nodes with subintervals on a normalized scale \([0, 1]\).
2. Select a weight expression.
3. Assign weights of the parent nodes.
4. Assign a variance parameter.
5. Compute a CPT using the above settings.
6. Verify the CPT and refine it if needed.

The first four and the sixth step require involvement of the expert, whereas the fifth step is carried out using a computational procedure.

In RNM, a given state \(x_i\) of a ranked node \(X_i\) is associated with a subinterval \(z_i\) of a normalized scale \([0, 1]\). Let \(z_i\) be called a state interval. Consecutive states of the node \(X_i\) are associated with consecutive state intervals that are of equal width, do not intersect, and their union covers the whole normalized scale. For example, if the states are \{Low, Medium, and High\}, they are associated with the state intervals \([0, 1/3]\), \([1/3, 2/3]\), and \([2/3, 1]\) respectively. In the fifth step in RNM is to associate the states of the nodes to the state intervals so that the direction of the influence of the parent nodes \(X_1, ..., X_n\) to the child node \(X_C\) is correctly portrayed. For example, suppose that the parent node \(X_i\) and the child node \(X_C\) both have the states \{Low, Medium, and High\}. If low states of \(X_i\) promote high states of \(X_C\), the state Low of \(X_i\) and the state High of \(X_C\) should both be associated with either the state interval \([0, 1/3]\) or \([2/3, 1]\).

The second step in RNM is to select a weight expression \(f\). Given a combination of the states of the parent nodes \((x_1, ..., x_n)\), \(f\) maps points from the corresponding state intervals \((z_1, ..., z_n)\) to points on the normalized scale of \(X_C\). There are four alternatives for \(f\) [23]: 1) weighted mean (WMEAN), 2) weighted minimum (WMIN), 3) weighted maximum (WMAX), and 4) mixture of minimum and maximum (MIXMINMAX).

The selection of the weight expression can be supported by, e.g., asking the expert about the central tendency in the form of the mode of the child node for different combinations of the extreme states of the parent nodes [23].

In the third step, one assigns weights \(w\). If the weight expression \(f\) is WMEAN, WMIN, or WMAX, \(w = (w_1, ..., w_n)\), i.e., a weight \(w_i\) is assigned to each parent node \(X_i\). If \(f\) is MIXMINMAX, \(w = (w_{MIN}, w_{MAX})\), i.e., only two weights are assigned. In AgenaRisk, the default range of the weights is \([1, 5]\), although any weight value can be added manually.

Assigning a variance parameter \(\sigma^2\) is the fourth step. The larger \(\sigma^2\) is, the flatter each conditional probability distribution \(P(X_C|X_1 = x_1, ..., X_n = x_n)\) in the CPT will be. In practice, \(\sigma^2\) is the variance parameter of doubly truncated normal distributions that are integrated over the state intervals of \(X_C\) when computing \(P(X_C|X_1 = x_1, ..., X_n = x_n)\). In AgenaRisk, the default range for \(\sigma^2\) is \([5 * 10^{-4}, 0.5]\).

In the fifth step, the CPT of the child node \(X_C\) is generated by calculating repeatedly conditional probabilities of the form \(P(X_C = x_C|X_1 = x_1, ..., X_n = x_n)\), where \(x_C\) is the state of the child node \(X_C\), and \(x_1, ..., x_n\) are the states of the parent nodes \(X_1, ..., X_n\) respectively. In the computational procedure, sample points are taken from each of the state intervals of the parent nodes \(z_i\). In AgenaRisk, \(s = 5\) by default but the value can be changed manually. The first and last sample points are always the lower and upper bounds of \(z_i\). Based on the \(n * s\) sample points, a set of combinations of the sample points \(\{z_{i,k}\}_{k=1}^n\) is constructed so that each combination contains one sample point from each parent node. For each combination in the set, a mean parameter \(\mu_k\) is calculated by

\[
\mu_k = f(z_{1,k}, ..., z_{n,k}, w),
\]

where \(f\) refers to the weight expression selected and \(w\) contains the weights assigned. The functional forms of the weight expressions are

\[
WMEAN(z_{1,k}, ..., z_{n,k}, w_1, ..., w_n) = \frac{\sum_{i=1}^n w_i * z_{i,k}}{\sum_{i=1}^n w_i},
\]

\[
WMIN(z_{1,k}, ..., z_{n,k}, w_1, ..., w_n) = \min_{i=1,...,n} \left\{ \frac{w_i * z_{i,k} + \sum_{j \neq i} w_j z_{j,k}}{w_i + n - 1} \right\},
\]

\[
WMAX(z_{1,k}, ..., z_{n,k}, w_1, ..., w_n) = \max_{i=1,...,n} \left\{ \frac{w_i * z_{i,k} + \sum_{j \neq i} w_j z_{j,k}}{w_i + n - 1} \right\},
\]

\[
MIXMINMAX(z_{1,k}, ..., z_{n,k}, w_{MIN}, w_{MAX}) = \frac{w_{MIN} * \min_{i=1,...,n} \left\{ z_{i,k} \right\} + w_{MAX} * \max_{i=1,...,n} \left\{ z_{i,k} \right\}}{w_{MIN} + w_{MAX}}.
\]

WMEAN is the weighted average of the sample points, and MIXMINMAX is the weighted average of the minimum and maximum of the sample points. WMIN and WMAX provide a value that is smaller or larger than the average of the sample points, respectively. The characteristics of the weight expressions are discussed in more detail in [23] and [42].

With each \(\mu_k\), a value \(p_{zC}^{\mu_k}\) is calculated by

\[
p_{zC}^{\mu_k} = \int_{z_C} TNormpdf(u, \mu_k, \sigma^2, 0, 1)du,
\]
where $z_C$ is the state interval corresponding to the state $x_C$ of the child node $X_C$, and $TNormpdf(u, \mu_k, \sigma^2, 0, 1)$ is the probability density function of a normal distribution with mean $\mu_k$ and variance $\sigma^2$ truncated to the interval $[0, 1]$. Using the values $p_{x_C}^k$, the probability $P(X_C = x_C | X_1 = x_1, \ldots, X_n = x_n)$ is calculated as the average

$$P(X_C = x_C | X_1 = x_1, \ldots, X_n = x_n) = \frac{1}{s^n} \sum_{k=1}^{s^n} p_{x_C}^k. \tag{7}$$

The whole CPT of the child node is generated by repeating the computations presented above for all states of the child node with each combination of the states of the parent nodes. The parameters $f$, $w$, and $\sigma^2$ can be fixed throughout the CPT. On the other hand, Fenton et al. [23] note the CPT may be partitioned with respect to the parameters. That is, different values of the parameters are used to generate different parts of the CPT. This feature is implemented in AgenaRisk software.

The final step in RNM is to verify the CPT constructed. That is, one analyzes whether for some representative combinations of the states of the parent nodes, the conditional probability distributions of the child node reflect the views of the expert. If the correspondence is deemed inadequate, the CPT has to be refined by adjusting the parameters used in RNM or possibly changing elements in the CPT by hand.

3 Challenges on RNM with Interval Scales

Although RNM is typically used with continuous quantities that lack a well-established interval scale, some of its applications concern quantities readily measured on interval scales, see, e.g., [29], [30], [31]. In such cases, the ordinal scales of ranked nodes are defined by discretizing the interval scales into consecutive subintervals. Even though applying RNM in this way may seem uncomplicated, there are challenges that can make the use of the method inefficient. As these challenges are not addressed in the existing literature, an illustrative example is next used to demonstrate them.

The BN presented in Fig. 1 describes how the rent of a studio apartment is dependent on its surface area, its distance from the city centre, and the time since its last overhaul. Each of the nodes in the BN can be considered to represent a continuous quantity that is measurable with a well-established interval scale. The surface area is measured in square meters, the distance in kilometers, the time in years, and the rent in euros. Suppose that the rent generally increases for an increasing surface area and decreases for an increasing distance or time. This type of probabilistic relationship between the parent nodes and the child node Rent is in line with the functioning of RNM. Hence, RNM can be applied for constructing a CPT.

In order to apply RNM, finite sets of states are first formed for the nodes by discretizing the interval scales. As states of ranked nodes typically have descriptive labels associated with them, e.g., {Very Low, Low, Medium, High, Very High}, the discretization can be carried out by asking the expert which subintervals of the interval scales he or she associates with certain type of labels. Let the subintervals and the states identified by the expert be those presented in Fig. 1.

![Fig. 1. Example BN with interval scales discretized into ordinal scales, and the corresponding state intervals.](image)

After the states of the nodes are defined, RNM is applied to construct a CPT for the node Rent as described in Section 2. First, the states of the nodes are associated with state intervals on the normalized scale $[0, 1]$. The way the states are associated with the state intervals in Fig. 1 is consistent with the above description concerning the probabilistic relationship between the nodes.

Next, a weight expression is determined. As mentioned in Section 2, this can be supported by asking the expert to assess the mode of the child node for different combinations of the extreme states of the parent nodes. With the example BN, there are eight combinations of the extreme states of the parent nodes that are presented in Table 1 along with the mode assessments of the expert. Rows 1 and 8 of Table 1 indicate that when all the parent nodes are in the highest or lowest states, the mode of Rent is also the highest or the lowest state. These probabilistic characteristics of Rent are achieved by using any of the weight expressions. Considering Row 1 as an initial combination, Rows 2 to 4 indicate that when the state of a single parent node shifts from the highest state to the lowest state, the mode of Rent shifts towards lower states. This effect is stronger in the case of shifting Surface Area or Distance to Centre than when shifting Time since Overhaul. On the other hand, considering Row 8 as the initial combination, Rows 5 to 7 indicate that when the state of a single parent node shifts from the lowest state to the highest state, the mode of Rent shifts towards higher states. However, the shifts of the mode are now generally smaller than in the cases related to Rows 2 to 4. Overall, Table 1 indicates that the mode of Rent is inclined to go more towards the low than the high states and that the parent nodes have different strengths of the influence on Rent. Hence, the most suitable weight expression is WMIN.

The next step is to elicit weights of the parent nodes and a variance parameter from the expert. However, no guideline for determining them is given in the literature concerning RNM. It is clear, e.g., that the larger the weight of a parent node is, the more it influences the central tendency of the child node. Thus, in the example, the weights of Surface Area and Distance to Centre should be larger than that of Time since Overhaul. It is also evident that by increasing the variance parameter, the conditional probability distributions in the CPT become flatter. However, nothing more specific can be stated about its value. Hence, trial and error is used.
Expert Assessments about the Mode of the Child Node Rent for the Combinations of the Extreme States of the Parent Nodes Surface Area, Distance to Centre, and Time since Overhaul

<table>
<thead>
<tr>
<th>Row</th>
<th>Surface Area (m²)</th>
<th>Distance to Centre (km)</th>
<th>Time since Overhaul (pert)</th>
<th>Rent (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[45, 50]</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>[20, 25]</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>X</td>
</tr>
<tr>
<td>3</td>
<td>[45, 50]</td>
<td>[20, 30]</td>
<td>[0, 5]</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
<td>[45, 50]</td>
<td>[20, 30]</td>
<td>[20, 25]</td>
<td>X</td>
</tr>
<tr>
<td>5</td>
<td>[45, 50]</td>
<td>[20, 30]</td>
<td>[20, 25]</td>
<td>X</td>
</tr>
<tr>
<td>6</td>
<td>[20, 25]</td>
<td>[20, 30]</td>
<td>[20, 25]</td>
<td>X</td>
</tr>
<tr>
<td>7</td>
<td>[20, 25]</td>
<td>[20, 30]</td>
<td>[20, 25]</td>
<td>X</td>
</tr>
</tbody>
</table>

Black and white stars indicate the states of the parent nodes associated with the state intervals (0.8, 1) and [0, 0.2), respectively. Checkmarks indicate the mode assessments of the expert.

Values of the Parameters for the Combination (Surface Area=Medium, Distance to Centre=Medium, Time since Overhaul=Medium)

<table>
<thead>
<tr>
<th>Row</th>
<th>w₁</th>
<th>w₂</th>
<th>w₃</th>
<th>Rent (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5</td>
<td>2</td>
<td>2</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>10</td>
<td>1</td>
<td>2000</td>
</tr>
<tr>
<td>3</td>
<td>2.5</td>
<td>2</td>
<td>1</td>
<td>2000</td>
</tr>
</tbody>
</table>

Shades of gray highlighting the probabilities represent their magnitudes. The probabilities are rounded to two decimal places.

In addition to the futility of altering the weights or the variance parameter, the use of an alternate weight expression would not resolve the problem at hand either. The reason for the failure of all these means is a feature of RNM that is stated as follows:

**Proposition 1.** Let the nodes \(X₁, \ldots, Xₙ, Xₖ\) all have the same amount of states. When the parent nodes \(X₁, \ldots, Xₙ\) are in the states \(x₁, \ldots, xₙ\), associated with the same state interval \(z\), the mode of the child node \(Xₖ\) is necessarily the state \(xₖ\) associated with \(z\).

**Proof:** See Online Appendix A (available as supplemental material at http://...).

Proposition 1 explains why RNM cannot now produce a CPT that would correctly represent the view of the expert. When Surface Area, Distance to Centre, and Time since Overhaul are all in the state Medium associated with the state interval (0.4, 0.6), the mode of Rent necessarily becomes the state Medium that is associated with the same state interval. If the expert thinks that the values of rent corresponding to the state High are now more probable than those associated with the state Medium, the discretizations of the interval scales should be changed.

As it has been exemplified, when RNM is applied to nodes with interval scales, the discretizations of the scales define the probabilistic relationship between parent nodes and a child node in a way that cannot be altered by the parameters assigned. If this feature is not taken into account, time may be spent in vain in the search of suitable values of the parameters. Alternatively, one might reject RNM as an unsuitable method for generating a required CPT.

Another challenge that can make the use of RNM inefficient is that suitable values of parameters are determined by trial and error. This may be problematic especially concerning the weights of parent nodes. Because the variance parameter solely defines the flatness of all conditional probability distributions generated, the effect of changing its value is straightforward to observe. Therefore, a convenient value for the variance parameter may be discovered by trial
and error. Indeed, Fenton et al. [23] note that in a case study concerning RNM, experts did not mind searching a value for the variance parameter by trial and error. As opposed to the flatness of the probability distributions, the central tendency of a child node on the ordinal scale depends on three factors: the weight expression, the states of the parent nodes, and the weights. Consequently, the effect of altering a single weight is not as unambiguous to observe and comprehend as the effect of changing the variance parameter. Thus, determining appropriate weights by trial and error can be difficult.

A possible challenge is related also to the selection of the weight expression. The deduction of a suitable weight expression based on mode assessments of the expert requires understanding of the functioning of RNM at a level that might be lacking from some of its users. An unskilled user might not, e.g., recognize a situation when no single weight is identified with the boundary points of the subintervals on the interval scales as consecutive subintervals on the interval scales are as-

In RNM, a given state $x$ is associated with both a specific subinterval $Z_i = [a_i, b_i]$ on the interval scale and with the state interval $z_i = [a_i, b_i]$ on the normalized scale. Hence, $Z_i$ and $z_i$ also become associated with each other. As consecutive subintervals on the interval scales are associated with consecutive state intervals on the normalized scales, the boundary points of $Z_i$ become identified with the boundary points of $z_i$. For example, in Fig. 1, points €800 and €1000 on the interval scale of Rent are identified with the points 0.6 and 0.8 on the normalized scale, respectively.

Next, assume that the parent nodes $X_1, ..., X_n$ are in the states $x_{1}, ..., x_{n}$ so that each $x_i$ is associated with the same state interval $z = [a, b]$ on the normalized scale. When the conditional probability distribution $P(X_C|X_1 = x_1, ..., X_n = x_n)$ is calculated in RNM, one of the combinations of sample points formed is $(z_{1,k} = a, ..., z_{n,k} = a)$. In this case, the mean parameter $\mu_k$ obtained with (1) is

$$\mu_k = f(z_{1,k} = a, ..., z_{n,k} = a) = a,$$

which can be verified by substituting $(z_{1,k} = a, ..., z_{n,k} = a)$ to each of (2)-(5). Equation (8) implies that if all the parent nodes have $a$ as their value on the normalized scale, the mode of the child node on the normalized scale is also $a$. As the boundary points of the state intervals $z = [a, b]$ are identified with the boundary points of the subintervals $Z_i = [A_i, B_i]$, the following proposition concerning the interval scales is obtained.

**Proposition 2.** Let the nodes $X_1, ..., X_n, X_C$ all have the same amount of states. Furthermore, for all $i = 1, ..., n, C$, let the state $x_i$ be associated with the subinterval $Z_i = [A_i, B_i]$ on the interval scale and with the state interval $z = [a, b]$ on the normalized scale so that $A_i$ is identified with $a$. If the parent nodes $X_1, ..., X_n$ are known to have the values $A_1, ..., A_n$ on the interval scales, the most probable value for the child node $X_C$ on the interval scale is $A_C$.

For example, the discretizations displayed in Fig. 1 imply that a 40 m² apartment located 5 km from the centre and with 10 years since overhaul is most likely to have a rent of €800. This is because the values $A_1 = 40$ m², $A_2 = 5$ km, $A_3 = 10$ y, and $A_C = €800$ on the interval scales of the nodes are all identified with the value $a = 0.6$ on the normalized scales.

### 4.1.2 Guideline for Discretization of Interval Scales

Based on Proposition 2, the discretization of interval scales is carried out according to the following guideline. First, $m$ states are formed for each node by discretizing the interval scales freely. One may define the states, e.g., by dividing the scales into subintervals of equal width or into subintervals associated with descriptive labels like Low, Medium, etc. Then, the boundary points of a state interval $z^j = [a^j, b^j]$ are examined by asking the expert revision questions. For a given boundary point $a^j$, the expert is asked to consider a situation where the parent nodes $X_1, ..., X_n$ have the values $A_1^j, ..., A_n^j$ on their interval scales so that each $A_i^j$ is identified with $a^j$. The expert is then asked whether the mode of the child node on the interval scale is $A_C^j$ that is also identified with $a^j$. If the expert does not think so, the discretizations are to be revised.

In the revision of the discretizations, it is sufficient to alter the discretization of only one node. The expert may, e.g., decide to keep the discretizations of the parent nodes intact and revise only the discretization of the child node. That is, given $A_1^j, ..., A_n^j$, if the expert thinks that $A_C^j$ rather than $A_C^j$ is the most probable value on the interval scale of the child node, its discretization is changed so that $A_C^j$ becomes identified with $a^j$. Alternatively, the value $A_i^j$ of a given parent node $X_i$ can be changed to $\tilde{A}_i^j$ so that given $A_1^j, ..., \tilde{A}_i^j, ..., A_n^j$ as the values of the parent nodes, $A_C^j$ is accepted as the mode of the child node. Instead of changing the discretization of a single node, one can also modify that of several if that is considered preferable.
By going through all the $m+1$ boundary points $a^1, ..., a^m$ and $b^m$, note that $a^{j+1} = b^j$ for all $j = 1, ..., m - 1$, one obtains discretizations that are compatible with Proposition 1.

4.1.3 Illustration of Guideline
To illustrate the above guideline, consider the example BN presented in Fig. 1. Now, the number of states for all nodes is $m = 5$ and the state intervals are indexed as $[a^1, b^1] = [1, 0.8], [a^2, b^2] = [0.8, 0.6], etc.$ Let the discretizations displayed in Fig. 1 be the ones formed freely. Then, concerning the boundary point $a^1 = 1$, the expert is asked whether €1200 is the most probable rent for a just overhauled apartment with 50 m$^2$ of surface area and located right in the centre. In the second question, concerning the boundary point $a^2 = 0.8$, the expert is asked whether €1000 is the most probable rent for a 45 m$^2$ apartment that is located 2 km from the centre and has 5 y since overhaul. Continuing this way, the expert is asked $m + 1 = 6$ similar questions concerning the boundary points of the subintervals. If the expert agrees with the most probable rent stated in each question, there is no need to change the discretizations. However, if the expert thinks that there are questions in which the value of the most probable rent is incorrect, the discretizations have to be revised.

Suppose that the expert disagrees with the value of the most probable rent stated in the revision questions concerning the boundary points $a^3 = 0.6$ and $a^4 = 0.4$. That is, the expert does not think that €800 is the most probable rent for a 40 m$^2$ apartment that is located 5 km from the centre and has 10 years since overhaul. In addition, the expert does not think that €600 is the most probable rent for a 30 m$^2$ apartment that is located 10 km from the centre and has 15 years since overhaul. Now, one way to revise the discretizations is to keep the discretizations of the parent nodes intact and change only the discretization of the child node. This means that the expert specifies his or her opinion about the most probable rent for the combinations (40 m$^2$, 5 km, 10 y) and (30 m$^2$, 10 km, 15 y) of area, distance from the centre, and time since overhaul. For example, if the expert says that €900 and €750 are the most probable rents, the revised discretization of Rent becomes the one presented in Fig. 2.

Fig. 2. Revised discretization of the example BN.

It should be noted that the guideline presented instructs experts to define the same amount of states for all nodes on their ordinal scales. If the amounts of the states are non-equal, Propositions 1 and 2 cannot be applied, and the effect of discretizations becomes harder to interpret. This may in turn complicate the construction of a CPT with RNM. Hence, it is recommended to use the same amount of states for all nodes, which already is the practice in most applications of RNM, see, e.g., [24], [25], [29], [30].

4.2 Determination of Weight Expression and Weights
After interval scales of nodes have been discretized, the second phase of the new approach is the determination of a weight expression and weights for the parent nodes. The determination is based on four concepts. The first one is interpretations of weights that are derived for each weight expression and concern values of the nodes on the normalized scale $[0, 1]$. The second one is piecewise linear mappings defined between the interval and normalized scales of the nodes. The third concept is interpretations of the weights concerning values of the nodes on the interval scales. These interpretations are based on the use of the mappings and the normalized scale interpretations. The interval scale interpretations enable the determination of the weights for each weight expression by utilizing elicitation questions that deal with scenarios defined on the interval scales of the nodes. The fourth concept is feasibility conditions of the weights. If the weights determined for a given weight expression fulfill the feasibility conditions, the weight expression and the feasible weights are used to generate a CPT.

4.2.1 Interpretations of Weights Related to Normalized Scales
The interpretations of weights related to normalized scales are next presented for each weight expression. The derivations of the interpretations are given in detail in Online Appendix B. Previously, the $k$th combination of sample points related to $n$ parent nodes has been denoted by $(z_{1,k}, ..., z_{n,k})$. The notation $z = (z_1, ..., z_n)$ is used in this section for such a combination for brevity. Furthermore, $µ$ defined by $µ = f(z, w)$ is used to denote the mean parameter of the child node $X_C$ obtained with $z$, a given weight expression $f$, and weights $w$. Now, $w = (w_1, ..., w_n)$ for all other weight expressions except MIXMINMAX for which $w = (w_{MIN}, w_{MAX})$.

4.2.1.1 WMEAN: Let $z' = (z_1, ..., z_k + ∆z_k, ..., z_n)$ be a combination of sample points that is otherwise identical to $z$ defined above, but there is a change $∆z_k$ in the sample point of the $k$th parent node $X_k$. Moreover, let $∆µ = WMEAN (z', w_1, ..., w_n) - WMEAN (z, w_1, ..., w_n)$ denote the difference between the mean parameters obtained with $z'$ and $z$.

Then, one obtains

$$w_k^N := \frac{w_k}{\sum_{i=1}^{n} w_i} = \frac{∆µ}{∆z_k},$$

which defines the weight of $X_k$ relative to the sum of the weights. That is, the relative weight $w_k^N$ equals the ratio between the change $∆µ$ in the mean parameter of $X_C$ and the change $∆z_k$ in the value of $X_k$. It is enough to have the result only for the relative weights as only they matter in (2).
4.2.1.2 WMIN: Let $z$ fulfill

$$z_k < \frac{1}{n} \sum_{j=1}^{n} z_j \leq z_i, \forall \ i = 1, ..., n, \ i \neq k. \quad (10)$$

In other words, the sample point related to $X_k$ is smaller than the average of the sample points while the sample points related to the other parent nodes are larger than or equal to the average. Then, it applies

$$w_k = 1 + \sum_{i=1}^{n} \frac{z_i - n \mu}{\mu - z_k}, \quad (11)$$

which defines the weight of $X_k$ in terms of $z_1, ..., z_n$ and $\mu$. Note that, as opposed to WMEN, only one combination of sample points is required to reveal the weight of $X_k$. Moreover, contrary to (9), (11) provides the interpretation for the absolute size of the weight $w_k$. This is because with WMIN, it is not enough to know the magnitudes of the weights relative to each other — their values must be known absolutely. This is implied by (3).

As an example, let $z = (z_1 = 0, z_2 = 1, z_3 = 1)$ be sample points related to $Surface Area$, $Distance to Centre$, and $Time since Overhaul$ in Fig. 2, respectively. Moreover, let $\mu = 0.4$ be the corresponding mean parameter of $Rent$. It now applies $z_1 = 0 < (z_1 + z_2 + z_3) / 3 = 2/3 < z_2 = z_3 = 1$ and hence (10) is fulfilled with $k = 1$. Then, $w_1$ obtained from (11) is

$$w_1 = 1 + \frac{2 - 3 \times 0.4}{0.4 - 0} = 3.0. \quad (12)$$

4.2.1.3 WMAX: The interpretation of weights for WMAX is obtained in a similar way as for WMIN. Let $z$ fulfill

$$z_i \leq \frac{1}{n} \sum_{j=1}^{n} z_j < z_k \forall \ i = 1, ..., n, \ i \neq k. \quad (13)$$

That is, the sample point related to $X_k$ is larger than the average of the sample points whereas the sample points related to the other parent nodes are smaller than or equal to the average. In this case, the weight of $X_k$ is given by

$$w_k = 1 + \frac{n \mu - \sum_{i=1}^{n} z_i}{\mu - z_k}, \quad (14)$$

which is equivalent to (11). This reflects the similar structure of WMIN and WMAX as functions. It also indicates that similarly to WMIN, the absolute sizes of the weights must be known with WMAX as well. However, note that $z$ is defined differently in (10) and (13).

4.2.1.4 MIXMINMAX: Similarly to WMEAN, the values of weights $w_{MIN}$ and $w_{MAX}$ must be known only relative to each other when applying MIXMINMAX. Considering $z$ and $\mu$, the relative weights for MIXMINMAX are obtained by

$$\begin{align*}
    w_{MIN}^{N} &= \frac{\max_{i=1, ..., n} \{z_i\} - \mu}{\max_{i=1, ..., n} \{z_i\} - \min_{i=1, ..., n} \{z_i\}}, \\
    w_{MAX}^{N} &= 1 - w_{MIN}^{N}.
\end{align*} \quad (15)$$

Hence, analogically to (11) and (14), (15) provides an interpretation to the weights in terms of $z_1, ..., z_n$ and $\mu$.

4.2.2 Mappings between Interval Scales and Normalized Scales

In Section 4.1, it is discussed how the boundary points of state intervals of a ranked node are associated with boundary points of the corresponding subintervals on the interval scale. This idea can be generalized to concern all points on the scales, see Online Appendix C for further discussion. Based on this discussion, a piecewise linear mapping is defined between the interval and normalized scales of each node.

For the node $X_i$ with $m$ states, the piecewise linear mapping $h_i(x)$ is defined as

$$\begin{align*}
    h_i(A_j^1) &= a_j^1 \forall j = 1, ..., m \text{ and } h_i(B_j^m) = b_j^m, \\
    h_i(x) &= h_i(A_j^1) + \left( \frac{x - A_j^1}{B_j^m - A_j^1} \right) (h_i(B_j^m) - h_i(A_j^1)), \\
    \forall x &\in [A_j^1, B_j^1] \forall j = 1, ..., m
\end{align*} \quad (16)$$

where $A_j^1$ is the boundary point of a subinterval on the interval scale identified with the boundary point $a_j^1$ of a state interval on the normalized scale.

As an example, Fig. 3 displays the piecewise linear mapping $h_C(x)$ between the interval scale of $Rent$ and the normalized scale $[0, 1]$. Here, $x$ denotes the rent measured in euros.

4.2.3 Interpretations of Weights Related to Interval Scales

By using piecewise linear mappings, the interpretations of weights related to normalized scales can be extended to concern values of nodes on interval scales. The interval scale interpretations are next presented for each weight expression. Their derivations are given in Online Appendix D.

Let $y = (y_1, ..., y_n)$ be a scenario in which $y_i$ denotes the value of the parent node $X_i$ on the interval scale. Furthermore, let $\hat{y}_C$ denote the corresponding mode of the child node $X_C$ on the interval scale, and let $h_i(\cdot)$ denote the piecewise linear mapping related to the node $X_i$.  

4.2.3.1 WMEN: Let $y' = (y_1 + \Delta y_1, ..., y_n + \Delta y_n)$ be a scenario that is otherwise identical to $y$ defined above but the value of the $k$th parent node $X_k$ differs by $\Delta y_k$. Moreover, let $\hat{y}_C'$ denote the mode of $X_C$ in the scenario $y'$.

If the probabilistic relationship between the parent nodes and the child node corresponds to WMEN, the relative weight $w_k^N$ is obtained by

$$w_k^N = \frac{h_C(\hat{y}_C') - h_C(\hat{y}_C)}{h_k(\hat{y}_k + \Delta y_k) - h_k(\hat{y}_k)}. \quad (17)$$

Thus, (17) interprets the relative weight of $X_k$ in terms of the values of $X_k$ as well as the modes of $X_C$ in the scenarios $y$ and $y'$. 

4.2.3.2 WMIN: Let the scenario \( y \) be such that
\[
h_k(y_k) < \frac{1}{n} \sum_{j=1}^{n} h_j(y_j) \quad \forall \, i = 1, \ldots, n, \ i \neq k. \tag{18}
\]
That is, the interval scale value \( y_k \) of \( X_k \) maps to a normalized scale point \( h_k(y_k) \) that is smaller than the average of all the mapped values \( \sum_{j=1}^{n} h_j(y_j)/n \). On the other hand, the interval scale values of all the other parent nodes map to normalized scale points equal to or larger than this average. If the probabilistic relationship between the nodes corresponds to WMIN, the weight \( w_k \) is given by
\[
w_k = 1 + \frac{\sum_{j=1}^{n} h_j(y_j) - nh_c(\hat{y}_C)}{h_c(\hat{y}_C) - h_k(y_k)}. \tag{19}
\]
Now, (19) defines the weight of \( X_k \) for WMIN in terms of \( y_1, \ldots, y_n \) and \( \hat{y}_C \).

As an example, let piecewise linear mappings be defined for each node in Fig. 2 according to (16). Consider then a scenario which Surface Area, Distance to Centre, and Time since Overhaul have the values \( y_1 = 25 \text{ m}^2, y_2 = 2 \text{ km}, y_3 = 5 \text{ m} \), respectively, and \( \hat{y}_C = \mathcal{E}800 \) is identified to be the most probable rent. Now, \( h_1(y_1) = 0.2, h_2(y_2) = 0.8, \) and \( h_3(y_3) = 0.8 \) satisfy (18) for \( k = 1 \). Moreover, as now \( h_c(\hat{y}_C) = 0.4667 \), the weight \( w_1 \) obtained from (19) is
\[
w_1 = 1 + \frac{1.8 - 3 \times 0.4667}{0.4667 - 0.2} = 2.5. \tag{20}
\]

4.2.3.3 WMAX: Assume that the scenario \( y \) fulfills
\[
h_i(y_i) \leq \frac{1}{n} \sum_{j=1}^{n} h_j(y_j) \quad \forall \, i = 1, \ldots, n, \ i \neq k. \tag{21}
\]
Now, the interval scale value \( y_k \) of \( X_k \) maps to a normalized scale point \( h_k(y_k) \) that is larger than the average of all the mapped values. On the other hand, the interval scale values of all the other parent nodes map to normalized scale points smaller than or equal to the average. If WMAX is the weight expression characterizing the probabilistic relationship between the nodes, the weight \( w_k \) is obtained by
\[
w_k = 1 + \frac{nh_c(\hat{y}_C) - \sum_{i=1}^{n} h_i(y_i)}{h_k(y_k) - h_c(\hat{y}_C)}, \tag{22}
\]
which is equivalent to (19). However, one should note that \( y \) is defined differently in (18) and (21).

4.2.3.4 MIXMINMAX: Assuming MIXMINMAX is the weight expression to be used, \( y \) and \( \hat{y}_C \) yield the relative weights \( w_{MIN}^N \) and \( w_{MAX}^N \) given by
\[
\begin{align*}
w_{MIN}^N & = \max_{i \neq k} \{ h_i(y_i) - h_c(\hat{y}_C) \} / \left( h_k(y_k) - h_c(\hat{y}_C) \right), \\
w_{MAX}^N & = 1 - w_{MIN}^N.
\end{align*} 
\tag{23}
\]
Now, (23) explains the relative weights for MIXMINMAX in terms of \( y_1, \ldots, y_n \) and \( \hat{y}_C \).

4.2.4 Feasibility Conditions of Weights

Given the modes of a child node in specific types of scenarios, the interval scale interpretations of weights enable one to calculate the weights for each weight expression. If the probabilistic relationship between the nodes corresponds to a given weight expression, the weights calculated for it fulfill specific feasibility conditions that follow from the characteristics of the weight expression. Thus, the weight expression and the weights may be determined simultaneously. The scenarios and the feasibility conditions are discussed in detail below. The derivations of the conditions are presented in Online Appendix E.

The examination of (18) and (21) indicates that the scenario types that allow the calculation of weights for WMIN and WMAX are mutually exclusive. On the other hand, the scenarios that are used to determine the weights for either WMIN or WMAX can also be used to discover the relative weights for WMEAN. With MIXMINMAX, a single scenario reveals the weights provided that it corresponds to unequal values of parent nodes on the normalized scale. Hence, in order to determine the weights of all \( n \) parent nodes for all weight expressions, two sets \( S_a \) and \( S_b \) of \( n \) scenarios are required. Now, \( S_a \) consists of scenarios that fulfill the conditions (18) whereas \( S_b \) consists of scenarios that fulfill the conditions (21). Formally, \( S_a \) and \( S_b \) are defined as
\[
\begin{align*}
S_a & = \{ y_{a,k} \mid 0 < h_i(y_{a,k}) \leq c,a,k \leq 1 \ \forall \, i = 1, \ldots, n, \ i \neq k, \\
S_b & = \{ y_{b,k} \mid 0 < h_i(y_{b,k}) < c,b,k \leq 1 \ \forall \, i = 1, \ldots, n, \ i \neq k, \}
\end{align*} 
\tag{24}
\]
where \( c,a,k \) and \( c,b,k \) are values on the normalized scales. Thus, in a given scenario \( y_{a,k} \) or \( y_{b,k} \), the values of all other parent nodes except \( X_k \) on the interval scales are identified with the same value \( c,a,k \) or \( c,b,k \) on the normalized scale. In \( y_{a,k} \), the value of \( X_k \) on the interval scale maps to a value smaller than \( c,a,k \) on the normalized scale. On the other hand, in \( y_{b,k} \), the value of \( X_k \) on the interval scale is identified with a value on the normalized scale that is larger than \( c,b,k \).

The feasibility conditions of weights for different weight expressions are given below. Here, \( \hat{y}_{a,C}^k \) and \( \hat{y}_{b,C}^k \) denote the mode of the child node on the interval scale in the scenarios \( y_{a,k} \) and \( y_{b,k} \), respectively.

4.2.4.1 WMEAN: For WMEAN, both \( \hat{y}_{a,C}^k \) and \( \hat{y}_{b,C}^k \) yield a value of the relative weight \( w_k^N \) of the parent node \( X_k \). The value produced by \( \hat{y}_{a,C}^k \) is
\[
w_{k,a}^N = \frac{h_c(\hat{y}_{a,C}^k) - c,a,k}{h_k(y_{a,k}) - c,a,k}. \tag{25}
\]
Similarly, the value produced by \( \hat{y}_{b,C}^k \) is
\[
w_{k,b}^N = \frac{h_c(\hat{y}_{b,C}^k) - c,b,k}{h_k(y_{b,k}) - c,b,k}. \tag{26}
\]
If the probabilistic relationship between the nodes corresponds to WMEAN, \( w_{k,a}^N \) and \( w_{k,b}^N \) must fulfill the feasibility conditions
\[
w_{k,a}^N = w_{k,b}^N = w_k^N \in [0, 1] \ \forall \, k = 1, \ldots, n, \tag{27a}
\]
\[
\sum_{k=1}^{n} w_k^N = 1. \tag{27b}
\]
When (27a) is fulfilled, the modes \( \hat{y}_{a,C}^k \) and \( \hat{y}_{b,C}^k \) are consistent from the point of view of WMEAN. On the other hand,
(27b) is the normalization condition that the relative weights by definition must fulfill.

4.2.4.2 WMIN: With WMIN, \( y_{C}^{a,k} \) produces a value to the weight \( w_{k} \) given by

\[
w_{k} = \frac{(n-1)(c^{a,k} - h_{C}(y_{C}^{a,k}))}{h_{C}(y_{C}^{a,k}) - h_{k}(y_{k}^{a,k})}.
\]

(28)

On the other hand, based on \( y_{C}^{b,k} \), an auxiliary weight \( v_{k} \) is calculated. Note that \( v_{k} \) does not refer to the weight of \( X_{k} \). Rather, the superscript of \( v_{k} \) refers to the fact that the scenario \( y_{k}^{b,k} \) is used to determine it.

If the probabilistic relationship between the nodes corresponds to WMIN, \( w_{k} \) and \( v_{k} \) must satisfy the feasibility conditions

\[
w_{1}, ..., w_{n} \geq 1, \quad v_{k} = \max_{i=1, ..., n} \left\{ w_{i} \right\} \forall k = 1, ..., n.
\]

(30a)

(30b)

Here, (30a) reflects a requirement that follows straight from the definition of WMIN, see Online Appendix B. In turn, when (30b) is fulfilled, the modes \( y_{C}^{a,k} \) and \( y_{C}^{b,k} \) are consistent from the point of view of WMIN.

4.2.4.3 WMAX: In the case of WMAX, the use of the scenarios and the modes is analogous to WMIN. The mode \( y_{C}^{b,k} \) yields a value to the weight \( w_{k} \) that is

\[
w_{k} = \frac{(n-1)(h_{C}(y_{k}^{b,k}) - c^{b,k})}{h_{k}(y_{k}^{b,k}) - h_{C}(y_{C}^{b,k})}.
\]

(31)

Based on \( y_{C}^{b,k} \), an auxiliary weight \( v_{k} \) is calculated by

\[
v_{k} = \frac{c^{b,k} - h_{k}(y_{k}^{b,k})}{c^{a,k} - h_{C}(y_{C}^{a,k})} - n + 1.
\]

(32)

The feasibility conditions of \( w_{k} \) and \( v_{k} \) for WMAX are the same as for WMIN defined in (30).

4.2.4.4 MIXMINMAX: With MIXMINMAX, each scenario \( y_{C}^{a,k} \) and \( y_{C}^{b,k} \) is used to calculate values to the relative weights \( w_{MIN}^{N,a,k} \) and \( w_{MAX}^{N,a,k} \). The values obtained with \( y_{C}^{a,k} \) and \( y_{C}^{b,k} \) are

\[
\begin{align*}
w_{MIN}^{N,a,k} &= \frac{c^{a,k} - h_{C}(y_{C}^{a,k})}{c^{a,k} - h_{k}(y_{k}^{a,k})}, \\
w_{MAX}^{N,a,k} &= 1 - w_{MIN}^{N,a,k}
\end{align*}
\]

(33)

whereas \( y_{C}^{b,k} \) and \( y_{C}^{b,k} \) produce

\[
\begin{align*}
w_{MIN}^{N,b,k} &= \frac{h_{k}(y_{k}^{b,k}) - h_{C}(y_{C}^{b,k})}{h_{k}(y_{k}^{b,k}) - c^{b,k}}, \\
w_{MAX}^{N,b,k} &= 1 - w_{MIN}^{N,b,k}
\end{align*}
\]

(34)

If MIXMINMAX portrays the probabilistic relationship between the nodes, \( w_{MIN}^{N,a,k} \) and \( w_{MIN}^{N,b,k} \) must satisfy the feasibility conditions

\[
w_{MIN}^{N,a,k} = w_{MIN}^{N,b,k} = \lambda \in [0, 1] \quad \forall k = 1, ..., n,
\]

(35)

where \( \lambda \) is a constant. If (35) is fulfilled, the modes in all the scenarios are consistent from the point of view of MIXMINMAX.

4.2.5 Guideline for Determination of Weight Expression and Weights

A guideline for the determination of a weight expression and weights based on expert elicitation is next presented. Assume that interval scales of nodes are discretized according to the guideline presented in Section 4.1.2 and piecewise linear mappings \( h_{i}(\cdot) \) are defined according to (16).

With \( n \) parent nodes, the expert is asked to assess the mode of the child node on the interval scale in \( 2n \) scenarios. In each scenario, the parent nodes have extreme values on the interval scales so that all except one have their values identified with the same end of the normalized scale \([0, 1]\). One can think that in each scenario, all parent nodes have originally had the extreme values identified with this end of the normalized scale. Correspondingly, the mode of the child node on the interval scale has originally been the extreme value identified with this end. Then, the value of a single parent node has either fallen or risen to the extreme value opposite to its original one. This changes the mode of the child node, and the expert has to consider the magnitude of the change.

The scenarios for which mode assessments of the expert are asked for correspond to those defined in (24) with the substitutions

\[
c^{a,k} = 1, \quad h_{k}(y_{k}^{a,k}) = 0, \quad c^{b,k} = 0, \quad h_{k}(y_{k}^{b,k}) = 1.
\]

(36)

Therefore, the scenarios are represented as

\[
\begin{align*}
S_{a} &= \{ y_{i}^{a,k} = ( y_{i}^{a,k} ) \mid h_{i}(y_{i}^{a,k}) = 1 \forall i = 1, ..., n, i \neq k, \\
h_{k}(y_{k}^{a,k}) = 0 \}_{i=1}^{n-1}, \\
S_{b} &= \{ y_{i}^{b,k} = ( y_{i}^{b,k} ) \mid h_{i}(y_{i}^{b,k}) = 0 \forall i = 1, ..., n, i \neq k, \\
h_{k}(y_{k}^{b,k}) = 1 \}_{k=1}^{n-1}
\end{align*}
\]

(37)

The consideration of full range shifts in the values of the parent nodes eases up distinguishing differences in the relative strengths by which the parent nodes affect the child node. The use of the extreme values is also analogous to how Fenton et al. [23] use questions concerning combinations of the extreme states of the parent nodes on the ordinal scales to support the selection of a weight expression.

Let \( y_{C}^{a,k} \) and \( y_{C}^{b,k} \) denote the mode assessments of the expert related to the scenarios \( y_{C}^{a,k} \) and \( y_{C}^{b,k} \), respectively. By using the substitutions (36) in the equations presented in the previous section, the mode assessments yield the weights given in Table 4. If the weights for a given weight expression satisfy the feasibility conditions, the weight expression and the piecewise linear mappings explain the mode assessments of the expert. Therefore, the weight expression and the weights are considered appropriate for generating a CPT for the child node.

The mode assessments of the expert concerning various scenarios produce multiple point values for a given weight. Some of the feasibility conditions of weights require that the point values obtained are equal with each other. The point values depend on the mode assessments, the functional forms of the weight expressions, and the piecewise linear mapping of the child node defined by the discretization of its interval scale. When the subjective views of the expert
about a real-life phenomenon are combined with such functional relations, it is likely that the point values are not equal and feasibility conditions are violated.

Because the fulfillment of feasibility conditions is challenging with point-valued mode assessments, it is advisable to search feasible weights by asking the expert to assess the mode of Rent presented in Fig. 2 with the aim of constructing a CPT to deal with this kind of case are discussed in Section 5.

Let the piecewise linear mapping $h_C(x)$ be defined for Rent according to Fig. 3. Then, by substituting the values in Table 5 to the formulas of Table 4, the weight intervals presented in Table 6 are obtained. Based on these intervals, feasible point values of weights do not exist for WMEAN, WMAX, and MIXMINMAX. For example, with WMEAN, $[w_1^{N,a}, w_1^{N,a}]$ and $[w_1^{N,b}, w_1^{N,b}]$ do not intersect which leads to violation of (27a). With WMAX, e.g., the whole interval $[w_1, w_3]$ is below 1 which means that (30a) cannot be satisfied. With MIXMINMAX, (35) is violated because, e.g., the intervals $[w_1^{N,a}, w_1^{N,a}]$ and $[w_1^{N,a}, w_1^{N,a}]$ do not overlap.

### Table 4

<table>
<thead>
<tr>
<th>Weight expression</th>
<th>Weight obtained with $\hat{y}_{C,a,k}^w$</th>
<th>Weight obtained with $\hat{y}_{C,b,k}^w$</th>
<th>Feasibility conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>WMEAN</td>
<td>$w_k^{N,a} = 1 - h_C(\hat{y}_{C,a,k}^w)$</td>
<td>$w_k^{N,b} = h_C(\hat{y}_{C,b,k}^w)$</td>
<td>Equation (27)</td>
</tr>
<tr>
<td>WMIN</td>
<td>$w_k = \frac{(n-1)(1-h_C(\hat{y}<em>{C,k}^w))}{h_C(\hat{y}</em>{C,k}^w)}$</td>
<td>$v_k = \frac{1}{h_C(\hat{y}_{C,k}^w)} - n + 1$</td>
<td>Equation (30)</td>
</tr>
<tr>
<td>WMAX</td>
<td>$v_k = \frac{1}{1-h_C(\hat{y}_{C,k}^w)} - n + 1$</td>
<td>$w_k = \frac{(n-1)h_C(\hat{y}<em>{C,k}^w)}{1-h_C(\hat{y}</em>{C,k}^w)}$</td>
<td>Equation (30)</td>
</tr>
<tr>
<td>MIXMINMAX</td>
<td>$w_k^{MIN} = 1 - h_C(\hat{y}_{C,a,k}^w)$</td>
<td>$w_k^{MIN} = 1 - h_C(\hat{y}_{C,b,k}^w)$</td>
<td>Equation (35)</td>
</tr>
</tbody>
</table>

### Table 5

| Scenarios and Mode Assessments |
|-------------------|----------------|----------------|----------------|
| Scenario Index   | Surface Area (m²) | Distance to Centre (km) | Time since Overhaul (v) | Rent (€) |
| k                | $y_1^{a,k}$ | $y_2^{a,k}$ | $y_3^{a,k}$ | $y_4^{a,k}$ |
| 1                | 20         | 0           | 0           | [600, 650] |
| 2                | 50         | 30          | 0           | [625, 675] |
| 3                | 50         | 0           | 25          | [850, 900] |
| $k^{b,k}$       | $y_1^{b,k}$ | $y_2^{b,k}$ | $y_3^{b,k}$ | $y_4^{b,k}$ |
| 1                | 50         | 30          | 25          | [450, 500] |
| 2                | 20         | 0           | 25          | [400, 450] |
| 3                | 20         | 30          | 0           | [375, 425] |

### Table 6

| Weight Intervals for each Weight Expression |
|-------------------|----------------|----------------|----------------|
| Scenario Index   | WMEAN          | WMAX           | MIXMINMAX      |
| k                | $[w_1^{N,a}, w_1^{N,a}]$ | $[w_1, w_2]$ | $[w_1^{N,a}, w_1^{N,a}]$ |
| 1                | [0.66, 0.72]  | [4.3, 5.1]     | [0.66, 0.72]  |
| 2                | [0.66, 0.70]  | [3.9, 4.0]     | [0.66, 0.70]  |
| 3                | [0.60, 0.47]  | [1.3, 1.8]     | [0.40, 0.47]  |

| k                | $[w_1^{N,b}, w_1^{N,b}]$ | $[w_2, w_3]$ | $[w_1^{N,a}, w_1^{N,a}]$ |
| 1                | [0.17, 0.20]  | [3.0, 4.0]     | [0.80, 0.83]  |
| 2                | [0.13, 0.17]  | [4.0, 5.2]     | [0.83, 0.87]  |
| 3                | [0.12, 0.15]  | [4.7, 5.6]     | [0.85, 0.90]  |

As opposed to other weight expressions, feasible point values of weights can be discovered for WMIN. Because the interval $[w_1, w_3]$ is above 1 with all $i = 1, 2, 3$ (30a) can be satisfied.

Considering (30b), because $w_2 > w_3$ for any plausible combination of the weights. Hence, $v_1 = w_2$ and the plausible range of $v^1$ and $w_2$ is the intersection of $[u_1^{1}, v^1]$ and $[w_2, w_2]$, i.e., $[3.9, 4.0]$. Similarly, because $w_1 > w_3$, it must be $w_1 > w_3$ and $v^2 = w_1$. Based on this, the plausible range for $v^2$ and $w_1$ is the intersection of $[u_2^{2}, v^2]$ and $[w_1, w_1]$, i.e., $[4.3, 5.1]$. Finally, considering $v^3$, the plausible ranges $w_i / [4.3, 5.1] / w_2 / [3.9, 4.0]$ imply that $w_1 > w_2$. Thus, it must be $v^3 = w_1$ and the plausible range of $v^3$ and $w_1$ becomes the intersection of $[4.3, 5.1]$ and $[u_3^{3}, v^3]$, i.e., $[4.7, 5.1]$.

To summarize, the feasible ranges of the weights $w_1$, $w_2$, and $w_3$ for Surface Area ($X_1$), Distance to Centre ($X_2$), and Time since Overhaul ($X_3$) are $w_1 \in [4.7, 5.1]$, $w_2 \in [3.9, 4.0]$, and $w_3 \in [w_1, w_3] = [1.3, 1.8]$. By using the combination...
(w₁, w₂, w₃) = (5.0, 4.0, 1.5) and the variance parameter σ² = 3×10⁻³, one obtains the CPT providing the conditional probability distributions of Rent displayed in Table 7 for the combinations of the extreme states of the parent nodes.

### Table 7: Conditional Probability Distributions of Rent for the Combinations of the Extreme States of the Parent Nodes

<table>
<thead>
<tr>
<th>Row</th>
<th>Surface Area (km²)</th>
<th>Distance to Centre (km)</th>
<th>Time area (vehicle per hour)</th>
<th>Rent (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(45, 50)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.66</td>
</tr>
<tr>
<td>2</td>
<td>(20, 25)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>(45, 50)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.62</td>
</tr>
<tr>
<td>4</td>
<td>(45, 50)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
<tr>
<td>5</td>
<td>(45, 50)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
<tr>
<td>6</td>
<td>(20, 25)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>(20, 25)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
<tr>
<td>8</td>
<td>(20, 25)</td>
<td>[0, 2]</td>
<td>[0, 5]</td>
<td>0.60</td>
</tr>
</tbody>
</table>

Shades of gray highlighting the probabilities represent their magnitudes. The probabilities are rounded to two decimal places.

The distributions displayed in Table 7 imply that a low state of any parent node pulls the central tendency of Rent towards a low state but with different magnitudes of strength. This reflects the use of WMIN as the weight expression with the alternate weights of the parent nodes.

### 4.3 Refinement of CPTs

After the CPT of a child node is generated with a weight expression and feasible weights, it is verified according to the final step of RNM. That is, one inspects that for given combinations of the states of the parent nodes considered relevant by the expert, the conditional probability distributions of the child node reflect the views of the expert. If there are distributions that are deemed faulty, the CPT is to be refined. Next, ways for performing the refinement are suggested.

While refining a CPT, it should be recognized that all conditional probabilities do not necessarily require the same level of accuracy. By performing sensitivity analysis on a BN, see, e.g., [35], [36], and [37], one can identify which conditional probabilities in the CPT pose much effect to the specific probabilities of interest and which do not. Therefore, conducting sensitivity analysis on the BN should be a preliminary step for the refinement of the CPT.

If only individual probabilities in a CPT require modification, they may be altered by hand, as pointed out by Fenton et al. [23]. However, if there are lots of changes to be made, manual modification of the CPT becomes laborious. Moreover, it may introduce to the CPT indiscriminate changes that are observed as inconsistent probabilistic behaviour of the child node. Therefore, when the CPT requires much refinement, it is preferable to construct a more appropriate one by using RNM with adjusted parameters.

Consider first that verified conditional probability distributions are deemed to be too flat or too peaked. In this case, a more suitable CPT is obtained by using an alternate value of the variance parameter. As discussed in the end of Section 3, trial and error is a befitting way to determine a more convenient value.

Suppose then that the central tendency of a child node is considered misplaced in some of the distributions verified. Having such distributions indicates that the weight expression and the weights used do not correspond to the probabilistic relationship between the nodes throughout the CPT. In this case, it is suggested to vary the values of the weights within the plausible ranges obtained in the second phase of the approach and check whether a CPT generated with the altered weights reflects the views of the expert better. If more suitable weights are not discovered, the expert is asked to consider shifting or expanding the intervals of the mode assessments. The modification of these intervals can change the plausible ranges of weights which may help to find suited weights for the initial or for some other weight expression.

Next, assume that desirable changes to a CPT are not achieved by adjusting the variance parameter or weights, or even by using another weight expression. Then, the CPT is refined by constructing it in parts using alternate values of the parameters. The partition of the CPT can be based on the states of an additional [23] or already existing parent node [41]. Thus, parts of the CPT corresponding to different states of the conditioning parent node are constructed with alternate weight expressions, weights, and values of the variance parameter. For each part, the determination of the weight expression and the weights can be performed by asking the expert about the mode of the child node in the types of scenarios presented in Section 4.2.4. In this case, the values of ωₐ,k and ωᵦ,k are defined by the state of the conditioning parent node. Moreover, the refinement of the parts can be conducted with the same means as the refinement of the whole CPT, i.e., manual modification, adjustment of parameters, and further partitioning.

### 5 Discussion

The new approach is currently being applied in a real-life case study concerning the performance model of an air surveillance network, and the preliminary experiments on the use of the approach have been encouraging. In the course of the study, practical means to support the execution of the approach have been recognized along with some limitations concerning its use. These findings, which provide topics for further research, are discussed below.

Recall that feasible weights are not necessarily found for any weight expression. That is, based on the scenarios used in the elicitation, the probabilistic relationship between parent nodes and a child node does not correspond to any single weight expression. In this case, the modification of the intervals of mode assessments may enable the discovery of feasible weights. Otherwise, the generation of the CPT in parts should be considered.

If plausible ranges of weights are not found for a given weight expression, it would be beneficial to have a semi-automated way to detect which mode assessments are inconsistent regarding the fulfillment of feasibility conditions. Likewise, if the CPT has to be generated in parts, it could be useful to have semi-automated means to support the elicitation of a suitable partition from the expert. In addition, any implementation of the approach should have an automatic routine that calculates plausible ranges of the weights for each weight expression based on the mode assessments. These features would facilitate and speed up the execution of the approach. It might also be worthwhile to develop
more sophisticated means than trial and error to discover the value of the variance parameter.

The main weakness of the new approach is that it requires the same amount of states to be defined for each node. Allowing nodes to have unequal numbers of states would extend the application scope of the approach. Note that this weakness is not restricted to the new approach alone. In general, when RNM is applied to nodes with unequal number of states, the result of Proposition 1 is of no use. Then, it can be difficult to verify whether RNM with any configuration of parameters provides means to construct a CPT that would represent the probabilistic relationship between the nodes satisfyingly.

To deal with the above issue, one can try to define an equal amount of states to the child node and as many parent nodes as possible. The parent nodes with unequal number of states then serve as the nodes whose states define the partition of the CPT. The fragment of the CPT corresponding to a given part is constructed by applying the new approach to the nodes with the same amount of states. Possibly a fragment generated like this can be reused in the construction of the full CPT by, e.g., shifting the probability distributions it contains towards better or worse states of the child node with a constant term.

Alternatively, one might try to initially define an equal amount of states for all nodes and apply the new approach normally to construct a CPT. Then, selected states of given nodes would be either merged together or divided into two and the CPT would be updated accordingly. If states of a parent node are merged, the related conditional probability distributions can be combined by, e.g., taking a weighted sum of them. If a state of a parent node is divided, a fragment of the CPT corresponding to the undivided state can be duplicated. Then, the duplicates are altered by, e.g., introducing shifts to the probability distributions they contain, as discussed above. If states of a child node are merged, the related probabilities can be summed up to form the probability of the merged state. If a state of a child node is divided, the probabilities of the resulting states can be formed by dividing the probability related to the original state appropriately.

Though possibly helpful in certain situations, the means described above are potentially laborious ways to allow nodes to have unequal numbers of states. Possibly a more effective way to enhance the new approach could be unifying it together with approaches based on dynamic discretization of interval scales of nodes, see, e.g., [32] and [33]. However, this idea, as the whole issue in general, requires further consideration.

Another line of future work is to investigate the idea of determining a weight expression and weights by data fitting. As there is an explicit functional relationship between the mode of a child node and the weights of parent nodes, one could formulate for each weight expression a minimization problem whose decision variables are weights and objective function represents the error between mode assessments and modes indicated by the weights. The solution of the problem gives the weights providing the best fit to the mode assessments of the expert. This optimization formulation would also enable the construction of CPTs by utilizing several mode values dealing with a single scenario which are assessed by one or more experts or estimated based on data. On the other hand, by applying data fitting to different segments of a CPT, one could support the determination of a suitable partition of the CPT for constructing it in parts. In addition, a potential avenue for future research is to generalize the approach presented to handle nodes that have only ordinal scales. Then, the weights and the weight expression would be determined based on scenarios and mode assessments defined on the ordinal scales of the nodes.

6 Conclusion

This paper demonstrated through an illustrative example challenges concerning the application of the ranked nodes method (RNM) to nodes expressed with interval scales. In order to resolve those challenges, the paper presented a novel approach for applying RNM to such nodes. First, there is a guideline for discretizing the interval scales compatibly with the functioning of RNM. Second, there is a guideline for determining the weight expression and the weights of the parent nodes by using assessments of the expert about the mode of the child node in various scenarios defined on the interval scales of the nodes. The determination is based on interpretations and feasibility conditions of the weights that are derived in the paper for each weight expression. Third, there are suggestions of ways concerning the refinement of a CPT after its verification.

The new approach responds to the challenges recognized in RNM. By following the discretization guideline, a basis is set for the generation of sensible CPTs with RNM. In turn, the guideline for the determination of a weight expression and weights provides a transparent and comprehensible way to elicit them from the expert. As a single mode assessment yields a single weight for each weight expression, the origins of the weights obtained are straightforward to verify and explain. Hence, the interpretations of the weights ease up the elicitation in RNM in a similar way as the interpretations of criteria weights ease up the elicitation in the context of multi-criteria decision analysis. Moreover, for an expert involved in the daily work with a phenomenon or a system to be described as a BN, the consideration of the mode of the child node in various scenarios may be a familiar form of reasoning. This can decrease the cognitive strain posed on the expert in the elicitation. It should also be noted that because mode assessments are given as intervals, group elicitation is also enabled by forming the intervals based on the views of several experts.

References
